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GRADE 12 EXAMINATION
NOVEMBER 2019

**ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA**

MARKING GUIDELINES

Time: 2 hours

200 marks

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QUESTION 1

1.1 Solve for $x \in \mathbb{R}$ without using a calculator and showing all working:

(a) Solve $|x^2 - 12| = x$
 $\therefore x^2 - 12 = x$ or $x^2 - 12 = -x$
 $\therefore x^2 - x - 12 = 0$ or $x^2 + x - 12 = 0$
 $\therefore (x - 4)(x + 3) = 0$ or $(x + 4)(x - 3) = 0$
 $\therefore x = 4$ or -3 or -4 or 3
A check reveals $x = 4$ or 3

(b) $e^x + 12e^{-x} = 8$
 $\therefore e^{2x} - 8e^x + 12 = 0$
 $\therefore (e^x - 2)(e^x - 6) = 0$
 $\therefore e^x = 2$ or $e^x = 6$
 $\therefore x = \ln 2$ or $x = \ln 6$
 $\therefore x = 0,693$ or $1,792$

1.2 If $z = a + bi$ and $z^2 = 23 - 6z$ then find all possible values of a and b .

$$(a + bi)^2 = 23 - 6(a + bi)$$

$$\therefore a^2 + 2abi + b^2i^2 = 23 - 6a - 6bi$$

$$\therefore (a^2 - b^2) + (2ab)i = (23 - 6a) - (6b)i$$

$$\therefore 2ab = -6b$$

$$\therefore a = -3 \text{ or } b = 0$$

also $(a^2 - b^2) = 23 - 6a$

so, if $a = -3$ then $9 - b^2 = 42$

$$\therefore b^2 = -32 \text{ (not possible since } b \text{ is real)}$$

$$\therefore b = 0$$

$$\therefore a^2 = 23 - 6a$$

$$\therefore a = -3 + 4\sqrt{2}$$

$$\therefore a = 2,66 \text{ or } -8,66 \text{ and } b = 0$$

1.3 Solve $f(x) = x^4 + x^3 - 2x^2 + 2x + 4 = 0$ in $\mathbb{C}$ if it is given that $f(1-i) = 0$

if $1-i$ is a root then so is $1+i$

so $(x-(1-i))(x-(1+i))$ is a factor

so $(x-1)^2 - i^2$ is a factor

so $(x^2 - 2x + 2)$ is a factor

by inspection :

$$x^4 + x^3 - 2x^2 + 2x + 4 = (x^2 - 2x + 2)(x^2 + 3x + 2) = (x^2 - 2x + 2)(x+1)(x+2)$$

$$\therefore x = 1-i \text{ or } 1+i \text{ or } -1 \text{ or } -2$$

QUESTION 2

Use Mathematical Induction to prove that:

$$\sum_{i=1}^n 2^i = 2^{n+1} - 2$$

we wish to prove that : $2 + 2^2 + \dots + 2^n = 2^{n+1} - 2$

first, let's consider if $n = 1$

$$LHS = 2 \text{ and } RHS = 2^2 - 2 = 2$$

so, it is true for $n = 1$

Assume it is true for $n = k$

$$\therefore 2 + 2^2 + \dots + 2^k = 2^{k+1} - 2 \quad (*)$$

adding the next term to each side gives :

$$\begin{aligned} 2 + 2^2 + \dots + 2^k + 2^{k+1} &= 2^{k+1} - 2 + 2^{k+1} \\ &= 2 \times 2^{k+1} - 2 \\ &= 2^{k+2} - 2 \\ &= 2^{(k+1)+1} - 2 \end{aligned}$$

but this is just with $n = k + 1$

so we have proved that it is true for $n = k + 1$

$\therefore$ by the principle of mathematical induction the result is true for $n \in \mathbb{N}$

QUESTION 3

Determine $f'(x)$ by first principles if $f(x) = \sqrt{x+3}$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+3+h} - \sqrt{x+3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+3+h} - \sqrt{x+3}}{h} \times \frac{\sqrt{x+3+h} + \sqrt{x+3}}{\sqrt{x+3+h} + \sqrt{x+3}} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+3+h} + \sqrt{x+3})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+3+h} + \sqrt{x+3}} \\
 &= \frac{1}{2\sqrt{x+3}}
 \end{aligned}$$

QUESTION 4

4.1 Consider the function: $f(x) = \frac{x^2 + bx - 6}{2x - a}$

Determine the values of a and b if the function has a vertical asymptote at $x = 4$ and an oblique asymptote of $y = \frac{1}{2}x + 4$

For a vertical asymptote at $x = 4$ the denominator must be zero when $x = 4$ so $a = 8$

Now,

$$f(x) = (2x - 8)\left(\frac{1}{2}x + 4\right) + \text{rem (where rem is a constant)}$$

$$\therefore f(x) = x^2 + 4x - 32 + \text{rem}$$

$$\text{so, } b = 4$$

4.2 Determine the values of a and b if the function $f(x) = \frac{x^2 + ax + b}{2x - 3}$ has a stationary point at $(1; 2)$

we know that $f(1) = 2$ and $f'(1) = 0$

$$\text{so } \frac{x^2 + ax + b}{2x - 3} = \frac{1 + a + b}{-1} = 2 \text{ or } a + b = -3$$

$$\therefore f'(x) = \frac{(2x + a)(2x - 3) - 2(x^2 + ax + b)}{(2x - 3)^2}$$

$$\text{now } f'(1) = (2 + a)(-1) - 2(1 + a + b) = 0$$

substituting the value of $a + b$ gives :

$$-2 - a - 2(1 - 3) = 0$$

$$\therefore a = -2 - 2(-2) = 2 \text{ and } b = -5$$

QUESTION 5

Consider the function f defined as follows:

$$f(x) = \begin{cases} 0.5x + 4 & x < -4 \\ 3 & -4 \leq x < -2 \\ 2 & x = -2 \\ 0.5x^2 + 1 & -2 < x < 2 \\ g(x) & x \geq 2 \end{cases}$$

Answer the following questions paying careful attention to the **notation** you use:

5.1 Determine $\lim_{x \rightarrow -4} f(x)$ if it exists. If not, explain why.

$$\lim_{x \rightarrow -4^-} f(x) = 2 \text{ but } \lim_{x \rightarrow -4^+} f(x) = 3$$

$\therefore \lim_{x \rightarrow -4} f(x)$ d.n.e. since they are unequal

5.2 Why is f discontinuous at $x = -2$?

$$\lim_{x \rightarrow -2} f(x) = 3 \text{ but } f(-2) = 2$$

$\therefore$ discontinuous since they are unequal

5.3 What type of discontinuity occurs at $x = -2$?

Removable

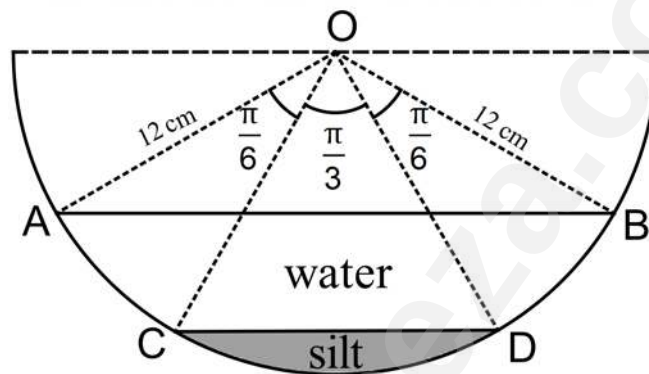
5.4 Determine $g(x)$ if $g(x)$ is a **linear function** and f is to be differentiable at $x = 2$.

$$\begin{aligned} \text{we need } g(2) &= \lim_{x \rightarrow 2^-} f(x) = 3 \\ \text{but we also need } \lim_{x \rightarrow 2^+} g'(x) &= \lim_{x \rightarrow 2^-} f'(x) = \lim_{x \rightarrow 2^-} x = 2 \\ \text{so, } g(x) &= 2x + c \\ \text{but } g(2) &= 2(2) + c = 3 \\ \text{so } c &= -1 \\ \therefore g(x) &= 2x - 1 \end{aligned}$$

QUESTION 6

Consider the diagram below. It represents the cross-section of a semi-circular gutter with O the centre of the semi-circle. There is silt at the bottom of the gutter. The surface area of the silt CD is parallel to the surface area of the water AB. Angles in radians are as shown. If the radius of the gutter is 12 cm and the gutter is 2 m long then determine the volume of water in the gutter to the nearest litre.

Remember: $1 \text{ cm}^3 = 1 \text{ ml}$ and $1 \text{ litre} = 1000 \text{ ml}$.



Area of water = minor segment AB – minor segment CD

Now minor segment AB = sector AOB – $\triangle AOB$

$$= \frac{1}{2} 12^2 \frac{2\pi}{3} - \frac{1}{2} 12^2 \sin \frac{2\pi}{3}$$

$$= 48\pi - \frac{72\sqrt{3}}{2}$$

and minor segment CD = sector OCD – $\triangle OCD$

$$= \frac{1}{2} 12^2 \frac{\pi}{3} - \frac{1}{2} 12^2 \sin \frac{\pi}{3}$$

$$= \frac{72\pi}{3} - \frac{72\sqrt{3}}{2}$$

$$\text{So, area of water} = 48\pi - \frac{72\sqrt{3}}{2} - \left(\frac{72\pi}{3} - \frac{72\sqrt{3}}{2} \right)$$

$$= 24\pi \text{ cm}^2$$

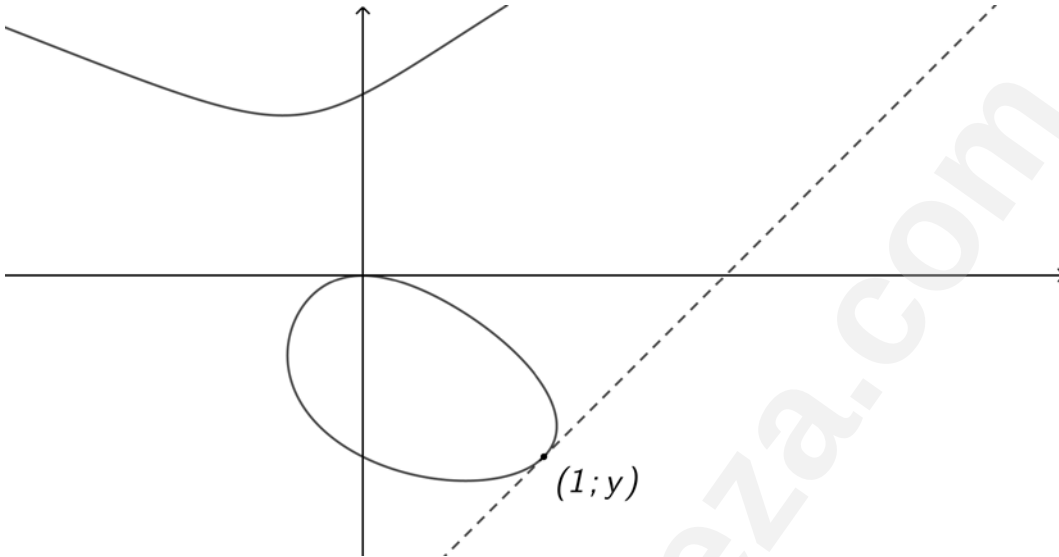
$$\text{So, volume} = 200 \times 24\pi \text{ cm}^3$$

$$= 15079,6 \text{ cm}^3$$

$$= 15 \text{ litres to the nearest litre}$$

QUESTION 7

Below is the graph of the implicitly defined relationship: $y^3 - xy = y + x^2$.



Find the equation of the tangent (indicated with a dotted line) if it is known that the x-coordinate of the point of contact is 1.

$$y^3 - xy = y + x^2$$

$$\text{When } x = 1, y^3 - y = y + 1$$

$$\therefore y^3 - 2y - 1 = 0$$

$$\therefore y = -1$$

$$\text{so, the pt. of contact is } (1; -1)$$

$$\therefore 3y^2 \frac{dy}{dx} - \left(y + x \frac{dy}{dx} \right) = \frac{dy}{dx} + 2x$$

$$\therefore 3y^2 \frac{dy}{dx} - y - x \frac{dy}{dx} = \frac{dy}{dx} + 2x$$

$$\therefore \frac{dy}{dx} (3y^2 - x - 1) = y + 2x$$

$$\therefore \frac{dy}{dx} = \frac{y + 2x}{3y^2 - x - 1}$$

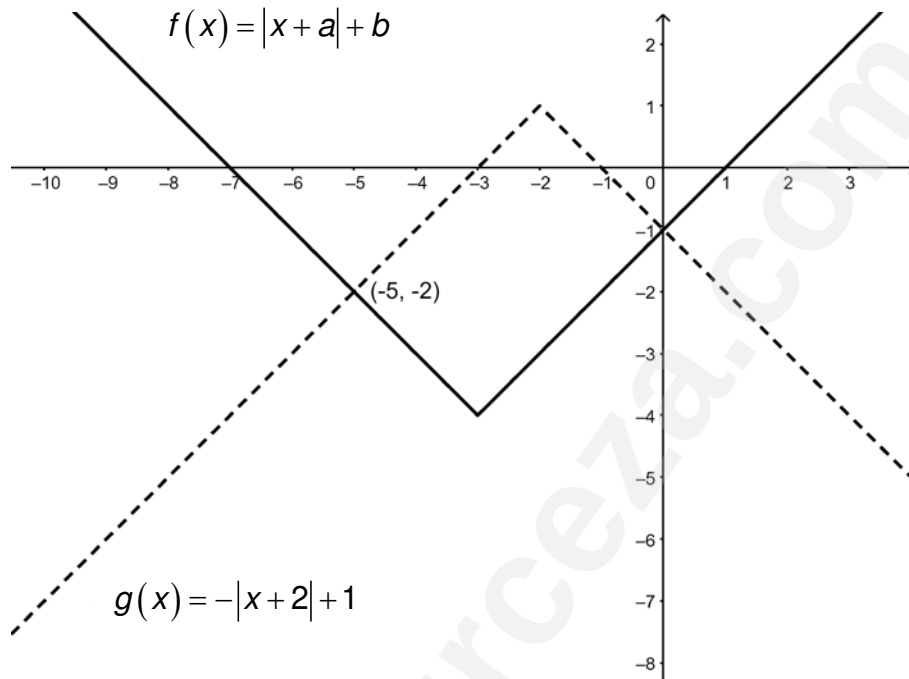
$$\therefore \frac{dy}{dx} = \frac{-1 + 2}{3 - 1 - 1} = 1$$

$$\therefore y - (-1) = 1(x - 1)$$

$$\therefore y = x - 2$$

QUESTION 8

8.1 Consider the functions $f(x) = |x + a| + b$ and $g(x) = -|x + 2| + 1$



(a) Determine the values of a and b

$$a = 3 \text{ and } b = -4$$

(b) Hence or otherwise solve:

$$|x + 3| + |x + 2| \leq 5.$$

$$|x + 3| + |x + 2| \leq 5$$

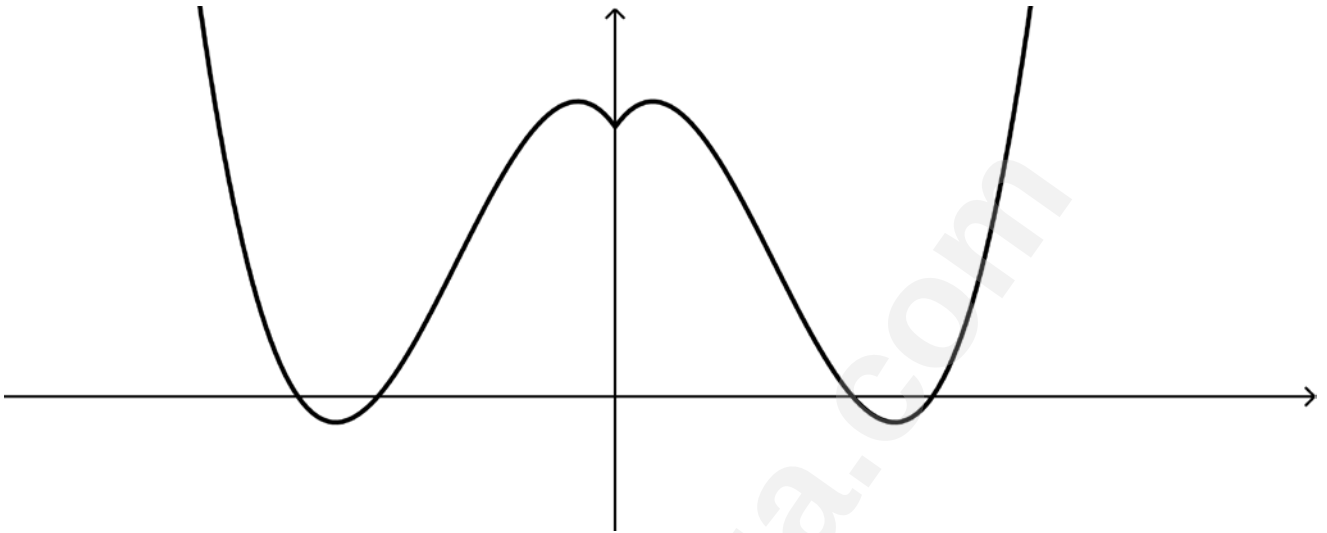
$$\therefore |x + 2| \leq 5 - |x + 3|$$

$$\therefore -|x + 2| \geq -5 + |x + 3|$$

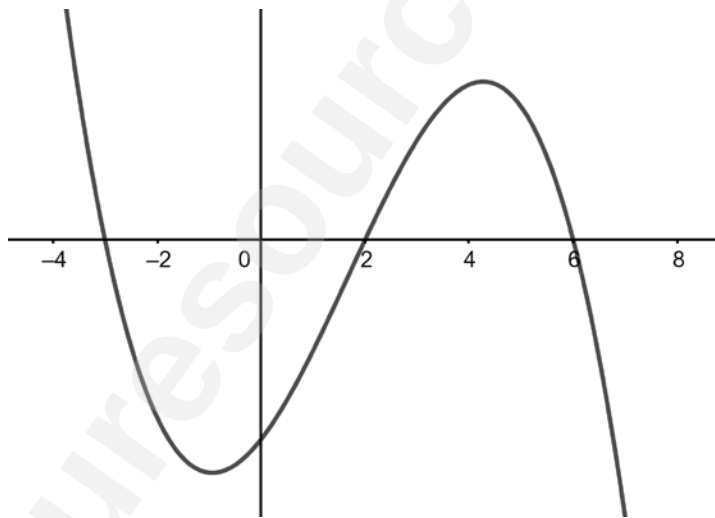
$$\therefore -|x + 2| + 1 \geq |x + 3| - 4$$

$$\therefore -5 \leq x \leq 0$$

- 8.2 Given the graph $y = f(x)$, draw on your own set of axes in your Answer Book a rough sketch of $y = f(|x|)$.



- 8.3 Consider the function, f drawn below.



Given that $\int_0^2 f(x) dx = -38,7$ and $\int_2^6 f(x) dx = 74,7$ determine:

(a) $\int_0^6 f(x) dx = -38,7 + 74,7 = 36$

(b) $\int_0^6 |f(x)| dx = 38,7 + 74,7 = 113,4$

QUESTION 9

Consider the function $f(x) = x \ln(x) - \sqrt{x^2 + 4}$, $x > 0$

- 9.1 Given that f is continuous at every value in its domain, justify why f has at least one root on the interval $x \in [1; 5]$

$$f(1) = -\sqrt{5}$$

$$f(5) = 2,6$$

since $f(1) < 0$ and $f(5) > 0$

and f is continuous on $[1; 5]$

f must cross the x -axis at least once

on $[1; 5]$ so there is at least one root on $[1; 5]$

- 9.2 Use Newton-Raphson iteration to find this root. You should:

- use an initial guess of 1
- show the iterative formula you use
- show your first two approximations
- give your answer to 5 d.p.

$$f(x) = x \ln(x) - \sqrt{x^2 + 4}$$

$$\therefore f'(x) = \ln(x) + x \left(\frac{1}{x} \right) - \frac{1}{2} (x^2 + 4)^{-\frac{1}{2}} (2x)$$

$$\therefore f'(x) = \ln(x) + 1 - x(x^2 + 4)^{-\frac{1}{2}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n \ln(x_n) - \sqrt{x_n^2 + 4}}{\ln(x_n) + 1 - x_n(x_n^2 + 4)^{-\frac{1}{2}}}$$

$$x_0 = 1$$

$$x_1 = 5,045085...$$

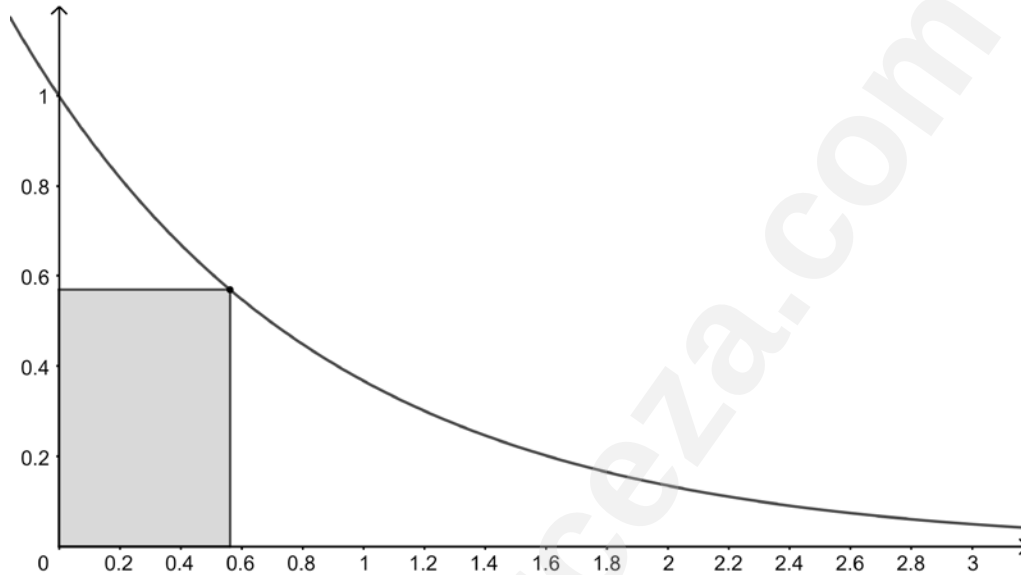
$$x_2 = 3,423819...$$

$$x \approx 3,23903 \text{ (to 5 d.p.)}$$

QUESTION 10

Consider the diagram below where rectangles are being formed in the first quadrant. The bottom left corner is on the origin while the top right corner is on the curve $y = e^{-x}$

Find, to 3 decimal places, the maximum area which can be achieved in this way.



$$A = xe^{-x}$$

$$\therefore \frac{dA}{dx} = e^{-x} - xe^{-x} = 0$$

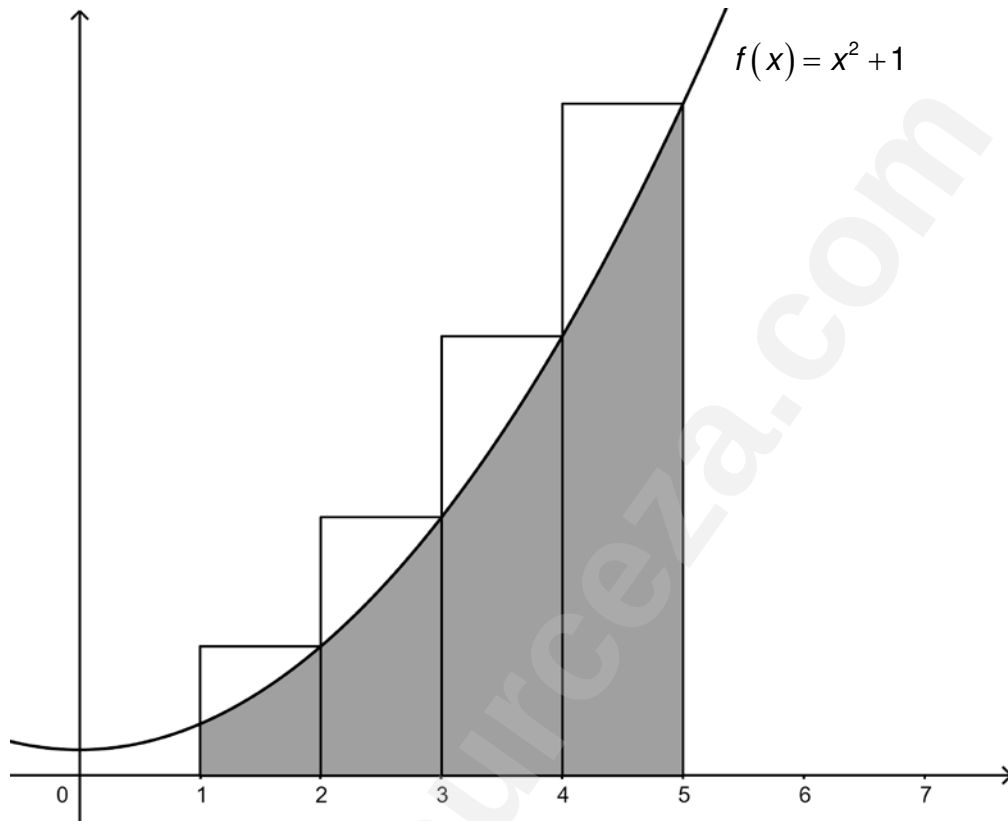
$$\therefore e^{-x}(1-x) = 0$$

$$\therefore x = 1$$

$$\therefore A_{\max} = e^{-1} = 0,368 \text{ units}^2$$

QUESTION 11

11.1 Robyn is using rectangles to estimate the shaded area. Calculate her percentage error to one decimal place.



Robyn's estimate is $(1 \times 5) + (1 \times 10) + (1 \times 17) + (1 \times 26) = 58 \text{ units}^2$

$$\text{exact answer} = \int_1^5 x^2 + 1 \, dx = \frac{136}{3} \text{ or } 45\frac{1}{3} \text{ units}^2$$

$$\text{percentage error is } \frac{58}{45\frac{1}{3}} \times 100 = 27,9\%$$

11.2 (a) Resolve $\frac{3x^2 + 11x - 5}{x^3 + 3x^2 - 4}$ into partial fractions.

$$\begin{aligned}\frac{3x^2 + 11x - 5}{x^3 + 3x^2 - 4} &= \frac{3x^2 + 11x - 5}{(x-1)(x^2 + 4x + 4)} \\ &= \frac{3x^2 + 11x - 5}{(x-1)(x+2)^2} \\ &= \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\end{aligned}$$

$A = 1$ by cover up method

$$\text{so } 1(x^2 + 4x + 4) + B(x-1)(x+2) + C(x-1) = 3x^2 + 11x - 5 \quad \text{adding}$$

$$\begin{aligned}x^2 + 4x + 4 + Bx^2 + Bx - 2B + Cx - C &= 3x^2 + 11x - 5 \\ &= (1+B)x^2 + (4+B+C)x + (4-2B-C)\end{aligned}$$

so, $B = 2$ and $C = 5$

$$= \frac{1}{x-1} + \frac{2}{x+2} + \frac{5}{(x+2)^2}$$

(b) Hence, or otherwise, determine $\int \frac{3x^2 + 11x - 5}{x^3 + 3x^2 - 4} dx$

$$\begin{aligned}\int \frac{3x^2 + 11x - 5}{x^3 + 3x^2 - 4} dx &= \int \frac{1}{x-1} + \frac{2}{x+2} + \frac{5}{(x+2)^2} dx \\ &= \ln|x-1| + 2\ln|x+2| - \frac{5}{(x+2)} + c\end{aligned}$$

11.3 Determine $\int xe^{2x} dx$

$$\text{let } f(x) = x \text{ then } f'(x) = 1$$

$$\text{let } g'(x) = e^{2x} \text{ then } g(x) = \frac{1}{2}e^{2x}$$

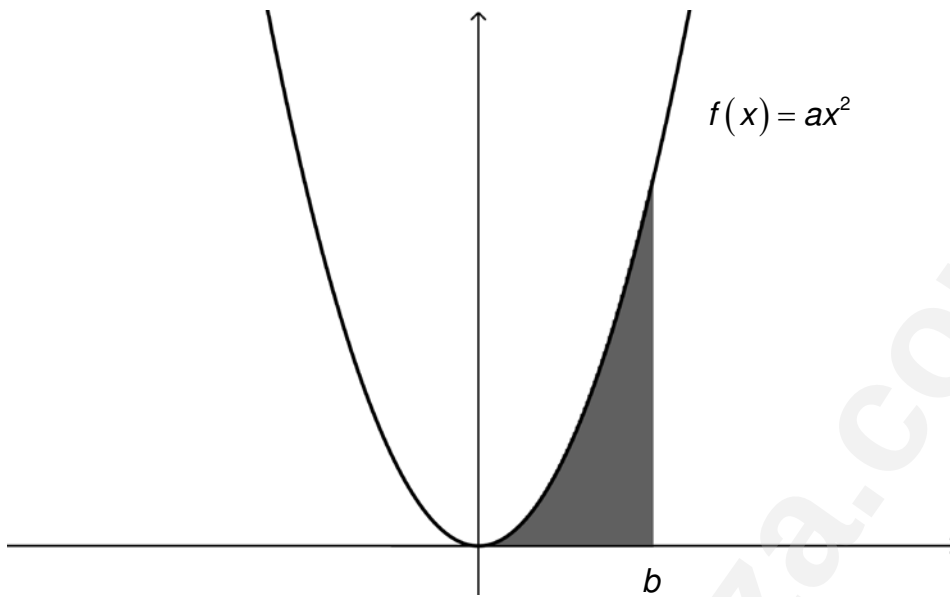
$$\begin{aligned}\text{then } \int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \frac{1}{2} \int e^{2x} dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c\end{aligned}$$

11.4 Determine $\int \operatorname{cosec}^2 x \cot^2 x dx$

$$= -\frac{\cot^3 x}{3} + c$$

QUESTION 12

Consider the diagram below.



The area bounded by the function f , the x -axis, the lines $x = 0$ and $x = b$ is $\frac{160}{3} \text{ units}^2$.

When this area is rotated around the x -axis the resulting volume is $1280\pi \text{ units}^3$.

Determine the values of a and b .

$$\int_0^b ax^2 \, dx = \frac{160}{3}$$

$$\therefore \left[\frac{ax^3}{3} \right]_0^b = \frac{160}{3}$$

$$\therefore \frac{ab^3}{3} - 0 = \frac{160}{3}$$

$$\therefore ab^3 = 160 \quad (1)$$

$$\pi \int_0^b (ax^2)^2 \, dx = 1280\pi$$

$$\therefore \left[\frac{a^2 x^5}{5} \right]_0^b = 1280$$

$$\therefore \frac{a^2 b^5}{5} = 1280$$

$$\therefore a^2 b^5 = 6400 \quad (2)$$

now, squaring both sides of equation (1) gives $a^2 b^6 = 25600 \quad (3)$

dividing equation (3) by equation (2) gives $b = 4$

substituting this value into (1) gives $a = \frac{5}{2}$

Total: 200 marks