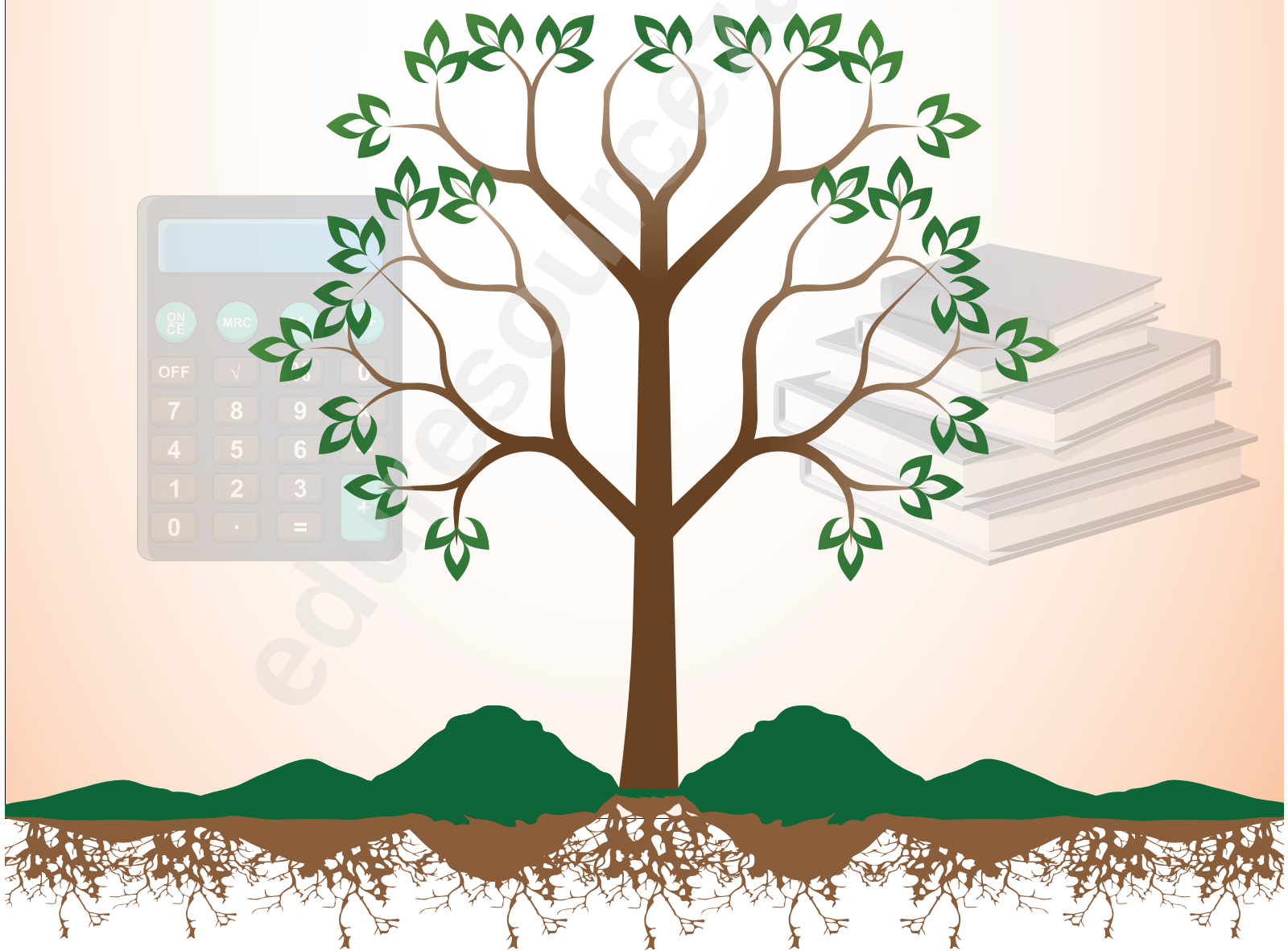




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MATHEMATICS
TRAINING MATERIAL



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NSC SUPPORT PROGRAMME

MATHEMATICS

2019 - 2021

NICPD CHIEF DIRECTORATE

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NSC Support Programme

Mathematics and Physical Science

Introduction

In the national spirit and thrust to improve performance in Mathematics, Science and Technology, the DBE has embarked on a range of programmes, two of which are (i) the Mathematics Indaba, and (ii) Intervention in twelve (12) identified districts. The former started in Dec 2016 to explore various causes of poor participation and performance in Mathematics and to propose possible strategies to remedy the situation. The latter on the other hand, focused on content and pedagogy teacher development programmes for districts and teachers that need more support in Mathematics and Physical Science.

Both DBE-initiated programmes resonate with the 2013 UNESCO Roadmap for implementing Education for Sustainable Development. The Roadmap lifts up five priority action areas, three of which are restated herein to shed more light on the programme for ten districts. The priority action areas are:

- (i) Transforming learning and training environments: *integrating sustainability principles into education and training settings*
- (ii) Building capacities of educators and trainers: *increase the capacity of educators to more effectively deliver ESD*
- (iii) Accelerating sustainable solutions at local levels: *at community level, scale up programmes and networks*

The profound intentions of the UNESCO roadmap and the DBE interventions interweave so pleasantly towards the same end: capacity for effectiveness. The subject specialists at ground level have worked together to produce the Training Manuals in Mathematics and Physical Science that will be used initially in ten identified districts, and then by any other districts. We want to express deep-felt gratitude to these subject specialists, who generously offered their time and expertise in the development of these Training Manuals. It is further worth stating the names of the writers of these manuals:

NO	MATHEMATICS	PROVINC E	PHYSICAL SCIENCE	PROVINCE
1	Mr Steven Muthige	LP	Ms Lerato Mphachoe	MP
2	Ms Nonhlanhla Mthembu	KZN	Ms Pfuluwani Mashapa	GP
3	Ms Thandi Mgwenya	MP	Mr Mthandeni Mkhize	KZN
4	Mr. Mlungiseleli Njomeni	EC	Mr. Mosala Leburu	NW
5	Mr LB Mashiane	FS	Mr LH makinta	LP
6	Mr STP Mokoale	NW	Ms MS Mangope	NW
7	Mr TE Mphahama	FS	Mr. Israel Hlatshwayo	MP
8	Mr SM Tebeila	GP	Mr NP Mukwevho	LP
9	Mr Leonard Mudau	DBE	Mr KL Motswadira	NW
			GME Monyatsi	NW

This Manual has been submitted to SACE for endorsement, and can then be used by all provinces, districts, and schools towards building teacher capacities, competencies and confidence. It is important to note that this is not a textbook, and therefore the format and the purpose are different from that of a textbook. Further, the ideas to further strengthen and improve the concepts are welcomed and can be sent to email address given hereunder.

Wishing all the trainers, teachers and other users of this training manual all the best. We earnestly hope that the manual will remain a valuable user-friendly tool in enhancing the requisite mathematical and scientific competencies for subject advisors and classroom-based educators.



Mr. Phillip K Dikgomo
Director: Teacher Development Implementation
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Department of Basic Education
January 2019

1. STATISTICS

Statistics is the branch of Mathematics dealing with collection, organization, presentation, analysis, and interpretation of data.

Data can either be quantitative or qualitative.

- Quantitative data is information about quantities; that is, information that can be written down with numbers.
- Qualitative data is information about qualities; information that cannot be measured and represented in the form of numbers

Activity 1

Give three examples of qualitative and quantitative data.

Two main statistical methods are used in data analysis: descriptive statistics (which summarize data from a sample using indexes such as the mean or standard deviation, and inferential statistics (which draws conclusions from data that are subject to random variation, e.g. observational errors, sampling variation).

N.B. that data can either be discrete or continuous.

- Discrete data is counted (number of learners in a class, number of people in the household)
- Continuous data is measured (height of a person, time in a race)

Activity 2

Indicate whether the data below is discrete or continuous:

- 1 Distance travelled by a learner to school
- 2 Time taken by person to prepare breakfast
- 3 Length of a leaf
- 4 Speed of the train
- 5 Number of languages a person can speak

Activity 3

Give two other examples discrete and continuous data

UNGROUPEd DATA

Ungrouped data is in the form of number list and it is raw. We simply say ungrouped data is an array of number, e.g. ages of learners in a class, marks of learners in a test etc. Ungrouped data can be represented in different forms, e.g. frequency tables, bar graphs, pie charts, stem-and-leaf, etc.

Data below represents marks (out of 50) of learners in a Mathematics Test:

17	38	10	17	44	26	20	22	9	17	20	38	10
22	20	9	44	38	17	10	22	20	9	44	5	

Marks of learners in a frequency table

Marks	Tally	Frequency
5		1
9		3
10		3
17		4
20		4
22		3
26		1
38		3
44		3
		$n = 25$

Activity 4

Represent the given data in a bar graph and stem-and-leaf.

MEASURES OF CENTRAL TENDENCY

A measure of central tendency (also referred to as measures of centre or central location) is a summary measure that attempts to describe a whole set of data with a single value that represents the middle or centre of its distribution.

There are three main measures of central tendency: mode, median and mean.

MODE

Mode is the most commonly occurring value/item in a distribution.

Advantage of **mode**:

The **mode** has the advantage over the median and mean as it can be found in both quantitative and qualitative data.

Limitation of **mode**:

- There can be more than one mode in the distribution, thus one single measure of central tendency cannot be identified.

MEDIAN

The median is the middle value in the distribution when the values are arranged in ascending or descending order. The median divides the distribution in half.

Make use of only ascending order when teaching.

Advantage of **median**:

Median is less affected by outliers than the mean.

An outlier is an extremely high or extremely low value in a data. The outlier can be identified if it is greater than $Q_3 + 1.5(IQR)$ or lower than $Q_1 - 1.5(IQR)$

Limitation of **median**:

Median cannot be identified for categorical nominal data as it cannot be logically ordered.

$$median_{pos} = \frac{n+1}{2}$$

Use the median position to determine the estimated median.

MEAN

Mean is the sum of the values of each observation in data set divided by the number of observations. This also known as the arithmetic average. **Avoid using the word average when referring to mean as the word is vague. Average could mean arithmetic mean, the median, or mode.**

Advantage of **mean**:

The mean can be used for both continuous and discrete numeric data.

Limitations of the **mean**:

- The mean cannot be calculated for categorical data, as the values cannot be summed.
- The mean is influenced by the outliers and skewness of data.

Sample mean: $\bar{x} = \frac{\sum x}{n}$, where:

- $\bar{x}$ is the mean
- $\sum x$ is the sum of the data values
- n is a sample space

MEASURES OF DISPERSION

In statistics, dispersion (also called variability, scatter, or spread) is the extent to which a distribution is stretched or squeezed. Common examples of statistical dispersion are the **variance, standard deviation, and interquartile range**.

VARIANCE:

Variance is the average of the squared differences from the mean.

Take note of the following:

- If the variance is 0, then there is no variability; that is all the numbers in the data set are the same (e.g. 5 5 5 5 5 5)
- A small change in the numbers equals a very small variance (e.g. 5 5 5 5 5 6)
- A large change in the numbers equals a very large variance (5 5 5 5 10300)
- The smallest value of the variance is 0, but it can be infinite with numbers in the millions or even billions and beyond

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}$$

STANDARD DEVIATION

Standard deviation is a measure of dispersement in statistics. Dispersement just means how much your data is spread out. Specifically, it shows how much data is spread out around the mean or average.

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

INTERQUARTILE RANGE

The interquartile range (IQR) is a measure of variability, based on dividing a data set into quartiles. The values that separate parts are called first quartile (Q_1), second quartile (Q_2), and third quartile (Q_3).

Quartile are the values that divide a list of numbers into quarters.

$$IQR = Q_3 - Q_1$$

Quartiles Positions:

Lower quartile	Median	Upper quartile
Position:	Position:	Position:
$Q_1 = \frac{n+1}{4}$	$Q_2 = \frac{n+1}{2}$	$Q_3 = \frac{3(n+1)}{4}$

FIVE NUMBER SUMMARY AND BOX AND WHISKERS PLOT

The five-number summary includes five items:

- The minimum
- The lower quartile
- The median
- The upper quartile
- The maximum

For the five-number summary to exist, the data must meet the following two requirements:

- Data must be univariate
- Data must be ordinal, interval, or ratio.

BOX AND WHISKERS PLOT

The box and whiskers plot is a visual representation of the five-number summary

- $\bar{x} > Q_2$, data is positively skewed/ skewed to the right
 - $\bar{x} < Q_2$, data is negatively skewed/ skewed to the left
- Or

- $Q_3 - Q_2 > Q_2 - Q_1$ (positive skewness)
- $Q_3 - Q_2 < Q_2 - Q_1$ (negative skewness)

Worked example:

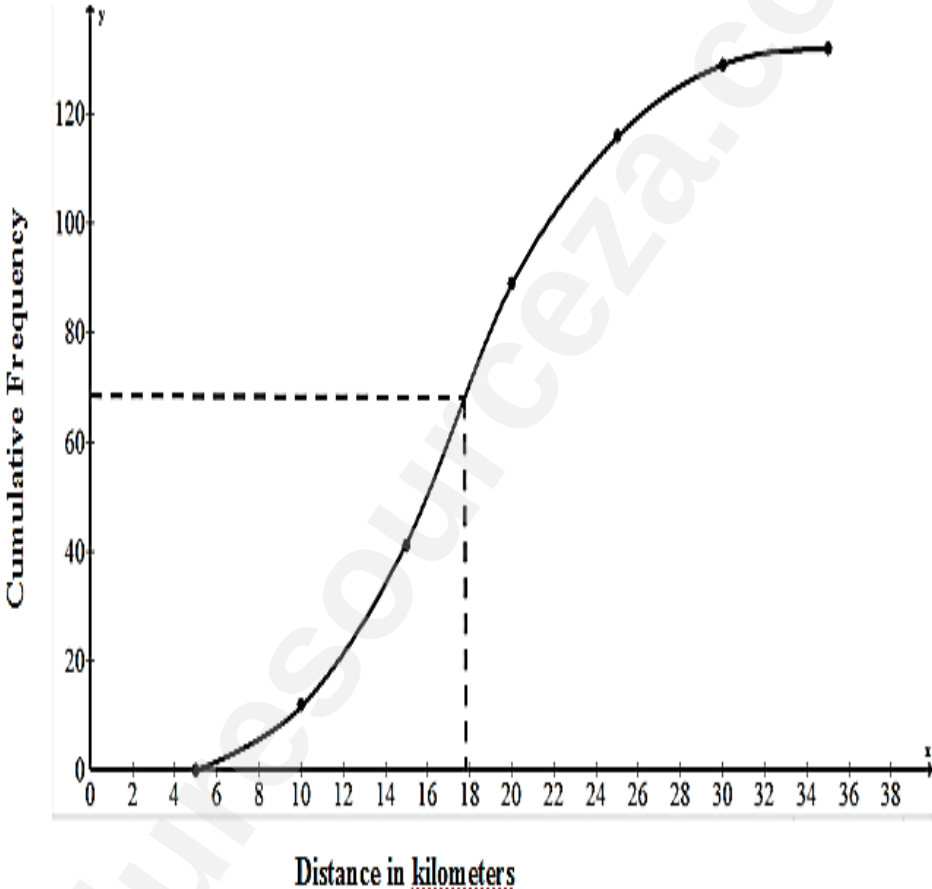
QUESTION: 1

The ordered data below lists the fees (in thousands of Rands) paid by 30 students for their first year of study at university.

11	11,9	12	12,52	13,125	13,125	13,15	13,2	13,25	13,25
13,25	13,25	13,4	13,5	13,7	13,7	14	14,2	14,2	14,5
14,5	15	15,5	15,6	16,1	17,3	17,5	17,8	19	19,7

- 1.1 Calculate the mean and standard deviation of the given data. (4)
- 1.2 Complete the cumulative frequency column in the table provided. (3)
- 1.3 Draw an Ogive (cumulative frequency curve) of the given data on the grid provided. (5)
- 1.4 Draw the box and whisker diagram for the given data. (3)

1.5 Comment on the skewness of the data. (1)

1.1	$\bar{x} = \frac{2355}{132}$ $= 17,84$	✓ 2355 ✓ 132 ✓ 17,84 (3)
1.2	$\delta x = 5,88$	✓ ✓ answer (2)
1.3		✓ plotting of points using upper limits/ ✓ using cumulative values/ ✓ groundin g at/begin by(5;0) (3)
1.4	Median/ mediaan = 17,8	✓ Accept between 17,5 and 18,5 ✓ shown on graph

1.5	$\bar{x} \cong \text{median}$ More or less symmetrical.	$\checkmark \bar{x} \cong \text{median/median}$ CA answers CA $\checkmark$ answer
-----	--	--

ACTIVITY 5

QUESTION 1

At a certain school, 15 learners are taking Mathematics and Physical Sciences. The marks (in %) of the learners in June examinations are recorded in the table below:

Mathematics	41	96	60	29	43	36	58	60	45	39	40	55	42	33	30
P. Sciences	60	62	88	90	72	89	67	75	52	60	48	57	44	70	92

- 1.1 Determine the mean of Physical Sciences marks.
- 1.2 Write down the mode of Physical Sciences marks.
- 1.3 Which measure of central tendency is not best to use to calculate the average of Mathematics marks. Substantiate your answer.
- 1.4 Calculate the median position of Mathematics marks, hence write down the median.
- 1.5 Determine the standard deviation of both Mathematics and Physical Sciences marks.
- 1.6 Explain what will happen to mean and standard deviation if the marks of Physical Sciences are increased by 5.
- 1.7 Draw a box-and whiskers plot for Mathematics marks.
- 1.8 How many Mathematics learners scored marks that fall within one standard deviation from the mean?

Question 2

May/ June 2016

On a certain day, a tour operator sent 11 tour buses to 11 different destinations. The table below

shows the number of passengers on each bus.

8	8	10	12	16	19	20	21	24	25	26
---	---	----	----	----	----	----	----	----	----	----

- 2.1 Calculate the mean number of passengers travelling in a tour bus.
- 2.2 Write down the five-number summary of the data
- 2.3 Draw a box and whisker diagram for the data.
- 2.4 Refer to the box and whisker diagram and comment on the skewness of the data set.
- 2.5 Calculate the standard deviation of this data set.
- 2.6 A tour is regarded as popular if the number of passengers on a tour bus is one standard deviation above the mean. How many destinations were popular on this particular day?

Question 3

June/ July 2015

The data below shows the ages in years of people who visited the library between 08:00 and 09:00 on a certain morning.

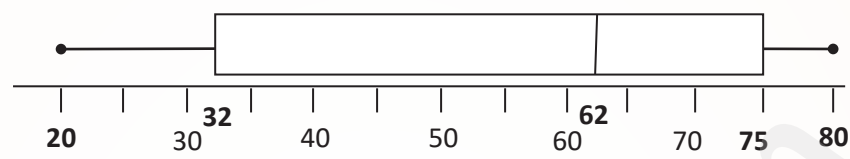
3	4	4	5	23	29	32	36	40	47	56	66	68	76	82
---	---	---	---	----	----	----	----	----	----	----	----	----	----	----

- 3.1 Determine:
 - 3.1.2 The mean of the visitors
 - 3.1.2 The median of the data
 - 3.1.3 The interquartile range of the data
- 3.2 Draw a box and whisker diagram for the data
- 3.3 By making reference to the box and whisker diagram, comment on the skewness of the data set.

4.1 Question 4

Feb/March 2016

The box and whisker diagram below shows the marks (out of 80) obtained in a History test by a class of nine learners.



4.1.1 Comment on the skewness of the data.

4.1.2 Write down the range of the marks obtained.

4.1.3 If the learners had to obtain 32 marks to pass the test, estimate the percentage of the class that failed the test.

4.2 In ascending order, the second mark is 28, the third mark 36 and the sixth mark 69. The seventh and eighth marks are the same. The average mark for this test is 54.

	28	36			69			
--	----	----	--	--	----	--	--	--

Fill in the marks of the remaining learners in ascending order.

Question 5

A group of 30 learners each randomly rolled two dice once and the sum of the values on the uppermost faces of the dice was recorded. The data is shown in the frequency table below.

Sum of the values	Frequency
2	0
3	3
4	2
5	4
6	4
7	8
8	3
9	2
10	2
11	1
12	1

- 5.1 Calculate the mean of the data.
- 5.2 Determine the median of the data.
- 5.3 Determine the standard deviation of the data.
- 5.4 Determine the number of times that the sum of the recorded values of the dice is within ONE standard deviation from the mean. Show your calculations.

BEAULIEU COLLEGE

QUESTION 6

Consider the following set of four positive whole numbers:

$$x + 3; 2x; x-1; 6$$

6.1 Show that the variance can be written as $\frac{1}{2}(x^2 - 6x + 15)$

6.2 Determine the standard deviation if $x = 2$

GROUPED DATA

Grouped data is data that has been grouped together in categories. Histograms, bar graphs and frequency tables are some of the tools to use to represent grouped data

MEASURES OF CENTRAL TENDENCY

About grouped data, measures of central tendency are not exact values but estimated ones.

MODE:

In a case of a mode, learners are expected to only determine the modal class and not estimated mode.

Modal class is the group/class/interval with the highest frequency.

MEDIAN:

Median position formula will help to identify the interval in which the median lies.

$$median_{pos} = \frac{n + 1}{2}$$

MEAN:

Estimated mean can be calculated using the midpoints of intervals.

$$\bar{x} = \frac{\sum fx}{\sum f}, \text{ where:}$$

- x is the class (category) midpoint
- f is the class frequency

WORKED EXAMPLE

Alex timed 21 people in the sprint race, to the nearest second. The information is represented in the frequency table below:

Second	Frequency
51-55	2
56-60	7
61-65	8
66-70	4

- 1 Write down the modal class.
- 2 Determine the estimated mean.
- 3 Determine the estimated median.

ANSWERS:

- 1 Remember that modal class is the one with the highest frequency. Therefore, the modal class is 61 - 65
- 2

Second	Frequency (<i>f</i>)	Midpoint of interval (<i>x</i>)	<i>fx</i>
51-55	2	53	106
56-60	7	58	406
61-65	8	63	504
66-70	4	68	272
	21		1288

$$\bar{x} = \frac{\sum fx}{\sum f}$$

$$\therefore \bar{x} = \frac{1288}{21} = 61.33$$

- 3 $median_{pos} = \frac{21+1}{2} = 11$, and the estimated median is in the range 61-63.

OGIVE (CUMULATIVE FREQUENCY GRAPH)

Ogive is a graphical representation of cumulative frequency.

Cumulative frequency distribution is the sum of the class and all classes below it in a frequency distribution.

DRAWING OGIVE (CUMULATIVE FREQUENCY GRAPH)

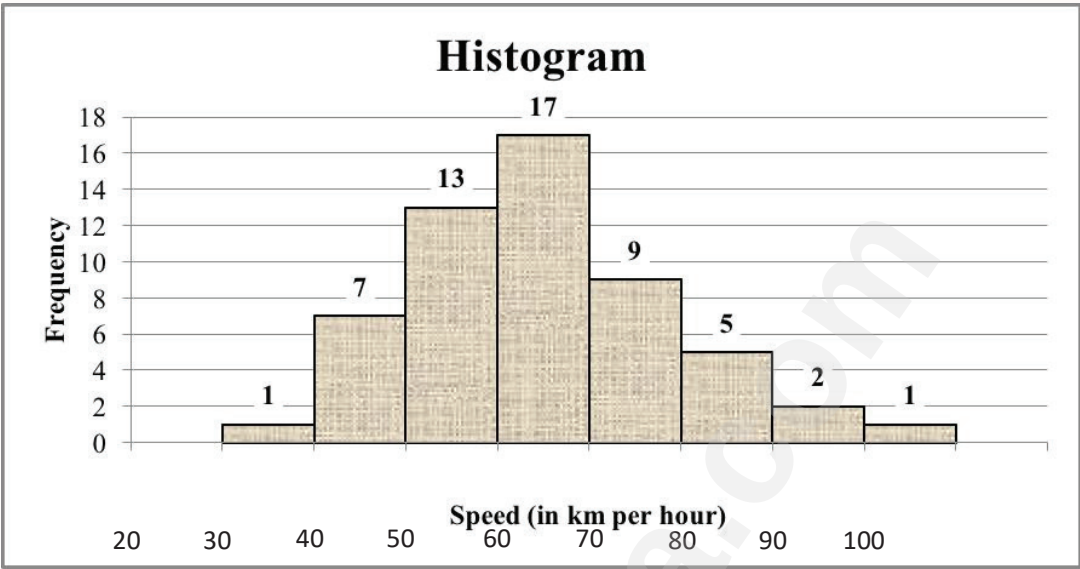
- Complete the cumulative frequency column
- Plot the points of the cumulative column on the upper limit of the intervals
- Join the points for an ogive
- Grounding on the lower limit of the first class must also be drawn.
- Please take note that interpretation of drawn ogive to determine estimated quartiles, mean, and median is very important.

WORKED ACTIVITY:

QUESTION 1

The speeds of 55 cars passing through a certain section of a road are monitored for one hour. The speed limit on this section of road is 60 km per hour. A histogram is drawn to represent

this data.



- 1.1 Identify the modal class of the data. (1)
- 1.2 Use the histogram to:
- 1.2.1 Complete the cumulative frequency column in the table on DIAGRAM SHEET 1 (2)
- 1.2.2 Draw an ogive (cumulative frequency graph) of the above data on the grid on DIAGRAM SHEET 1 (3)
- 1.3 The traffic department sends speeding fines to all motorists whose speed exceeds 66 km per hour. Estimate the number of motorists who will receive a speeding fine. (2)
- [8]

Solution

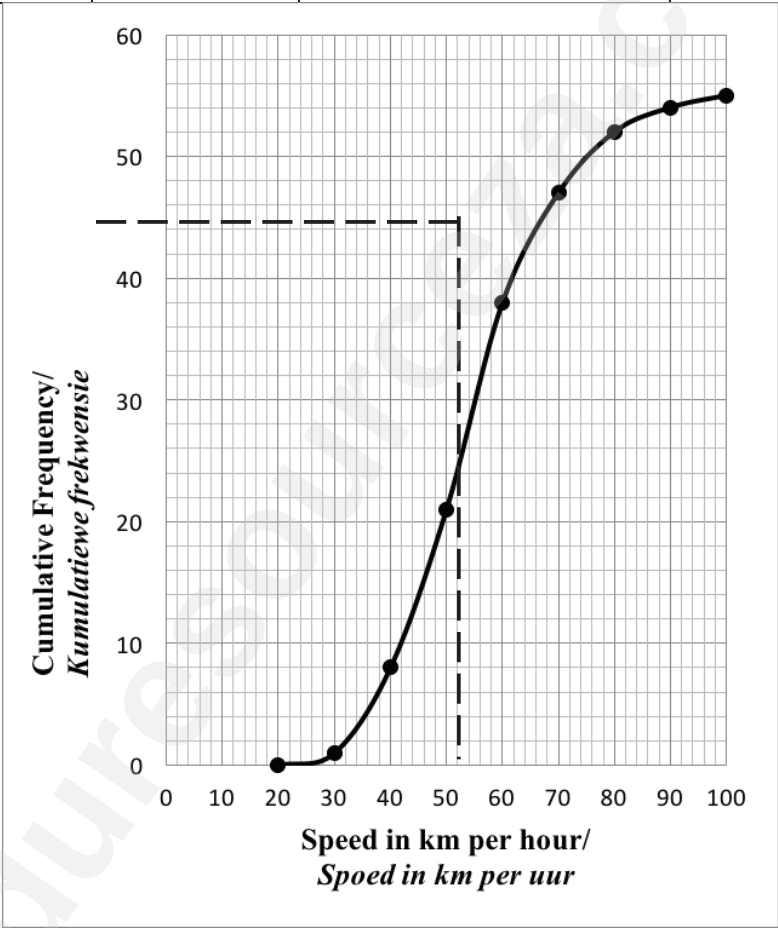
- 1.1 $50 < x \leq 60$ OR $50 \leq x < 60$ OR between 50 and 60 ✓ answer (1)

1.2.1

Class	Frequency	Cumulative frequency
$20 < x \leq 30$	1	1

$30 < x \leq 40$	7	8
$40 < x \leq 50$	13	21
$50 < x \leq 60$	17	38
$60 < x \leq 70$	9	47
$70 < x \leq 80$	5	52
$80 < x \leq 90$	2	54
$90 < x \leq 100$	1	55

1.2.2



✓✓
cumulative
frequency/
(2)

✓ grounding
at (20 ; 0)
✓ plotting at
upper limits

✓ smooth
shape
of curve

1.3 55 – 44 (accept 43 – 45)
≈ 11 motorist
(accept 10 – 12 motorist)

✓ 43
✓ answer
(2)
[8]

ACTIVITY 6

QUESTION 1

A company has 132 employees in their Polokwane branch. The distance (x) in kilometres which they travel to work each day, is summarised in the following frequency table:

Distance (in kms)	Frequency
$5 < x \leq 10$	12
$10 < x \leq 15$	29
$15 < x \leq 20$	48
$20 < x \leq 25$	27
$25 < x \leq 30$	13
$30 < x \leq 35$	3

- 1.1 Determine the estimated mean distance covered by the employees. (3)
- 1.2 Determine the standard deviation for the data, to two decimal places. (2)
- 1.3 Draw an ogive for this data on a set of axes provided in the answer book. (3)
- 1.4 Determine the median distance travelled, showing on the graph where your answer is read off. (2)
- 1.5 By referring to the relationship between the mean and median, discuss the distribution of the data. (2)
- [12]

QUESTION 2

The ordered data below lists the fees (in thousands of Rands) paid by 30 students for their first year of study at university.

11 11,9 12 12,52 13,125 13,125 13,15 13,2 13,25 13,25
13,25 13,25 13,4 13,5 13,7 13,7 14 14,2 14,2 14,5
14,5 15 15,5 15,6 16,1 17,3 17,5 17,8 19 19,7

- 2.1 Calculate the mean and standard deviation of the given data. (4)

- 2.2 Complete the cumulative frequency column in the table provided. (3)
- 2.3 Draw an Ogive (cumulative frequency curve) of the given data on the grid provided. (5)
- 2.4 Draw the box and whisker diagram for the given data. (3)
- 2.5 Comment on the skewness of the data. (1)

QUESTION 3

The ages of 500 people attending a concert are given in the table below.

Age in years	Number of people	Cumulative frequency
$0 \leq A < 10$	20	
$10 \leq A < 20$	130	
$20 \leq A < 30$	152	
$30 \leq A < 40$	92	
$40 \leq A < 60$	86	
$60 \leq A < 80$	18	
$80 \leq A < 100$	2	

- 3.1 Complete the cumulative frequency column on DIAGRAM SHEET above. (1)
- 3.2 Draw an ogive (cumulative frequency graph) of the above data. (3)
- 3.3 Use your cumulative frequency graph to estimate:

3.3.1 the median age (1)

3.3.2 the percentage of people at the concert who are 16 years and older. (3)

BIVARIATE DATA

Data for two variables (usually two types of related data).

The other variable is a dependent and the other is independent variable. It is important to identify the **dependent** and **independent variables** in case they are not indicated in the information.

Example: Ice cream sales (dependent variable) versus temperature (independent variable) on the day.

SCATTER PLOT

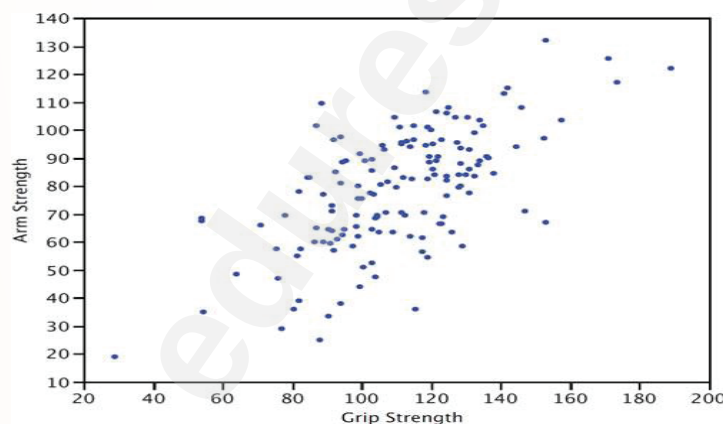
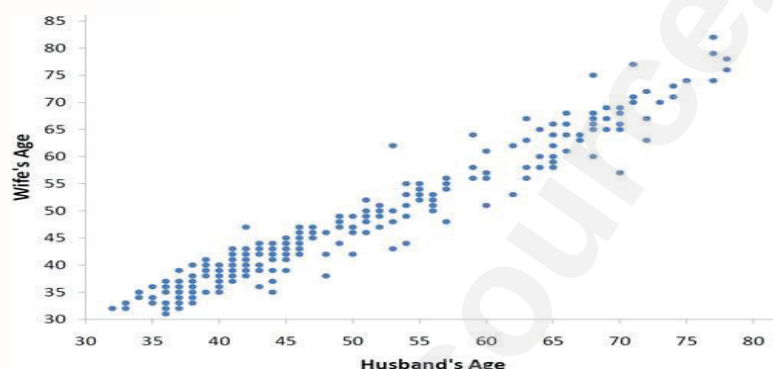
A graph in which the values of two variables are plotted along two axes, the pattern of the resulting points revealing any correlation present.

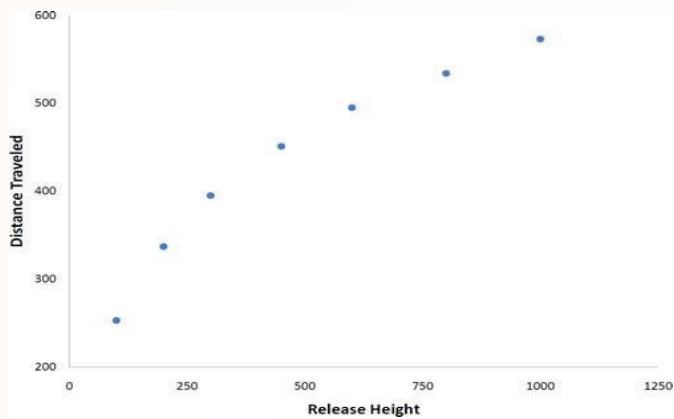
Correlation is a measure of how things are related.

Correlation coefficient is a statistical measure of the degree to which changes to the value of one variable predict change to the value of another.

- In positively correlated variables, the value increases or decreases at the same time.
- In negatively correlated variables, the value of one increases as the value of the other decreases.
- Correlation coefficient are expressed as values between +1 and -1.
- A coefficient of +1 indicates a perfect positive correlation.
- A coefficient of -1 indicates a perfect negative correlation.
- A coefficient of 0 indicates that there is no relationship between fluctuations of the variables.

Examples of scatter plots:





Note from the graphs, that not all scatter plots are linear. In this section, we will pay special attention to linear scatter plots for the sake of Regression line and correlation coefficient.

Regression line is the “best fit” line for your data. It is like an average of where all the points line up.

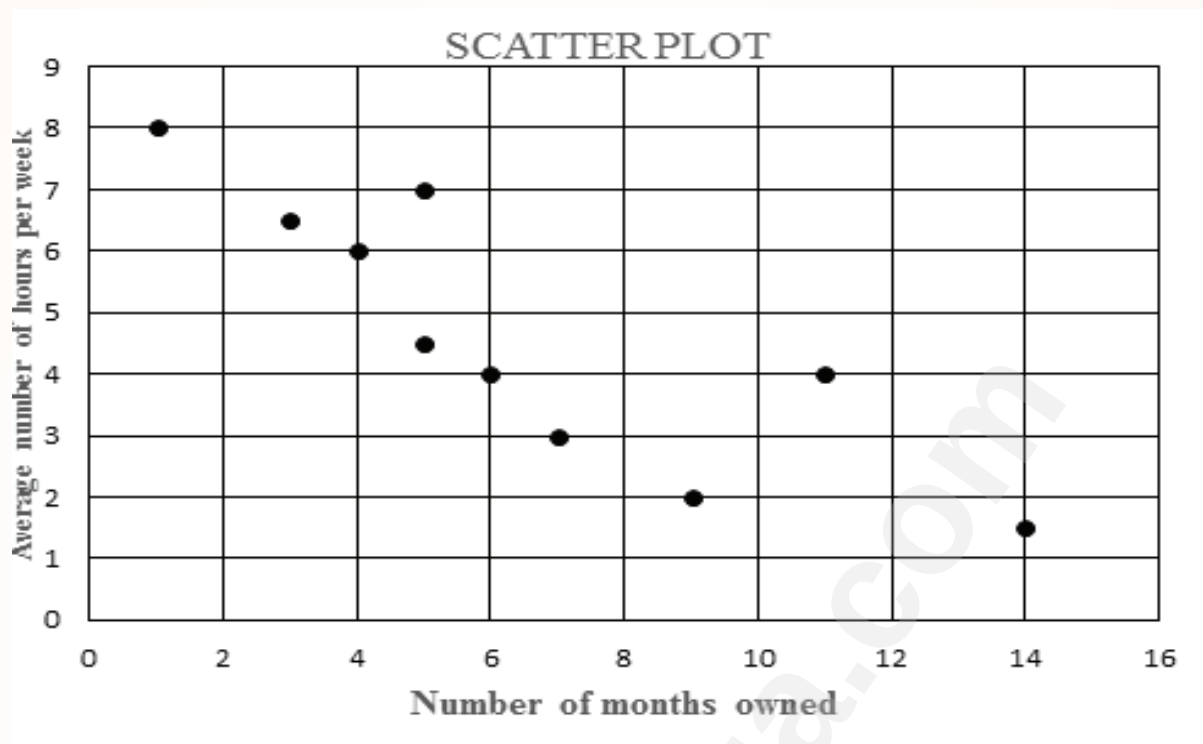
Regression equation is used in statistics to find out what relation, if any, exists between sets of data.

WORKED ACTIVITY

QUESTION: 4

A study was conducted to investigate the average number of hours per week an exercise bicycle is used in comparison to the number of months the bicycle has been owed. The data below was collected.

Number of months owned	1	5	7	4	9	14	5	11	3	6
Average number of hours per week used	8	4,5	3	6	2	1,5	7	4	6,5	4



2.1 Determine the equation of the least square regression line. (3)

2.2 Predict the number of hours the exercise bicycle will be used per week if it has been owned for 8 months. (2)

2.3 Comment on the trend of the data. (1)

[6]

SOLUTIONS

- 2.1

$y = a + bx$
 $y = 7,75 - 0,478x$

✓a
✓b
✓equation (3)
- 2.2

$y = 7,75 - 0,478(8)$
 $y = 3.93\text{hrs}$

✓subs
✓answer (2)
- 2.3

The value of r is negative so there is a strong negative linear association.

✓
- [6]

EXAM TYPE QUESTIONS

QUESTION 1

The table below shows the total fat (in grams, rounded off to the nearest whole number) and energy (in kilojoules, rounded off to the nearest 100) of 10 items that are sold at a fast-food restaurant.

Fat (in grams)	9	14	25	8	12	31	28	14	29	20
Energy (in kilojoules)	1 100	1 300	2 100	300	1 200	2 400	2 200	1 400	2 600	1 600

- 1.1 Represent the information above in a scatter plot on the grid provided in the ANSWER BOOK. (3)
- 1.2 The equation of the least squares regression line is $\hat{y} = 154,60 + 77,13x$.

1.2.1 An item at the restaurant contains 18 grams of fat. Calculate the number of kilojoules of energy that this item will provide. Give your answer rounded off to the nearest 100 kJ. (2)

1.2.2 Draw the least squares regression line on the scatter plot drawn for QUESTION 1.1. (2)
- 1.3 Identify an outlier in the data set. (1)
- 1.4 Calculate the value of the correlation coefficient. (2)
- 1.5 Comment on the strength of the relationship between the fat content and the number of kilojoules of energy. (1)

QUESTION 2

A training manager investigates the correlation between hours spent on the training of an employee (x) and consequent productivity (y units delivered per day). The data collected from the files of 10 employees was recorded in the following table:

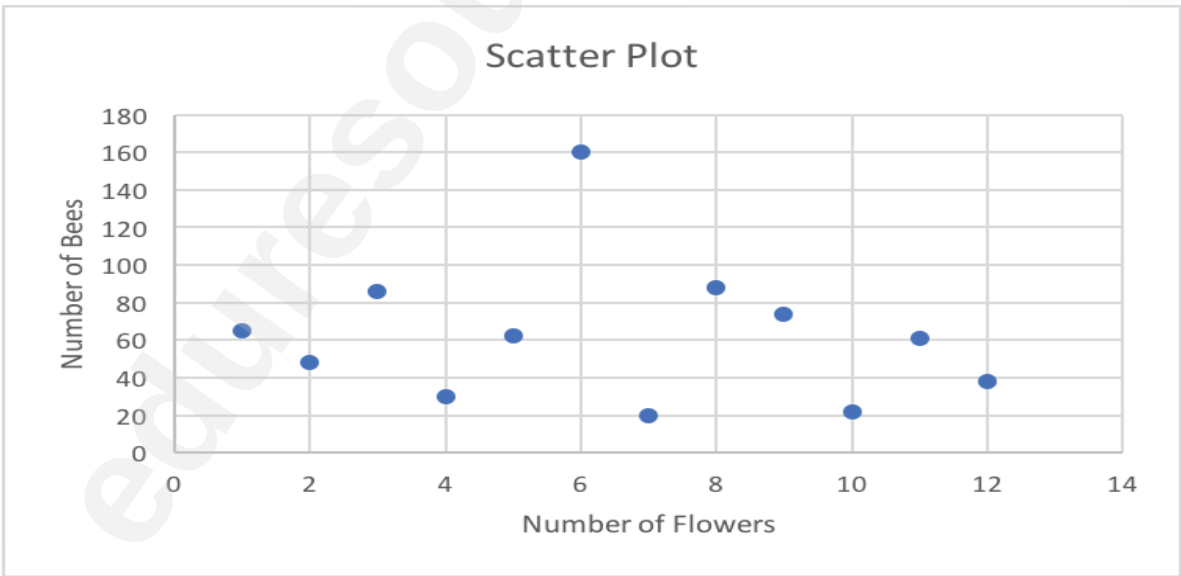
Employee	1	2	3	4	5	6	7	8	9	10
Hours training (x)	16	36	20	38	40	30	35	22	40	24
Productivity (y units delivered per day)	45	70	44	56	60	48	75	60	63	38

- 2.1 Calculate the equation of the least squares regression line. (3)
- 2.2 Calculate the correlation coefficient for the data. (2)
- 2.3 Describe the strength of the correlation. (2)
- 2.4 Use your regression line to determine the units produced by an employee who received 28 hours of training. (2)
- [9]

QUESTION 3

A survey was conducted indicating the number of bees that visited flowers over a period of 12 days. The information is represented in the table and in the scatter plot below.

No. of Flowers	4	10	7	12	1	6	2	5	11	9	8	3
No. of bees	30	22	20	38	65	160	48	62	61	74	88	86



- 3.1 Write down the coordinates of the outlier. (2)
- 3.2 Determine the equation for the least squares regression line. (4)

3.3 Calculate the correlation co-efficient if the outlier is excluded. (2)

QUESTION 4

The table below compares the number of hours spent by 7 Mathematics learners and the learners’ performance in the test.

Learners	1	2	3	4	5	6	7
Hours	1	3	5	6	8	10	11
Marks (%)	35	55	60	65	75	70	80

4.1 Represent the data in a scatter plot on DIAGRAM SHEET 1. (3)

4.2 Determine the equation of the line of best fit for the data. (3)

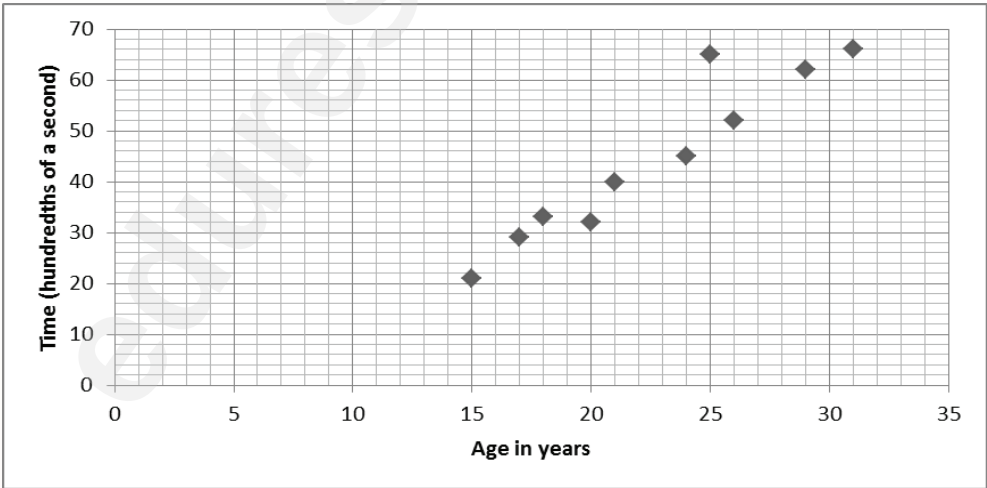
4.3 Hence, draw the line of best fit on the scatter plot. (2)

4.4 Determine the correlation coefficient of the data. Hence, comment about the strength of the relationship between the two variables. (3)

QUESTION 5

A hospital carries out a survey to compare the reaction time of patients of different ages, to a specific medication taken. The results are shown in the table and scatter plot below.

Age in years	15	17	18	20	21	24	25	26	29	31
Time (hundredths of a second)	21	29	33	32	40	45	65	52	62	66



5.1 Draw the line of best fit on the scatter plot on DIAGRAM SHEET 1. (2)

5.2 One of these patients reaction time is an outlier.

5.2.1 How old is this patient? (1)

5.2.2 Explain why this patient is an outlier. (1)

5.3 Calculate the equation of the least squares regression line (line of best fit) for the data. (3)

5.4 Calculate the correlation coefficient of the data. Comment on the strength of the relationship between the variables. (2)

2. PROBABILITY

This topic is in paper 1 and is out of ±20 marks in grades 10 and 11 and ±15 marks in grade 12.

2.1 Overview and Curriculum Statement

Overview of Probability		
GRADE 10	GRADE 11	GRADE 12
a) Compare the relative frequency of an experimental outcome with the theoretical probability of the outcome b) Venn diagrams as an aid to solving probability problems c) Mutually exclusive events and complementary events. d) The identity for any two events A and B: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$	a) Dependent and independent events b) Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	a) Generalisation of the fundamental counting principle. b) Probability problems using the fundamental counting principle
Curriculum Statement		
1. The use of probability models to compare the relative frequency of events with the theoretical probability. 2. The use of Venn diagrams to solve probability problems, deriving and applying the following for any two events A and B in a sample space S: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ A and B are mutually exclusive if $P(A \text{ and } B) = 0$; A and B are complementary if they are mutually exclusive; and $P(A) + P(B) = 1$. Then $P(B) = P(\text{not}(A)) = 1 - P(A)$.	1. Revise the addition rule for mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$, the complementary rule: $P(\text{not } A) = 1 - P(A)$ and the identity $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ 2. The use of Venn diagrams to solve probability problems, deriving and applying formulae for any three events A, B and C in a sample space S. 3. Identify dependent and independent events and the product rule for independent events: $P(A \text{ and } B) = P(A) \times P(B)$. 4. Use tree diagrams for the probability of consecutive or	1. Revise: • dependent and independent events • product rule for independent events: $P(A \text{ and } B) = P(A) \times P(B)$. • the sum rule for mutually exclusive events A and B: $P(A \text{ or } B) = P(A) + P(B)$ • the identity: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ • the complementary rule: $P(\text{not } A) = 1 - P(A)$ 2. Probability problems using Venn diagrams, tree diagram, two-way contingency tables and other techniques to solve probability problems (where events are not necessarily independent).

	simultaneous events which are not necessarily independent.	3. Apply the fundamental counting principle to solve probability problems.
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2.2 Introduction

- **What is Probability?**
Probability is a measure of the likelihood of an event taking place OR a numerical measure of an event to occur.

$$P(\text{Event}) = \frac{\text{Number of favourable outcomes (E)}}{\text{Total number of Possible Outcomes (S)}}$$

And can be written in short as:
$$P(E) = \frac{n(E)}{n(S)}$$

- **Probability Scale**

In words:	Impossible	Very unlikely	Unlikely	Even chances	Likely	Very likely	Certain
As decimal fractions:	0	0,2	0,4	0,5	0,6	0,8	1
As fractions:	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{5}$	1
As Percentages:	0%	20%	40%	50%	60%	80%	100%

2.3 Relative frequency and Theoretical Probability

2.3.1 Relative Frequency

Probability that is estimated is calculated after performing a very large number of trials of an experiment or conducting a survey involving a very large number of items, and is called relative frequency.

For an example: Tossing a coin 100 times and counting the number of times it lands on tail.

2.3.2 Theoretical Probability

Theoretical probability (or just Probability) is used when each outcome in a sample space is equally likely to occur.

2.4 Probability Identities and/or Rules

2.4.1 Any two events (A and B) happening

Addition rule for any two events happening:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

2.4.2 Mutually Exclusive Events

Mutually exclusive events are events that cannot happen in the same trial.

e.g.: Passing and failing the same test cannot happen at the same time. Hence these events are mutually exclusive.

$$P(A \text{ and } B) = P(\emptyset) = 0$$

Therefore, **addition rule for mutually exclusive events** is:

$$P(A \text{ or } B) = P(A) + P(B)$$

2.4.3 For Complementary Events

Events A and B are complementary if:

- they are **mutually exclusive** and
- $P(A) + P(B) = 1$ (exhaustive)

NOTE: $P(\text{not } A) = P(B)$ [$P(\text{not } A)$ can be written as $P(A')$]

Therefore: $P(\text{not } A) = 1 - P(A)$

e.g.: $P(\text{not studying}) = 1 - P(\text{studying})$

2.4.4 Independent Events

Independent events are events where the outcome of the second event is not affected by the outcomes of the first event.

e.g.: drawing the first card from a pack of cards, then drawing the second one after replacing the first card.

If events are independent, $P(A \text{ and } B) = P(A) \times P(B)$

Example 1

Given the following probabilities: $P(A) = 0,45$, $P(B) = 0,3$ and $P(A \text{ or } B) = 0,615$.

- 1.1 Determine whether or not events A and B are mutually exclusive.
- 1.2 Determine $P(A \text{ and } B)$.
- 1.3 Are events A and B independent?

Solution

$$1.1 \quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$0,615 = 0,45 + 0,3 - P(A \text{ and } B)$$

$$P(A \text{ and } B) = 0,75 - 0,615 \\ = 0,135$$

$\therefore$ The events are not mutually exclusive.

$$1.2 \quad P(A \text{ and } B) = 0,135$$

$$1.3 \quad P(A \text{ and } B) = P(A) \times P(B) \text{ if the events are independent.}$$

$$P(A) \times P(B) = 0,45 \times 0,3 \\ = 0,135$$

and

$$P(A \text{ and } B) = 0,75 - 0,615 \quad \text{as was calculated above} \\ = 0,135$$

$\therefore$ The events are independent because $P(A \text{ and } B) = P(A) \times P(B)$

Activity 1

$$1.1 \quad P(A) = 0,3 \text{ and } P(B) = 0,5. \text{ Calculate } P(A \text{ or } B) \text{ if:}$$

- (a) A and B are mutually exclusive events.
- (b) A and B are independent.

$$1.2 \quad \text{Events A and B are mutually exclusive. It is given that:}$$

- $P(B) = 2P(A)$
- $P(A \text{ or } B) = 0,57$

Calculate $P(B)$.

1.3 For two events, A and B, it is given that:

- $P(A) = 0,2$
- $P(B) = 0,63$
- $P(A \text{ and } B) = 0,126$

Are the events, A and B, independent? Justify your answer with appropriate calculations.

2.5 Venn Diagrams, Tree Diagrams and Contingency Tables

2.5.1 Venn Diagrams

Venn diagrams were introduced in 1880 by John Venn as a way of picturing relationships between different groups of items.

Example 2

A study was done to determine the effect of 3 drugs A, B and C in relieving headache pain. 80 patients were given the chance to use all 3 drugs. The following results were obtained:

40 reported relief from drug A

35 reported relief from drug B.

40 reported relief from drug C.

21 reported relief from both drug A and C.

18 reported relief from drug B and C.

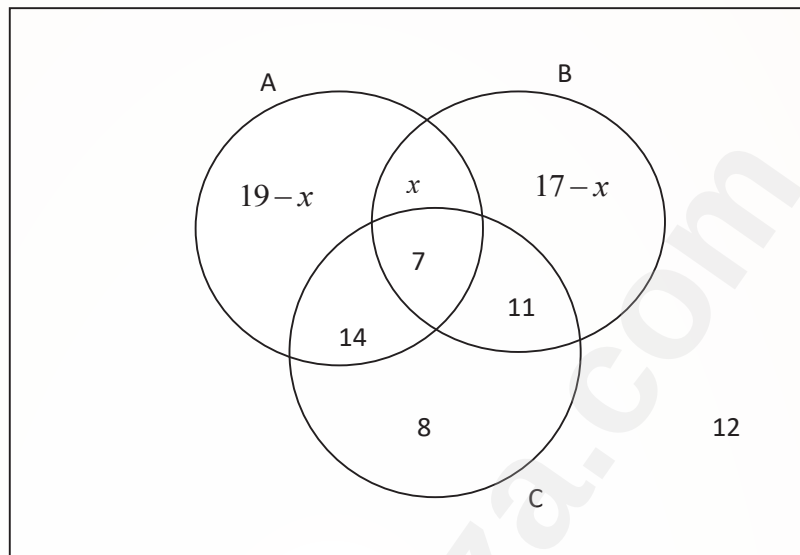
68 reported relief from at least one of the drugs.

7 reported relief from all graphs.

2.1 How many of the patients got relief from none of the drugs?

$$\begin{aligned} n(\text{relief from none of the drugs}) &= 80 - 68 \\ &= 12 \end{aligned}$$

2.2 How many patients got relief from drug A and B but not C?



$$\begin{aligned}
 n(\text{relief from A and B but not C}) &= x \\
 19 - x + x + 14 + 7 + 11 + 8 + 17 - x + 12 &= 80 \\
 -x + 88 &= 80 \\
 x &= 8
 \end{aligned}$$

- 2.3 What is the probability that a randomly chosen subject got relief from at least two of the drugs?

$$\begin{aligned}
 P(\text{a randomly chosen subject got relief from at least two of the drugs}) &= \frac{14 + 11 + 7 + 8}{80} \\
 &= \frac{40}{80}
 \end{aligned}$$

2.5.2 Tree Diagrams

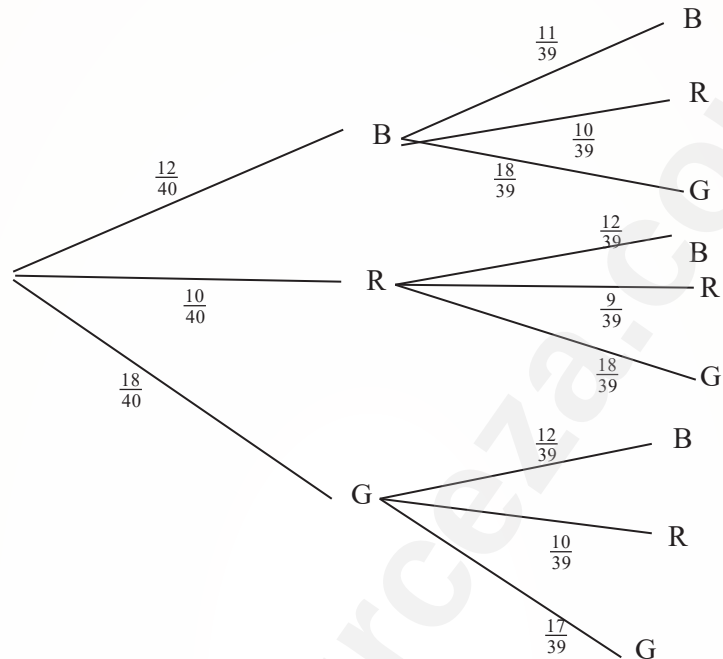
A tree diagram is a graphical method for obtaining the outcomes of a random experiment. Drawing a tree diagram provides a systematic way of generating all the possible outcomes.

Example 3

A bag contains 12 blue balls, 10 red balls and 18 green balls. 2 balls are then drawn without

replacement. Use the information provided to answer the questions that follow.

3.1 Draw a Tree diagram to represent the information



3.2 Determine the probability that both balls are green.

$$P(GG) = \frac{18}{40} \times \frac{17}{39} = \frac{51}{260}$$

3.3 Determine the probability that one blue and one red ball is chosen.

$$P(BR) = \frac{12}{40} \times \frac{10}{39} + \frac{10}{40} \times \frac{12}{39} = \frac{2}{13}$$

2.5.3 Contingency Tables

A two—way contingency table shows the observed frequencies for two variables in various categories. Each variable has two categories (choices) which are mutually exclusive (events

that do not occur at the same time) and exhaustive (the categories cover all possibilities so there is a category for everyone).

Example 4

A survey was conducted among 100 boys and 60 girls to determine how many of them watched TV in the period during which examinations were written. Their responses are shown in the partially completed table below.

	WATCHED TV DURING EXAMINATIONS	DID NOT WATCH TV DURING EXAMINATIONS	TOTALS
Male	80	a	
Female	48	12	
Totals	b	32	160

4.1 Calculate a and b .

	Watches TV during exams	Do not watch TV during exams	Total
Male	80	10	$80+a$
Female	48	12	60
Total	128	32	160

$a + 12 = 32$

$a = 20$

$b = 80 + 48$

$= 128$

4.2 Are the events 'being a male' and 'did not watch TV during examinations' mutually exclusive? Give a reason for your answer.

No

$P(\text{M and not watching TV}) = \frac{20}{160} \neq 0$

4.3 If a learner who participated in this survey is chosen at random, what is the probability that the learner:

4.3.1 Watched TV in the period during which the examinations were written?

$$P(\text{watching TV}) = \frac{128}{160} = \frac{4}{5} = 0,8 = 80\%$$

4.3.2 Is not a male and did not watch TV in the period during which examinations were written?

$$P(\text{female and not watching TV}) = \frac{12}{160} = \frac{3}{40} = 0,075 = 7,5\%$$

Activity 2

1 During a survey at a certain school, 350 learners were asked about their favourite sport. It was found that:

- 180 like Netball (N)
- 180 like Rugby (R)
- 240 like Soccer (S)
- 78 like all the three sports
- 60 like Netball and Rugby
- 54 like Rugby and Soccer but not Netball
- 30 like Netball and Rugby but not Soccer

1.1 Calculate how many learners:

- (a) Like Netball only
- (b) Like Soccer only
- (c) Do not like any of the three sports

1.2 Represent the given information in a Venn diagram

1.3 Calculate the probability that one of these learners, chosen at random

- (a) Likes Rugby only
- (b) Likes none of the sports

- 1.2 Suppose a family has three children. There are many arrangements of boys (B) and girls (G) that can make up these three children.
- 1.1.1 Suppose a family has three children. There are many arrangements of boys (B) and girls (G) that can make up these three children.
- 1.1.2 Draw a tree diagram to find all possible arrangements of the three children in the family
- 1.1.3 How many arrangements are there in 1.2.1?
- 1.1.4 Use the tree diagram to determine:
- (a) Probability that all children are girls.
 - (b) Probability that all three children are of the same gender.
 - (c) Probability that at least two of the three children are boys.
 - (d) Probability that only one of the children is a girl.

- 1.3 The data below was obtained from the financial aid office at a university.

	Receiving financial aid	Not receiving financial aid	Total
Undergraduates	4 222	3 898	8 120
Postgraduates	1 879	731	2 610
Total	6 101	4 629	10 730

- 1.3.1 Determine the probability that the student selected at random is...
- (a) receiving financial aid.
 - (b) a postgraduate student and not receiving financial aid.
 - (c) an undergraduate student and receiving financial aid
- 1.3.2 Are the events of being an undergraduate and receiving financial aid independent? Show ALL relevant workings.
- Are the events of being an undergraduate and receiving financial aid mutually exclusive? Justify your answer.

Exercise 1

- 1.1 Each of the 200 employees of a company wrote a competency test. The results are indicated in the table below.

	Pass	Fail	Total
Males	46	32	78
Females	72	50	122
Total	118	82	200

- 1.1.1 Are the events Pass and Fail mutually exclusive? Explain your answer.
- 1.1.2 Is passing the competency test independent of gender? Substantiate your answer with the necessary calculations.

- 1.2 The table summarises the results of all the language tests taken at a Language Centre in Cape Town during the first week of January.

	Male	Female	Totals
Pass	32	43	75
Fail	8	15	23
Total	40	58	98

A person is chosen at random from those who took their test during the first week of January.

- 1.2.1 Find the probability that the person was a male who failed.
- 1.2.2 The person chosen is a female. Find the probability that she passed the test.
- 1.3 A rare kidney disease affects only 1 in 1000 people and the test for this disease has a 99% accuracy rate.
- 1.3.1 Draw a two-way contingency table showing the results, if 100 000 of the general population are tested.
- 1.3.2 Calculate the probability that a person who tests positive for this rare kidney disease is sick with the disease correct to two decimal places.
- 1.4 Three cards are selected at random (without replacement) from a standard full pack of playing cards. There are 52 cards in the pack, jokers are excluded. Find the probability that the cards are all the same colour.
- 1.5 A study of numbers of male and female offspring in a certain population is being carried out. It is found that the first child in any family is equally likely to be male or female, but that for any subsequent offspring, the probability that they will be of the same sex as the previous child is $\frac{3}{5}$. No twins, triplets etc., are possible
- 1.5.1 Find the probability that the first child of a family will be female
- 1.5.2 Find the probability that the first two children of a family will be female
- 1.5.3 Find the probability that a family will have two females followed by two males (in that order). Leave your answer in simplified fraction form.
- 1.6 There are 20 boys and 15 girls in a class. The teacher chooses individual learners at random to deliver a speech
- 1.6.1 Calculate the probability that the first learner chosen is a boy.
- 1.6.2 Draw a tree diagram to represent the situation if the teacher chooses three learners, one after the other. Indicate on your diagram ALL possible outcomes.
- 1.6.3 Calculate the probability that a boy, then a girl and then another boy is chosen in that order.
- 1.6.4 Calculate the probability that all three learners chosen are girls.
- 1.6.5 Calculate the probability that at least one of the learners chosen is a boy.
- 1.6.6 What is the probability that 5 learners chosen are of the same gender?

- 1.7 There are t orange balls and 2 yellow balls in a bag. Craig randomly selects one ball from the bag, records his choice and returns the ball to the bag. He then randomly selects a second ball from the bag, records his choice and returns it to bag. It is known that the probability that Craig will select two balls of the same colour from the bag is 52%.

Calculate how many orange balls are in the bag.

- 1.8 There are four black balls and y yellow balls in a bag. Thandi takes out a ball, notes its colour and then puts it back in the bag. She then takes out another ball and also notes its colour. If the probability that both balls have the same colour is $\frac{5}{8}$, determine the value of y .

- 1.9 At a school for boys there are 240 learners in Grade 12. The following information was gathered about participation in school sport.

- 122 boys play rugby (R)
- 58 boys play basketball (B)
- 96 boys play cricket (C)
- 16 boys play all three sports
- 22 boys play rugby and basketball
- 26 boys play cricket and basketball
- 26 boys do not play any of these sports

Let the number of learners who play rugby and cricket only be x

- 1.9.1 Draw a Venn diagram to represent the above information.

- 1.9.2 Determine the number of boys who play rugby and cricket.

- 1.9.3 Determine the probability that a learner in Grade 12 selected at random:

- 1.9.4 (a) does not play cricket
(b) participates in at least 2 of these sports.

- 1.10 Given that:

$$\begin{aligned} P(A \text{ only}) &= x \\ P(A \text{ and } B) &= 0,1 \\ P(B) &= 0,4 \\ P(\text{not } (A \text{ or } B)) &= y \end{aligned}$$

- 1.10.1 Represent this information in a Venn Diagram.

- 1.10.2 If A and B are independent events, find the values of x and y .

- 1.11 Given that:

- A and B are independent events
- $P(B) = 2P(A)$
- $P(A \text{ or } B) = 0,6$

Calculate $P(B)$

- 1.12 Given that: $P(A) = \frac{3}{5}$; $P(B) = \frac{2}{5}$; $P(C) = \frac{3}{10}$

A and B are independent events

B and C are independent events

1.11.1 Calculate $P(A \text{ or } B)$

1.11.2 Calculate $P(C \text{ only})$

- 1.13 In a Physical Science quiz, two teams work independently on a problem. They are allowed a maximum of 10 minutes to solve the problem. The probabilities that each team will solve the problem are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. Calculate the probability that the problem will be solved in the ten minutes allowed.

- 1.14 A local club has facilities that include tennis courts and a golf course. A survey of the club members indicated that 504 regularly use the golf course and 336 regularly use the tennis courts. Some members regularly use both while 56 use neither of the facilities. The club has 700 members.

1.14.1 Determine the number of members that regularly use at least one of the facilities. (NB: A Venn diagram may be useful).

1.14.2 What is the probability that a club member selected at random uses exactly (only) one facility?

1.14.3 Given that: $P(\text{using the golf course}) \times P(\text{using the tennis courts}) = 0,3456$. Validate statistically whether these events are independent or not.

2.6 Fundamental Counting Principles

At times, instead of determining the total number of possible outcomes in the sample space using a tree diagram where occurrences/events are more than two then the Fundamental counting principle is used as a short (and fast) way of finding the number of all elements in the sample space and also the subset where favorable outcomes are found.

The **Fundamental Counting Principle** is the guiding rule for finding the number of ways to accomplish two or more tasks.

If there are m choices for the first decision and n choices for the second decision, then there are $m \times n$ choices for the first decision followed by the second decision.

2.6.1 Multiple events with p , q and r outcomes: pqr

Example 1

Khanyi has 3 pairs of shoes, 4 trousers and 6 jerseys. Use counting principles to:

- 1.1 Determine how many different combinations he has. (2)

$$n(E) = 3 \times 4 \times 6 = 72$$

- 1.2 Give the probability that he will randomly choose his favourite pair of shoes. (1)

$$P(\text{favourite pair of shoes}) = \frac{1}{3}$$

- 1.3 Give the probability that he will randomly choose his favourite pair of shoes **and** his favourite jersey. (3)

$$\begin{aligned} P(\text{favourite pair of shoes and favourite jersey}) &= \frac{1}{3} \times \frac{1}{6} \\ &= \frac{1}{18} \end{aligned}$$

- 1.4 Give the probability that he will randomly choose his favourite pair of shoes **or** his favourite jersey. (2)

$$\begin{aligned} P(\text{favourite pair of shoes or favourite jersey}) &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{3}{6} \end{aligned}$$

- 1.5 Give the probability that he will randomly choose his favourite pair of shoes and trousers, or jersey. (3)

$$\begin{aligned} P(\text{favourite pair of shoes jersey and trousers or jersey}) &= \frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \\ &= \frac{1}{4} \end{aligned}$$

2.6.2 Arranging n elements, with elements repeated: $\frac{n!}{p!q!}$

Write down all possible four letter arrangements that may be formed using the following words: (a) DEED (b) EZEE (c) ANNA (d) BULL (e) MIST

	Word	Different words	No. of words	No of identical letters	
(a)	DEED	DEED;DEDE ; DDEE ; EDED; EDDE ; EEDD	6	2D's and 2E's (2 × 2)	$\frac{4!}{2!2!}$

(b)	EZEE	EZEE ; EEZE; EEEZ ; ZEEE	4	3E's (1 × 3)	$\frac{4!}{1!3!}$
(c)	ANAN	ANAN;ANNA; AANN ; NANA NNAA ; NAAN	6	2A's and 2N's (2 × 2)	$\frac{4!}{2!2!}$
(d)	BULL	BULL; BLUL; BLLU ; UBLL ULBL ; ULLB; LLUB ; LLBU LBLU; LBUL; LULB ; LUBL	12	2L's (1 × 1 × 2)	$\frac{4!}{1!1!2!}$
(e)	MIST	MIST ; MITS; MSIT ; MSTI MTIS ; MTSI; IMST ; IMTS ISMT ; ISTM; ITMS ; ITSM SMIT ; SMTI; SITM ; SIMT STIM ; STMI; TMIS ; TMSI TSIM ; TSMI; TISM ; TIMS	24	0 (1 × 1 × 1 × 1)	$\frac{4!}{1!1!1!1!}$ $\left(\frac{4!}{0!}\right)$

The number of distinct permutations of n objects, of which p are identical to each other, and then q of the remainder are identical, and r of the remainder are identical, and so on is

$\frac{n!}{p! \times q! \times r! \times \dots}$ where $p + q + r + \dots = n$

Example 2

The following words are given: 1. MISSISSIPPI 2. STATISTICS 3. RIGIDITY

- 2.1 Find the number of distinct random arrangements of the words above.
- 2.2 Determine the probability that each of the words above will start and end with a vowel.
- 2.3 Determine the probability that each of the words above will start and end with the same letter.

Solution:

Word	No. of letters & id letters	(2.1) No. of arrangements $n(S)$	(2.2)P(Start & end with vowel)	(2.3)P(Start & end with same letter)
MISSISSIPPI	11 with 4 I's; 4 S's & 2 P's	$\frac{11!}{4! \times 4! \times 2!} = 34\ 650$	We want I _ _ _ _ _ _ I	We want I _ _ _ _ _ _ I or S _ _ _ _ _ S or P _ _ _ _ _ P

			$n(E) = \frac{9!}{2! \times 4! \times 2!}$ $= 3\,780$ $\therefore P(E) = \frac{3\,780}{34\,650} = \frac{6}{55}$	$n(E) = \frac{2 \times 9!}{2! \times 2! \times 4!} + \frac{9!}{4! \times 4!}$ $= 8\,190$ $\therefore P(E) = \frac{8\,190}{34\,650} = \frac{13}{55}$
STATISTICS	10 with 3 S's; 3 T's & 2 I's	$\frac{10!}{3! \times 3! \times 2!} = 50\,400$	We want I _____ _I or I _____ A or _____ A _____ _____ I	We want I _____ _I or S _____ S or _____ T _____ _____ T
			$n(E) = \frac{3 \times 8!}{3! \times 3!} = 3\,360$ $\therefore P(E) = \frac{3\,360}{50\,400} = \frac{1}{15}$	$n(E) = \frac{2 \times 8!}{3! \times 2!} + \frac{8!}{3! \times 3!}$ $= 7\,840$ $\therefore P(E) = \frac{7\,840}{50\,400} = \frac{7}{45}$
RIGIDITY	8 with 3 I's	$\frac{8!}{3!} = 6\,720$	We want I _____ _I	We want I _____ _I
			$n(E) = 6! = 720$ $\therefore P(E) = \frac{720}{6\,720} = \frac{3}{28}$	$n(E) = 6! = 720$ $\therefore P(E) = \frac{720}{6\,720} = \frac{3}{28}$

2.6.3 Fitting n elements in n slots with no repeats: $n!$

NOTE: Factorial: $n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$

2.6.4 Fitting n elements in m slots with repeats: n^m

Example 3

- 3.1 Eight top athletes must each choose a lane to run in out of the 8 lanes. Calculate the number of different arrangements possible.

$n(E) = 8 \times 7 \times 6 \times 5 \times 4 \times 4 \times 3 \times 2 \times 1 = 8! = 40\,320$

- 3.2 Calculate the number of different four-slot combination codes that can be created from using only the digits 1 to 4. Digits may be used more than once.

$n(E) = 4 \times 4 \times 4 \times 4 = 4^4 = 256$

- 3.3 Gauteng has about 7 million registered vehicles, with numbers in the form of three letters (no vowels and Q, no repeats) and three numbers (repeats are allowed). Explain why Gauteng is running out of license numbers.

$$n(E) = 20 \times 19 \times 18 \times 10 \times 10 \times 10 = 6\,840\,000$$

Only a maximum of 6 840 000 number plates can be formed which is less than 7 million.

- 3.4 A bracelet can be formed from 8 different amulets.
Calculate the number of different arrangements possible.

$$n(E) = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = (8 - 1)! = 5\,040$$

2.6.5 Fitting n elements in m slots, no repeats: ${}^nP_m = \frac{n!}{(n-m)!}$

Example 4

- 4.1 Seven passengers need to be transported in a car that legally has room for only four.
Calculate the number of ways that the driver can select to transport and seat the seven people.

$$\text{arrangements} = 7 \times 6 \times 5 \times 4 = 840$$

OR

$$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 840$$

OR

$$\frac{7!}{(7-4)!} = 840$$

OR

$${}^7P_4 = 840 \text{ (using a calculator)}$$

- 4.2 A four-slot combination lock is created using the digits 0 to 9.

- 4.2.1 Calculate the number of combinations available with repeats, and also without repeats. (2)

$$\text{With repeats: } 10 \times 10 \times 10 \times 10 = 10^4 = 10\,000$$

$$\text{With no repeats: } 10 \times 9 \times 8 \times 7 = 5\,040 \text{ OR } \frac{10!}{(10-4)!} = 5\,040 \text{ OR } {}^{10}P_4$$

- 4.2.2 Calculate the probability of selecting a combination without repeats that has a 6 in it. (2)

no 6's = $9 \times 8 \times 7 \times 6 = 3\,024$ $\therefore 2\,016$ with at least one 6

$$\frac{2\,016}{5\,040} = 0,4$$

2.6.6 Fitting n groups of p and q elements: $p!q!n!$

Example 5

- 5.1 A cutlery set has 3 spoons, 4 knives and 5 forks. Each of the spoons, forks and knives are similar in design. Calculate:

- 5.1.1 How many different arrangements of the set there are

$$\text{arrangements} = \frac{12!}{3! \times 4! \times 5!} = 27\,720$$

- 5.1.2 How many different arrangements there are so that each of the three cutlery types are together

$$\text{Arrangements} = 3! = 6$$

- 5.1.3 The probability of selecting an arrangement where there is no grouping of cutlery items at all.

$$\begin{aligned} P(\text{no grouping of cutlery}) &= 1 - P(\text{grouping of cutlery}) \\ &= 1 - \frac{6}{27\,720} \\ &= 0,999784 \end{aligned}$$

- 5.1.4 The probability of selecting an arrangement where two forks are placed first, and two forks are placed last.

$$\text{arrangements} = \frac{8!}{3! \cdot 4! \cdot 1!} = 280$$

$$\text{probability} = \frac{280}{27\,720} = 0,010\,101$$

Exercise 2

- 2.1 Ryan packs his suitcase for his holiday with 3 caps, 5 shirts, 3 pairs of jeans and 2 pairs of takkies:
 - 2.1.1 How many different outfits can he put together if when he dresses, he must wear a shirt, a pair of jeans, a pair of takkies and a cap?
 - 2.1.2 Ryan reaches his destination and hangs all the 5 shirts and the three pairs of jeans (each item separately) on a different hanger, on the rail in the cupboard.
 - (a) How many different arrangements are possible?
 - (b) What is the probability that the shirts are all hanging together next to each other in the cupboard?
 - 2.1.3 While on holiday Ryan decides to buy a pair of sandals in addition to his outfit items, on a given day what is the probability that Ryan will wear a pair of sandals or a pair of takkies?
- 2.2 Every client of Future Bank has a personal identity number (PIN) which is made up of 5 digits chosen from the digits 0 to 9.
 - 2.2.1 How many personal identity numbers (PINs) can be made if:
 - (a) Digits can be repeated?
 - (b) Digits cannot be repeated?
 - 2.2.2 Suppose a PIN can be made up by selecting digits at random and that the digits can be repeated. What is the probability that such a PIN will contain at least one 9?
- 2.3 Each of the digits 1; 1; 2; 3; 4; 7 is written on a separate card. The cards are then placed next to each other to make a 6-digit number.
 - 2.3.1 How many different 6-digit numbers can be formed from these digits?
 - 2.3.2 How many numbers start and end with the same digit?
 - 2.3.3 What is the probability of getting a number that starts and ends with the same digit?
 - 2.3.4 Find the probability that a number is 112347 or 743211?
- 2.4 In Gauteng the number plates consists of 3 alphabets, excluding **the five vowels**, next to each other followed by 3 digits from 0 to 9. All number plates end with GP. An example: TDG 234 GP. The alphabets and digits are allowed to repeat.
 - 2.4.1 Determine the number of unique number plates.
 - 2.4.2 Determine the probability that the number plate starts with a Y.
 - 2.4.3 Calculate the probability that the number plate contains an E.
 - 2.4.4 Determine the number of number plates that will contain one 5.

- 2.5 Consider the word “SIMPLIFY”
 - 2.5.1 How many six letter words can be made?
 - 2.5.2 Calculate the probability of the word starting and ending with the same letter
- 2.6 Six cars are parked alongside each other, three are silver. How many ways can the cars be arranged if the silver cars have to be next to each other.
- 2.7 Three men (Peter, Jabu and Les) and two women (Liz and Kate) are to stand in a straight line to have their group photograph taken. Find the probability that Peter stands next to Liz and Jabu stands next to Kate.
- 2.8 A music group plans a concert tour in South Africa. They will perform in Cape Town, port Elizabeth, Pretoria, Johannesburg, Bloemfontein, Durban and East London.
 - 2.8.1 In how many different orders can they plan their tour if there are no restrictions?
 - 2.8.2 In how many different orders can they plan their tour if there are no restrictions?
 - 2.8.3 In how many different orders can they plan their tour if their tour begins in Cape Town and ends in Durban?
 - 2.8.4 If the tour cities are chosen at random, what is the probability that their performances in Cape Town, Port Elizabeth, Durban and East London happen consecutively? Give your answer correct to 3 decimal places.
- 2.9 Landline telephone numbers are 10 digits long. Numbers begin with a zero followed by 9 digits chosen from the digits 0 to 9. Repetitions are allowed.
 - 2.9.1 How many different phone numbers are possible?
 - 2.9.2 The first three digits of a number form an area code. The area code for Cape Town is 021. How many different phone numbers are available in the Cape Town area?
 - 2.9.3 What is the probability of the second digit being an even number?
 - 2.9.4 Ignoring the first digit, what is the probability of a phone number consisting of only odd digits? Write your answer correct to three decimal places.
- 2.10 The code to a safe consists of 10 digits chosen from 0 to 9. None of the digits are repeated. Determine the probability of a code where the first digit is odd and none of the first three digits may be a zero. Write your answer as a percentage correct to two decimal places.
- 2.11 A group of 10 people consisting of 5 married couples is arranged at random in a line for a photograph. Determine the probability that each wife is standing next to her husband.
- 2.12 Assuming that it is equally likely to be born in any of the 12 months of the year, what is the probability that in a group of six, at least two people have birthdays in the same month?

QUESTIONS FROM PREVIOUS YEARS' QUESTION PAPERS

A. DBE/EXEMPLAR 2014**QUESTION 11**

11.1 Events A and B are mutually exclusive. It is given that:

$$\bullet \quad P(B) = 2P(A) \quad \& \quad P(A \text{ or } B) = 0,57$$

Calculate $P(B)$. (3)

11.2 Two identical bags are filled with balls. Bag A contains 3 pink and 2 yellow balls.

Bag B contains 5 pink and 4 yellow balls. It is equally likely that Bag A or Bag B is chosen. Each ball has an equal chance of being chosen from the bag. A bag is chosen at random and a ball is then chosen at random from the bag.

11.2.1 Represent the information by means of a tree diagram. Clearly indicate the probability associated with each branch of the tree diagram and write down all the outcomes. (4)

11.2.2 What is the probability that a yellow ball will be chosen from Bag A? (1)

11.2.3 What is the probability that a pink ball will be chosen? (3)

[11]

QUESTION 12

Consider the word M A T H S.

12.1 How many different 5-letter arrangements can be made using all the above letters? (2)

12.2 Determine the probability that the letters S and T will always be the first two letters of the arrangements in QUESTION 12.1. (3)

[5]

B. DBE/NOVEMBER 2014**QUESTION 11**

A survey concerning their holiday preferences was done with 180 staff members.

The options they could choose from were to:

- Go to the coast
- Visit a game park
- Stay at home

The results were recorded in the table below:

	Coast	Game Park	Home	Total
Male	46	24	13	83
Female	52	38	7	97
Total	98	62	20	180

- 11.1 Determine the probability that a randomly selected staff member:

11.1.1 is male (1)

11.1.2 does not prefer visiting a game park (2)
- 11.2 Are the events 'being a male' and 'staying at home' independent events. Motivate your answer with relevant calculations. (4)
- [7]

QUESTION 12

- 12.1 A password consists of five different letters of the English alphabet. Each letter may be used only once. How many passwords can be formed if:

12.1.1 All the letters of the alphabet can be used (2)

12.1.2 The password must start with a 'D' and end with an 'L' (2)
- 12.2 Seven cars of different manufacturers, of which 3 are silver, are to be parked in a straight line.

12.2.1 In how many different ways can ALL the cars be parked? (2)

12.2.2 If the three silver cars must be parked next to each other, determine In how many different ways the cars can be parked. (3)
- [9]

C. JUNE 2015 (AMENDED CAPS:SCE)

D. DBE/NOVEMBER 2015

E. DBE/MARCH 2016

QUESTION 10

- 10.1 Each passenger on a certain Banana Airways flight chose exactly one beverage from tea, coffee or fruit juice. The results are shown in the table below.

	MALE	FEMALE	TOTAL
Tea	20	40	60
Coffee	b	c	80
Fruit juice	d	e	20
TOTAL	60	100	a

- 10.1.1 Write down the value of a . (1)
- 10.1.2 What is the probability that a randomly selected passenger is male? (2)
- 10.1.3 Given that the event of a passenger choosing coffee is independent of being a male, calculate the value of b . (4)

- 10.2 A Banana Airways aeroplane has 6 seats in each row.

- 10.2.1 How many possible arrangements are there for 6 people to sit in a row of 6 seats? (2)
- 10.2.2 Xoliswa, Anees and 4 other passengers sit in a certain row on a Banana Airways flight. In how many different ways can these 6 passengers be seated if Xoliswa and Anees must sit next to each other? (2)
- 10.2.3 Mary and 5 other passengers are to be seated in a certain row. If seats are allocated at random, what is the probability that Mary will sit at the end of the row? (4)
- [15]

F. DBE/JUNE 2016(AMENDED CAPS: SCE)

QUESTION 10

- 10.1 A tournament organiser conducted a survey among 150 members at a local sports club to find out whether they play tennis or not. The results are shown in the table below.

	PLAYING TENNIS	NOT PLAYING TENNIS
Male	50	30
Female	20	50

- 10.1.1 What is the probability that a member selected at random is:
- (a) Female (2) (4) [10]
- (b) Female and plays tennis (1)
- 10.1.2 Is playing tennis independent of gender? Motivate your answer with the necessary calculations. (3)
- 11.1 One boy and girl are a couple and want to sit next to each other at any end of the row of friends. In how many different ways can the entire group be seated? (3)
- 11.2 If all the friends are seated randomly, calculate the probability that all the girls are seated next to each other. (3) [6]

G. DBE/NOVEMBER 2016

QUESTION 11

A survey was conducted among 100 boys and 60 girls to determine how many of them watched TV in the period during which examinations were written. Their responses are shown in the partially completed table below.

	WATCHED TV DURING EXAMINATIONS	DID NOT WATCH TV DURING EXAMINATIONS	TOTALS
Male	80	a	
Female	48	12	
Totals	B	32	160

- 11.1 Calculate the values of a and b . (2)
- 11.2 Are the events 'being a male' and 'did not watch TV during examinations' mutually exclusive? Give a reason for your answer. (2)
- 11.3 If a learner who participated in this survey is chosen at random, what is the probability that the learner:

11.3.1 Watched TV in the period during which the examinations were written? (2)

11.3.2 Is not a male and did not watch TV in the period during which examinations were written? (2)
- [8]

QUESTION 12

The digits 1 to 7 are used to create a four-digit code to enter a locked room. How many different codes are possible if the digits may not be repeated and the code must be an even number bigger than 5 000? [5]

3. EUCLIDEAN GEOMETRY

Euclid was a Greek mathematician who lived about 2 300 years ago. He developed a form of geometry that measures straight lines and flat shapes. He uses axioms (given facts) to prove theorems in a logical way. You need to be able to prove these theorems and use them to solve geometry problems using logical reasoning.

Overview and Curriculum Statement

Overview of Probability		
Grade 10	Grade 11	Grade 12
(a) Revise basic results established in earlier	(a) Investigate and prove theorems of the	(a) Revise earlier (Grade 9) work on the necessary and

grades. (b) Investigate line segments joining the midpoints of two sides of a triangle. (c) Properties of special quadrilaterals. Solve problems involving volume and surface area of solids studied in earlier grades as well as spheres, pyramids and cones and combinations of those objects.	geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles. (b) Solve circle geometry problems, providing reasons for statements when required. (c) Prove riders. Revise Grade 10 work.	sufficient conditions for polygons to be similar. (b) Prove (accepting results established in earlier grades): • that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Mid-point Theorem as a special case of this theorem); • that equiangular triangles are similar; • that triangles with sides in proportion are similar; • the Pythagorean Theorem by similar triangles; and riders.
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Curriculum Statement		
1. Revise basic results established in earlier grades regarding lines, angles and triangles, especially the similarity and congruence of triangles. 2. Investigate line segments joining the midpoints of two sides of a triangle. 3. Define the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and trapezium. Investigate and make conjectures about the	Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. Then investigate and prove the theorems of the geometry of circles: • The line drawn from the centre of a circle perpendicular to a chord bisects the chord; • The perpendicular bisector of a chord passes through the centre of the circle; • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); • Angles subtended by a chord of the circle, on the same side of the	1. Revise earlier work on the necessary and sufficient conditions for polygons to be similar. 2. Prove (accepting results established in earlier grades): • that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Mid-point Theorem as a special case of this theorem) • that equiangular triangles are similar; • that triangles with sides in proportion are similar; and • the Pythagorean Theorem by similar triangles.

properties of the sides, angles, diagonals and areas of these quadrilaterals. Prove these conjectures.	chord, are equal; • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length; • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. Use the above theorems and their converses, where they exist, to solve riders.	
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This topic is assessed in P2 and carries 30 ± 3 in grade 10, 50 ± 3 in grade 11 and 50 ± 3 in grade 12

Revision from lower grades corollaries, theorems and axioms.

Activity 1

Draw a relevant diagram (figure) to illustrate the following statements/theorems/axioms

1.1 POINT, LINES AND ANGLES ON PARALLEL LINES

- The angles around a point add up to 360° .
- Adjacent angles on a straight line add up to 180° .
- Vertically opposite angles are equal
- Corresponding angles are equal:
- Co-interior angles add up to 180°
- Alternate angles are equal

1.2 TRIANGLES

- The sum of the angles of a triangle add up to 180° .
- The exterior angle of a triangle is equal to the sum of the two interior opposite angles.
- An **isosceles triangle** has two sides equal in length and the angles opposite the equal sides are equal

Converse:

- The sides opposite equal angles of a triangle are equal in length.
- An **equilateral triangle** has all three sides equal in length then each angle equals 60° .
- Triangles on the same(equal) base and between the same parallel lines are equal in area

The theorem of Pythagoras

- In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Midpoint theorem

- The line joining the midpoints of the triangle is parallel to the third side and is half the length of the third side

Similar triangles

- Triangles are similar if they are either equiangular **or** their corresponding pairs of sides are in proportion

Congruency axioms for triangles

Two triangles are **congruent** if:

- three sides of one triangle are respectively equal to three sides of the other triangle (SSS)
- two sides and the *included* angle of one triangle are respectively equal to two sides and the included angle of the other triangle (SAS)
- two angles and a side of one triangle are respectively equal to two angles and a corresponding side of the other triangle (AAS or ASA)
- in two right angle triangle, the hypotenuse and a side of one triangle are equal to the hypotenuse and side of the other triangle (RHS)

1.3 QUADRILATERALS

Activity 2

2.2 Define the quadrilateral

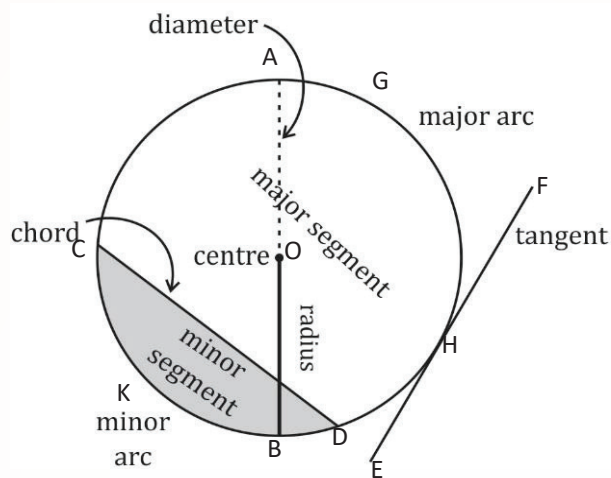
2.3 Define and write down the properties of the following quadrilaterals

- 2.2.1 Parallelogram 2.2.2 Rhombus 2.2.3 Rectangle 2.2.4 Square 2.2.5 Kite
2.2.6 Trapezium

1.4 CIRCLE: Parts of the circle

Activity 3

Describe each of the labelled part of the circle



1.4.1 Circle theorems

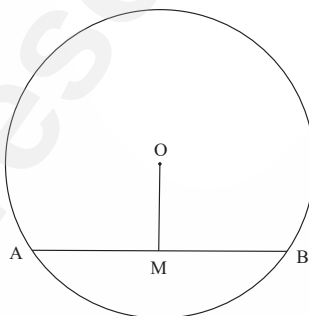
Theorem 1 (Proof is examinable)

The line drawn from the centre of a circle perpendicular to a chord bisects the chord. (*line from centre $\perp$ to chord*)

Converse of Theorem 1

The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord (*line from centre to midpt of chord*)

Deductions from the two theorems



Given circle with centre O and chord AB. OM is drawn such that M is on AB,

Theorem 1: If $OM \perp AB$ then $AM = MB$

Converse: If $AM = MB$ then $OM \perp AB$

Theorem 2

The perpendicular bisector of a chord passes through the centre of the circle

Example 1

$AB = 8 \text{ cm}$ is the chord of the circle with centre O .

OCD is the radius of the circle with C on AB such that C is the midpoint of AB . If $DC = 2 \text{ cm}$, Calculate the radius of the circle.

Solution

$$CB = 4 \text{ cm} \quad [C \text{ is the midpoint of } AB]$$

$$OC \perp AB \quad [\text{line from centre to midpoint of chord}]$$

$$OC^2 + CB^2 = OB^2 \quad [\text{Pythagoras}]$$

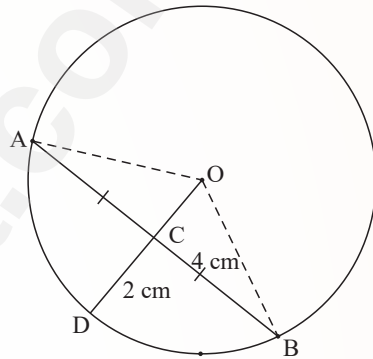
$$\text{But } OC = OB - 2 \quad [\text{radii}]$$

$$(OB - 2)^2 + 4^2 = OB^2$$

$$OB^2 - 4OB + 4 + 16 = OB^2$$

$$-4OB = -20$$

$$OB = 5 \text{ cm}$$



Theorem 3 (Proof is examinable)

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc on the same segment at the circle ($\angle \text{ at centre} = 2 \times \angle \text{ at circumference}$)

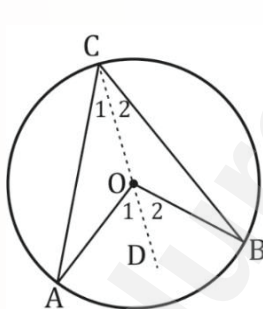


Figure 1

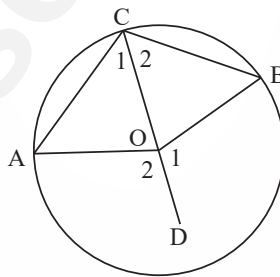


Figure 2

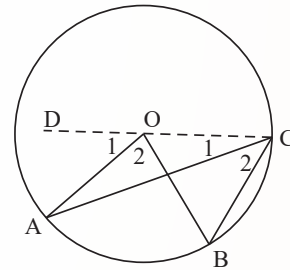


Figure 3

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a) $\hat{A}_1$

b) $\hat{D}_2$

4.2.2 Show that AE bisect $\hat{DAB}$

Theorem 4

Angles subtended by a chord of the circle on the same side of the chord, are equal.

($\angle$ s in the same seg)

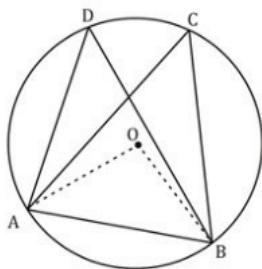


Figure 1

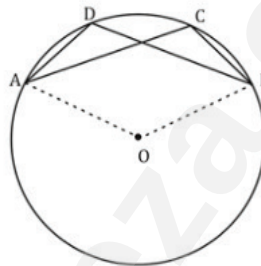


Figure 2

In each of the figures above, AB subtends $\hat{D}$ and $\hat{C}$ on the same segment

In each case $\hat{D} = \hat{C}$

Corollaries: (Deductions) (Results) from the above theorem

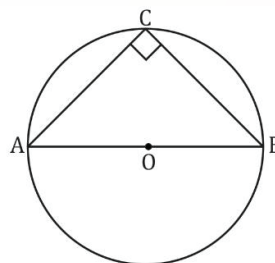
You do not need to learn the proofs of the following corollaries, but they are very useful for solving riders.

1. Equal arcs subtend equal angles on the circle (*equal arcs*)
2. Equal chords subtend equal angles on the corresponding arcs of the circle (*equal chords*)
3. An angle subtended on the circle by a diameter is a right angle ($\angle$ in semi-circle)

This fact is very useful. When you see a diameter in a diagram, you know that there is a right angle that will be useful to you!

AOB is a diameter.

$$\therefore \hat{ACB} = 90^\circ \text{ (}\angle \text{ in semi-circle)}$$

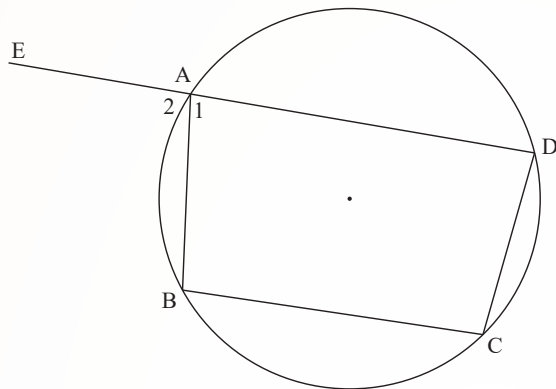


4. If a chord of the circle subtends an angle of 90° on the circle then the chord is a diameter of the circle. ($\angle$ at circle = 90°)

1.4.2 Cyclic quadrilaterals

A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of the same circle. All the sides of the quadrilateral are chords of the circle.

Example of a cyclic quadrilateral and its parts



ABCD is a cyclic quadrilateral because its vertices are on the circumference of the circle

$\hat{A}_1$ and $\hat{C}$ are opposite interior angles

$\hat{A}_2$ is an exterior angle of cyclic quadrilateral ABCD

Theorem 5 (Proof is examinable)

The opposite angles of a cyclic quadrilateral are supplementary. (*opp $\angle$ s of cyclic quad*)

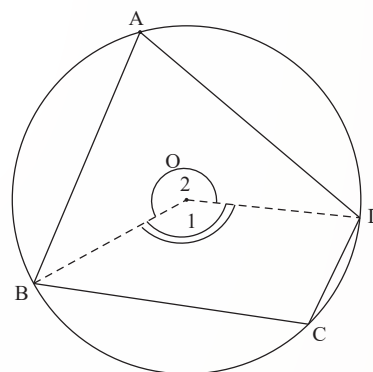
From ($\angle$ at centre = $2 \times \angle$ at circumference)

Construction (join BO and OD)

It follows that $\hat{O}_1 = 2\hat{A}$ and

$$\hat{O}_2 = 2\hat{C}$$

$$\therefore \hat{O} = 2(\hat{A} + \hat{C})$$



But $\hat{O} = 360^\circ$

$$\therefore A + C = \frac{360^\circ}{2} = 180^\circ$$

Corollary

An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (*The proof is not examinable but its application is important to solve riders*).

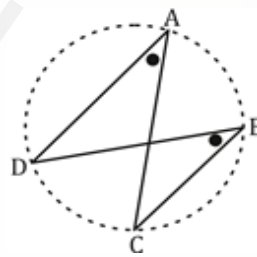
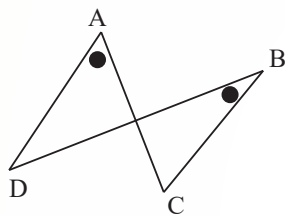
Converse of the corollary:

If an exterior angle of a quadrilateral is equal to the interior opposite angle of that quadrilateral, then the quadrilateral is cyclic. (*ext $\angle$ = int opp $\angle$*)

Conditions to prove that a quadrilateral is cyclic.

Given quadrilateral ABCD.

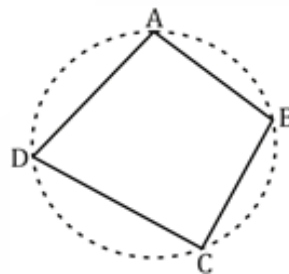
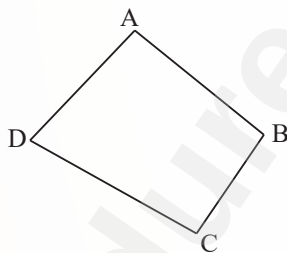
1.



If $\hat{A} = \hat{B}$ or $\hat{C} = \hat{D}$

(converse of $\angle$ s in same seg/line subt equal $\angle$ s)
ABCD is a cyclic quadrilateral

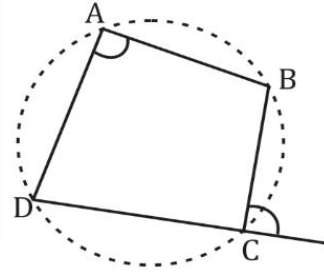
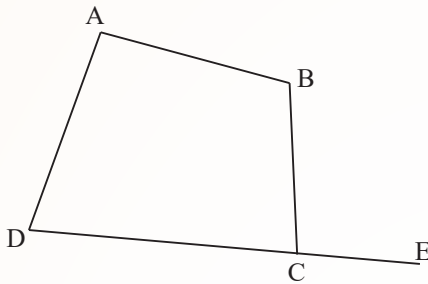
2.



If $\hat{A} + \hat{C} = 180^\circ$ or $\hat{B} + \hat{D} = 180^\circ$

(converse of opp $\angle$ s of cyclic quad)
ABCD is a cyclic quadrilateral

3.

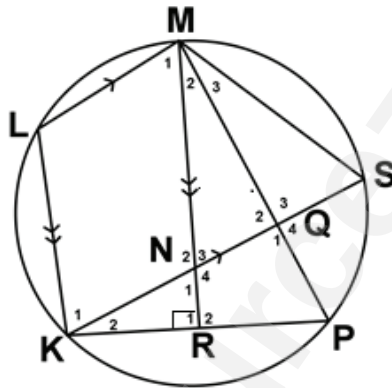


If DC is extended to E and $\hat{A} = \hat{BCE}$

(converse of ext $\angle$ of cyclic quad)
 $ABCD$ is a cyclic quadrilateral

Activity 6

In the diagram SK is a chord of the circle passing through L, M, S, P and K. LK, KP, PM, SM and LM are chords of the circle. $LM \parallel KS$ and $KL \parallel MR$. $MR \perp KP$, MNR and MQP are straight lines.



If $\hat{M}_1 = x$, Prove that:

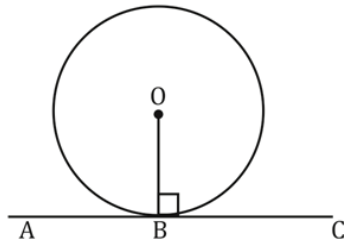
- 6.1 NRPQ is a cyclic quadrilateral.
- 6.2 MKRQ is a cyclic quadrilateral.

1.4.3 Tangent to circle

Axiom: (This is a fact that does not need to be proved)

A tangent to a circle is perpendicular to a radius at the point of contact.

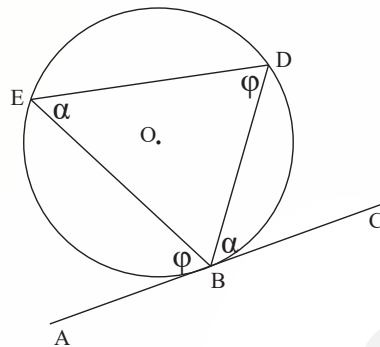
($\text{tan} \perp \text{rad}$)



If ABC is a tangent to the circle with centre O at B, then $OB \perp AC$

Theorem 6 (Proof is examinable)

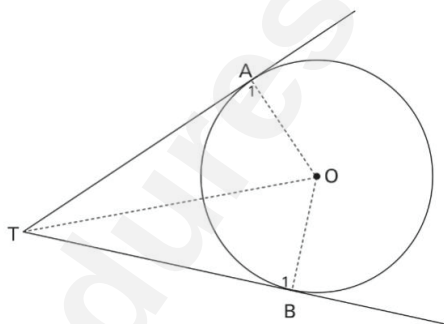
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. (*tan chord theorem*)



If ABC is a tangent to the Circle centre O at B with chords BD and BE, then $\hat{C}BD = \hat{E}$ and $\hat{A}BE = \hat{D}$

Theorem 7

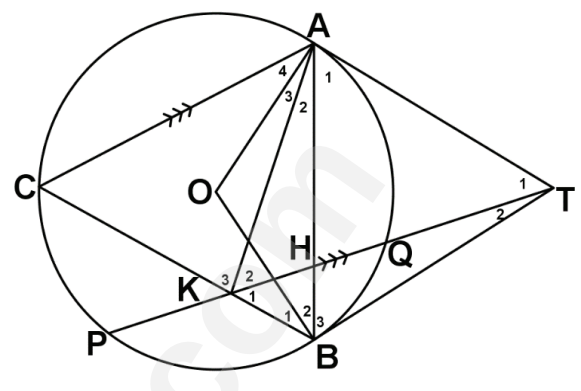
Two tangents drawn to a circle from the same point outside the circle are equal in length.
(tans from same pt)



If TA and TB are tangents to the circle with centre O at A and at B, then $TA = TB$

Activity 7

- 7.1 In the diagram O is the centre of the circle passing through C, A and B. TA and TB are two tangents to the circle at A and B. TQP cuts the circle at Q and P. CA $\parallel$ PT. QP cuts AB and BC at H and K respectively.



Prove that:

- 7.1.1 AOBT is a cyclic quadrilateral.
 7.1.2 $\hat{HKB} = \hat{A}_1$
 7.1.3 TA is a tangent to the circle through A, H and K.

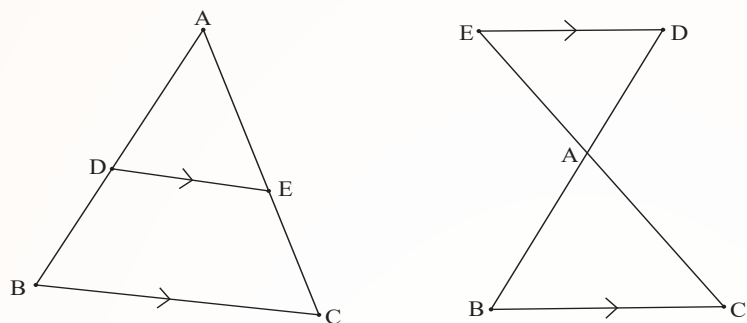
1.5 Ratio and Proportions

- Ratio: e.g. the ratio of 1 to 2 is written as $1 : 2 = \frac{1}{2}$
- Proportion: Four quantities are in proportion if the ratio of the first two is equal to the ratio of the last two.

1. Also note that if $\frac{a}{b} = \frac{c}{d}$ then
1. $\frac{a}{c} = \frac{b}{d}$
 2. $\frac{b}{a} = \frac{d}{c}$
 3. $\frac{a+b}{b} = \frac{c+d}{d}$
 4. $\frac{a-b}{b} = \frac{c-d}{d}$
 5. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$
 6. $ad = bc$

Theorem 8 (Proof is examinable)

A line drawn parallel to one side of a triangle, divides the other two sides proportionally. (*prop theorem; name $\parallel$ lines*)

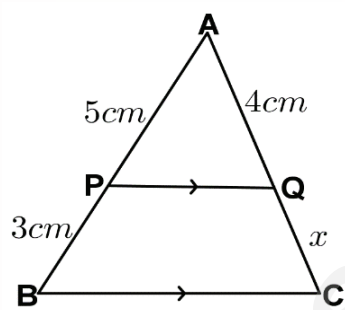


In $\triangle ABC$, If DE is drawn such that $DE \parallel BC$ with D on AB and E on AC, then $\frac{AD}{DB} = \frac{AE}{EC}$ and

$$\frac{AD}{DB} = \frac{AE}{EC}$$

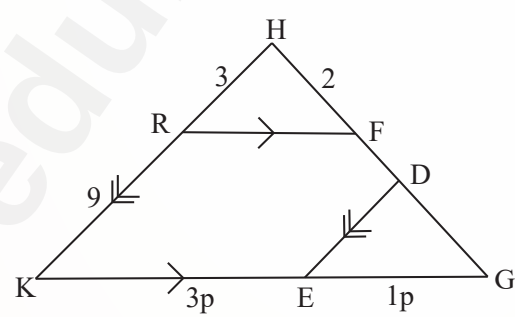
Activity 8

8.1 Determine the value of x if $PQ \parallel BC$



8.2 In the diagram, $RF \parallel KG$, $ED \parallel KH$,

$RH = 3$ units, $RK = 9$ units, $HF = 2$ units. $GE : EK = 1 : 3$



Calculate (stating reasons) the lengths of:

8.2.1 FG

8.2.2 FD

1.6 Similarity

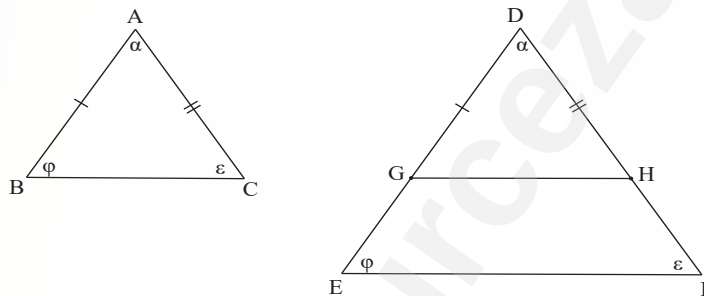
- Similar polygons: they are equiangular **and** their corresponding pairs of sides are in proportion.
- Triangles are the only polygons which are similar if they are either equiangular **or** their corresponding pairs of sides are in proportion.

From the above mentioned facts, answer the following questions? Give reasons for your answer.

- Is every rectangles similar to every other rectangle?
- Is every square similar to every other square?
- Is every equilateral triangle similar to every other equilateral triangle?
- Is every Isosceles triangle similar to every other Isosceles triangle?
- Is every rhombus similar to every other rhombus?

Theorem 9 (Proof is examinable)

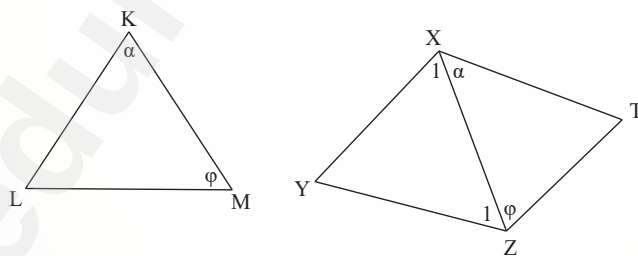
Equiangular triangles are similar ($\triangle \triangle$ **OR** equiangular $\triangle$ s)



If $\triangle ABC$ and $\triangle DEF$ are equiangular, then $\frac{AB}{DE} = \frac{AC}{DF}$ and therefore $\triangle ABC \sim \triangle DEF$

Theorem 10 (Converse of theorem 8)

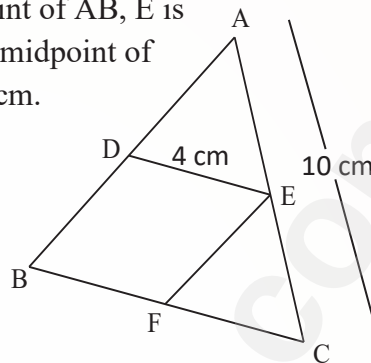
If two triangles have their sides in the same proportion, then the corresponding angles are equal and the triangles are similar. (Sides of $\triangle$ in prop)



If in $\triangle KLM$ and $\triangle XYZ$, $\frac{KL}{XT} = \frac{KM}{XZ} = \frac{LM}{TZ}$, then $\triangle KLM \parallel \triangle XYZ$

Activity 9

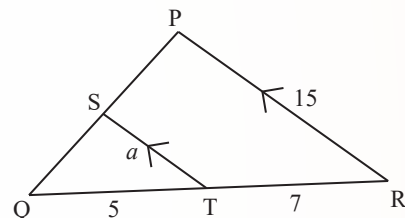
- 9.1 In triangle ABC, D is the midpoint of AB, E is the midpoint of AC and F is the midpoint of BC. If DE = 4 cm and AC = 10 cm.



Determine the length of:

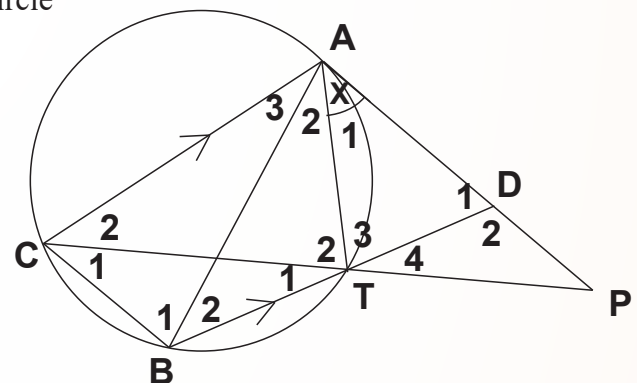
Give reason for your answers.

- 9.1.1 BC
9.1.2 FC
- 9.2 In the sketch below, PQR is a triangle. S and T are points on PQ and QR respectively. QT = 5 units, TR = 7 units and PR = 15 units



Determine the value of a stating reasons.

- 9.3 In the diagram DA is a tangent to the circle ACBT at A. CT and AD are produced to meet at P. BT is produced to cut PA at D. AC, CB, AB and AT are joined. AC $\parallel$ BD. Let $\hat{A}_1 = x$.

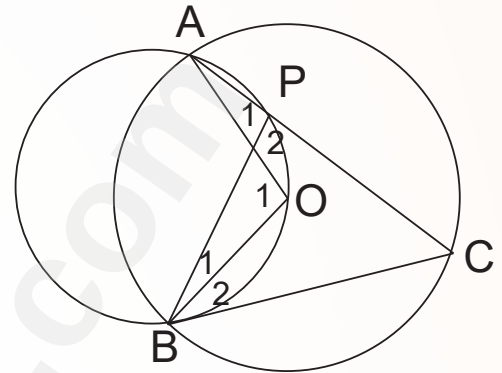


- 9.3.1 Prove that $\triangle ABC \parallel \triangle ADT$.
9.3.2 Prove that PT is a tangent to the circle ADT at T.
9.3.3 Prove that $\triangle APT \parallel \triangle TPD$.

9.3.4 If $AD = \frac{2}{3}AP$, show that $AP^2 = 3PT^2$

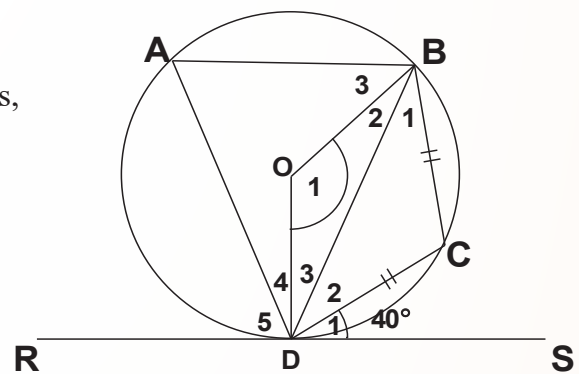
EXERCISES

1. In the diagram AP is produced to C and O is the centre of the circle ABC . A, P, O and B are on the circumference of the smaller circle.
 $\hat{C} = 30^\circ$.



Determine with reasons, the value of:

- 1.1 $\hat{P}_1$
 - 1.2 $\hat{PBC}$
 - 1.3 Prove that:
 $PB = PC$
2. In the figure, RDS is a tangent to circle O at D . if $BC = DC$ and $\hat{CDS} = 40^\circ$, calculate, with reasons, the size of:

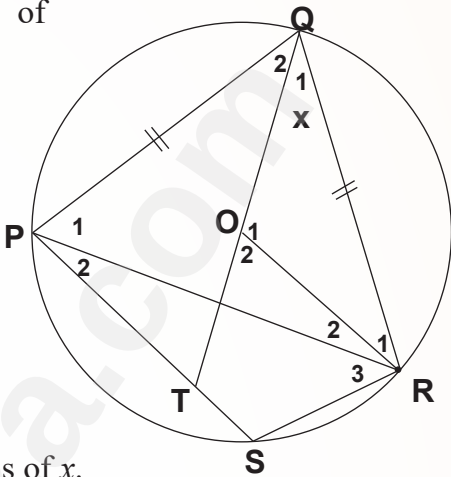


- 2.1 $\hat{BDC}$
- 2.2 $\hat{C}$

2.3 $\hat{A}$

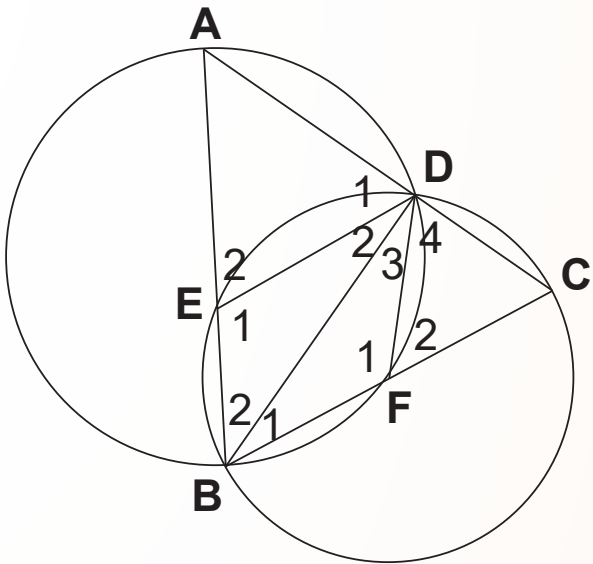
2.4 $\hat{O}_1$

3. In the diagram, O is the centre of the circle. P, Q, R and S are points on the circumference of the circle. TOQ is a straight line such that T lies on PS. PQ = QR and $\hat{Q}_1 = x$



- 3.1 Calculate, with reasons, $\hat{P}_1$ in terms of x.
- 3.2 Show that TQ bisects $\hat{PQR}$.
- 3.3 Show that STOR is a cyclic quadrilateral.

4. In the diagram below two circles intersect one another at D and B. AB is a straight line such that it intersects the circle BCD at point E. BC is a straight line such that it intersects the circle ABD at F. DE, DB and DF are joined. $\hat{F}_2 = 180^\circ - 2x$ FC = FD



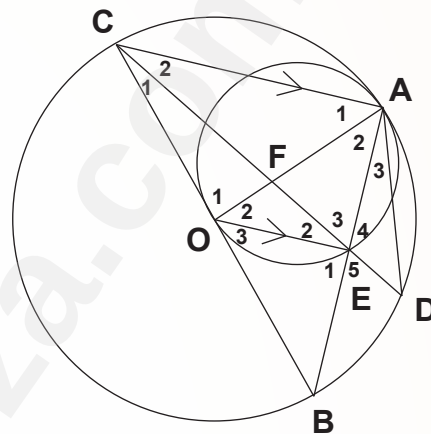
- 4.1 Calculate, with reasons and in terms of x:
- 4.1.1 $\hat{DEB}$

4.1.2 $\hat{A}$

4.2 Hence, or otherwise, prove

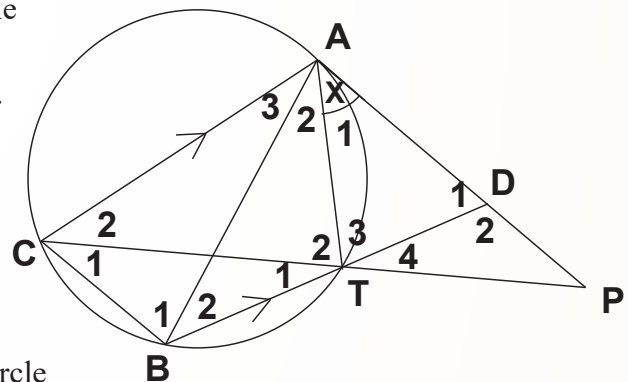
$$ED \parallel BC$$

5. Two circles touch each other at point A. The smaller circle passes through O, the centre of the larger circle. Point E is on the circumference of the smaller circle. A, D, B and C are points on the circumference of the larger circle. $OE \parallel CA$.



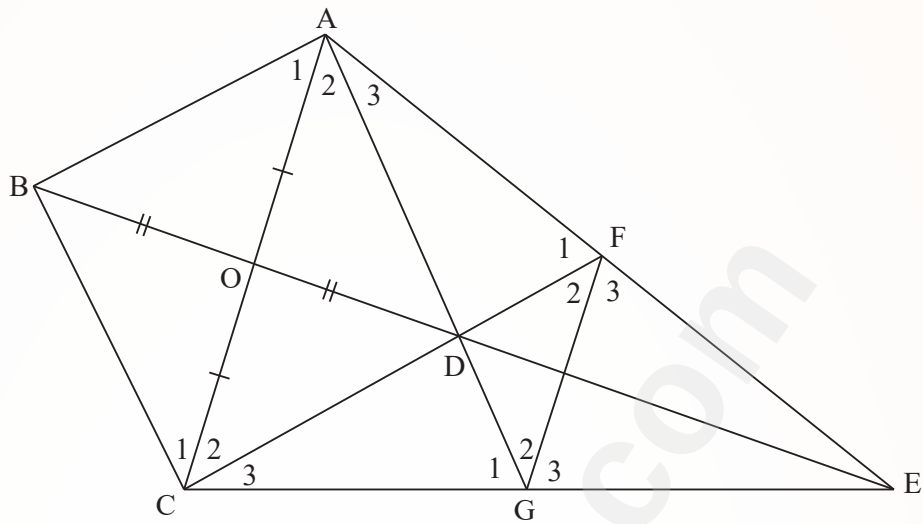
- 5.1 Prove, with reasons, that $AE = BE$.
- 5.2 Prove that $\triangle AED \parallel \triangle CEB$.
- 5.3 Hence, or otherwise, show that $AE^2 = DE \cdot CE$
- 5.4 If $AE \cdot EB = EF \cdot EC$, show that E is the midpoint of DF

6. In the diagram DA is a tangent to the circle ACBT at A. CT and AD are produced to meet at P. BT is produced to cut PA at D. AC, CB, AB and AT are joined. Let $\hat{A}_1 = x$

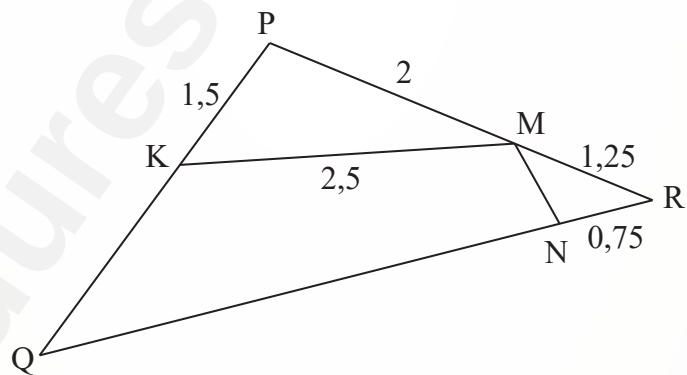


- 6.1 Prove that $\triangle ABC \parallel \triangle ADT$.
- 6.2 Prove that PT is a tangent to the circle ADT at T
- 6.3 Prove that $\triangle APT \parallel \triangle TPD$.
- 6.4 If $AD = \frac{2}{3}AP$, show that $AP^2 = 3PT^2$

7. In the diagram below, EO bisects side AC of $\triangle ACE$. EDO is produced to B such that $BO = OD$. AD and CD produced meet EC and EA at G and F respectively



- 7.1 Give a reason why ABCD is a parallelogram
- 7.2 Write down, with reasons, TWO ratios each equal to $\frac{ED}{DB}$.
- 7.3 Prove that $\hat{A}_1 = \hat{F}_2$.
- 7.4 It is further given that ABCD is a rhombus. Prove that ACGF is a cyclic quadrilateral.
8. In the diagram below, K, M and N respectively are points on sides PQ, PR and QR of $\triangle PQR$. $KP = 1,5$; $PM = 2$; $KM = 2,5$; $MN = 1$; $MR = 1,25$ and $NR = 0,75$.



- 8.1 Prove that $\triangle KPM \parallel \triangle RNM$.
- 8.2 Determine the length of NQ

❖ ACCEPTABLE REASONS: EUCLIDEAN GEOMETRY

In order to have some kind of uniformity, the use of the following shortened versions of the theorem statements is encouraged.

(ENGLISH) THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	$\angle$ s on a str line
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	adj $\angle$ s supp
The adjacent angles in a revolution add up to 360° .	$\angle$ s round a pt OR $\angle$ s in a rev
Vertically opposite angles are equal.	vert opp $\angle$ s =
If $AB \parallel CD$, then the alternate angles are equal.	alt $\angle$ s; $AB \parallel CD$
If $AB \parallel CD$, then the corresponding angles are equal.	corresp $\angle$ s; $AB \parallel CD$
If $AB \parallel CD$, then the co-interior angles are supplementary.	co-int $\angle$ s; $AB \parallel CD$
If the alternate angles between two lines are equal, then the lines are parallel.	alt $\angle$ s =
If the corresponding angles between two lines are equal, then the lines are parallel.	corresp $\angle$ s =
If the cointerior angles between two lines are supplementary, then the lines are parallel.	coint $\angle$ s supp
TRIANGLES	
The interior angles of a triangle are supplementary.	$\angle$ sum in $\triangle$ OR sum of $\angle$ s in $\triangle$ OR Int $\angle$ s $\triangle$
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	ext $\angle$ of $\triangle$
The angles opposite the equal sides in an isosceles triangle are equal.	$\angle$ s opp equal sides
The sides opposite the equal angles in an isosceles triangle are equal.	sides opp equal $\angle$ s
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	Pythagoras OR Theorem of Pythagoras
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	Converse Pythagoras OR Converse Theorem of Pythagoras
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	SSS
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	SAS OR S $\angle$ S
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the	AAS OR $\angle\angle$ S

triangles are congruent.	
If in two right angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	RHS OR 90°HS
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	Midpt Theorem
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	line through midpt $\parallel$ to 2 nd side
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	line $\parallel$ one side of Δ OR prop theorem; name $\parallel$ lines
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	line divides two sides of Δ in prop
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	$\parallel \Delta$ s OR equiangular Δ s
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	Sides of Δ in prop
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	same base; same height OR equal bases; equal height
CIRCLES	
The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	tan $\perp$ radius tan $\perp$ diameter
If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.	line $\perp$ radius OR converse tan $\perp$ radius OR converse tan $\perp$ diameter
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	line from centre to midpt of chord
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	line from centre $\perp$ to chord
The perpendicular bisector of a chord passes through the centre of the circle;	perp bisector of chord
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	$\angle$ at centre = $2 \times \angle$ at circumference
The angle subtended by the diameter at the circumference of the circle is 90°.	$\angle$ s in semi circle OR diameter subtends right angle OR $\angle$ in 21
If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter.	chord subtends 90° OR converse $\angle$ s in semi circle
Angles subtended by a chord of the circle, on the same side of the chord, are equal	$\angle$ s in the same seg
If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	line subtends equal $\angle$ s OR converse $\angle$ s in the same seg

Equal chords subtend equal angles at the circumference of the circle.	equal chords; equal $\angle$ s
Equal chords subtend equal angles at the centre of the circle.	equal chords; equal $\angle$ s
Equal chords in equal circles subtend equal angles at the circumference of the circles.	equal circles; equal chords; equal $\angle$ s
Equal chords in equal circles subtend equal angles at the centre of the circles.	equal circles; equal chords; equal $\angle$ s
The opposite angles of a cyclic quadrilateral are supplementary	opp $\angle$ s of cyclic quad
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	opp $\angle$ s quad supp OR converse opp $\angle$ s of cyclic quad
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	ext $\angle$ of cyclic quad
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	ext $\angle$ = int opp $\angle$ OR converse ext $\angle$ of cyclic quad
Two tangents drawn to a circle from the same point outside the circle are equal in length	Tans from common pt OR Tans from same pt
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	tan chord theorem
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	converse tan chord theorem OR $\angle$ between line and chord
QUADRILATERALS	
The interior angles of a quadrilateral add up to 360° .	sum of $\angle$ s in quad
The opposite sides of a parallelogram are parallel.	opp sides of $\parallel$ m
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	opp sides of quad are $\parallel$
The opposite sides of a parallelogram are equal in length.	opp sides of $\parallel$ m
If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.	opp sides of quad are = OR converse opp sides of a parm
The opposite angles of a parallelogram are equal.	opp $\angle$ s of $\parallel$ m
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	opp $\angle$ s of quad are = OR converse opp angles of a parm
The diagonals of a parallelogram bisect each other.	diag of $\parallel$ m
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	diags of quad bisect each other OR converse diags of a parm
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	pair of opp sides = and $\parallel$
The diagonals of a parallelogram bisect its area.	diag bisect area of $\parallel$ m
The diagonals of a rhombus bisect at right angles.	diags of rhombus
The diagonals of a rhombus bisect the interior angles.	diags of rhombus
All four sides of a rhombus are equal in length.	sides of rhombus
All four sides of a square are equal in length.	sides of square
The diagonals of a rectangle are equal in length.	diags of rect

The diagonals of a kite intersect at right-angles.	diags of kite
A diagonal of a kite bisects the other diagonal.	diag of kite
A diagonal of a kite bisects the opposite angles	diag of kite

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4. ANALYTICAL GEOMETRY

Analytical Geometry is a branch of algebra that is used to model geometric objects – points, (straight) lines, and circles. Analytical Geometry provides a connection between algebra and geometry through graphs of lines and curves. This enables geometric problems to be solved algebraically and provides insights into algebra.

This topic weighs 40 out of 150 marks in Mathematics P2.

Content covered in Analytical Geometry:

Grade 10	Grade 11	Grade 12
Represent geometric figures in a Cartesian co-ordinate system, and derive and apply, for any two points $(x_1; y_1)$ and $(x_2; y_2)$, a formula for calculating: • The distance between the two points; • The gradient of the line segment joining the points; • Conditions for parallel and perpendicular lines; and • The co-ordinates of the mid-point of the line segment joining the points.	Use a Cartesian co-ordinate system to derive and apply: the equation of a line through two given points. -the equation of a line through one point and parallel or perpendicular to a given line; and - the inclination of a line.	Use a two-dimensional Cartesian co-ordinate system to derive and apply: the equation of a circle (any centre) -the equation of a tangent to a circle given a point on the circle.

1. Distance between two points

The distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. It is used to:

- Calculate the distance/ length given two points
- Determine the x-coordinate or y-coordinate given the length.
- Show/Prove that a triangle is:
Isosceles: Two sides are equal
Equilateral: all sides are equal

Right-angled triangle: Square of the hypotenuse equal to sum of the squares on the other two sides

Scalene: all sides are not equal.

- Show/ Prove that a quadrilateral is:

Rectangle: opposite sides are equal

Parallelogram: opposite side are equal

Square : all sides are equal

Rhombus: all sides are equal

Kite: adjacent sides are equal

- Calculate the area of a triangle: $\text{-Area of } \Delta = \frac{1}{2}bh$
 $\text{-Area of a rectangle} = l \times b$
 $\text{-Area of a parallelogram} = b \times \perp h$
 $\text{-Area of a square} = s^2$
 $\text{-Area of a rhombus} = b \times \perp h$
 $\text{-Area of a kite} = \frac{1}{2}(d_1 \times d_2)$
 $\text{-Area of a trapezium} = \frac{1}{2}(\text{sum of // sides}) \times \perp h$

2. Midpoint of a line segment:

The midpoint formula is $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. It is used to:

- Determine the midpoint of line segment
- Find the coordinates given the midpoint of a line segment.
- Determine the coordinates of the fourth vertex of:
 - Rectangle
 - Parallelogram
 - Square
 - Rhombus
- Show that the quadrilateral is:
 - Rectangle
 - Parallelogram
 - Square
 - Rhombus since the mid-points of their diagonals bisect each other.

3. The Gradient or Slope is $m = \frac{y_2 - y_1}{x_2 - x_1}$. It is used to:

- Determine the gradient of the line segment.
- Show that two lines are parallel: since parallel line have equal gradient i.e. $m_1 = m_2$
- Show that two lines are perpendicular: since the product of their gradient is -1

i.e. $m_1 \times m_2 = -1$

- Show that the points are collinear – points that lie on the same straight line have the same gradient/slope.
- Show that a triangle is a right-angled triangle, since two sides are perpendicular.
- Show that a quadrilateral is;

Parallelogram: 2 pairs of opposite sides are parallel

OR

Diagonals bisect each other

OR

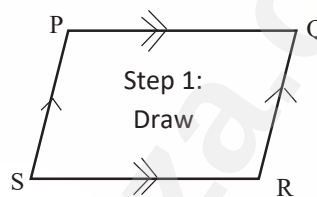
Opposite angles equal

OR

2 pairs of opposite sides equal

OR

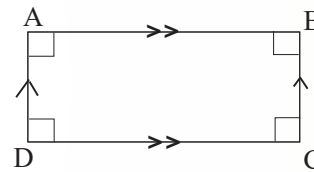
One pair of sides parallel and equal



Rectangle: Opposite sides are parallel and one interior angle equal to 90°

OR

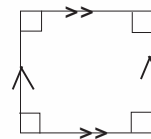
Diagonals equal in length and bisect each other



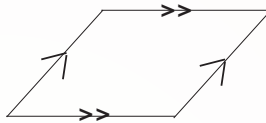
Square: Parallelogram with at least one interior angle equal to 90°

OR

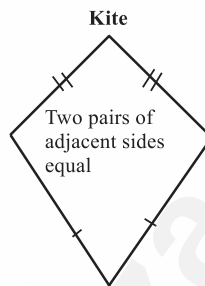
Diagonals equal in length and bisect each other at 90° .



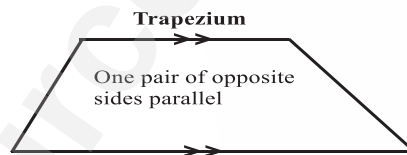
Rhombus: Parallelogram with one pair of adjacent sides equal
OR
 Parallelogram in which the diagonals bisect at right angles



Kite: Adjacent sides are equal in length
OR
 The longer diagonal bisect the shorter diagonal at 90°



Trapezium: One pair of opposite sides parallel



- Find the inclination of a straight line: $\tan \theta = m$ (inclination is an angle formed between the straight line and the x-axis in the positive direction of x-axis and is always positive. The inclination of lines can be used to find the sizes of angles inside a triangle using the fact that:
 - ✓ sum angles of a triangle are supplementary (add up to 180°)
 - ✓ Vertically opposite angles are equal.
 - ✓ Exterior angle of triangle is equal to the sum of two opposite interior angles.
- Find the equation of straight line : $y - y_1 = m(x - x_1)$
- A line parallel to the x-axis has a zero gradient.

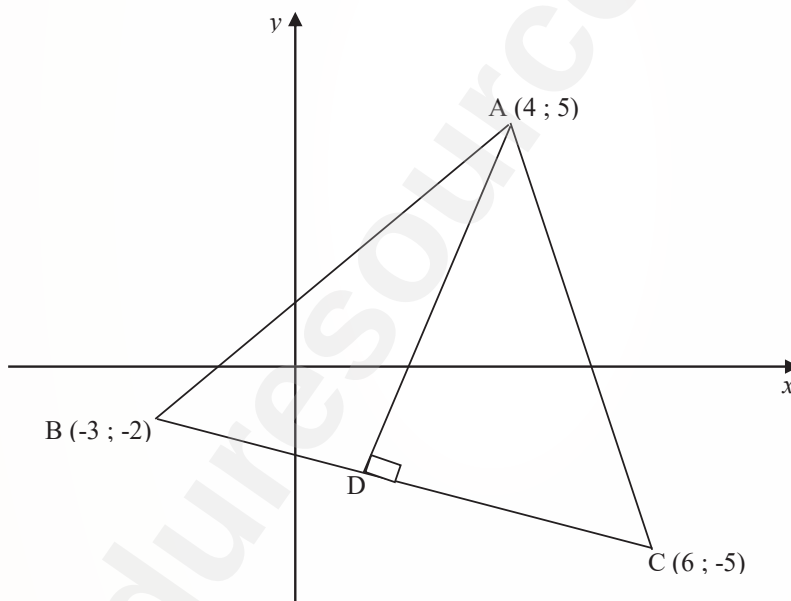
- A line parallel to the y-axis the gradient is undefined.
- Find the gradient of a straight line given the inclination of that line.

4 **The equation of the straight line:** $y = mx + c$ or $y - y_1 = m(x - x_1)$

- Determine the equation of the straight line
- Find the gradient of the straight line by expressing it in a form $y = mx + c$ where the coefficient of x is the gradient.
- Show that the point lies on straight line, where LHS = RHS
- Find the coordinates of the point of intersection of two line by solving them simultaneously.
- Find the coordinates of the intercept on the axes.
- Find the equation of the perpendicular bisector, by using the fact that $m_1 \times m_2 = -1$.
- Find the equation of a tangent to the circle: $m_{rad} \times m_{tan} = -1$ since the tangent is always perpendicular to the radius.

EXAMPLE 1:

1. In the diagram below, $A(4;5)$, $B(-3;-2)$ and $C(6;-5)$ are the vertices of $\triangle ABC$. AD is drawn perpendicular to BC .



- 1.1 Calculate the length of BC . (2)

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$BC = \sqrt{(6+3)^2 + (-5+2)^2}$$

$$BC = \sqrt{(9)^2 + (3)^2}$$

$$BC = \sqrt{90}$$

- 1.2 Determine the equation of BC. (3)

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y + 5 = -\frac{1}{3}(x - 6)$$

$$y = -\frac{1}{3}x - 3$$

- 1.3 Determine the equation of AD. (3)

$$m_{BC} = -\frac{1}{3} \text{ and } m_{AD} = 3 \quad (m_{AD} \times m_{BC} = -1; AD \perp BC)$$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 3(x - 4)$$

$$y - 5 = 3x - 12$$

$$y = 3x - 7$$

- 1.4 Determine the coordinates of D. (3)

$$-\frac{1}{3}x - 3 = 3x - 7$$

$$10x = 12$$

$$x = \frac{12}{10} = \frac{6}{5}$$

$$y = 3\left(\frac{6}{5}\right) - 7 = -3\frac{2}{5}$$

$$\therefore D\left(\frac{6}{5}; -3\frac{2}{5}\right)$$

- 1.5 Calculate the size of $\hat{BAD}$. (5)

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{-2 - 5}{-3 - 4} = \frac{-7}{-7} = 1$$

$$\therefore \tan \alpha = 1$$

$$\therefore \alpha = 45^\circ$$

$$\therefore m_{AD} = 3$$

$$\therefore \tan \theta = 3$$

$$\therefore \theta = 71,57^\circ$$

$$\theta = \alpha + \hat{BAD} \quad \text{ext. } \angle \Delta$$

$$\therefore \hat{BAD} = \theta - \alpha$$

$$\therefore \hat{BAD} = 26,57^\circ$$

- 1.6 Calculate the coordinates of a point E if the $\text{area of } \Delta EBC = \text{area of } \Delta ABC$ and E is a point on the positive x-axis. (4)

$$\therefore E(x;0); x > 0$$

$$\text{area } \Delta EBC = \text{area } \Delta ABC$$

$$\therefore AE \parallel BC \quad (\text{same base and area})$$

$$\therefore m_{AE} = m_{BC}$$

$$\frac{y_E - y_a}{x_E - x_A} = -\frac{1}{3}$$

$$\frac{0 - 5}{x_E - 4} = -\frac{1}{3}$$

$$\therefore 3(-5) = -1(x_E - 4)$$

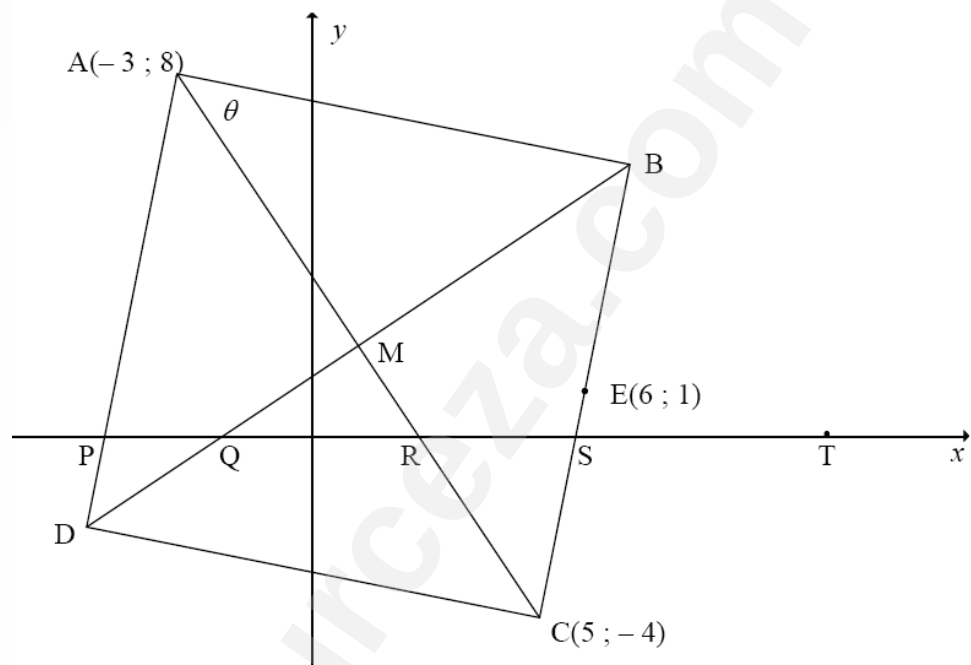
$$-15 = -x_E + 4$$

$$\therefore x_E = 19$$

$$\therefore E(19;0)$$

EXAMPLE 2 (Q5 NOVEMBER 2012)

ABCD is a rhombus with A(-3 ; 8) and C(5 ; -4). The diagonals of ABCD bisect each other at M. The point E (6; 1) lies on BC



- 2.1. Calculate the coordinates of M. Diagonals bisect each other at M: (2)

$$x_M = \frac{-3+5}{2} = 1$$

$$y_M = \frac{8+(-4)}{2} = 2$$

$$\therefore M(1;2)$$

- 2.2. Calculate the gradient of BC. (2)

$$m_{BC} = \frac{1+4}{6-5} = 5$$

- 2.3. Determine the equation of the line AD in the form $y = mx + c$. (3)

$$y - y_1 = m(x - x_1)$$

$$y - 8 = m(x + 3)$$

$$m_{AD} = m_{BC} = 5$$

Lines parallel

$$y - 8 = 5(x + 3)$$

$$y = 5x + 23$$

- 2.4. Determine the size of θ , that is $\hat{BAC}$. Show ALL calculations. (6)
 ABCD is a rhombus, therefore $AB = BC$

$$\theta = \hat{BCA} = \hat{ARS} - \hat{RSC}$$

$$= \hat{ARC} - \hat{BST}$$

$$\tan \hat{ARC} = m_{AC} = \frac{8+4}{-3-5}$$

$$\tan \hat{ARC} = -\frac{3}{2}$$

$$\hat{ARS} = 180^\circ - 56,3099\dots$$

$$\hat{ARS} = 123,69^\circ = 123,69^\circ$$

$$\tan \hat{BST} = m_{BC} = 5$$

$$\hat{BST} = 78,69^\circ$$

$$\theta = \hat{BCA} = 123,69^\circ - 78,69^\circ$$

$$\theta = 45^\circ$$

- 5 **The equation of the circle centred at point (a ; b)** $(x-a)^2 + (y-b)^2 = r^2$. In (a ; b) is at the origin. E.g. (0 ; 0), the circle equation becomes $x^2 + y^2 = r^2$.

- Determine the equation of the circle given: centre (a ; b) and a point (x ; y) on the circumference of the circle. Use the point to find the value of r^2 .
- Determine the centre and the radius given the equation of the circle in a form $ax^2 + by^2 + cx + dy + e = 0$, complete the square so that it in the form $(x-a)^2 + (y-b)^2 = r^2$ by grouping x and y together.
- To find a and b, given some points on the circumference of the circle always create two equations and solve them simultaneously, using the fact that radius of the same circle are equal in length.
- Points of intersection solve the two equations simultaneously.
- To show that a point lies inside, outside or on the circumference of the circle check : LHS = RHS : on the circumference of the circle

$$\text{LHS} > \text{RHS} \quad : \text{outside the circle}$$

$$\text{LHS} < \text{RHS} \quad : \text{inside the circle}$$

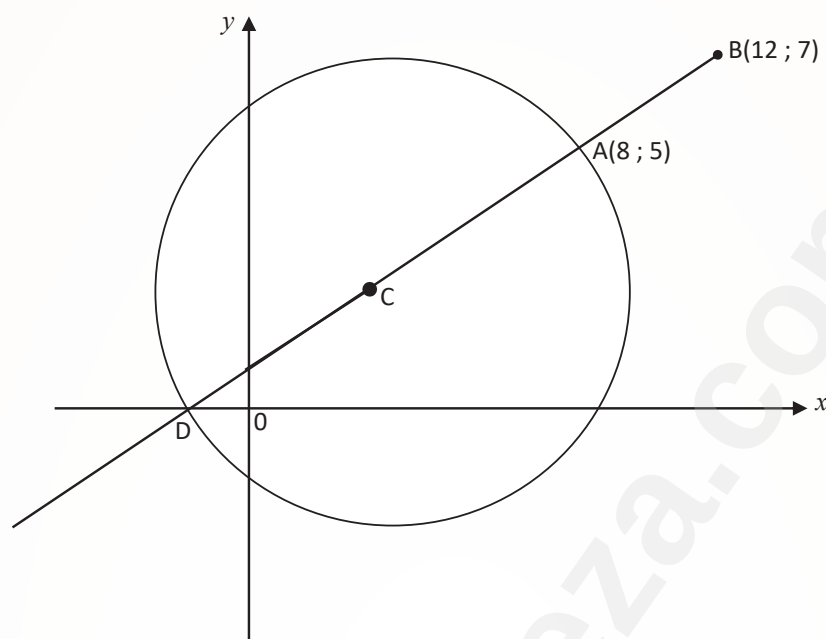
- Given two circles and to check whether they touch each other : is when the distance between their centres is equal to the sum of their radius.

$$d = C_1 \times C_2 = r_1 + r_2$$

- Two circles are CONCENTRIC when they have the same centre.

EXAMPLE 3

A(8 ; 5) and B(12 ; 7) Are two points in a Cartesian plane. BA produced intersects the x-axis at D. AD is a diameter of the circle centred at C.



- 3.1. Show that the equation of the line through A and B can be given as $x - 2y + 2 = 0$ (3)

$$m_{AB} = \frac{7-5}{12-8} = \frac{1}{2}$$

Equation of AB is

$$y - 5 = \frac{1}{2}(x - 8)$$

$$y = \frac{1}{2}x + 1$$

$$2y - x = 2$$

$$x - 2y + 2 = 0$$

- 3.2. Determine the coordinates of D. (2)

At D: $x - 2(0) + 2 = 0$

$$x = -2$$

$$D(-2 ; 0)$$

- 3.3 Determine the coordinates of C. (2)

$$C\left(\frac{8-2}{2}; \frac{5+0}{2}\right)$$

$$C\left(3; \frac{5}{2}\right)$$

- 3.4. Determine the equation of the circle. (5)

$$AC^2 = (8-3)^2 + \left(5 - \frac{5}{2}\right)^2$$

$$AC^2 = 25 + \frac{25}{4}$$

$$AC^2 = \frac{125}{4}$$

Equation of the circle is

$$(x-3)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{125}{4}$$

- 3.5. Determine the equation of the tangent passing through A(8 ; 5) (3)

Gradient of a tangent = -2 (tangent $\perp$ radius)

Equation of tangent is

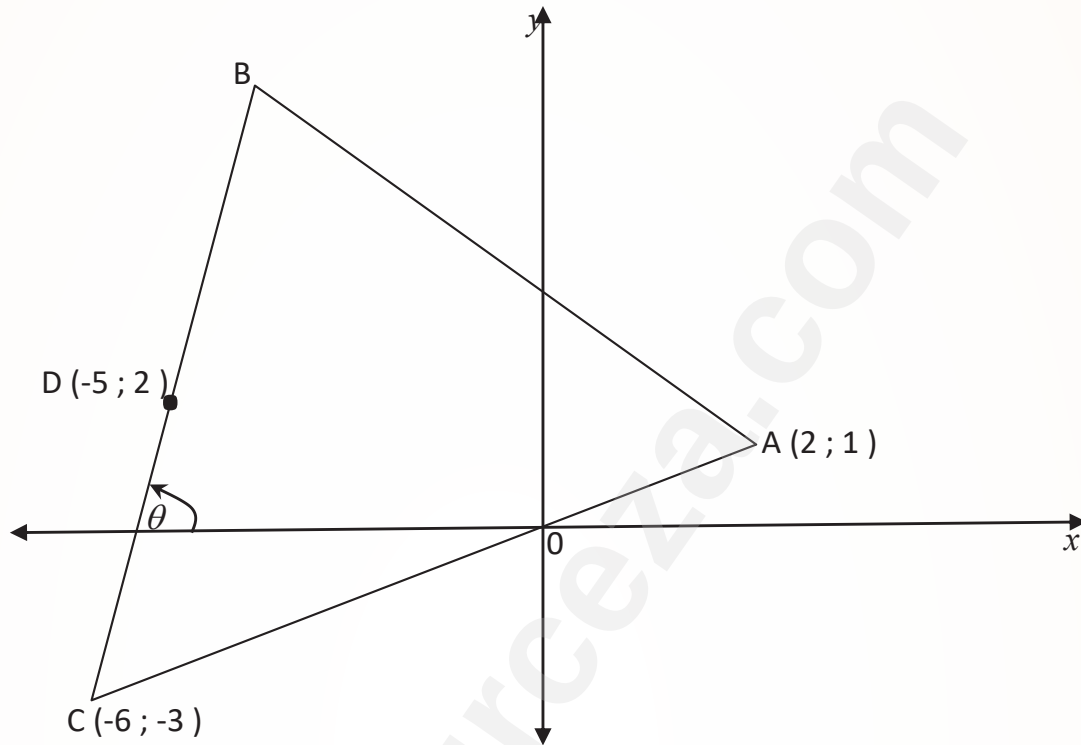
$$y - 5 = 2(x - 8)$$

$$y - 5 = -2x + 16$$

$$y = -2x + 21$$

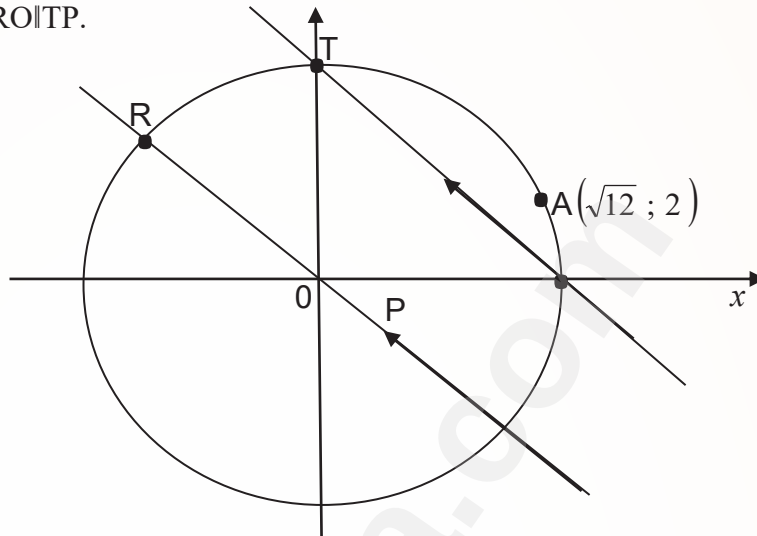
ACTIVITIES

1. In the diagram below, $A(2; 1)$, $C(-6; -3)$ and B are the vertices of $\triangle ABC$.
 $D(-5; 2)$ is the midpoint of BC . Straight line BC makes angle θ with the x -axis.

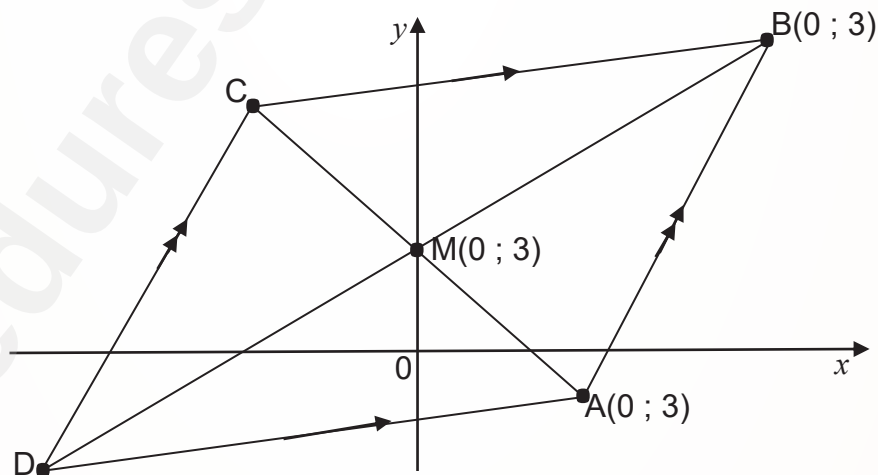


- 1.1 Calculate the length of AC , **without using a calculator**. (2)
- 1.2 Determine:
- 1.2.1 The gradient of DC (2)
- 1.2.2 The size of θ , rounded off to one decimal digit (2)
- 1.2.3 The value of k if D , C and $E(-3; k)$ are collinear (3)
- 1.2.4 The coordinates of B (2)
- 1.3 Determine whether point $(1; -3)$ lies on the straight line $2x + y + 5 = 0$ (2)

2. In the diagram below, $A(\sqrt{12}; 2)$ is a point on the circle with centre $O(0; 0)$. The circle cuts the x-axis and y-axis at P and T respectively. Straight line RO intersects the circle at R . $RO \parallel TP$.



- 2.1 Show that the equation of the circle is given by $x^2 + y^2 = 16$ (2)
- 2.2 Write down the coordinates of P and T . (4)
- 2.3 Hence write down the equation of the tangent to the circle at T . (2)
- 2.4 Determine:
- 2.4.1 The gradient of RO (3)
- 2.4.2 The equation of RO (2)
3. In the diagram below, $ABCD$ is a parallelogram, with $A(4; -1)$ and $B(6; 9)$. The diagonals AC and BD intersect at $M(0; 3)$.



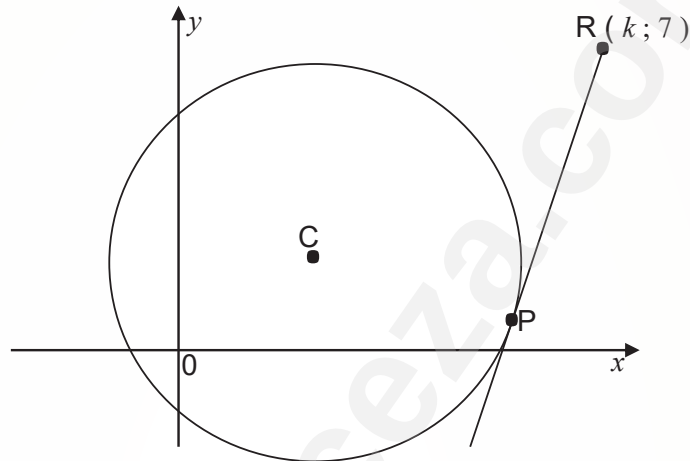
- 3.1 Calculate the coordinates of C . (2)
- 3.2 Determine the equation of the straight line CD . (4)

3.3 Prove analytically that $\hat{A}MB = 90^\circ$ (3)

3.4 Calculate the of $\hat{A}BM$ (round off to ONE decimal digit) (6)

3.5 Determine the value of k if A, B and P (k ; 5) are collinear. (3)

4. In the diagram below, a circle with centre C and with equation $x^2 + y^2 - 6x - 2y + 1 = 0$ is given. PR is tangent to the circle at P.



4.1 Determine the coordinates of C and the length of the radius. (4)

4.2 The length of a tangent from point R (k ; 7) to point P is $2\sqrt{13}$ units.

If $k > 0$ show that $k = 8$ (5)

4.3 B (-2 ; 3) is the centre of a circle and A (1 : 1) is a point on the circle.

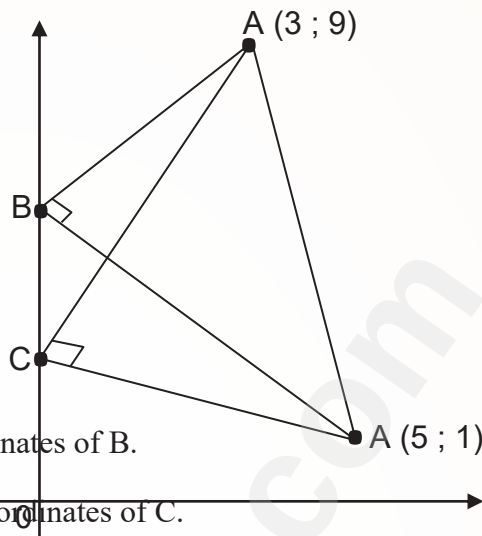
Calculate:

4.3.1 The length of the diameter of the circle (4)

4.3.2 The equation of the tangent to the circle at A (4)

5 In the diagram along side,
 $A(3; 9)$ and $D(5; 1)$ are points in a
 Cartesian plane. B and C are points
 on the y-axis such that

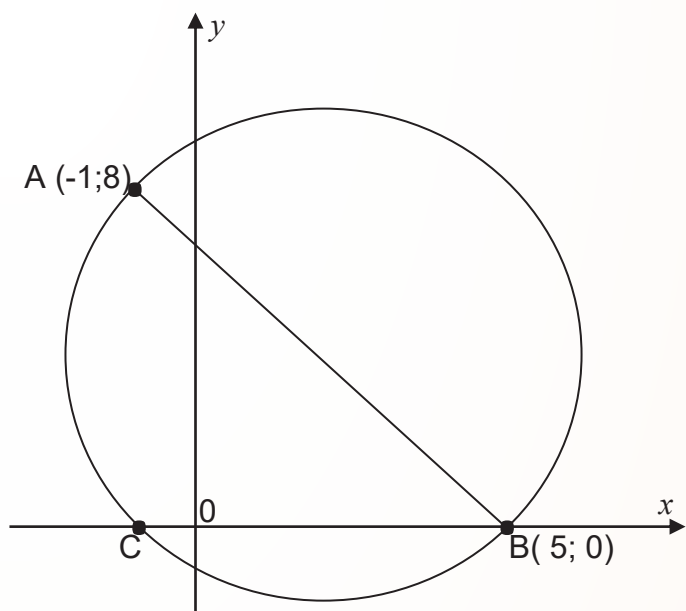
$$\angle ABD = \angle ACD = 90^\circ$$



- 5.1 Show that $(0; 6)$ are the coordinates of B. (6)
- 5.2 Hence, or otherwise, find the coordinates of C. (2)
- 5.3 Calculate the coordinate of M, the mid-point of AD. (2)
- 5.4 Calculate the length of radius of circle ABD,
 Leave your answer in simplified surd form. (2)
- 5.5 Will circle ABD pass through point C? Give reason for your answer. (2)
- 5.6 Calculate the size of $\angle BDA$, rounded off to the nearest degree. (7)
- 5.7 Find the equation of a straight line which is parallel to BD and which passes
 through point A. (4)

6 In the diagram alongside,
 diameter AB of circle ACB
 in a Cartesian plane is given.
 The coordinates of A and B are
 $(-1; 8)$ and $(5; 0)$
 respectively.

C is a point on the x-axis.



- 6.1 Determine:

6.1.1 The equation of circle ACB (5)

6.1.2 The coordinates of C (3)

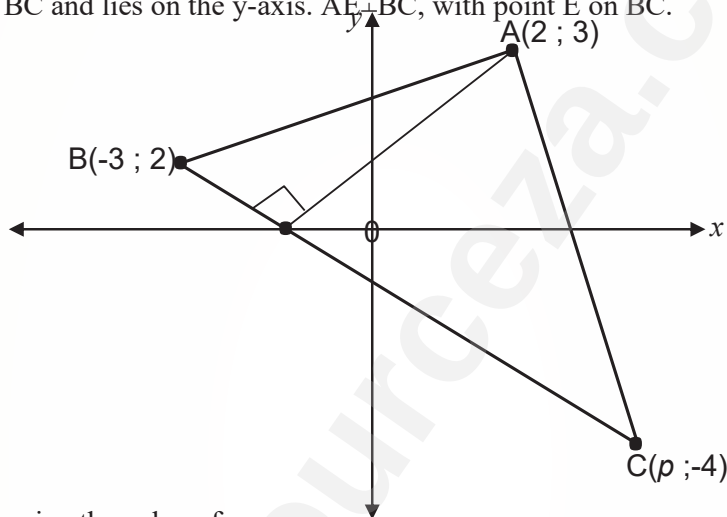
6.1.3 The equation of the tangent to the circle at C (4)

6.2 Another circle with the equation $x^2 - 4x + y^2 - 8y - 29 = 0$ is given.

6.2.1 Determine whether the two circles are concentric. (4)

6.2 Determine, with reasons, whether this circle lies inside or outside circle ACB. (2)

7. In the diagram points A (2 ; 3), B (- 3 ; 2) and C (p ; - 4) are drawn. D is the midpoint of BC and lies on the y-axis. $AE \perp BC$, with point E on BC.



7.1 Determine the value of p. (3)

7.2 Determine the equation of line BC. (3)

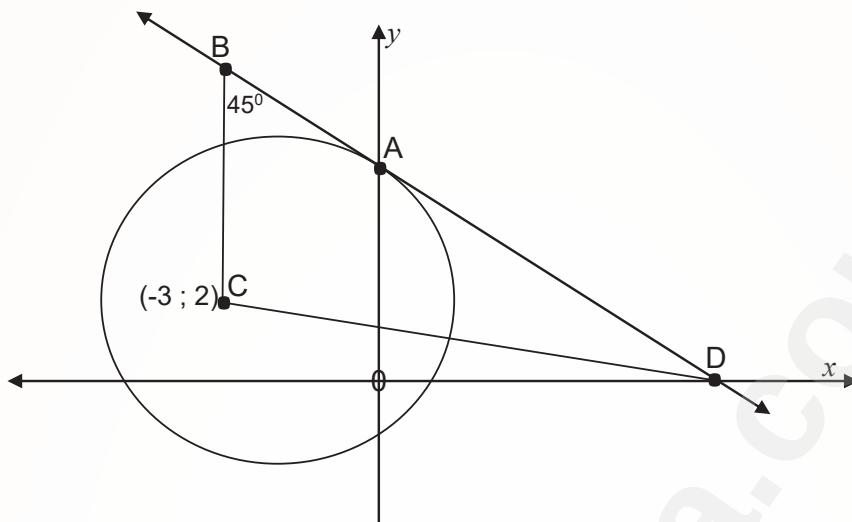
7.3 Determine the equation of line AE, the altitude of $\triangle ABC$ (4)

7.4 Determine the coordinates of E, the point of intersection of BC and AE. (4)

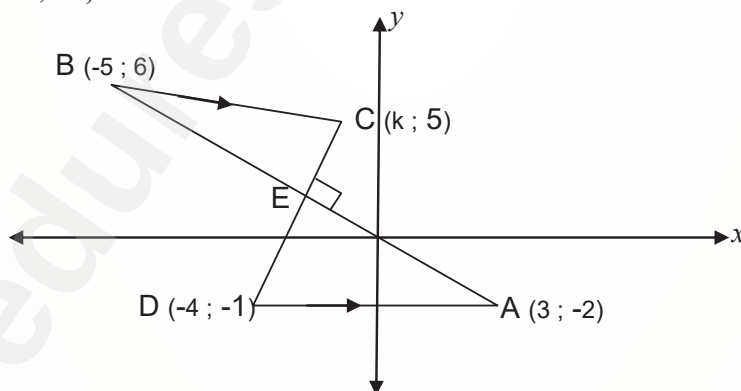
7.5 Calculate the area of $\triangle ABD$ (5)

7.6 Write down the coordinate of F so that ABFD is a parallelogram. (3)

- 8 The equation of the tangent AB to the circle is $4y + 3x = 24$. The point of contact, is also the y-intercept of the tangent. C (- 3 ; 2) is the centre of the circle. $\hat{ABC} = 45^\circ$.

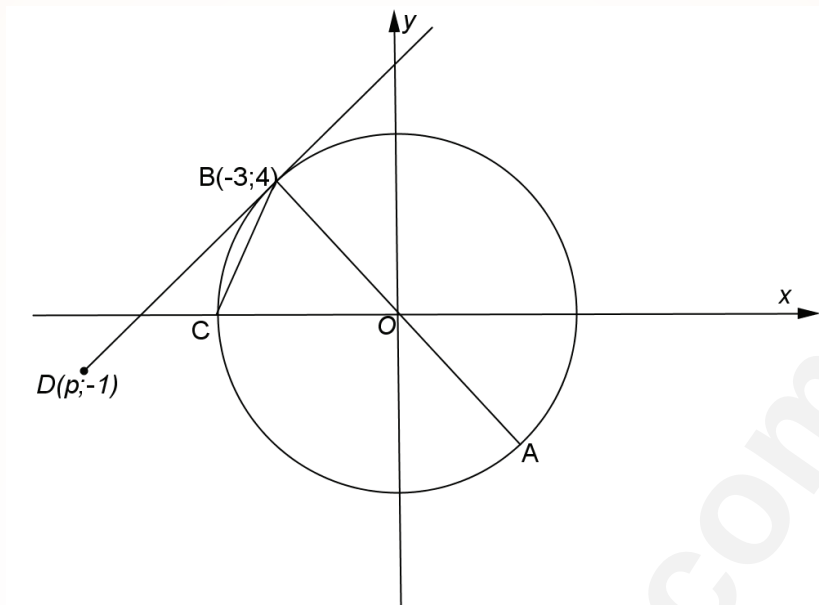


- 8.1 Determine the coordinates of A. (2)
- 8.2 Show that the equation of the circle is: $x^2 + 6x + y^2 - 4y - 12 = 0$ (3)
- 8.3 If $x + k = 0$ is a tangent to the circle, determine the value(s) of k. (2)
- 8.4 Determine the length of AB. (6)
- 8.5 Determine the coordinates of B. (5)
9. The diagram below is drawn to scale. It shows points A (3 ; - 2), B (- 5 ; 6), and D (- 4 ; - 1). $AD \parallel CB$ and $AB \perp CD$.



- 9.1 Determine the gradient of line AB. (2)

- 9.2 Determine the equation of line DC (3)
- 9.3 Determine the value of k. (2)
- 9.4 Determine θ , the acute angle formed between BC and BA, correct to 1 decimal place. (4)
- 9.5 What type of quadrilateral is BCAD? Give reasons. (3)
- 9.6 Determine the coordinates of E, the intersection of AB and CD. (2)
- 9.7 Determine the area of $\triangle ADE$. (6)
- 10 A circle with the equation $x^2 + y^2 - 4x + 2y + k = 0$ passes through the point (1 ; 3). Determine:
- 10.1 the centre of the circle (4)
- 10.2 the value of k (2)
- 10.3 the radius of the circle in (in surd form) (2)
- 10.4 the equation of the normal to the circle at (1 ; 3). (4)
- 11 Prove that the circles $x^2 + y^2 = 16$ and $(x - 4)^2 + (y + 3)^2 = 1$ touch each other. (5)
- 12 In the diagram below, the origin O is the centre of circle. A (x ; y) and B (- 3 ; 4) are points on the circle, C is an x-intercept and AOB is a diameter of the circle. D is the point (p ; -1) and BD is a tangent to the circle. at B.

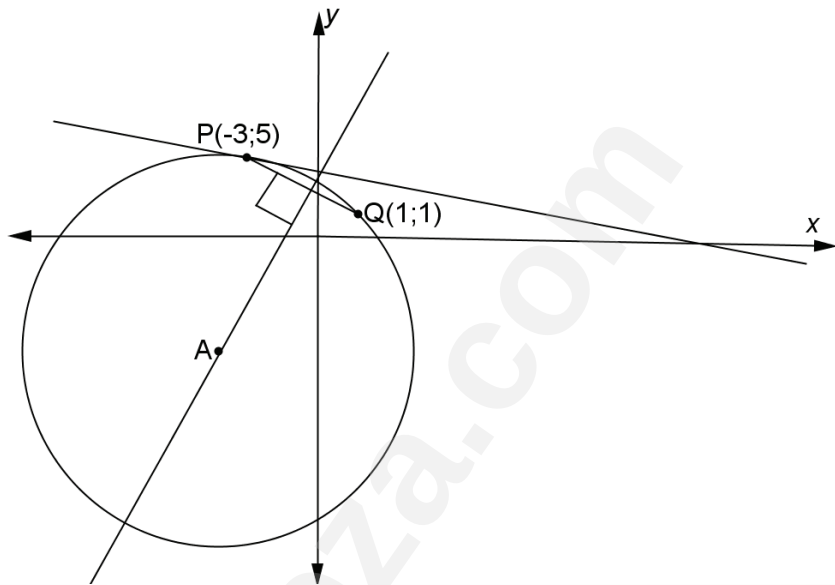


Determine:

- 12.1 the equation of the circle (3)
 - 12.2 the coordinates of A (2)
 - 12.3 the coordinates of C (2)
 - 12.4 the equation of AB (2)
 - 12.5 the equation of the tangent BD (3)
 - 12.6 the value of p. (2)
 - 12.7 the length of BC (in surd form). (3)
13. The equation of straight line L is $2y - 5x = 6$
 - 13.1 If p (4 ; a) is a point on L, prove that $a = 13$ (2)
 - 13.2 Q has coordinates (b ; 9) and PQ is perpendicular to L.
Calculate the value of b. (4)
 - 13.3 Determine the equation of a circle centred at P and passing through Q. (5)

- 14 P (- 3 ; 2) and Q (1 ; 1) are two points on a circle with centre A

The equation of the tangent to the circle at P is $2y + x - 7 = 0$.



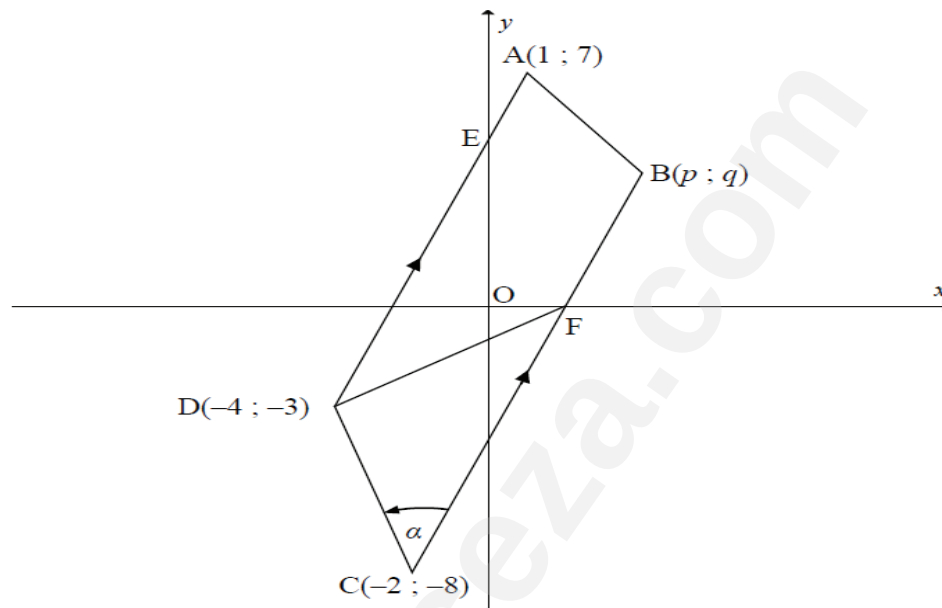
Determine:

- | | | |
|------|---|-----|
| 14.1 | the equation of PA | (4) |
| 14.2 | the equation the perpendicular bisector of PQ | (5) |
| 14.3 | the coordinates of A | (4) |
| 14.4 | the length of AQ | (2) |
| 14.5 | the equation of the circle | (3) |

QUESTION 1 (Q4 FEBRUARY/MARCH 2013)

In the diagram below, trapezium ABCD with $AD \parallel BC$ is drawn. The coordinates of the vertices are $A(1 ; 7)$; $B(p ; q)$; $C(-2 ; -8)$ and $D(-4 ; -3)$. BC intersects the x -axis at F.

$$\angle DCB = \alpha$$

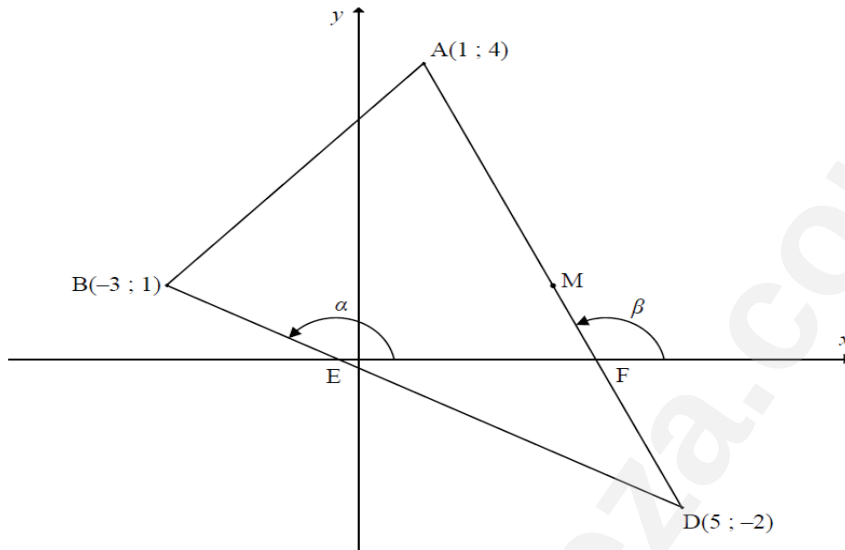


- 1.1. Calculate the gradient of AD. (2)
 - 1.2. Determine the equation of BC in the form of $y = mx + c$. (3)
 - 1.3. Determine coordinates of the point F. (2)
 - 1.4. $AB'CD$ is a parallelogram with B' on BC . Determine the coordinates of B' , using a transformation $(x; y) \rightarrow (x + a; y + b)$ that sends A to B' (2)
 - 1.5. Show that $\alpha = 48,37^\circ$ (4)
 - 1.6. Calculate the area of $\triangle DCF$ (6)
- [19]**

QUESTION 2 (Q5 FEBRUARY/MARCH 2012)

In the figure below, $A(1;4)$, $B(-3;1)$ and $D(5;-2)$ are the coordinates of the vertices $\triangle ABD$.

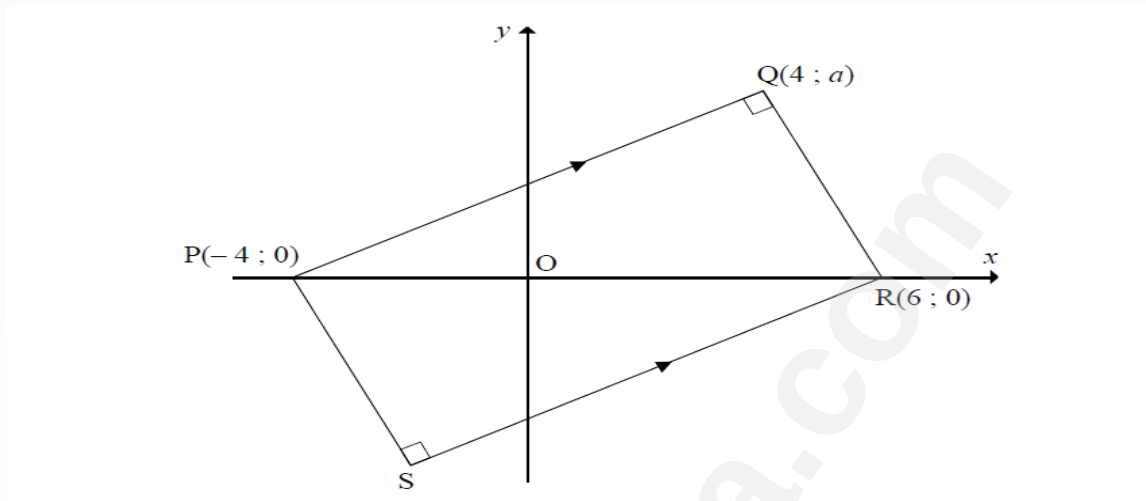
- BD and AD intersect the x axis at E and F respectively
- The angle of inclination of BD with the x axis at E is α
- The angle of inclination of AD with the x axis at F is β



- 2.1. Calculate the gradient of AD (2)
- 2.2. Determine the length of the line segment AD . (Leave your answer in surd form, if necessary) (2)
- 2.3. Determine the coordinates of M , the midpoint of AD (2)
- 2.4. C is a point such that the line BC is parallel to AD . Determine the equation of the line BC in the form of $ax + by + c = 0$ (3)
- 2.5.
 - 2.5.1. Calculate the size of β (2)
 - 2.5.2. Calculate ALL the angles of $\triangle DEF$. (5)
- 2.6. Determine the equation of a circle, with centre M , which passes through the point A and D . Give your answer in the form: $(x-a)^2 + (y-b)^2 = r^2$ (2)
- 2.7. Does the point B lie inside, outside or the circle in Question 2.6? Show ALL calculations to justify your answer. (2)

QUESTION 4 (Q5 NOVEMBER 2011)

In the diagram below, PQRS is a rectangle with vertices $P(-4;0)$, $Q(4;a)$, $R(6;0)$ and S. Q lies in the first quadrant

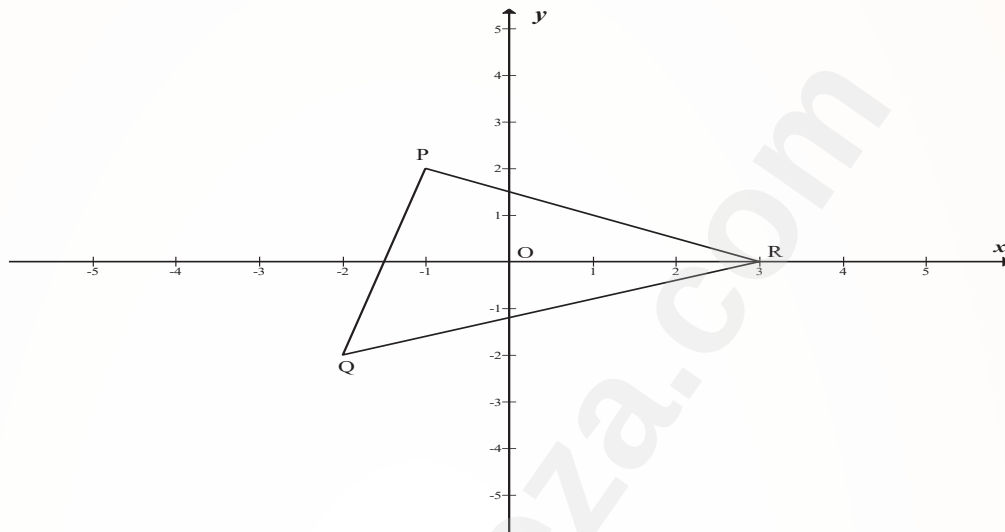


- 4.1. Show that $a = 4$ (4)
- 4.2. Determine the equation of the straight line passing through the points S and R in the form $y = mx + c$ (4)
- 4.3. Calculate the coordinates of S (4)
- 4.4. Calculate the length of PR (2)
- 4.5. Determine the equation of the circle that has diameter PR.
Give the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$ (3)
- 4.6. Show that Q is a point on the circle in Question 4.5. (2)
- 4.7. Rectangle PQRS undergoes the transformation $(x; y) \rightarrow (x+k; y+l)$ where k and l are numbers. What is the minimum value of $k+l$ so that the image of PQRS lies in the first quadrant (that is $x \geq 0$)? (3)

[22]

QUESTION 5 (Q4 FEBRUARY/MARCH 2011)

In the diagram below $\triangle PQR$ with vertices $P(-1; 2)$, $Q(-2; -2)$ and $R(3; 0)$ is given



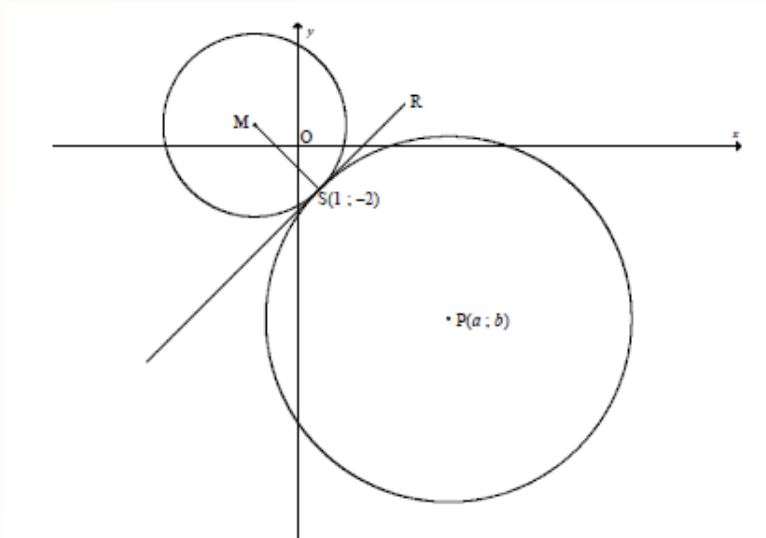
- 5.1. Calculate the angle that PQ makes with the positive x -axis. (3)
- 5.2. Determine the coordinates of M, the midpoint of PR. (2)
- 5.3. Determine the perimeter of $\triangle PQR$ to the nearest whole number. (5)
- 5.4. Determine an equation of the line parallel to PQ that passes through M. (3)

[13]**QUESTION 7 (Q6 FEBRUARY/MARCH 2012)**

In the figure below, a circle with centre M is drawn. The equation of the circle is

$$(x+2)^2 + (y-1)^2 = r^2$$

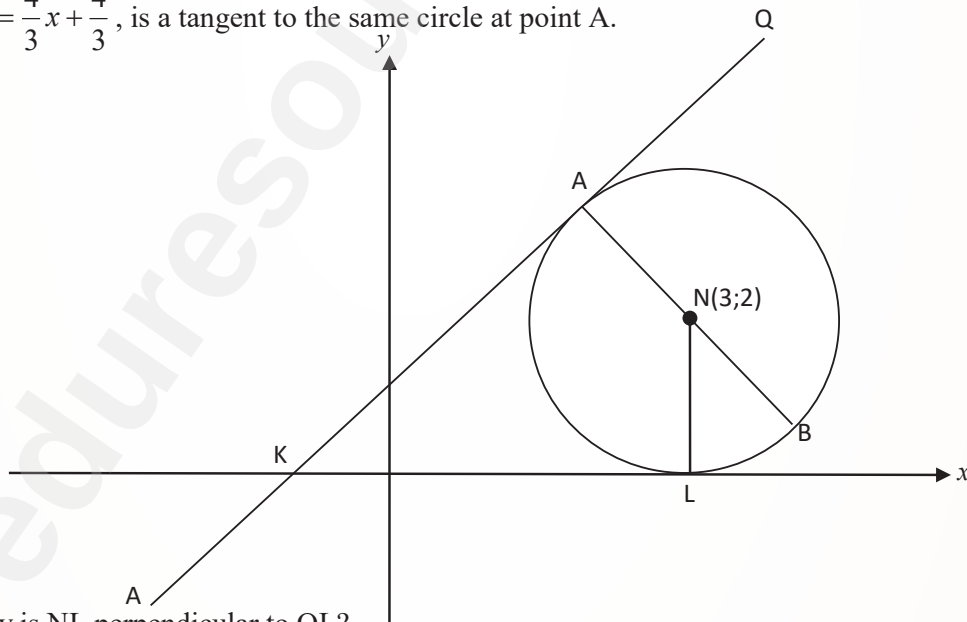
$S(1; -2)$ is a point on the circle. SR is a tangent to the circle.



- 7.1. Write down the coordinates of M (4)
- 7.2. Determine the equation of the tangent SR in the form $y=mx+c$ (4)
- 7.3. The circles having centres P and M touch externally at points S . SR is a tangent to both these circles. If $MS : MP = 1 : 3$, determine the coordinates $(a; b)$ of point P (8)
- [16]**

QUESTION 8 (Q6 NOVEMBER 2012)

A circle centred at $N(3; 2)$ touches the x -axis at point L . The line PQ , defined by the equation $y = \frac{4}{3}x + \frac{4}{3}$, is a tangent to the same circle at point A .



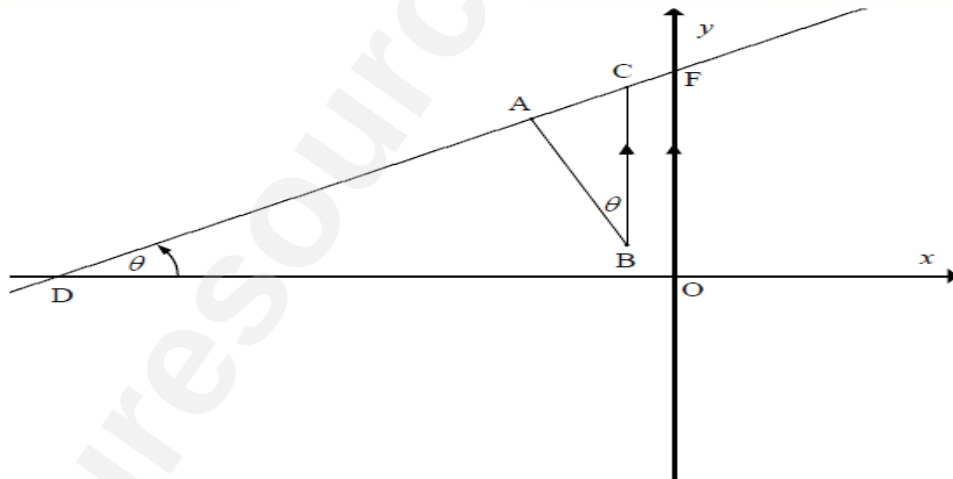
- 8.1. Why is NL perpendicular to OL ? (1)

- 8.2. Determine the coordinates of L. (1)
- 8.3. Determine the equation of the circle with centre N in the form $(x - a)^2 + (y - b)^2 = r^2$ (3)
- 8.4. Calculate the length of KL. (3)
- 8.5. Determine the equation of the diameter AB in the form $y = mx + c$. (4)
- 8.6. Show that the coordinates of A are $\left(\frac{7}{5}; \frac{16}{5}\right)$. (3)
- 8.7. Calculate the length of KA. (3)
- 8.8. Why is KLNA a kite? (2)
- 8.9. Show that $\hat{ABK} = 45^\circ$. (3)
- 8.10. If the given circle is reflected about the x-axis, give the coordinates of the centre of the new circle. (1)

[24]

QUESTION 9 (Q6 NOVEMBER 2011)

The circle with centre $B(-1;1)$ and radius $\sqrt{20}$ is shown. BC is parallel to the y -axis and $CB=5$
 The tangent to the circle at A passes through C
 $\hat{ABC} = \hat{ADO} = \theta$



- 9.1. Determine the coordinates of C. (2)
- 9.2. Calculate the length of CA. (3)
- 9.3. Write down the value of $\tan \theta$ (1)
- 9.4. Show that the gradient of AB is -2 . (2)
- 9.5. Determine the coordinates of A (6)
- 9.6. Calculate the ratio of the area of $\triangle ABC$ to the area $\triangle ODF$. Simplify your answer (5)

QUESTION 10 (Q 5 FEBRUARY/MARCH 2011)

10.1. The equation of a circle is $x^2 + y^2 - 8x + 6y = 15$.

9.1.1. Prove that the point $(2; -9)$ is on the circumference of the circle. (2)

9.1.2. Determine an equation of the tangent to the circle at the point $(2; -9)$. (7)

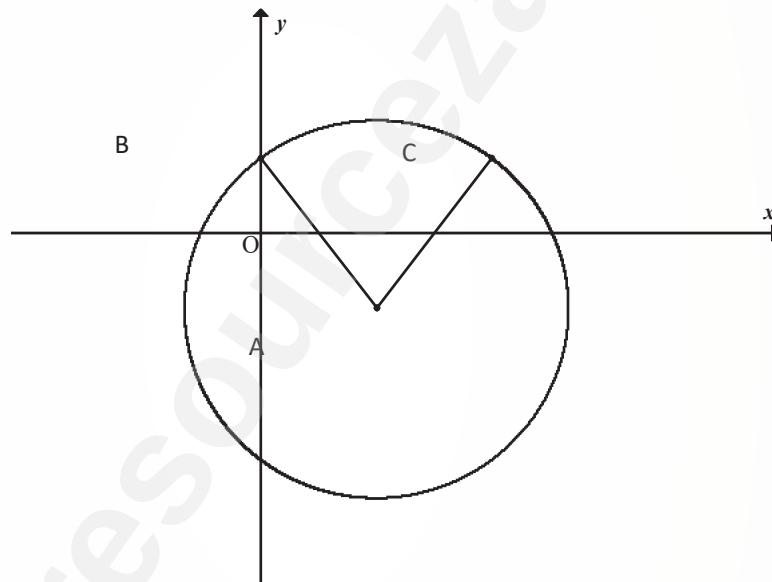
10.2. Calculate the length of the tangent AB drawn from the point $A(6; 4)$ to the circle with equation $(x - 3)^2 + (y + 1)^2 = 10$. (5)

[14]

QUESTION 11 (Q6 FEBRUARY/MARCH 2011)

The circle, with centre A and equation $(x - 3)^2 + (y + 2)^2 = 25$ is given in the following diagram.

B is a y-intercept of the circle



11.1. Determine the coordinates of B. (4)

11.2. Write down the coordinates of C, if C is the reflection of B in the line $x = 3$. (2)

11.3. The circle is enlarged through the origin by a factor of $\frac{3}{2}$. Write down the equation of the new circle, centre A' , in the form $(x - a)^2 + (y - b)^2 = r^2$. (2)

11.4. In addition to the circle with centre A and equation $(x - 3)^2 + (y + 2)^2 = 25$, you are given the circle $(x - 12)^2 + (y - 10)^2 = 100$ with centre B.

11.4.1. Calculate the distance between the centres A and B. (2)

11.4.2. In how many points do these two circles intersect? Justify your answer. (2)

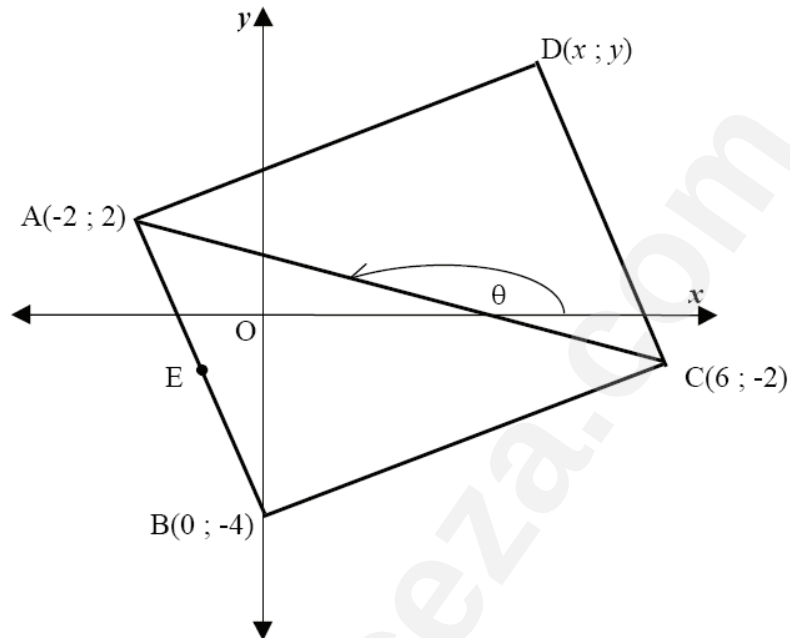
[12]

PREP

(WCED)

QUESTION 1

In the diagram below, ABCD is a quadrilateral with vertices
 $A(-2; 2)$, $B(0; -4)$, $C(6; -2)$ and $D(x; y)$



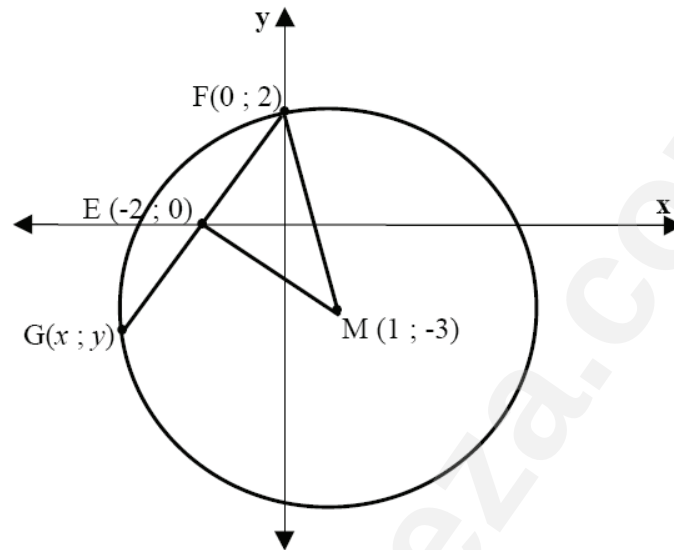
- 1.1 Calculate the length of BC. (Leave answer in surd form.) (2)
- 1.2 Show that $\triangle ABC$ is a right angled triangle. (5)
- 1.3 Calculate the area of $\triangle ABC$. (3)
- 1.4 Determine the equation of AC. (4)
- 1.5 Determine θ , the angle of inclination of AC. (Round off to the nearest degree) (3)
- 1.6 If E is the midpoint of AB, determine the coordinates of E. (3)
- 1.7 If ABCD is a rectangle, determine the coordinates of D. (4)

[24]

QUESTION 2

In the diagram below, $F(0 ; 2)$ and $G(x ; y)$ are points on the circle with midpoint $M(1 ; -3)$.

$E(-2 ; 0)$ is the midpoint of FG .

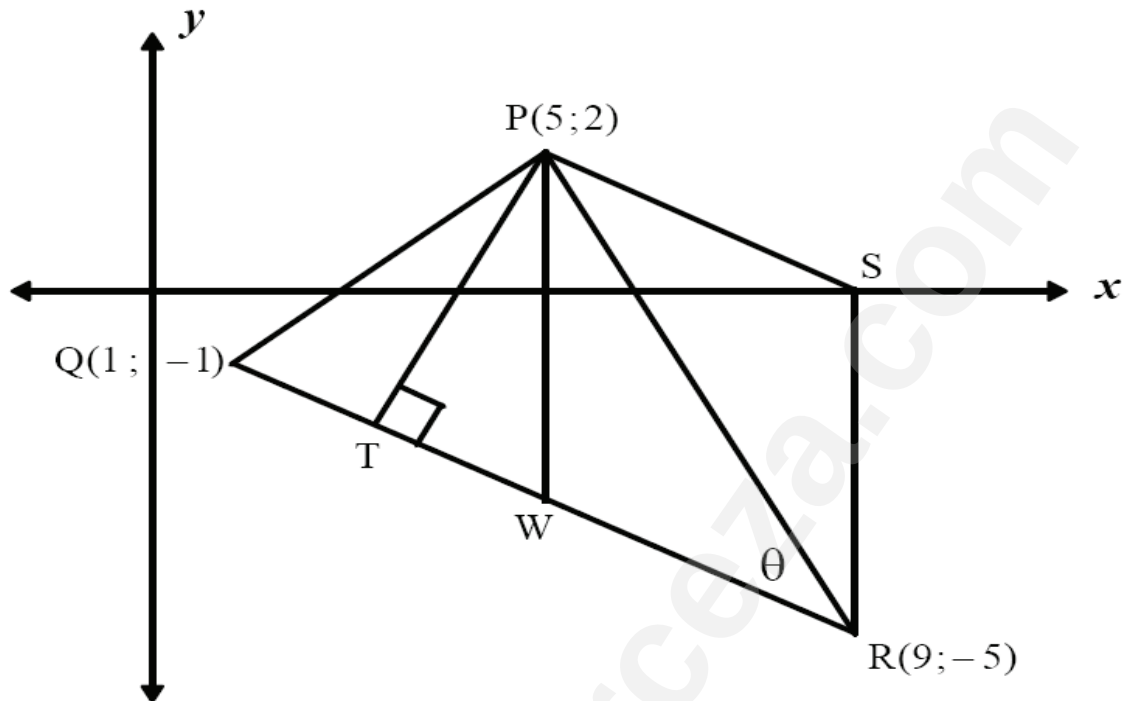


- 2.1 Determine the equation of the circle. (4)
- 2.2 Calculate the coordinates of G . (4)
- 2.3 Determine the equation of the tangent to the circle at F . (4)
- 2.4 Does the point $H(4 ; 7)$ lie inside or outside the circle? Show all your calculations to justify your answer. (4)

[16]

QUESTION 1

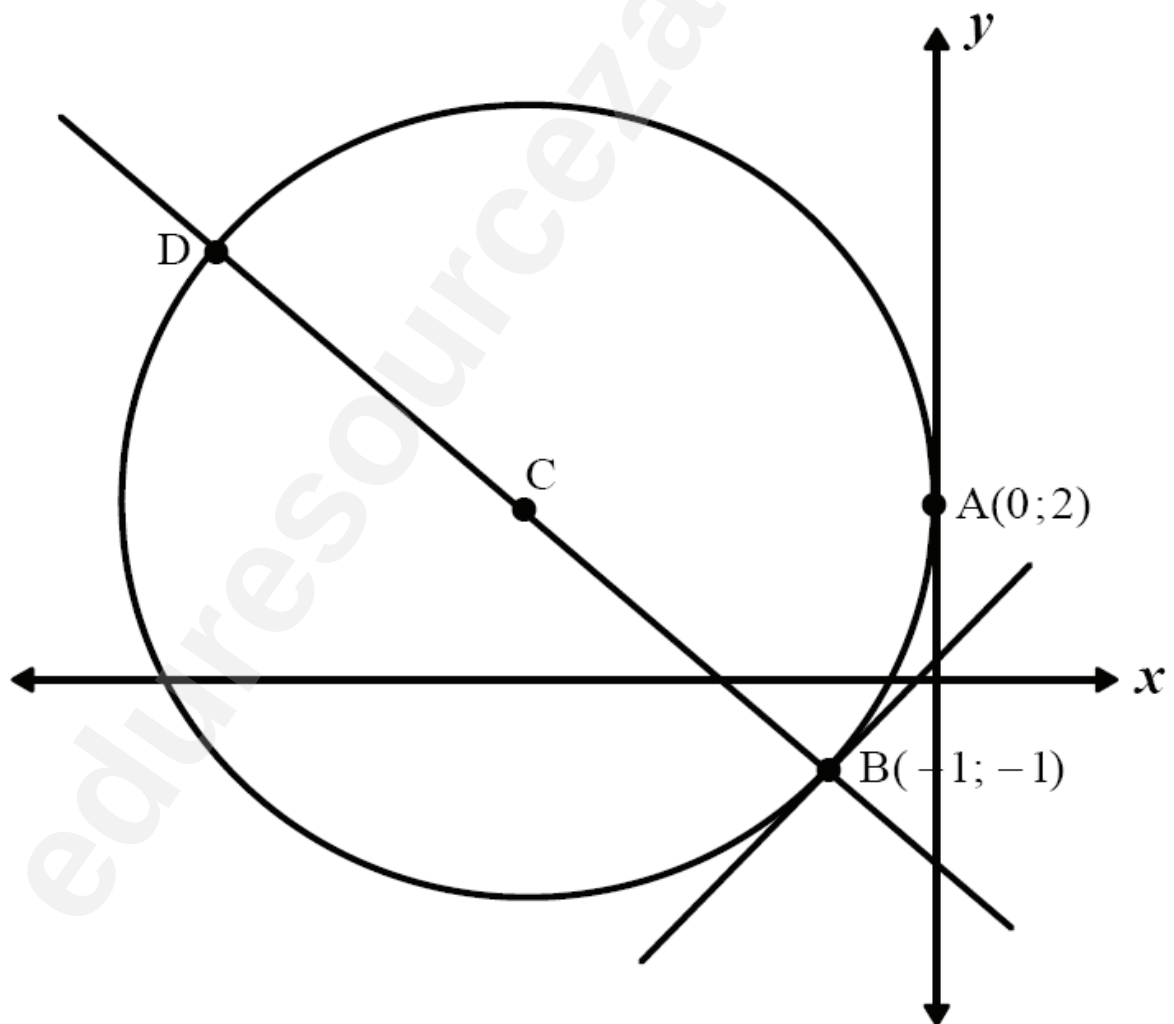
In the diagram, PQRS is a trapezium with vertices $P(5;2)$, $Q(1;-1)$, $R(9;-5)$ and S . PT is the perpendicular height of PQRS and W is the midpoint of QR . Point S lies on the x -axis and $\hat{P}RQ = \theta$.



- 1.1 Determine the equation of PW if W is the midpoint of QR . (2)
- 1.2 Determine the equation of PS . (4)
- 1.3 Determine the equation of PT . (3)
- 1.4 Determine the coordinates of T . (5)
- 1.5 Show that $QT = \frac{1}{3}TR$. (5)
- 1.6 Calculate the size of θ rounded off to two decimal places. (5) [24]

QUESTION 2

- 2.1 Calculate the value of k if the points $A(6;5)$, $B(3;2)$ and $C(2k;k+4)$ are collinear. (3)
- 2.2 A circle with equation $x^2 + y^2 - 4x + 6y + 4 = 0$ is rotated 90° clockwise about the origin and then enlarged through the origin by a scale factor of 2 units. Determine the equation of the new circle formed under this transformation. (6)
- 2.3 In the diagram below, a circle centre C touches the y -axis at $A(0;2)$. A straight line with equation $3x + 4y = -7$ cuts the circle at $B(-1;-1)$ and D .



2.3.1 Determine the equation of the tangent to the circle at B. (4)

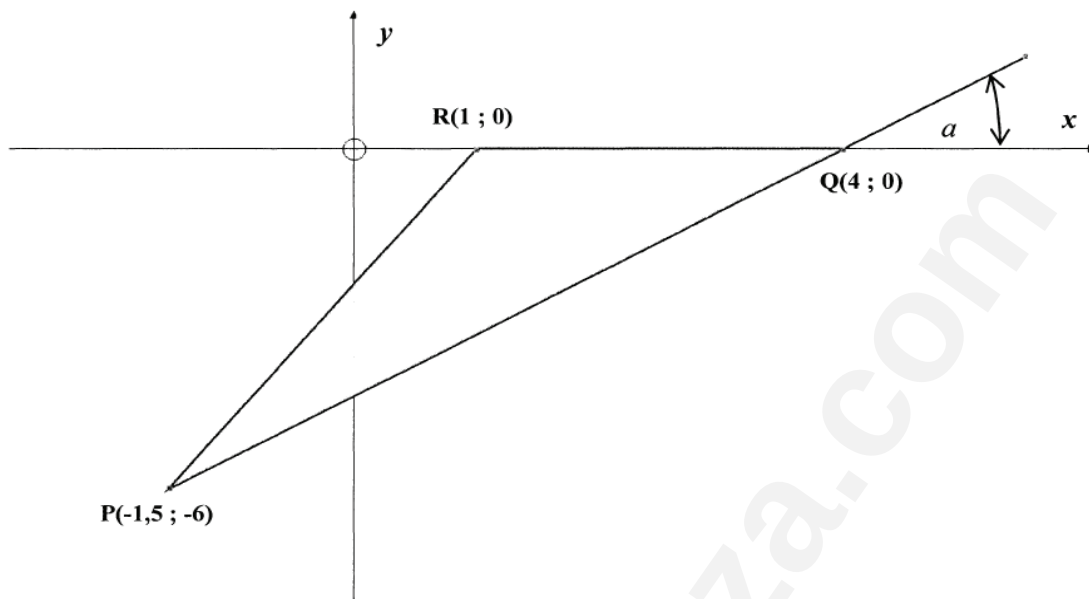
2.3.2 Determine the equation of the circle in the form
 $(x - a)^2 + (y - b)^2 = r^2$. (5)

2.3.3 Determine the coordinates of D. (4) [22]

PREP (GAUTENG) 2010

QUESTION 4

In the diagram below the vertices of $\triangle PQR$ are $P(-1,5;-6)$; $Q(4;0)$ and $R(1;0)$.
Inclination of PQ is α .



- 4.1 Calculate (2)
- 4.1.1 the gradient of PQ . (2)
- 4.1.2 the size of $\hat{PQR}$, correct to two decimal places. (3)
- 4.1.3 the distance PQ , correct to two decimal places. (3)
- 4.1.4 the area of $\triangle PQR$. (3)
- 4.2 Determine the equation of the line that bisects $\hat{PQR}$. (4)
- 4.3 Write down the coordinates of a point $T(x; y)$ with $x > 0$, such that $PTQR$ is a parallelogram. (2)
- 4.4 Write down the coordinates of M , the mid-point of PQ . (2)
- 4.5 4.5.1 Given the point $W(x; y)$ such that $WQ = 2WR$. Use analytical methods to find an expression in terms of x and y that will satisfy the given relationship. (4)
- 4.5.2 Describe fully the shape traced by the point W . (3)
- [26]

QUESTION 5

- 5.1 AB is a tangent to the circle defined by $x^2 + y^2 + 2x - 4y - 3 = 0$ at B. Point B(x ; y) lies on the circumference of the circle. A is the point (8 ; 4).
- 5.1.1 Calculate the coordinates of the centre P and the length of the radius of the circle. (Leave answer in the simplest surd form.) (4)
- 5.1.2 Hence or otherwise, calculate the length of AB. (Leave answer in the surd form.) (6)
- 5.2 Calculate the value of k if the straight line defined by $y = 3x + 2$ is a tangent to the circle $x^2 + y^2 = k$, at P. (4)
- [14]**

GN (D1) STRATEGY FOR LEARNER ATTAINMENT

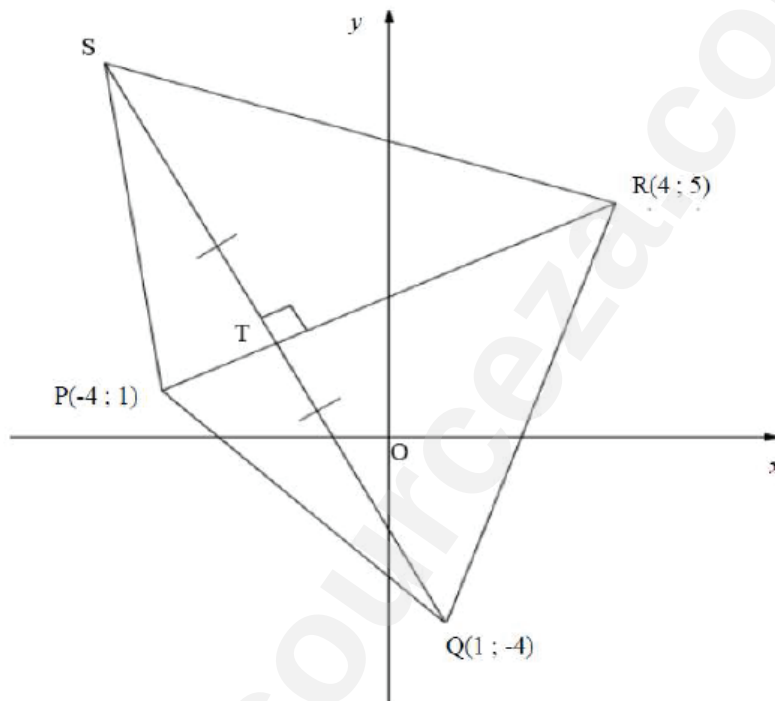
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TOPIC REVISION_ANALYTICAL GEOMETRY

PREP 2016_EASTERN CAPE

QUESTION 3

In the diagram below $P(-4; 1)$, $Q(1; -4)$, $R(4; 5)$ and S are the vertices of quadrilateral PQRS. $SQ \perp PR$ with T on PR . T is the midpoint of line SQ .



- 3.1 Determine the gradient of PR . (2)
 - 3.2 Hence, determine the equation of line SQ . (3)
 - 3.3 Show that the coordinates of T are $(-2; 2)$. (5)
 - 3.4 Hence, determine the coordinates of S . (3)
 - 3.5 Calculate the area of $\triangle PQS$. (5)
- [18]**

Mathematics

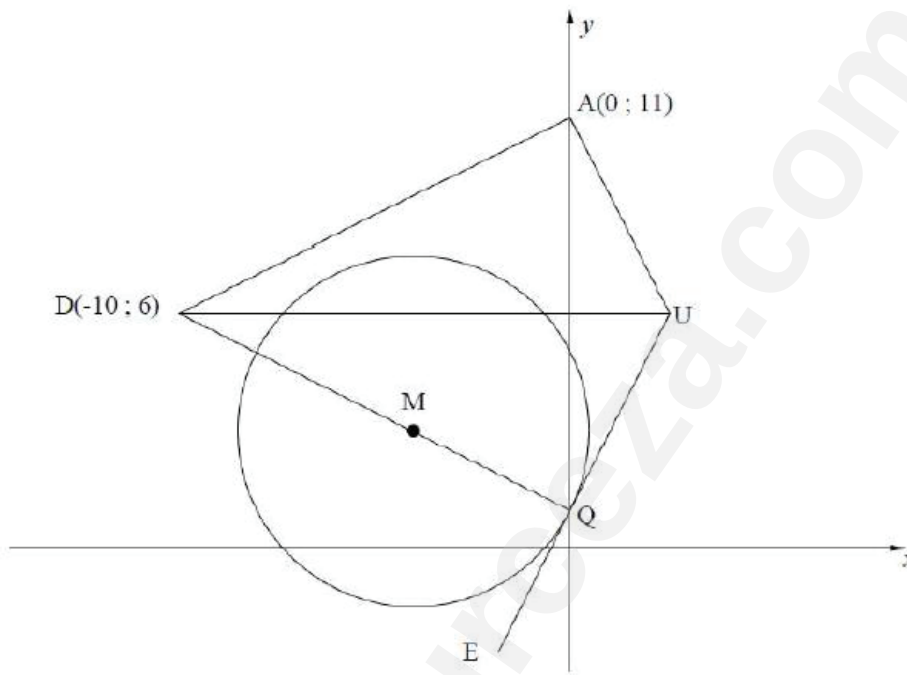
Grades 11 & 12

Gauteng North Strategy for Learner Attainment

2

QUESTION 4

$x^2 + y^2 + 8x - 6y = -5$, is the equation of the circle with centre M. UE is a tangent to the circle at Q. QMD, DA, AU and UQE are straight lines. DU is parallel to the x-axis.



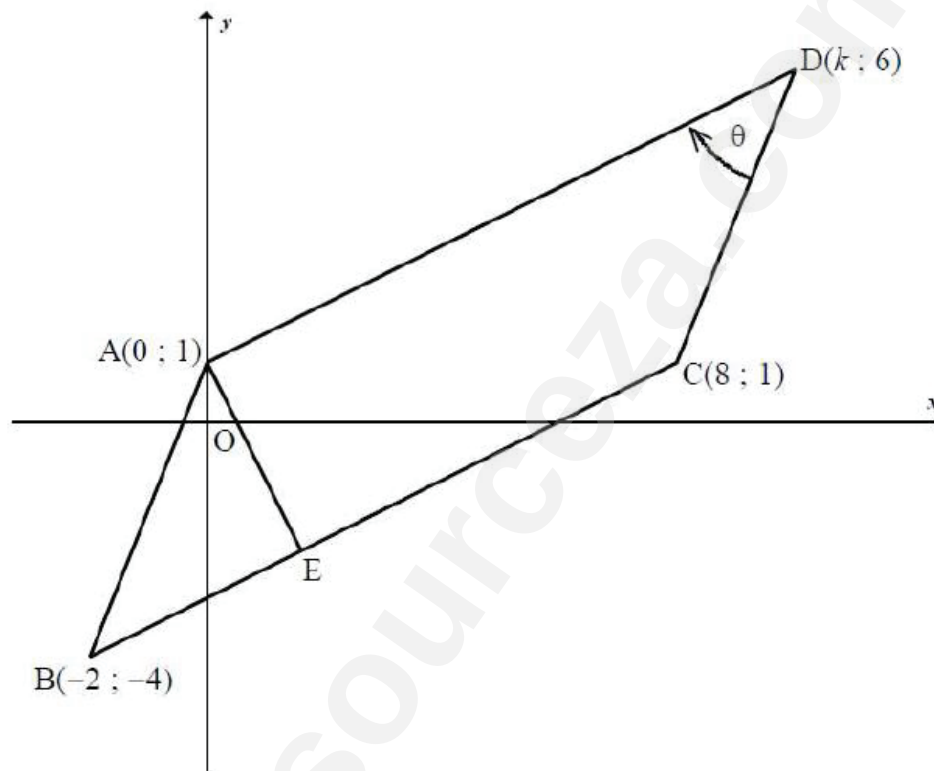
- 4.1 Determine the coordinates of M, the centre of the circle. (4)
- 4.2 Calculate the coordinates of Q, if $y < 2$. (3)
- 4.3 Calculate the equation of tangent UE. (4)
- 4.4 Write down the equation of DU. (1)
- 4.5 Calculate the coordinates of U. (2)
- 4.6 Prove that QUAD is a cyclic quadrilateral. (6)

[20]

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QUESTION 3

In the figure below, ABCD is a parallelogram with vertices:
 $A(0; 1)$, $B(-2; -4)$, $C(8; 1)$ and $D(k; 6)$. AE is perpendicular to BC.

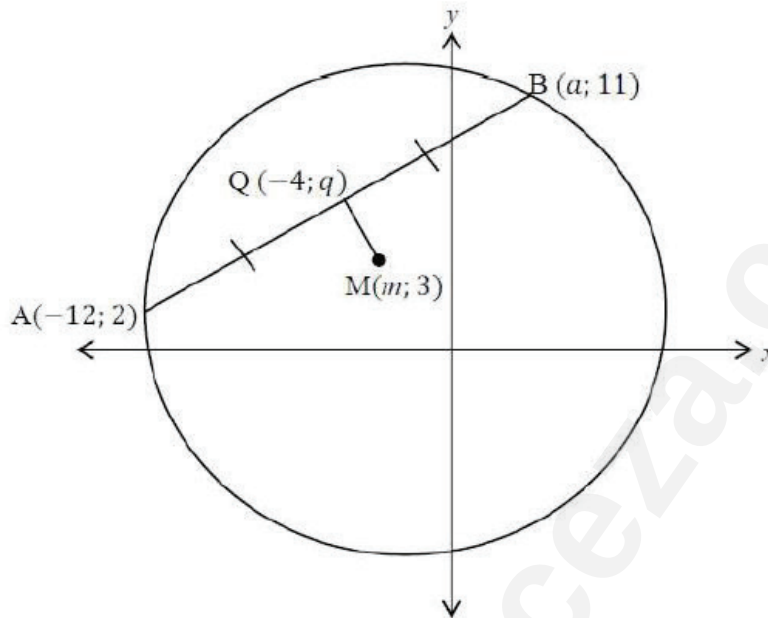


- 3.1 Calculate the length of BC. Leave your answer in simplest surd form. (3)
- 3.2 Determine the value of k . (3)
- 3.3 Determine the equation of AE. (4)
- 3.4 Calculate the size of θ rounded off to two decimal places. (6)

[16]

QUESTION 4

In the diagram below, $M(m; 3)$ is the centre of the circle. $Q(-4; q)$ is the midpoint of chord AB with $A(-12; 2)$ and $B(a; 11)$. The length of the radius of the circle is 10.



- 4.1 Calculate the values of a and q . (3)
- 4.2 Calculate the value of m . (3)
- 4.3 If $M(-2; 3)$, determine the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$. (2)
- 4.4 Determine the equation of a tangent to the circle at point A. (3)
- [11]**

QUESTION 5

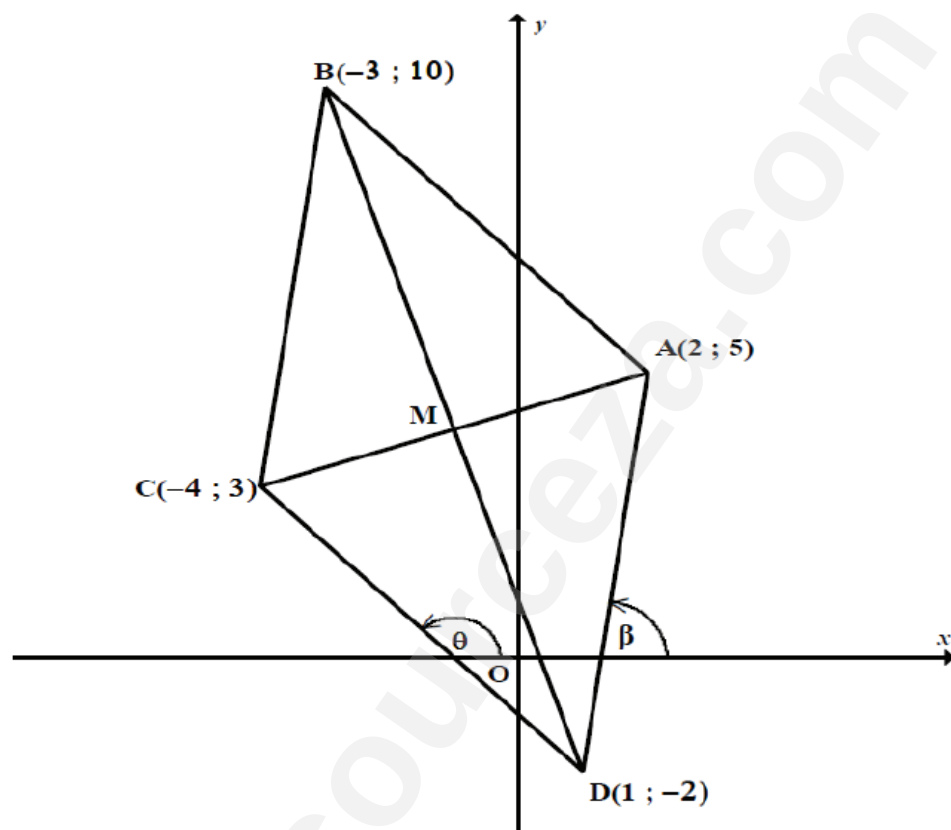
- 5.1 Calculate the value of k if the points $A(6; 5)$, $B(3; 2)$ and $C(2k; k + 4)$ are collinear. (3)
- 5.2 The equation of circle is given: $x^2 + y^2 - 4x + 4y + 3 = 0$.
- 5.2.1 Determine the coordinates of the centre of the circle and the length of the radius. (4)
- 5.2.2 Determine whether the point $T(3; -3)$ lies inside, outside or on the circle. Show all your calculations. (2)

[9]

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QUESTION 3

ABCD is a quadrilateral with vertices $A(2 ; 5)$, $B(-3 ; 10)$, $C(-4 ; 3)$ and $D(1 ; -2)$.

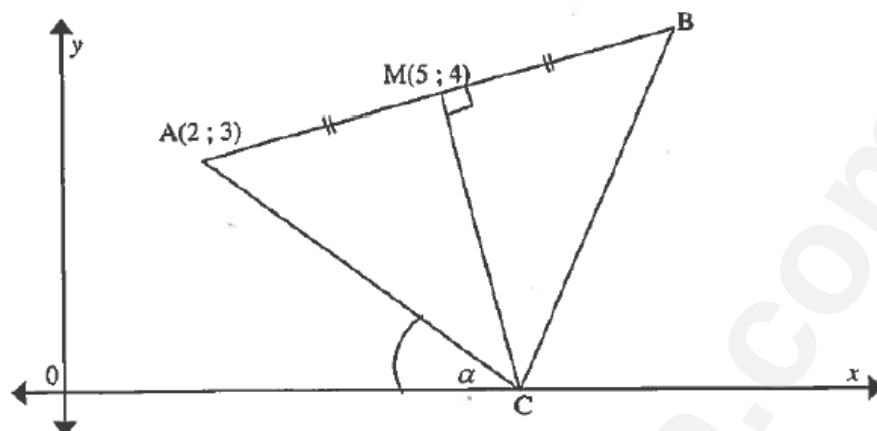


- 3.1 Calculate the length of AC. (Leave the answer in simplest surd form.) (2)
- 3.2 Determine the coordinates of M, the midpoint of AC. (2)
- 3.3 Show that BD and AC bisect each other perpendicularly. (5)
- 3.4 Calculate the area of $\triangle ABC$. (4)
- 3.5 Determine the equation of DC. (3)
- 3.6 Determine θ , the angle of inclination of DC. (3)
- 3.7 Calculate the size of $\widehat{ADC}$. (3)

[22]

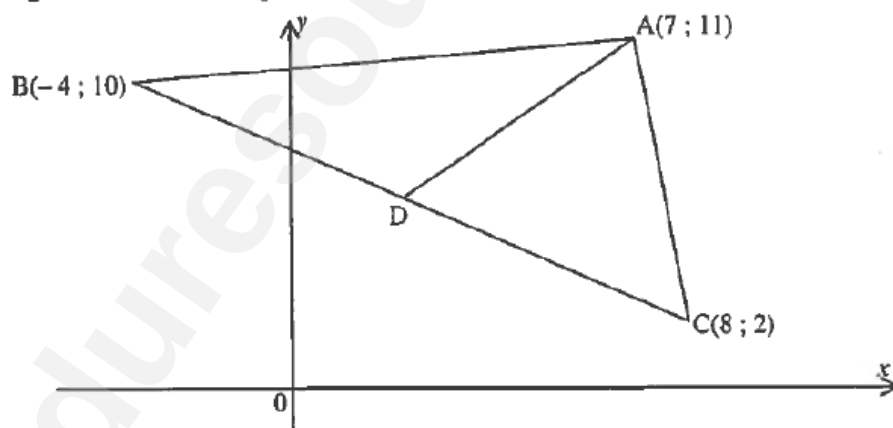
QUESTION 3

3.1 In the following sketch AMB is a straight line with $AM = MB$ and $CM \perp AB$.



- 3.1.1 Calculate the coordinates of B. (2)
 3.1.2 Determine the equation of MC. (4)
 3.1.3 Calculate the size of α , correct to one decimal place. (5)
 3.1.4 State with a reason why the vertices of triangle BCM lie on the circumference of a circle. (2)

3.2 In the figure below, $A(7; 11)$, $B(-4; 10)$ and $C(8; 2)$ are the vertices of a triangle and D is the midpoint of BC.



3.2.1 Calculate the length of AD and leave the answer in its simplest surd form. (4)

3.2.2 If AD is produced to the point $E(x; -4)$ in the fourth quadrant such that

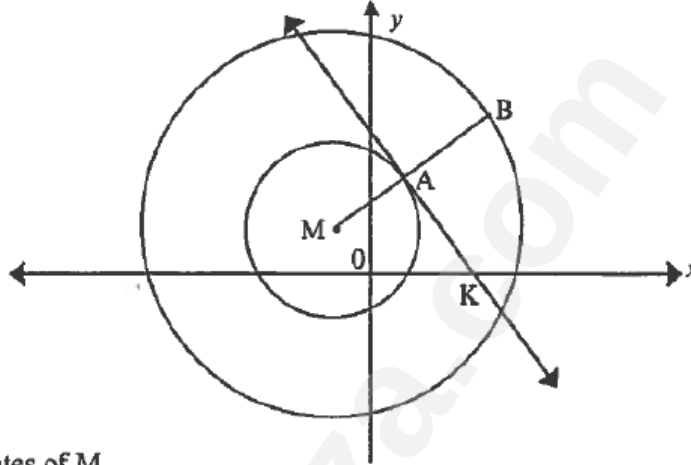
$$AD = \frac{1}{2} DE \text{ calculate the value of } x. \quad (5)$$

3.3 If $A(-7; 4)$, $B(-3; y)$ and $C(5; 2)$ are collinear, calculate the value of y . (3)

[25]

QUESTION 4

In the figure, M is the centre of two concentric circles (i.e. the circles have a common centre). The larger circle has equation $x^2 + y^2 = 4y - 2x + 44$. The line $y + x - 5 = 0$ is the tangent to the smaller circle at A. K is the x-intercept of the tangent AK. B is a point on the larger circle such that MAB is a straight line.



- 4.1 Determine the coordinates of M. (3)
- 4.2 Show that the coordinates of A are A(1 ; 4). (3)
- 4.3 Determine the equation of the smaller circle. (3)
- 4.4 Calculate the length of AB. (2)
- 4.5 The straight line $y + x - 5 = 0$ meets the straight line $y = 0$ at point K.
Calculate the area of ΔAMK . (5)

[16]

5. FINANCE, GROWTH AND DECAY

Overview from the CAPS document

Grade 10	Grade 11	Grade 12
Use simple and compound growth formulae $A = P(1 + in)$ $A = P(1 + i)^n$ to solve problems (including interest, higher purchase, inflation, population growth and other real life problems).	Use simple and compound decay formulae $A = P(1 - in)$ $A = P(1 - i)^n$ to solve problems (including straight line depreciation and depreciation on a reducing balance). Link to work on functions.	(a) Calculate the value on n in the formulae $A = P(1 + i)^n$ $A = P(1 - i)^n$ (b) Apply knowledge of geometric series to solve annuity and bond repayment problems.
The implications of fluctuating foreign exchange rates.	The effect of different periods of compounding growth and decay (including effective and nominal interest rates).	Critically analyze different loan options.

The weighting of this topic in the final examination :

- Grade 10: 10 ± 3
- Grade 11: 15 ± 3
- Grade 12: 15 ± 3

Concepts to be covered are simple growth, compound growth, simple decay, compound decay, effective interest rate, nominal interest rate, annuities (future value, present value, loan repayments, balance outstanding, final payment, deferred annuity)

1. Simple and Compound growth

Concepts: Simple and compound growth/ interest, higher purchase, foreign exchange rates.

Skill: Calculate A ; P and i in the formulae $A = P(1 + in)$ and $A = P(1 + i)^n$

Example 1

Sam deposited R1 000 into a savings account at 9% p.a. for one year. Calculate the total amount at the end of the year if:

- (a) Interest is compounded annually
- (b) Interest is compounded monthly
- (c) Calculate percentage increase in

Solution

$$\begin{aligned}
 \text{(a)} \quad A &= P(1+i)^n \\
 &= 1000(1+0,09)^1 \\
 &= R\ 1090
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad A &= P(1+i)^n \\
 &= 1000\left(1+\frac{0,09}{12}\right)^{12} \\
 &= R\ 1093,806898
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \% \text{ increase} &= \frac{1093,806898 - 1000}{1000} \times 100 \\
 &= 9,3806898\%
 \end{aligned}$$

This is the actual interest
“effective interest”
received by Sam

$$1000(1 + 9,3806898\%) = R1093,806898$$

$$\therefore 1000(1 + 0,093806898) = 1000\left(1 + \frac{0,09}{12}\right)^{12}$$

$$(1 + 0,093806898) = \left(1 + \frac{0,09}{12}\right)^{12}$$

$$\therefore (1 + i_e) = \left(1 + \frac{i_m}{m}\right)^m$$

i_e = effective interest,

i_m = nominal interest

m = compounding period.

Derivation is not examinable

2. Simple and Compound decay

Concepts: straight line depreciation and depreciation on a reducing balance/ diminishing balance.

Skills: Use of formulae $A = P(1 - in)$ and $A = P(1 - i)^n$ to calculate A ; P and i

3. Converting between nominal and effective interest rates.

Concepts: nominal and effective interest rate

Nominal interest rate – is the quoted interest rate.

Effective interest rate – is the actual interest one receives on an investment, loan or other financial product as a result of compounding over a given time period.

Skills: Convert between different compounding periods e.g. nominal to effective interest rates.

Example 2

Convert 8% p.a. compounded quarterly to an effective interest rate.

$$(1 + i_e) = \left(1 + \frac{i_m}{m}\right)^m$$

$$(1 + i_e) = \left(1 + \frac{0,08}{4}\right)^4$$

$$\begin{aligned} i_e &= (1,02)^4 - 1 \\ &= 0,08243216 \\ &= 8,24\% \end{aligned}$$

Example 3

Convert an interest rate of 10 p.a., compounded monthly, to an annual interest rate, compounded semi-annually.

$$\left(1 + \frac{i_m}{m}\right)^m = \left(1 + \frac{i_n}{n}\right)^n$$

$$\left(1 + \frac{i_m}{2}\right)^2 = \left(1 + \frac{0,1}{12}\right)^{12}$$

$$i_m = 2 \times \sqrt{\left(1 + \frac{0,1}{12}\right)^{12}} - 1$$

$$\begin{aligned} i_m &= 0,1021 \\ &= 10,21\% \end{aligned}$$

Activity 1

- 1.1 Convert an effective annual interest rate of 14,2% p.a. to a nominal annual interest rate compounded monthly.
- 2.2 Convert a nominal interest rate of 11% p.a. compounded quarterly to nominal interest rate per annum compounded monthly.
- 3.3 Convert an interest rate of 10 p.a., compounded monthly, to an annual interest rate, compounded semi-annually.

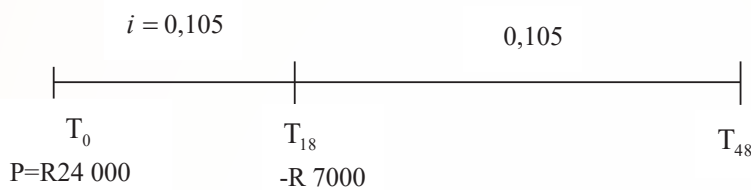
4. Time Lines

Time lines are useful when there are transactions made in between or when there are changes in interest rates.

Example 4.1

A savings account was opened with an initial deposit of R24 000. 18 months later R7 000 was withdrawn from the account. Calculate how much money will be in the savings account at the end of 4 years if the interest rate was 10,5% p.a. compounded monthly.

Solution



$$A = 24000 \left(1 + \frac{0,105}{12} \right)^{30} - 7000 \left(1 + \frac{0,105}{12} \right)^{30}$$

$$= R27\,369,56$$

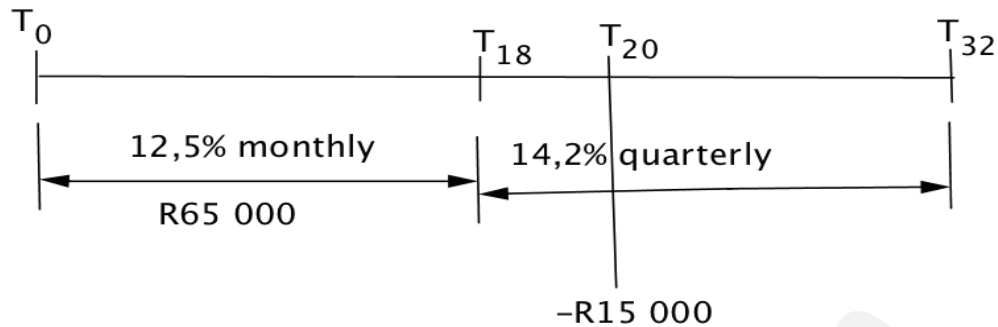
OR

$$A = \left[24000 \left(1 + \frac{0,105}{12} \right)^{18} - 7000 \right] \left(1 + \frac{0,105}{12} \right)^{30}$$

$$= R27\,369,56$$

Example 4.2

Shirley inherits R65 000 from her mother. She invests the money in a savings account. The interest rate is 12,5% p.a. compounded monthly for 18 months, and it is then increased to 14,2% p.a. compounded quarterly. At the end of two years she withdraws R15 000. Calculate how much money she has in his savings account at the end of 5 years.



After 18 months, the interest was added quarterly, T_{19} marks the end of 21 months, T_{20} marks the end of 24 months and T_{32} marks the end of 60 months.

$$\begin{aligned} \text{Accum amount} &= 65000 \left(1 + \frac{0,125}{12}\right)^{18} \left(1 + \frac{0,142}{4}\right)^{14} - 15000 \left(1 + \frac{0,142}{4}\right)^{12} \\ &= R104\,853,11 \end{aligned}$$

Activity 2

2.1 A man will need R40 000 in 5 years' time. He invests into an account that offers 12% p.a. compounded monthly. He makes the following payments:

- R20 000 now
- Rx after 2 years
- R2x another year later

Calculate x.

2.2 A certain amount of money was invested ten years ago and is now worth R200 000. The interest rate during the first 6 years was 3.5% per annum compounded annually and for the remaining four years, the interest rate was 4.6% per annum compounded monthly. How much money was invested ten years ago?

5. Annuities

A number of equal payments made at the regular intervals or time and subjected to a rate of interest.

Skills: Calculate future value, present value annuity and monthly repayment or deposits using formula or knowledge of geometric series.

5.1 Future value annuity

Nomsa deposits R500 at the end of each month into an investment fund. Calculate the accumulated amount at the end of 12 months if interest is 11% p.a. compounded monthly.

$$A_1 = 500(1 + 0,11)^{11}$$

$$A_2 = 500(1 + 0,11)^{10}$$

$$A_3 = 500(1 + 0,11)^9$$

⋮

$$A_{12} = 500(1 + 0,11)^0$$

$$= 500$$

first deposit will accumulate 11 months' interest since it was deposited at the end of the first month.

last deposit will not accumulate any interest.

$$A_t = A_1 + A_2 + A_3 + \dots + \text{up to 12 terms}$$

$$= 500(1 + 0,11)^{11} + 500(1 + 0,11)^{10} + 500(1 + 0,11)^9 + \dots + 500$$

$$= 500 + 500(1 + 0,11)^1 + 500(1 + 0,11)^2 + \dots + 500(1 + 0,11)^{11}$$

$$\text{Geometric Series } S_n = \frac{a(r^n - 1)}{r - 1}$$

$$A_t = \frac{500[(1 + 0,11)^{12} - 1]}{(1 + 0,11) - 1}$$

$$\therefore F = \frac{x[(1 + i)^n - 1]}{i}$$

x is the monthly deposit

5.1 Present value

Noluthando takes out a loan of R10 000. She repays the loan over 4 years with equal installments at the end of each month. Determine her monthly repayments if the interest is calculated at 16% p.a. compounded monthly.

$$A_1 = x(1+i)^{-1}$$

$$A_2 = x(1+i)^{-2}$$

$$A_3 = x(1+i)^{-3}$$

⋮

$$A_n = x(1+i)^{-n}$$

$$A_t = A_1 + A_2 + A_3 + \dots + A_n$$

$$= x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-3} + \dots + x(1+i)^{-n}$$

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{x(1+i)^{-1}[1-(1+i)^{-n}]}{1-\frac{1}{1+i}} \\ &= \frac{x(1+i)^{-1}[1-(1+i)^{-n}]}{\frac{1+i-1}{1+i}} \\ &= \frac{x(1+i)^{-1}[1-(1+i)^{-n}]}{i(1+i)^{-1}} \\ &= \frac{x[1-(1+i)^{-n}]}{i} \end{aligned}$$

$$a = x(1+i)^{-1}, \quad r = (1+i)^{-1}$$

$$\therefore P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$10000 = \frac{x \left[1 - \left(1 + \frac{0,16}{12} \right)^{-48} \right]}{\frac{0,16}{12}}$$

$$x = \text{R}283,40$$

$$A = P(1+i)^n, \quad P = A(1+i)^{-n}$$

Derivation of Future value and Present value formulae are not examinable

Example 5.1

Saloshni wants to buy a house for R700 000. She has a deposit of R50 000 and takes out a loan for the balance at a rate of 18% p.a. compounded monthly.

- 5.1.1 How much must Saloshni borrow from the bank?
- 5.1.2 Calculate the monthly repayment if she wishes to settle the loan in 15 years.
- 5.1.3 Saloshni won a lottery and wishes to settle the loan after the 50th payment. What is the outstanding balance?

Solution:

- 5.1.1 She must borrow R700 000 – 50 000
= R 650 000

- 5.1.2
$$P = \frac{x[1 - (1+i)^{-n}]}{i}$$

$$650\,000 = \frac{x[1 - (1,015)^{-180}]}{0,015}$$

$$x = R10\,467,74$$

$$\text{OB} = \frac{10\,467,74[1 - (1,015)^{-130}]}{0,015}$$

$$= R597\,115,31$$

note: n = number of months remaining, 10 467,74 is the monthly repayment

OR

$$\text{OB} = 650\,000(1 + 0,15)^{50} - \frac{10\,467,74[(1,015)^{50} - 1]}{0,015}$$

$$= R597\,114,89$$

value after 50 months minus payments over 50 months. The diff in cents is due to early rounding off of monthly payments

Example 6.1

A farmer has just bought a new tractor for R800 000. He has decided to replace the tractor in 5 years' time, when its trade-in value will be R200 000. The replacement cost of the tractor is expected to increase by 8% per annum.

- 6.1.1 The farmer wants to replace his present tractor with a new one in 5 years' time. The farmer wants to pay cash for the new tractor, after trading in his present tractor for R200 000. How much will he need to pay
- 6.1.2
- One month after purchasing his present tractor, the farmer deposited x rands into an account that pays interest at a rate of 12% p.a. compounded monthly.
 - He continued to deposit the same amount at the end of each month for a total of 60 months.
 - At the end of 60 months he has exactly the amount that is needed to purchase a new tractor, after he trade in his present tractor.
Calculate the value of x .
- 6.1.3 Suppose that 12 months after the purchase of the present tractor and every 12 months thereafter, he withdraws R5 000 from his account, to pay for maintenance of the tractor. If he makes 5 such withdrawals, what will the new monthly deposit be?

Solution

$$\begin{aligned}
 6.1.1 \quad A &= P(1+i)^n \\
 &= 800000(1,08)^5 \\
 &\quad \text{R1 175 462,46} \\
 \therefore \text{R } 1\,175\,462,46 - \text{R}200\,000 \\
 &= \text{R } 975\,462,46
 \end{aligned}$$

6.1.2

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$975462,46 = \frac{x[(1,01)^{60} - 1]}{0,01}$$

$$x = \frac{975462,46 \times 0,01}{[(1,01)^{60} - 1]}$$

$$= R11944,00$$

6.1.3

$$5000 = \frac{x[(1,01)^{12} - 1]}{0,01}$$

$$x = \frac{5000 \times 0,01}{[(1,01)^{12} - 1]}$$

$$= R394,24$$

Monthly deposits must be increased by R394,24

$$\text{Hence the new monthly deposit} = R11\,994 + R394,24$$

$$= R12\,338,24$$

OR

$$(1 + i_{eff}) = (1 + 0,01)^{12}$$

$$i_{eff} = 0,12682503$$

$$F = \frac{500[(1 + 0,12682503)^5 - 1]}{0,12682503}$$

$$= R32\,197,77$$

$$\therefore \text{money needed} = R975462,46 + R32197,77 = R1007660,23$$

$$1007660,23 = \frac{x[(1,01)^{60} - 1]}{0,01}$$

$$x = \frac{1007660,23 \times 0,01}{[(1,01)^{60} - 1]}$$

$$= R12338,24$$

Example 6.2

A school purchases a photocopying machine for R150 000. The machine depreciates in value at 22% p.a. on a reducing balance. The school wants to buy a new machine in 5 years' time. A new machine will cost more in future and its cost will escalate at 19% p.a. effective. The old machine will be sold at scrap value after 5 years. A sinking fund is set up in order to save up for the new machine. The proceeds from the old machine will be used together with the sinking fund to buy the new machine. The school will pay equal monthly amounts into the sinking fund and the interest earned is 14,4% p.a. compounded monthly.

- 6.2.1 Find the scrap value of the old machine.
- 6.2.2 Find the cost of the new machine 5 years from now.
- 6.2.3 Find the amount required in the sinking fund 5 years from now.
- 6.2.4 Find the equal monthly payments made into the sinking fund.
- 6.2.5 The company then decides to use the sinking fund account as a means of financing half-yearly services of R6 000 to the old machine. The first service will take place 6 months after the sinking fund was opened and the last service will take place 6 months before the new machine is bought. In order to have enough money to finance these services, the company will need to increase their monthly premiums into the sinking fund.

Activity 2

Mr. Brown has just finished paying off his 20-year home loan of R400 000. During the first 5 years the interest rate was 24% p.a. compounded monthly. Thereafter, and for the rest of the term, the interest rate decreased to 18% p.a. compounded monthly.

- 2.1 Calculate his initial monthly payment.
- 2.2 Calculate his balance outstanding at the end of December in the 5th year.
- 2.3 When the interest rate changed after 5 years, Mr. Brown was able to pay a decreased monthly payment starting at the end of January in the 6th year. Calculate what this new repayment was.

2.4 How much did Mr. Brown actually pay for his home?

2.5 Suppose Mr. Brown decided to rather pay R9 000 per month starting from the end of January in the 6th year. How long would it have then taken him to pay off the loan?

6. Sinking fund

Sinking fund – money set aside in order to buy or replace a certain asset.

6.2.1

$$\begin{aligned} A &= P(1 - i)^n \\ &= 150\,000(1 - 0,22)^5 \\ &= R43\,307,62 \end{aligned}$$

6.2.2

$$\begin{aligned} A &= P(1 + i)^n \\ &= 150\,000(1 + 0,19)^5 \\ &= R357\,953,05 \end{aligned}$$

6.2.3 Amount required = R357 953.05 – R43 307,62
= R 314 645,43

6.2.4

$$F = \frac{x[(1 + i)^n - 1]}{i}$$

$$\begin{aligned} 314645.43 &= \frac{x[(1,012)^{60} - 1]}{0,012} \\ x &= R3610,92 \end{aligned}$$

$$\frac{14,4\%}{12} = 0,012$$

6.2.5 Note: The sinking fund earns interest at 14,4% p.a. compounded monthly whilst service withdrawals are done half-yearly.

Hence, 14,4% p.a. monthly must be converted to 14,4% p.a. half yearly.

$$(1 + i_{new}) = (1 + i_{nom})$$

$$\left(1 + \frac{i_{new}}{2}\right)^2 = \left(1 + \frac{0,144}{12}\right)^{12}$$

$$i_{new} = 2\left(1 + \frac{0,144}{12}\right)^6 - 2$$

$$i_{new} = 0,148389745$$

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$= \frac{6\,000[(1,148389745)^9 - 1]}{0,148389745}$$

$$= R100\,025,43$$

∴ money required = R 314 645,43 + R 100 025,43

$$= R 414\,670,86$$

$$F = \frac{x[(1+i)^n - 1]}{i}$$

$$414\,670,86 = \frac{x[(1,012)^{60} - 1]}{0,012}$$

$$= R4\,758,82$$

this is the increased monthly premiums.

7. Deferred annuities

To defer a payment is to make an arrangement to pay later.

Example 7

A car that costs R130 000 is advertised in the following way: 'No deposit necessary and first payment due three months after date of purchase.' The interest rate quoted is 18% p.a. compounded monthly.

Peter bought this car on 1 March 2009 and made his first payment on 1 June 2009. Thereafter he made another 53 equal payments on the first day of each month.

Calculate his monthly repayments

$$A = P(1+i)^n$$

$$= 130\,000 \left(1 + \frac{0,18}{12}\right)^2$$

$$= R133\,929,25$$

$$R133\,929,25 = \frac{x[1 - (1,015)^{-54}]}{0,015}$$

$$x = R3\,636,36$$

OR

$$130\,000 = x(1,015)^{-3} + x(1,015)^{-4} + x(1,015)^{-5} + \dots \text{up to 54 terms}$$

$$130\,000 = \frac{x(1,015)^{-3}[1 - (1,015)^{-54}]}{1 - (1,015)^{-1}}$$

$$x = R3\,636,36$$

This is the amount owed after 2 months. Note that if we skip an instalment the loan amount increases.

$$S_n = \frac{a(1-r^n)}{1-r}, a = x(1,015)^{-3}$$

$$r = (1,015)^{-1}$$

Activity 3

Edward takes out a loan of R840 000 from the bank to start his own business. The loan is paid off by means of an annuity 10 years after it was granted. The repayment started 12 months after the loan was granted. The interest rate is 18% per annum compounded monthly.

Calculate the monthly repayment. (answer: R18 491,07)

8. Final payment

Example 8

Jill negotiates a loan of R300 000 with a bank which has to be paid by means of R5 000 and a final payment which is less than R5 000. The repayments start one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly.

8.1 Determine the number of payments required to settle the loan.

8.2 Calculate the balance outstanding after Jill has paid the last R5 000.

8.3 Calculate the value of the final payment made by Jill to settle the loan.

Solution

$$\begin{aligned}
 8.1 \quad P &= \frac{x[1 - (1+i)^{-n}]}{i} \\
 300\,000 &= \frac{5\,000[1 - (1,015)^{-n}]}{0,015} \\
 0,9 &= 1 - (1,015)^{-n} \\
 0,1 &= (1,015)^{-n} \\
 -n &= \frac{\log 0,1}{\log 1,015} \\
 n &= 154,65
 \end{aligned}$$

$\therefore$ Number of payments = 155.

$$\begin{aligned}
 8.2 \quad \text{BO} &= \frac{5\,000[1 - (1,015)^{-0,6541086}]}{0,015} \\
 &= \text{R}3\,230,50
 \end{aligned}$$

$$\begin{aligned}
 8.3 \quad \text{Final Payment} &= 3\,320,5(1,015)^1 \\
 &= \text{R}3\,278,96
 \end{aligned}$$

OR

$$\begin{aligned}
 300\,000 &= \frac{5\,000[1 - (1,015)^{-154}]}{0,015} + x(1,015)^{-155} \\
 x &= \text{R}3\,278,96
 \end{aligned}$$

The first 154 terms form a geometric series, present value formula holds. The last term 155th is not part of the series since the last instalment is less than R5 000

Activity 4

- 4.1 Lindiwe secured a loan of R85 000 from her bank. She has agreed to repay the loan in monthly payments of R2 000 each, starting in three months' time. Interest is charged at 21% p.a. compounded monthly.
- 4.1.1 What is the value of the loan at the end of 2 months? (R88 001.03)
- 4.1.2 How long will it take before the loan is completely paid off? ($n = 85$ instalments)
- 4.1.3 Calculate Lindiwe's last instalment in the 85th month which is less than R2 000? (R1 436.44)
- 4.2 Mark wants to purchase a big screen television set at a cash price of R39000. The bank agrees to grant a loan at 23% p.a. compounded quarterly. Mark offers to pay back the loan making n equal payments of R2000 into this account, and a final payment which is less than the R2000.
- 4.2.1 Why did the bank not accept Mark's offer?
- 4.2.2 What is the minimum payment that the bank should accept from Mark?
- 4.2.3 If the bank agrees to accept R2500 per quarter, how long will it take Mark to pay back

Toolbox/Lunchbox

- Analyse the question by underlining the **keywords**.
- The keywords will assist in the **choice of the correct formula**.
- Identify if the amount is increasing or decreasing.
- Identify if **you have** the big amount (present value: **P**) or if you **are waiting** for the big amount (future value: **F**)
- Make sure that you can use the functions on the calculator

- Convert “ i ” and “ n ” to the appropriate values according to the given statement
- i – percentages to decimals/ $i = \text{quoted interest rate divided by the compounded periods}$.
- n - number of interest payments according to the given statements
- Substitute correct values of P ; A ; F ; i ; n in the appropriate formula chosen.

EXAMINATION TYPE QUESTIONS

- 1.1 A survey conducted in December 2008 determined that 5,7 million South Africans were living with HIV.
The researchers used a model of exponential growth $A = P(1 + i)^n$ to predict that there will be 6 million people living with HIV in December 2015.
Calculate as a percentage, the annual rate of increase that the researchers used for the 7 years.
- 1.2 A bank offers interest on investment at a rate of 15% p.a. compounded monthly.
- Calculate the effective interest rate equivalent to this.
 - Determine how much a person must invest now so that they have R 20 000 in four years’ time.
- 1.3 (a) Daniel has a new trailer worth R8 500. The depreciation rate on this item is 10% p.a. using the reducing balance method.
Calculate the value of the trailer after 4 years.
- (b) Draw a sketch graph showing the reducing value of Daniel’s trailer over the 4 year period.
- 1.4 A car, costing R210 000 depreciates at a rate of $a\%$ per annum, on a reducing balance basis. Calculate the value of a if it takes 8 years for the value of the car to decrease to R80 000.
- 1.5 Jenna invests R20 000 into a savings account. The interest paid on money in the account is 15% p.a. compounded monthly. Calculate how long (in years and months) it will take for the savings to reach R100 000.

- 1.6 Determine how long, in years, it will take for the value of a motor vehicle to decrease to 25% of its original value if the rate of depreciation, based on reducing balance method, is 21% per annum.

Feb/March 2016

- 1.7 Diane invests a lump sum of R5 000 in a savings account for exactly 2 years.
The investment earns interest at 10% p.a., compounded quarterly.
- (a) What is the quarterly interest rate for Diane's investment?
 - (b) Calculate the amount in Diane's savings account at the end of the 2 years.
- 1.8 Nomusa received a valuation of R130 000 for her car that has depreciated at a rate of 15% p.a. on a reducing balance over the last years.
Determine the value of the car 5 years ago. (Round off your answers to the nearest thousand rands).

Gauteng Prep Exam 2014

- 1.9 For two years, equipment worth R20 000, depreciates by 20% per annum according to the reducing balance method. Each year thereafter, the annual depreciation is 2% less than that of the previous year.
Determine the depreciated value of the equipment after 4 years.

IEB Nov 2012

- 1.10 Nomusa received a valuation of R130 000 for her car that has depreciated at a rate of 15% p.a. on a reducing balance over the last years.
Determine the value of the car 5 years ago. (Round off your answers to the nearest thousand rands)

IEB Nov 2015

- 1.11 Sipho bought a laptop for R12 500. It depreciates in value to R5 546,32 after 5 years.
Calculate the annual depreciation rate.

- 1.12 A company bought office furniture that cost R120 000. After how many years will the furniture depreciates to a value of R41 611,57 according to the reducing balance method, if the rate of depreciation is 12,4% p.a.?
- 1.13 Anisha and Lindiwe each received R12 000 to invest for a period of 5 years. They invested the money at the same time according to the following options:
- Anisha: 8,5% p.a. simple interest. At the end of the 5 years she will receive an additional bonus pay out of exactly 7,5% of the original amount invested.
 - Lindiwe: 8,5% p.a. compounded quarterly.

Who will have the larger final amount after 5 years? Justify your answer with appropriate calculations.

NB: Compounding period of interest rate should be the same as the interval at which payments are made e.g. if monthly payments are made and interest is added quarterly then the quarterly interest rate must first be changed to monthly interest rate.

IEB Nov 2015

- 3.1 When Sam started his first job, he decided to save R500 at the end of each month in an account earning interest at 6% p.a. compounded monthly.
- (a) Calculate the amount of money he could expect to be in his account at the end of 15 years.
- (b) At the end of the 15 years, Sam added money to his account so that the total was R150 000. He no longer made monthly payments but the money remained in the account earning interest at 8,5% p.a. compounded quarterly.
Calculate the amount in the account after a further 5 years.

IEB Nov 2012

- 3.2 Emily saves R300 from her monthly salary in an account earning interest at 8,5% p.a. compounded monthly. At the end of 10 years of working, Emily stops making payments but leaves the money in the account to continue earning interest.
- (a) Calculate how much Emily can expect to have in the account 20 years after she started her service plan.

- (b) Yerma starts working at the same time as Emily but does not have a saving fund for the first ten years. Calculate how much Yerma needs to save each month if she is to have R130 000 available at the end of the next ten years, using the same interest rate of 8,5% p.a. compounded monthly.

IEB Nov 2010

- 3.3 Diana decided when she started work that she would invest R500 each month into a retirement fund earning interest at 8,5% p.a. compounded monthly. The first payment is made 1 month after the final payment earns no interest.
- (a) Calculate how much she could expect to be in her account at the end of 30 years.
- (b) Alex also started work at the same time as Diana and invested in the same scheme but decided to deposit R600 per month. After n years, Alex's investment accumulated to A, where A is the answer obtained in 2.1 above. Calculate the value of n , to the nearest whole number.
- 3.4 Steven is 20 years old and plans to retire with R8 Million when he turns 60. He starts paying money into a pension fund one month after his 20th birthday, and his last payment will be made on his 60th birthday. Calculate his monthly payments if he is to achieve his goal. The interest rate of money accumulated in the pension fund is 13% p.a. compounded monthly.

Durban Girls' College Prep Exam 2015

- 3.5 Ms Samuel has finally decided to buy an apartment. She takes out a loan of R500 000 from the bank and plans to repay the loan in monthly instalments over a period of 20 years. The interest rate charged is 9% p.a. compounded monthly.
- (a) Calculate her monthly repayments.
- (b) What is the total amount she will have paid for the apartment after 20 years?
- (c) After 15 years, Ms Samuel wins the lotto and decides to clear the account. What will the outstanding balance on the loan be?
- 3.6 Patrick purchases a house for R1 200 000. He pays a deposit of 10% of the value of the house. The bank grants him a loan for the outstanding amount, at an interest rate of 8,4% p.a. compounded monthly, payable over a period of 20 years.
- (a) Calculate the deposit Patrick pays on the house.
- (b) Calculate Patrick's monthly repayments.

- (c) Calculate the balance outstanding at the end of 15 years.
- (d) Calculate the total amount Patrick would have paid for the house at the end of 20 years.

3.7 Thulani buys a house and takes out a loan for R450 000. The interest is 9,5% p.a. compounded monthly.



- (a) Calculate his monthly repayments if the loan is repaid over 20 years.
- (b) How much money does he pay in total over 20 years to repay the loan?
- (c) Calculate the balance on the loan after 8 years.
- (d) Calculate his monthly repayments if he chose to repay the loan over 15 years.
- (e) How much money does he pay over 15 years to repay the loan and how much money does he save by reducing the repayment period by 5 years?

3.8 Mpumelelo negotiates a loan of R300 000 with a bank which has to be paid by means of R5 000 and a final payment which is less than R5 000. The repayments start one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly.

- (a) Determine the number of payments required to settle the loan.
- (b) Calculate the balance outstanding after Jill has paid the last R5 000.
- (c) Calculate the value of the final payment made by Jill to settle the loan.

Mathematics Handbook: Kevin Smith

3.9 A loan of R750 000 is taken out at an interest rate of 12,5% p.a. compounded bi-annually. It is repaid by equal bi-annual payments of R55 000 and a final payment less than R55 000.

- (a) How many payments are required to settle the loan?
- (b) What is the outstanding balance on the loan after the final of R55 000?

(c) What is the value of the final payment?

3.10 Peter purchases a car for R200 000. He expects to replace the car in 5 years' time. The replacement cost of the car increases at a rate of 11% p.a. compounded annually and the rate of depreciation of his current car is 18% p.a. on a reducing balance. Peter sets up a sinking fund to pay for a new car in 5 years. Calculate:

- (a) The trade-in value of Peter's car in 5 years' time.
- (b) The cost of a new car in 5 years' time.
- (c) The value of the sinking fund in 5 years' time if Peter trades in his old car.
- (d) The monthly payments into the sinking fund if it earns an interest of 7% p.a. compounded monthly.

3.11 Machinery is purchased at a cost of R550 000 and is expected to rise in cost at 15% per annum compounded annually and depreciate in value at a rate of 8% p.a. compounded annually. A sinking fund is started to make provision for replacing the old machine. The sinking fund pays 16% p.a. compounded monthly, and you make monthly payments into this account for 10 years. Determine:

- (a) The replacement cost in ten years' time.
- (b) The scrap value of the machine in ten years' time.
- (c) The monthly payment into the sinking fund that will make provision for the replacement of the new machine.

May/June 2016

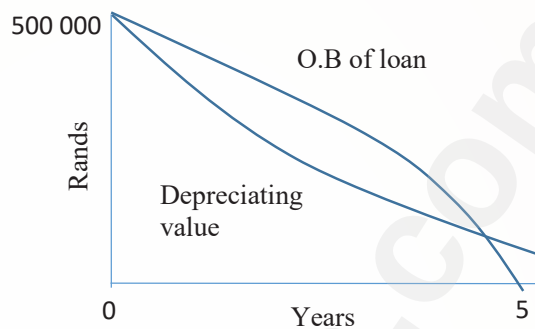
3.12 Lebogo bought a tractor for R x on 1 April 2016.

- She will trade in this tractor when she replaces with a similar one in 5 years' time on 1 April 2021.
 - The tractor depreciates by 20% p.a. according to the reducing balance method.
 - The price of a similar tractor increases by 18% annually.
 - Lebogo calculated that if she deposited R8 000 per month into a sinking fund, which paid interest at 10% p.a. compounded monthly, she would have enough money to cover the replacement cost of the tractor. She made the first deposit in this fund on 30 April 2016 and will continue to do so at the end of every month until 31 March 2021.
- (a) Determine, in terms of x , what the book value of the current tractor will be on 1 April 2021 (that is, 5 years after it was bought). Give your answer correct to FIVE decimal places.
- (b) Determine, in terms of x , what the price of a similar new tractor will be on 1 April 2021. Give your answer correct to FIVE decimal places.
- (c) Calculate the amount accumulated in the sinking fund on 1 April 2012.
- (d) Calculate the value of x , the price of the current tractor. Round off your answer to the nearest thousand.

IEB Nov 2014

- 3.13 Rachel bought a car for R500 000 on an agreement in which she had to repay it in monthly instalments at the end of each month for 5 years, with interest charged at 18% p.a. compounded monthly.

The diagram below shows the reducing balance over this time period as well as the depreciating market value of the car.



- Calculate the annual effective interest rate of the loan.
- Calculate Rachel's monthly instalment.
- Suppose that Rachel decided to pay R12 700 each month as her repayment. Calculate the outstanding balance of the loan after two years.
- At the end of the 2 years, the market value of Rachel's car had reduced to R304 200. Determine the annual interest rate of depreciation on the reducing value.
- Calculate what percentage of the original purchase price, the car's value had reduced to, at the end of the 5 years, when Rachel had finished paying it off.

3.14 Nov 2016

On 1 June 2016 a bank granted Thabiso a loan of R250 000 at an interest rate of 15% p.a. compounded monthly, to buy a car. Thabiso agreed to repay the loan in monthly instalments commencing on 1 July 2016 and ending 4 years later on 1 June 2020. However, Thabiso was unable to make the first two instalments and only commenced with the monthly instalments on 1 September 2016.

- Calculate the amount Thabiso owed the bank on 1 August 2016, a month before he paid his first monthly instalment.

- (b) Having paid the first monthly instalment on 1 September 2016, Thabiso will still pay his last monthly instalment on 1 June 2020. Calculate his monthly instalment.
- (c) If Thabiso paid R9 000 as his monthly instalment starting on 1 September 2016, how many months sooner will he repay the loan?
- (d) If Thabiso paid R9 000 as his monthly instalment starting on 1 September 2016, calculate the final instalment to repay the loan.

June/ July 2015

- 3.15 Tebogo opened a savings account with a single deposit of R5 000 at the beginning of June 2015. He then made 24 monthly deposits of R8 000 at the end of every month, starting at the end of June 2015. The account earned interest at 15% p.a. compounded monthly. Calculate the amount that should be in his savings account immediately after he makes the last deposit.

6. FUNCTIONS AND GRAPHS

Teaching approach

Straight line / linear graph

- Revise number system with emphasise on real numbers.
- use **table method** to plot your graph as your starting point
- pay attention to the intercepts (coordinates)
- introduce the **dual-intercept method**.
- effects of “a” and “q” in $y = ax + q$ or $y = mx + c$
- For $y = 2x + 0$, consider gradient **intercept method** : $y = \frac{2}{1}x$ (change in y over change in x) OR substitute x by any value and plot the coordinate with y-intercept as 0 already.
- Introduce graphs like : $y = 4$, $x = -3$, $y = x$, etc.
- Use the notation of : $f(x)$, $g(x)$, etc.
- domain and range
- finding the equation
- making deductions from the given sketch

Parabola

- Consider $y = x^2$, as the point of departure.
- use table method to investigate the effect of “a”, “p” and “q”.
- let learners draw rough sketches (if they know the values of a, p, and q. At this stage ignore the intercepts)
- introduce the intercepts and draw the graph considering all other characteristics including decreasing and increasing intervals.
- domain and range.
- completing the square
- finding the equation of the graph, when given intercepts and a point and when given the turning point and any other point.
 $y = a(x - x_1)(x - x_2)$ and $y = a(x - p)^2 + q$
- making deductions from the given sketch.
- combination of the two graphs- focus on the points of intersection and comparison e.g.
 $f(x) > g(x)$; $f(x) < 0$; $f(x).g(x) \leq 0$; $\frac{f(x)}{g(x)} > 0$, etc.
- Consider the transformations of $f(x)$ to : $f(-x)$; $-f(x)$; $f(x + a)$; $f(x) + a$; $a.f(x)$

Hyperbola

- use table method to investigate the effect of “ a ”, “ p ” and “ q ”. encourage self-discovery and explain asymptotes.
- let learners draw rough sketches (if they know the values of a , p , and q . At this stage ignore the intercepts)
- introduce the **intercepts** $\left(0 = \frac{6}{x-2} + 1\right)$ and draw the graph
- Write asymptotes in equation form.
- Lines of symmetry : $y = x + c$ and $y = -x + c$
- domain and range
- finding the equation of the graph
- making deductions from the given sketch
- combination of the two or more graphs - focus on the points of intersection.
- Hyperbola has two arms at all times and show at least one point on the other arm with intercepts on the other.

Exponential function

- use table method to illustrate the effect of “ a ”, “ p ” and “ q ”. encourage self discovery
- asymptote
- increasing/decreasing function
- let learners draw rough sketches (if they know the values of a , p , and q . At this stage ignore the intercepts)
- introduce the intercepts and draw the graph
- asymptote
- domain and range
- finding the equation of the graph
- making deductions from the given sketch

N.B

- starting point must be the “mother function”
- introduce a , p and q gradually
- let learners participate
- encourage self discovery e.g. investigations,
- make use of available technology e.g HeyMath! software
- always link lessons with real life examples
- assessment must be done regularly

INVERSE FUNCTIONS

- Notation : $f(x) = \dots\dots\dots$
 $f^{-1}(x) = \dots\dots\dots$
- Interchange values of x and y and make y the subject of the formula.
- Reflection about the line $y = x$.

Challenges:

Learners should:

- Guard against : $f^{-1}(x) = \frac{1}{f(x)}$, **its incorrect.**
- Learners will want to introduce log on both sides of the equation.

Teaching Approach.

- Definition of a function: discussing one-to-one and many-to-one mapping using sets as examples.
- Table for original graph of $y = mx + c$ and inter change the values of x and y and lead to the conclusion that this is **reflection** of original graph about the line $y = x$
- Use table for the original graph and then exchange the values of x and y . Draw the graph of the inverse and find the equation of the second graph which should be the same as the equation of the inverse when you substituted/exchanged the value of x and y .
- Use table method for inverse of exponential graph: $x = 3^y$.
- Domain / range of the original graph become the range / domain of the inverse.
- Draw the graph $y = x^2$ and its inverse, then ask your class if the inverse is a function or not with reasons.
- Original graph $y = x^2$ is a function by vertical line test but be careful for the inverse to be a function you will need to **restrict the domain of the original graph** :i.e. $x \geq 0$ or $x \leq 0$. Put condition for inverse that $y = \sqrt{x}$, $x \geq 0$.
- When drawing the inverse of an exponential graph plot three points to show you the direction of the graph.
- Changing from exponential form to log form must be revised particularly for exponential graphs.
- Make learners aware of different ways to translate the graph (variations).
- Use coloured chalk when discussing this topics.

EXAMPLE

Please begin by sketching $y = \frac{8}{x}$, then explain the following worked example.

Consider $g(x) = \frac{8}{x-2} - 3$ has the **horizontal asymptote** at $y = -3$ and $x - 2 \neq 0 \therefore x \neq 2$ because if

$x = 2$ the denominator of the expression $\frac{8}{x-2}$ would be $\frac{8}{2-2} = \frac{8}{0}$ which is undefined because the denominator is zero.

Thus the graph is undefined for $x - 2 = 0 \therefore x = 2$ is the **vertical asymptote**

The graph $y = \frac{8}{x}$ shift 2 units to the right and 3 units down to form the graph $g(x) = \frac{8}{x-2} - 3$

SKETCHING THE HYPERBOLA OF THE FORM: $y = \frac{a}{x-p} + q$

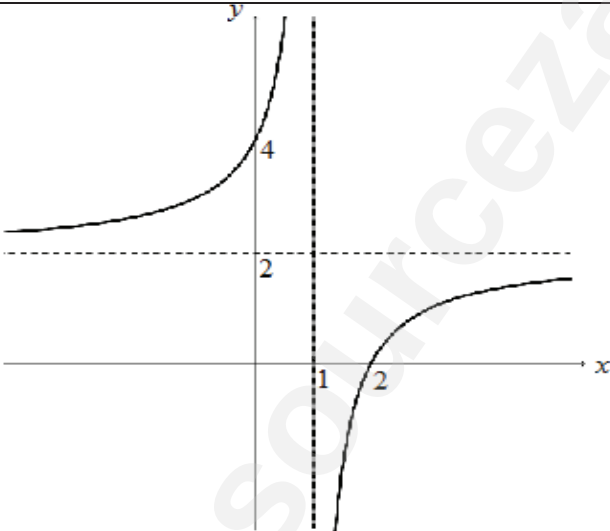
- Write down the equations of the asymptotes
- Draw the asymptotes on the set of axes as dotted lines
- Use a to determine the two quadrants where the graph will be drawn
- Determine the x - intercept(s) let $y = 0$
- Determine the y - intercept(s) let $x = 0$
- Plot the points and then draw the graph using free hand

EXAMPLE 1

Given: $f(x) = -\frac{2}{x-1} + 2$

- 1.1 Write down the equation of the asymptotes of f (2)
- 1.2 Find the intercepts of the graph of f with the axes (3)
- 1.3 Draw the graph of f on the system of axes (3)
- 1.4 Determine the equation of the axis of symmetry of the graph of f if $m < 0$ (3)
- 1.5 Write down the values of x for which $f(x) \leq 0$ (2)

SOLUTIONS

1.1	$x = 1$ $y = 2$
1.2	x - intercept : $y = 0$ $0 = -\frac{2}{x-1} + 2$ $2(x-1) = 2$ $\therefore x = 2$ y - intercept : $x = 0$ $y = -\frac{2}{0-1} + 2$ $\therefore y = 4$
1.3	
1.4	$y = -x + c$ $2 = -1 + c$ $\therefore c = 3$ $\therefore y = -x + 3$
1.5	$1 < x \leq 2$

Exercises

1. Consider the function $f(x) = \frac{2}{x-3} + 1$

- Write down the equations of the asymptotes of f
- Calculate the coordinates of the x and y -intercepts of f
- Write the domain and range
- Sketch the graph of f clearly showing ALL asymptotes and intercepts with the axes.

2. Consider the function $f(x) = \frac{5-2x}{x-1}$

- Write $f(x)$ in the form : $f(x) = \frac{a}{x+p} + q$
- Write down the equation of the asymptotes.
- Calculate the coordinates of the intercept of the graph of f with the axes.
- Sketch the graph of f clearly showing the intercepts with the axes and the asymptotes
- Write down the range of $y = -f(x)$.
- Determine the equations of the axes of symmetry of f
- Describe, in words the transformation of f to g if $g(x) = \frac{-3}{x+1} - 2$

SKETCHING THE EXPONENTIAL FUNCTION OF THE FORM: $y = ab^{x-p} + q$; $b > 0$, $b \neq 1$

- Write down the equation of the horizontal asymptote
- Draw the asymptote on the set of axes as dotted lines
- Use b to determine if the function is increasing or decreasing
- Determine the x – intercept(s) let $y = 0$
- Determine the y – intercept(s) let $x = 0$
- Draw the graph using free hand, indicate at least one point

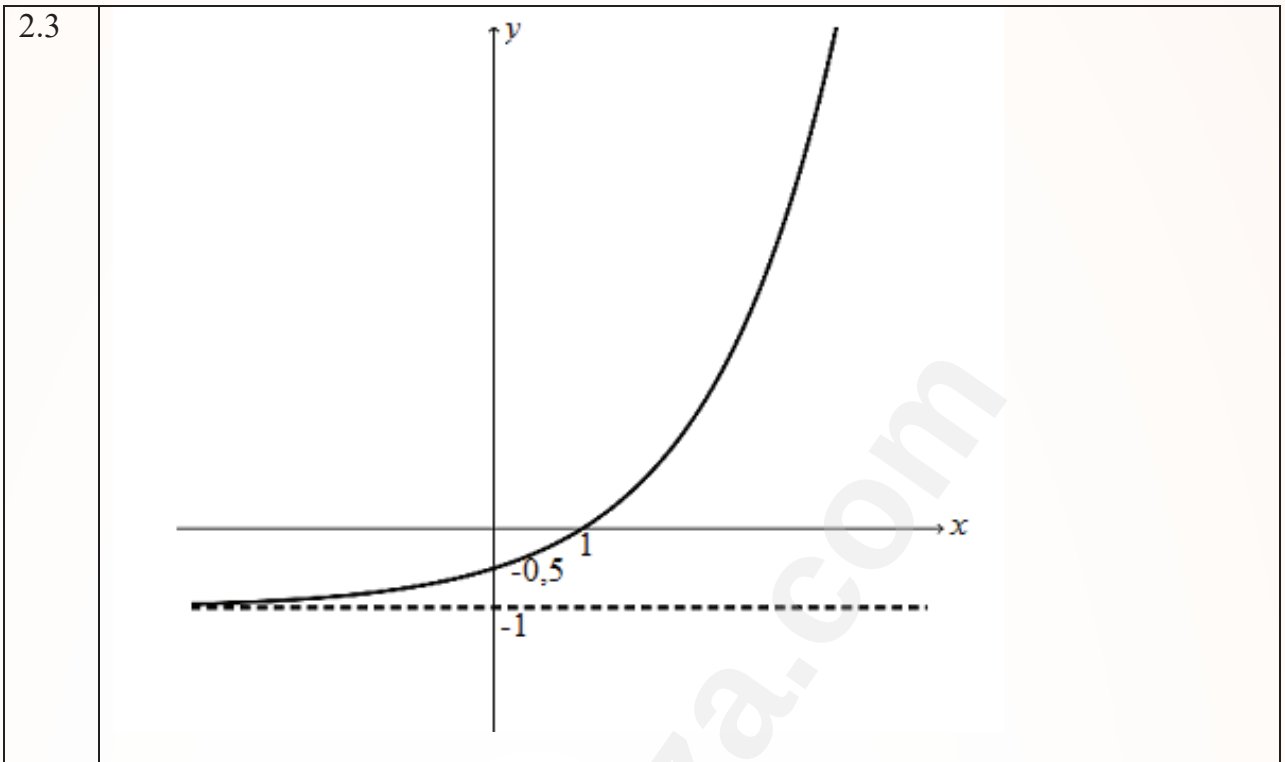
EXAMPLE 2

Given: $h(x) = 2^{x-1} - 1$

- 2.1 Write down the equation of the asymptote of h
- 2.2 Determine the intercepts of h with the axes
- 2.3 Draw a neat sketch of the graph of h

SOLUTIONS

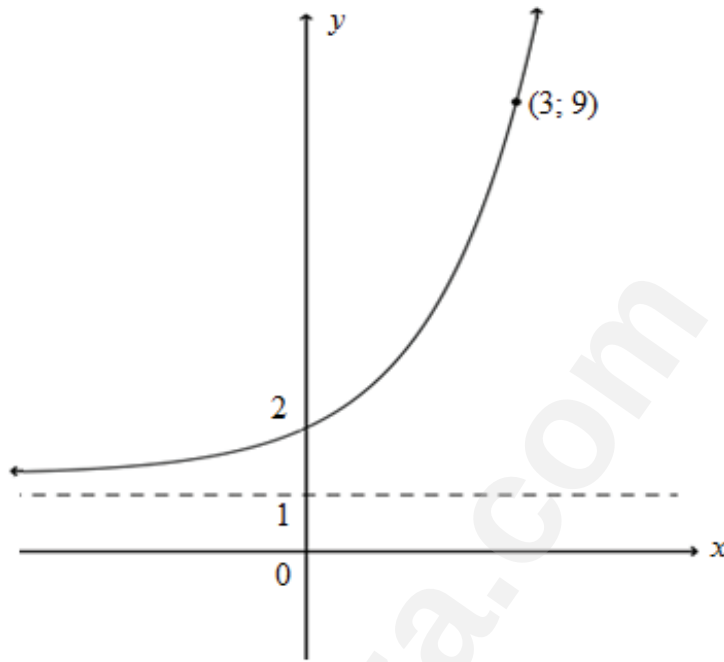
2.1	$y = -1$
2.2	x – intercept; $y = 0$ $0 = 2^{x-1} - 1$ $2^{x-1} = 1$ $2^{x-1} = 2^0$ $\therefore x - 1 = 0$ $\therefore x = 1$ y – intercept; $x = 0$ $y = 2^{0-1} - 1$ $\therefore y = -\frac{1}{2}$



Exercises

3. Consider the function: $f(x) = 4^{-x} - 2$
 - 3.1 Write down the equation of the asymptote of f
 - 3.2 Calculate the coordinates of the intercepts of f with the axes
 - 3.3 sketch the graph of f
 - 3.3 Write down the equation of g if g is the graph of f shifted 2 units upwards

4. The figure below shows the graph of $f(x) = a^x + q$



- 4.1 Find the values of a and q
- 4.2 Write down the range of the graph of f
- 4.3 Write down the equation of the asymptote of the graph of $g(x) = f(x) - 1$
- 4.4 Sketch the graph of g on the DIAGRAM SHEET 2 provided.

SKETCHING THE PARABOLA OF THE FORM: $y = a.(x - p)^2 + q$

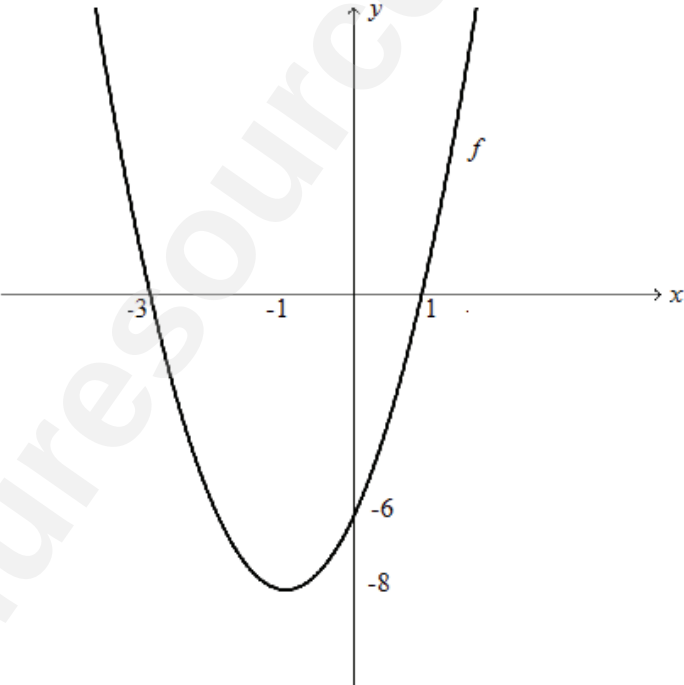
- Write down the coordinates of the turning points
- Use a to determine if the function is concave up or concave down
- Determine the x – intercept(s) let $y = 0$
- Determine the y – intercept(s) let $x = 0$
- Draw the graph using free hand

EXAMPLE 3

Given: $f(x) = 2(x + 1)^2 - 8$

- 3.1 Write down the coordinates of the turning point.
- 3.2 Determine the x and y intercepts
- 3.3 Sketch the graph of f

SOLUTIONS

3.1	$(-1;-8)$
3.2	y – intercept; $x = 0$ $y = 2(x + 1) - 8$ $y = 2(0 + 1) - 8$ $y = -6$ x – intercepts ; $y = 0$ $2(x + 1)^2 - 8 = 0$ $2(x + 1)^2 = 8$ $\sqrt{(x + 1)^2} = \sqrt{4}$ $x + 1 = \pm 2$ $x = -1 \pm 2$ $x = -3$ or 1
3.3	

Exercises

5 Given: $f(x) = -(x+2)^2 + 6$

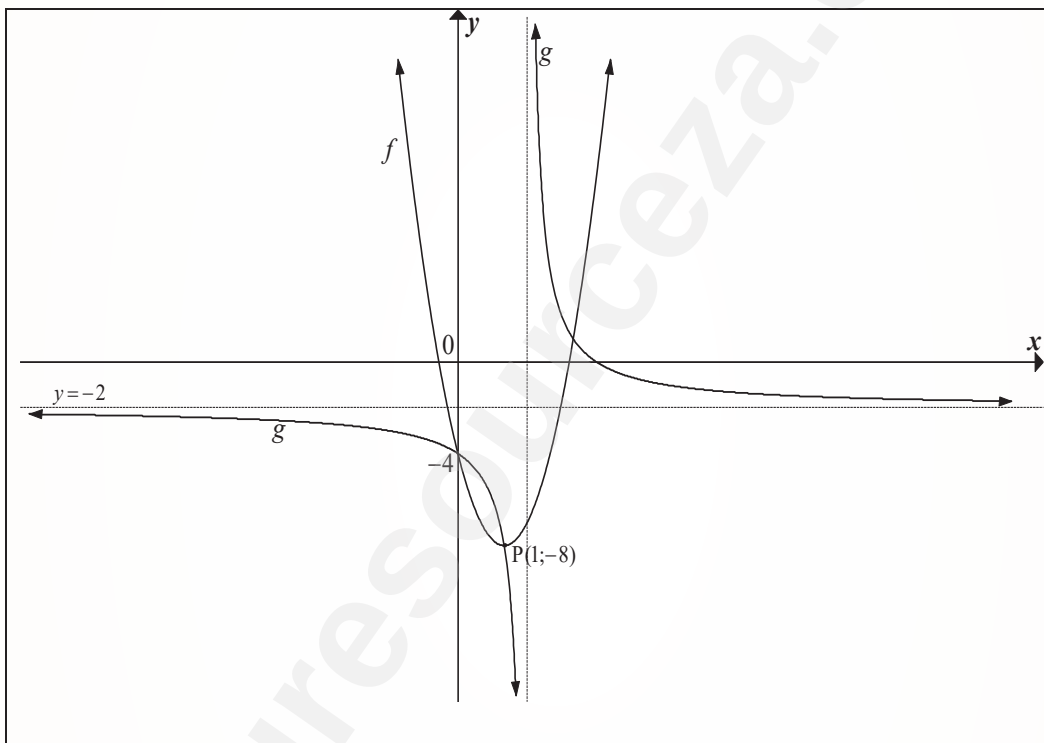
(a) Draw graph of f . Clearly show the intercepts with the axes as well as the turning point

(b) Write down the value(s) of t if $f(x) = t$ has:

- (i) equal roots
- (ii) one root equal to 0

6. The graphs of the functions $f(x) = a(x + p)^2 + q$ and $g(x) = \frac{k}{x + r} + d$ are sketched

below. Both graphs cut the y -axis at -4 . One of the points of intersection of the graphs is $P(1; -8)$, which is also the turning point of f . The horizontal asymptote of g is $y = -2$.



Calculate the values of a , p and q

Calculate the values of k , r and d .

Determine the value(s) of x in the interval $x \leq 1$ for which $g(x) \geq f(x)$.

Determine the value(s) of k for which $f(x) = k$ has two, unequal positive roots.

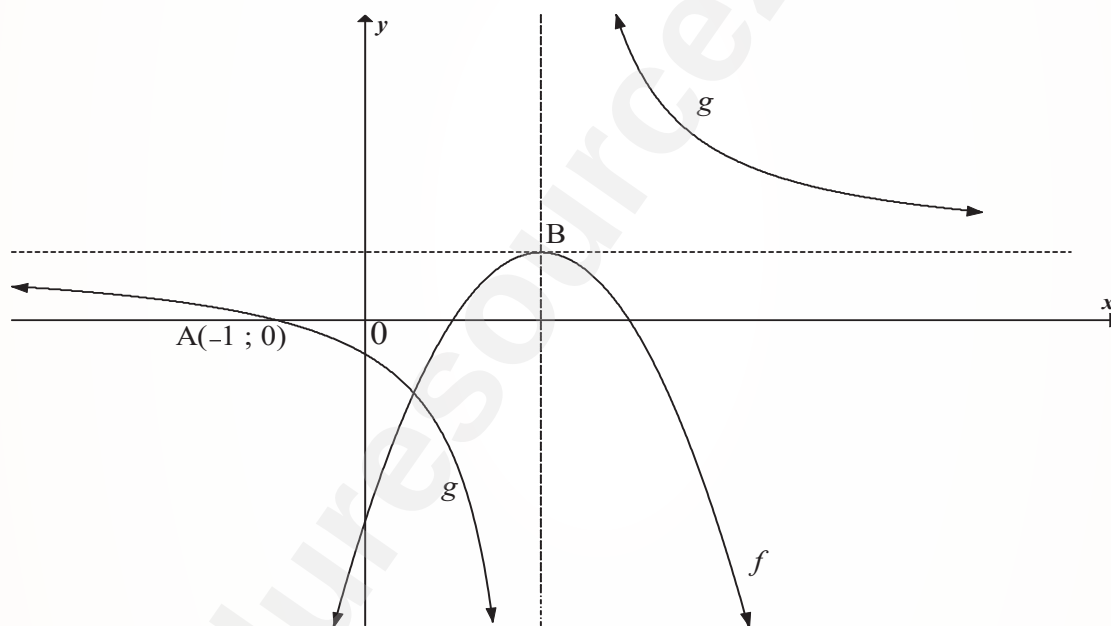
Write down an equation for the axis of symmetry of g that has a negative gradient

The point P is reflected in the line determined in (f) to give the point Q. Write down the coordinates of Q.

7. Consider the function: $f(x) = \frac{-6}{x-3} - 1$

- Calculate the coordinates of the y -intercept of f .
- Calculate the coordinates of the x -intercept of f .
- Sketch the graph of f in your ANSWER BOOK, showing clearly the asymptotes and the intercepts with the axes.
- For which values of x is $f(x) > 0$?
- Calculate the average gradient of f between $x = -2$ and $x = 0$.

8. Sketched below is the parabola f , with equation $f(x) = -x^2 + 4x - 3$ and a hyperbola g , with equation $(x - p)(y + t) = 3$. B, the turning point of f , lies at the point of intersection of the asymptotes of g . A(-1 ; 0) is the x -intercept of g .



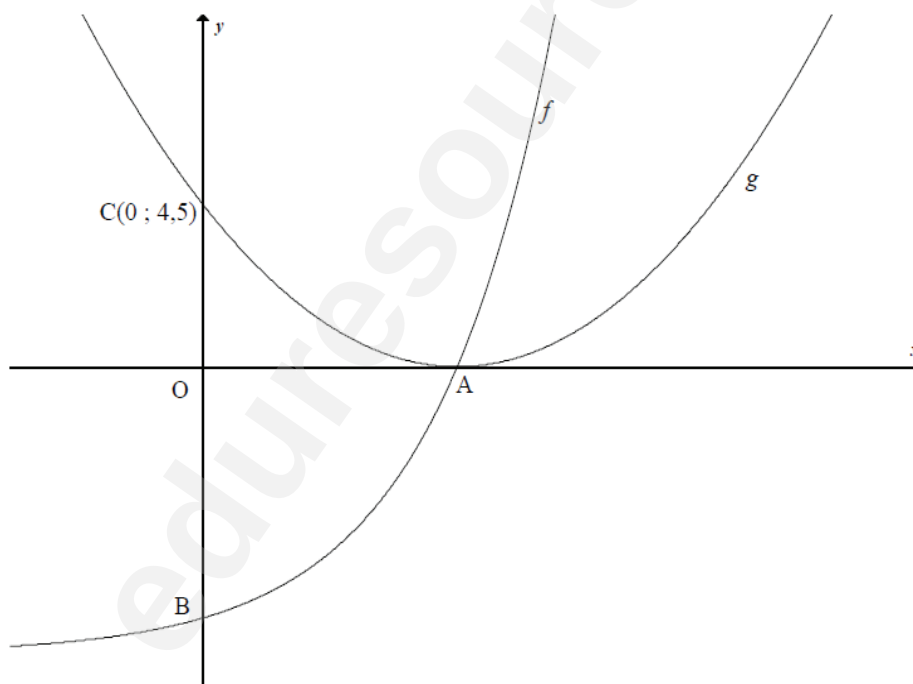
- Show that the coordinates of B are (2 ; 1)
- Write down the range of f .

- (c) For which value(s) of x will $g(x) \geq 0$?
- (d) Determine the equation of the vertical asymptote of the graph of h if $h(x) = g(x+4)$
- (e) Determine the values of p and t .
- (f) Write down the values of x for which $f(x) \cdot g'(x) \geq 0$

9. If $t(x) = 2(2^x)$:

- (a) draw a sketch graph of t , indicate clearly the intercepts with the axes;
- (b) write down the equation of the asymptote of t ;
- (c) determine the equation of m if m is the reflection of t about the x -axis; and
- (d) write down the range of m .

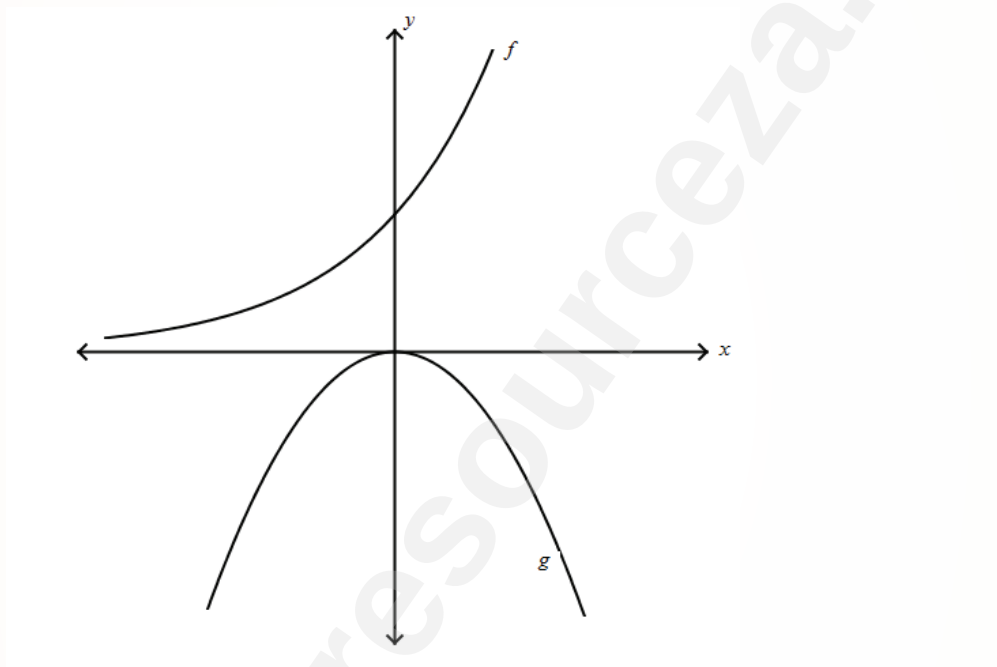
10. The graphs of $f(x) = 2^x - 8$ and $g(x) = ax^2 + bx + c$ are sketched below. B and C (0; 4,5) are the y -intercepts of the graphs of f and g respectively. The two graphs intersect at A, which is the turning point of the graph of g and the x -intercept of the graphs of f and g .



- Identify and assemble information from the graphs.
- Read the information given to verify your interpretation.
- What characteristics as reflected in the work schedule come to mind?
- Remember that -8 is not the y -intercept.

- (a) Determine the coordinates of A and B.
- (b) Write down the equation of the asymptote of the graph of f.
- (c) Determine the equation of h if $h(x) = f(2x) + 8$.
- (d) Determine an equation of h^{-1} in the form $y = \dots$
- (e) Write down an equation of p, if p is the reflection of h^{-1} about the x-axis.
- (f) Calculate $\sum_{k=0}^3 g(k) - \sum_{k=4}^5 g(k)$. Show your working.

11 Sketched below are the graphs of $f(x) = 3^x$ and $g(x) = -x^2$



- (a) Write down the equation of f^{-1} the inverse of f in the form $y = \dots$
- (b) Draw the graph of f^{-1} on the system of axes
- (c) Write down the domain of the graph of g
- (d) Write down the coordinates of the turning point of $g(x)$ if $g(x) = f(x-1) + 3$

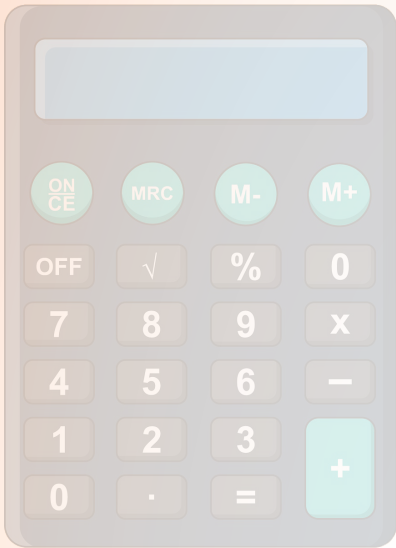
12. Given $h(x) = x^2$

- (a) Write down the equation of the inverse of h in the form $y = \dots$

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Published by the Department of Basic Education
222 Struben Street
Private Bag X895, Pretoria, 0001
Telephone: 012 357 3000 Fax: 012 323 0601
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