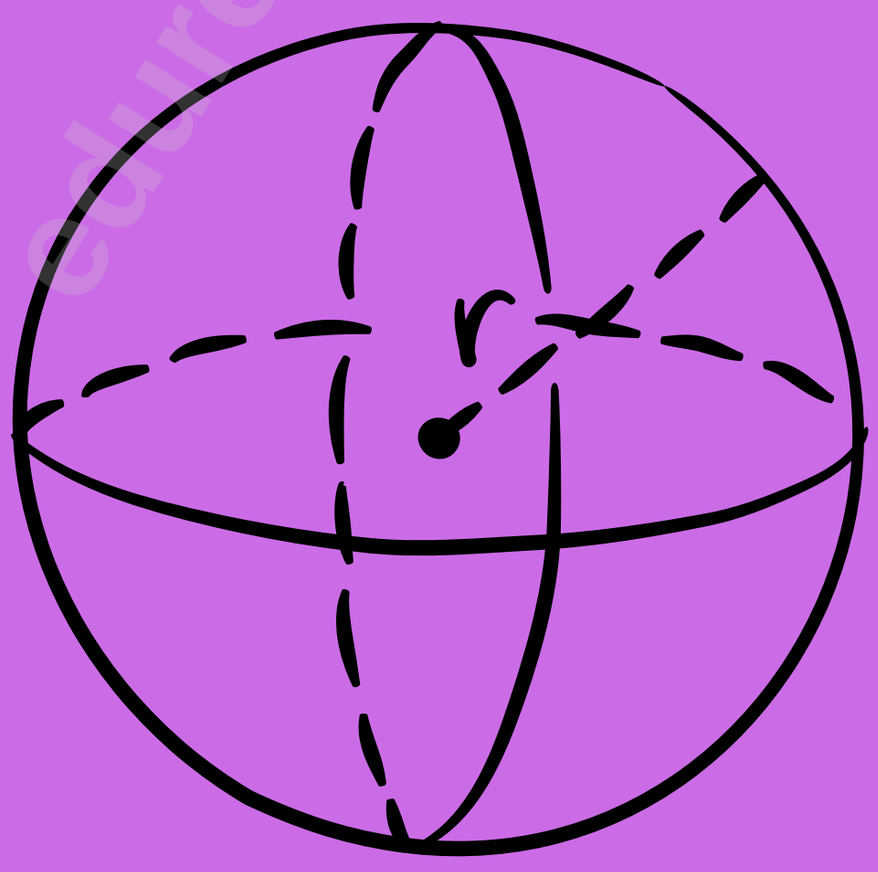


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THE CHEAT SHEET | Grade 12 CAPS

MATHEMATICS: PAPER 2



$$V = \frac{4}{3}\pi r^3$$

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Miss Angler

THE CHEAT SHEET:
GR. 12 MATHEMATICS PAPER 2

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That Maths Teacher
Miss Hanekom x Miss Angler

1ST EDITION
PUBLISHED 2023
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ACKNOWLEDGEMENTS
Questions taken from past NSC Examination Papers are cited individually.
Grade 12 NSC November 2022 Paper 2 pg. 18-19, 32-34, 36-37, 44, 46, 49, 70-71, 81-82, 104-106
Grade 12 NSC November 2021 Paper 2 pg. 9-10, 38-39, 43, 46, 48, 55, 79-80, 102-103, 107-109
Grade 12 NSC November 2020 Paper 2 pg. 43, 44, 57, 72-73, 83-84
Grade 12 NSC November 2019 Paper 2 pg. 14-15, 56, 100-101
Grade 12 NSC November 2018 Paper 2 pg. 58
All questions not cited have been created by the author.
Grade 12 Mathematics Examination Guidelines, pg. 4-5

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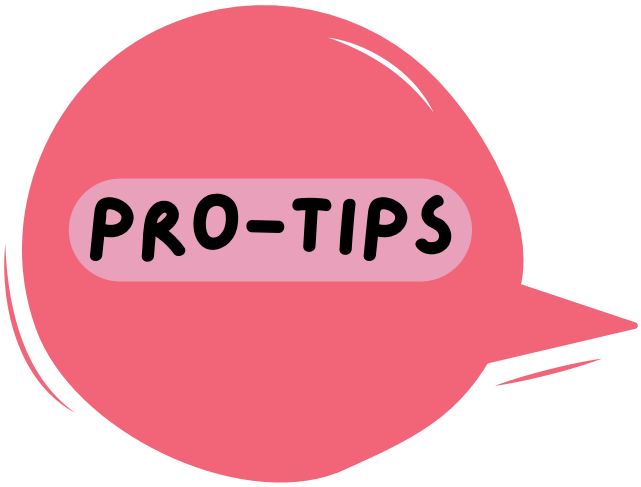
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HOW TO USE THIS STUDY GUIDE

PLEASE REMEMBER: This study guide does not replace your textbook or the instruction of your teacher. Its purpose is to make learning and revision EASY!

1. The Cheat Sheet is laid out in the same way that your final exam will be laid out.

- Keep this Cheat Sheet next to you when working through past papers for guided examples and explanations.

2. 

- Use these pro-tips for exam success!
- These tips are often the reason why my students get 80%+ in their exams and finals.
- They also point out common mistakes you should be careful of when answering questions.

3. ## QUESTION TYPES

Each question is also labelled with its difficulty level (Look out for either **K, R, C or P** next to each question!) so that you can prepare properly for the different levels of questions.

See **page 5** for a detailed explanation of the type of questions that each difficulty level could entail.

Each section is broken up into the **QUESTION TYPES** that most frequently show up in the final examinations.

REMEMBER: ALWAYS USE YOUR EXAM GUIDELINE WHEN PREPARING FOR AN EXAM OR TEST!

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PAPER 2: TOPICS & WEIGHTINGS

You will be writing a 3-hour paper out of 150 marks.
The 150 marks will be made up as follows:

Statistics and Regression	20 marks
Analytical Geometry	40 marks
Trigonometry	50 marks
Euclidean Geometry	40 marks

The questions in your final Paper 2 exam are also set up in this same order.

EXAM KEY

It's a good idea to plan your revision time per section according to these **weightings**.

25 marks = 25 min practice Qs

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EXPLAINING THE DIFFICULTY LEVELS OF QUESTIONS

Your exam will include questions covering **four** difficulty levels. Each level of question will require a different level of understanding and way of answering the question. It is helpful to know which questions fall in each of the following levels:

Knowledge (20 marks) - K
Recalling information, identifying correct formulae from the formula sheet, using mathematical facts and terminology, and rounding correctly.

Routine questions (52-53 marks) - R
Proving theorems and deriving formulae, performing familiar procedures and simple calculations that could involve a few steps, identifying and using the correct formulae from the formula sheet, answering questions that are similar to those taught in class.

Complex questions (45 marks) - C
Complex calculations that require higher-order thinking to solve. Being able to make connections between different representations of information and integrating different topics.

Problem-solving (22-23 marks) - P
Problems that are unfamiliar but not necessarily difficult. Questions that are asked differently from how they were in class, but can be solved using existing knowledge on the topic.

IMPORTANT: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.
Look out for either **K, R, C** or **P** next to each question!

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STATISTICS AND REGRESSION

<u>In order to achieve maximum marks for this section, you need to be able to:</u>

- calculate the **mean, median, mode, range** and **standard deviation** of a list of values.
- calculate the **approximate mean, modal class** and **quartiles** of these values (including the median) of grouped data.
- draw and interpret an **ogive** (cumulative frequency curve).
- calculate the **correlation coefficient** and the equation of the **least squares regression line** of bivariate data.
- **estimate** a value for bivariate data using the calculator or the equation of the least squares regression line (this is called interpolation or extrapolation).
- draw and interpret a **scatter plot** and **line of best fit.**
- comment on the **relationship between variables** and the **reliability of a prediction.**

There are three types of data groupings:

1. **Ungrouped data** - individual data values
2. **Grouped data** - data values that are arranged into equal intervals
3. **Bivariate data** - data values that compare two variables

PRO-TIPS

Grouped and bivariate data is most commonly tested.

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Question Type 1: Ungrouped Data

Ungrouped data is data that consists of individual values.
These values can be given in a **list** in which the values are either ordered or not ordered, or in a **frequency table**.

You need to be able to calculate the following measures of central tendency and spread:

1. **Range** - the difference between the maximum and minimum values.
2. **Mean** - the average of all of the data values.
3. **Mode** - the data value that occurs the most or the value with the highest frequency.
4. **Median** - the data value that is in the middle of all of the values, when the values are placed in order.
5. **Lower quartile** - the value that divides the values into the bottom 25% and the upper 75%, when the values are placed in order.
6. **Upper quartile** - the value that divides the values into the bottom 75% and the upper 25%, when the values are placed in order.
7. **Interquartile range** - the difference between the upper and lower quartile.
8. **Standard deviation** - the measure of how spread out the data values are in relation to the mean.
9. How many values are within/outside of one **standard deviation** from the mean.



You can use your CASIO scientific calculator to calculate the MEAN and the STANDARD DEVIATION:

- Press [MODE] and then select the [STAT] option.
- Select [1: 1 - VAR].
- Now type the data values in the "X" column. Press [=] after each value.
- Press [SHIFT] then [1].
- Select [4: Var].
- To calculate the mean, select [2].
- To calculate the standard deviation, select [3].

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Let's calculate each of these for a list of data values:

List: 1; 2; 2; 3; 5; 7; 7; 7; 9; 11; 11; 12; 13; 15

- Range = 15 − 1
  = 14

  Range = maximum − minimum

- Mean ($\bar{x}$) = $\frac{\text{sum of all values}}{\text{number of values}}$
  = $\frac{85}{14}$
  = 6,07

  You can use this formula to calculate the mean or you can use the STAT function on the calculator.

- Mode = 7

  There are three 7s in the list.

- Median, lower and upper quartile:
  1; 2; 2; ③; 5; 7; 7 / 7; 9; 11; ⑪; 12; 13; 15
  Q_1 Q_2 Q_3

  Make sure that the values are listed in order.
  First find the median: there are 14 values, so the median cuts the list into two equal halves of 7 values each.
  The lower quartile cuts the lower half of the list in half, and the upper quartile cuts the upper half of the list in half.

- Lower quartile (Q_1) = 3
  Median (Q_2) = 7
  Upper quartile (Q_3) = 11

- Interquartile range (IQR)
  = 11 − 3
  = 8

  IQR = $Q_3 - Q_1$

- Standard deviation (σ) = 4,34

  Use the STAT function on the calculator.

- How many values are within one standard deviation from the mean:
  $\bar{x} - \sigma$ = 6,07 − 4,34
  = 1,73
  $\bar{x} + \sigma$ = 6,07 + 4,34
  = 10,41
  ∴ 8 values (2; 2; 3; 5; 7; 7; 7; 9)

  Add and subtract the standard deviation to / from the mean.
  Then identify the values that lie between those boundaries.

- How many values are outside of one standard deviation from the mean:
  $\bar{x} - \sigma$ = 6,07 − 4,34
  = 1,73
  $\bar{x} + \sigma$ = 6,07 + 4,34
  = 10,41
  ∴ 6 values (1; 11; 11; 12; 13; 15)

  Add and subtract the standard deviation to / from the mean.
  Then identify the values that lie below or above those boundaries.

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Now let's look at a typical exam-type question!

Worked example (K + R + C)

A bakery kept a record of the number of loaves of bread a tuckshop ordered daily over the last 18 days. The information is shown in the table below:

10	11	13	14	14	15	16	18	18
19	19	20	21	35	35	37	40	41

Worked example (K)

1. Calculate the:

1.1. Mean number of loaves of bread ordered daily

$\bar{x} = \frac{396}{18}$

$= 22$ loaves of bread

$\bar{x} = \frac{\text{sum of all values}}{\text{number of values}}$

or use the STAT function of the calculator.

1.2. Standard deviation of the data

$\sigma = 10{,}17$ loaves Use the STAT function of the calculator.

Worked example (R)

1.3. Number of days on which the number of loaves of bread ordered was more than one standard deviation above the mean

$\bar{x} + \sigma = 22 + 10{,}17$

$= 32{,}17$

$\therefore$ 5 days

One standard deviation above the mean $= \bar{x} + \sigma$

Count the number of days on which more than 32,17 loaves of bread were ordered.

Worked example (C)

2. The tuckshop owner was not able to sell all the loaves of bread delivered daily. He calculated the mean number of loaves sold over the 18 days to be 20. Calculate the number of loaves of bread which were NOT sold over the 18 days.

$\bar{x}$ of loaves sold $= 20$

$\therefore$ Number of loaves sold

$= 20 \times 18$

$= 360$

To calculate the number of loaves sold, use the formula: $\bar{x}$ x number of days

$\therefore$ Number of loaves not sold

$= 396 - 360$

$= 36$

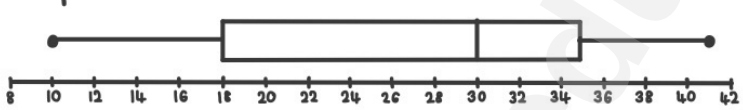
Now subtract the number of loaves sold from the number of loaves ordered.

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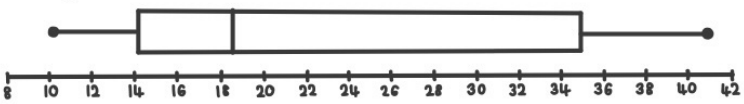
Worked example (K)

3. One of the two box-and-whisker diagrams drawn below represents the data given in the table above:

Graph A:



Graph B:



3.1. Which ONE of the two box-and-whisker diagrams, drawn above, correctly represents the data? Write down a reason for your answer.

Graph B.
The median of the data is 18,5 loaves, which is the same as indicated on Graph B.

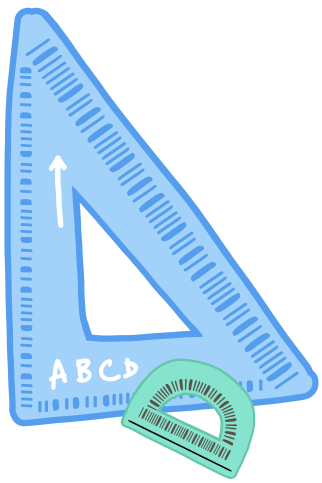
Compare the values in the table with the values shown in the box and whisker diagrams. Both diagrams have the same minimum and maximum values, but the medians are different.

3.2. Describe the skewness of the data.

Skewed to the right
or positively skewed

The data values are more spread out on the right of the median.

Question source: NSC Grade 12 November 2021 Paper 1



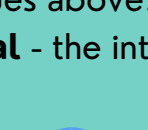
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Question Type 2: Grouped Data

Grouped data consists of data values that are organised into **equal intervals**. These values can be given in a **frequency table** or in a **cumulative frequency graph (ogive).**

You need to be able to calculate the ESTIMATED values of the following measures of central tendency and dispersion:

1. **Mean** - the average of all of the data values, but since we don't have the actual values, we use the midpoint of each interval as an estimate of the values in that interval.
2. **Median** - using an ogive.
3. **Lower quartile** - using an ogive.
4. **Upper quartile** - using an ogive.
5. **Interquartile range** - the difference between the upper and lower quartile, using the values above.
6. **Modal class/interval** - the interval with the highest frequency.



There are two types of data:

Discrete data is data that consists of items that can be **counted**, eg. number of people.

Continuous data is data that consists of values that can be **measured**, eg. litres of petrol.

When finding the midpoint of an interval of **discrete data**, we **do not include both** endpoints of the interval.

When finding the midpoint of an interval of **continuous data**, we **include both** endpoints of the interval.

Eg. if the interval is $10 < x \leq 20$ and the interval represents discrete data:

$$\text{Midpoint} = \frac{11 + 20}{2}$$

$$= 15{,}5$$

AND if the interval is $10 < x \leq 20$ and the interval represents continuous data.

$$\text{Midpoint} = \frac{10 + 20}{2}$$

$$= 15$$

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Calculating the estimated mean:

1. **Find the midpoint of each interval.** This is the value that we use to represent all of the data values in that interval.
2. **Now multiply the midpoint of each interval with the frequency of that interval.** This product is the value that we use to represent the total of all of the individual data values in that interval.
3. **Calculate the sum of each of these products.** This sum is the value that we use to represent the sum of all of the individual data values.
4. **Divide the sum of the products by the sum of the frequencies** (the sum of the data values). This is the estimated mean.

Worked example (R)

Calculate the estimated mean of the following discrete data represented in the table below.

Values	Frequency (f)	Midpoint (X)	$f \times X$
$0 < x \leq 10$	4	5,5	22
$10 < x \leq 20$	6	15,5	93
$20 < x \leq 30$	10	25,5	255
$30 < x \leq 40$	22	35,5	781
$40 < x \leq 50$	12	45,5	546
$50 < x \leq 60$	8	55,5	444
Total	62		2 141

$$\text{Estimated } \bar{x} = \frac{2\,141}{62}$$
$$= 34,53$$

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Drawing an ogive/cumulative frequency curve:

An ogive is a S-shaped curve drawn by connecting the following points:

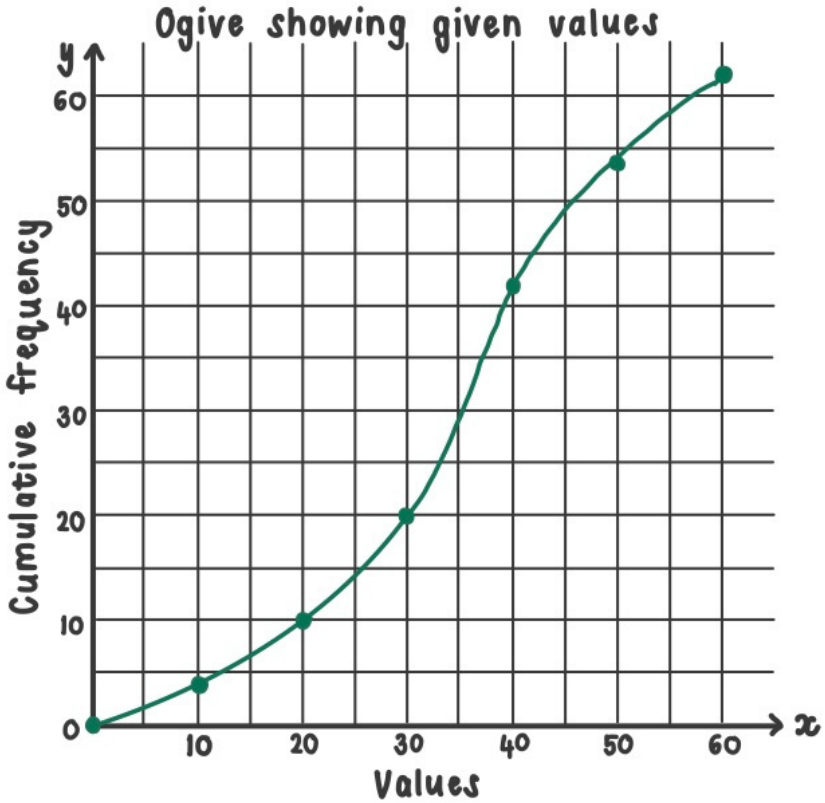
1. (lower value of first interval ; 0).
2. (endpoint of interval ; cumulative frequency)

The cumulative frequency of each interval is determined by adding the frequency to the cumulative frequency of the previous interval.

Worked example (R)

Draw an ogive of the data represented in the frequency table below.

Values	Frequency (f)	Cumulative frequency
$0 < x \leq 10$	4	4
$10 < x \leq 20$	6	10
$20 < x \leq 30$	10	20
$30 < x \leq 40$	22	42
$40 < x \leq 50$	12	54
$50 < x \leq 60$	8	62



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Worked example (K + R + C)

A survey was conducted among 100 people about the amount that they paid on a monthly basis for their cellphone contracts. The person carrying out the survey calculated the estimated mean to be R309 per month. Unfortunately, he lost some of the data thereafter. The partial results of the survey are shown in the frequency table below:

AMOUNT PAID (IN RANDS)	FREQUENCY
$0 < x \leq 100$	7
$100 < x \leq 200$	12
$200 < x \leq 300$	a
$300 < x \leq 400$	35
$400 < x \leq 500$	b
$500 < x \leq 600$	6

Worked example (R)

1. How many people paid R200 or less on their monthly cellphone contracts?

$7 + 12 = 19$ people

7 people paid between R0 and R100, and 12 people paid between R100 and R200.

Worked example (C)

2. Use the information above to show that $a = 24$ and $b = 16$.

$7 + 12 + a + 35 + b + 6 = 100$

100 people were questioned ∴ the total of all of the frequency values is 100.

$a + b = 40$

$a = 40 - b \ldots ①$

The estimated mean is 309 ∴ write an equation using the variables a and b and the method for calculating the estimated mean of grouped data. Then solve the resulting equation and the equation above simultaneously.

$$\frac{(7 \times 50) + (12 \times 150) + (a \times 250) + (35 \times 350) + (b \times 450) + (6 \times 550)}{100} = 309$$

$$\frac{17\,700 + 250a + 450b}{100} = 309$$

$17\,700 + 250a + 450b = 30\,900$

$250a + 450b = 13\,200$

$5a + 9b = 264 \ldots ②$

$\therefore 5(40 - b) + 9b = 264$

$200 - 5b + 9b = 264$

$4b = 64$

$b = 16$

$\therefore a = 40 - 16$

$= 24$

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Worked example (K)

3. Write down the modal class for the data.

Modal class = $300 < x \leq 400$ This interval has the highest frequency of 35.

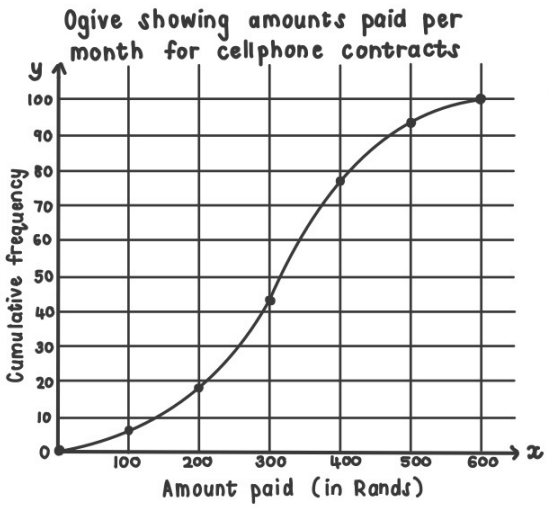
Worked example (R)

4. Draw an ogive (cumulative frequency graph) to represent the data.

The cumulative frequency of an interval is the sum of the frequency of that interval and the cumulative frequency of the previous interval.

Amount paid (in Rands)	Frequency	Cumulative frequency
$0 < x \leq 100$	7	7
$100 < x \leq 200$	12	19
$200 < x \leq 300$	24	43
$300 < x \leq 400$	35	78
$400 < x \leq 500$	16	94
$500 < x \leq 600$	6	100

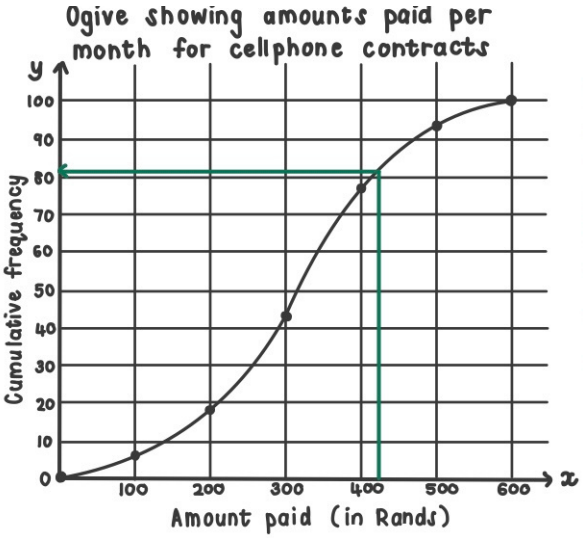
Cumulative f of first interval = frequency
12 + 7 = 19
24 + 19 = 43
35 + 43 = 78
16 + 78 = 94
6 + 94 = 100



Plot the following points:
(0; 0)
(100; 7)
(200; 19)
(300; 43)
(400; 78)
(500; 94)
(600; 100)
Then connect the points using a smooth curve.

Worked example (C)

5. Determine how many people paid more than R420 per month for their cellphone contracts.



Draw a line from R420 on the x-axis, connect the line to the curve, and then draw another straight line from the ogive to the y-axis.
The value on the y-axis is 81.
That means that 81 people spent less than R400 per month.
∴ 100 − 81 = 19 people spent more than R400.

Question source: NSC Grade 12 November 2019 Paper 2

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Question Type 3: Bivariate Data

Bivariate data is data that consists of two variables. The **strength** and **type** of relationship between these two variables is usually the focus of these questions.

You need to be able to **calculate** and **comment on** the following:

1. **The correlation coefficient (r)** - this is a measure of how strong the relationship/correlation between the two variables is.
2. The equation of the **least squares regression line (y = A + Bx)** - this is the exact equation of the "line of best fit" when the data is represented in a scatter plot.
3. The estimated value of one variable if the value of the other variable is given - this is called **interpolation** (if the value is within the range of the other values) or **extrapolation** (if the value is not within the range of the other values).

You can use your CASIO scientific calculator to calculate these values:



- Press [MODE] and then select the [STAT] option.
- Select [2: A + Bx].
- Now type the data values in the "X" and "Y" column. Press [=] after each value.
- Press [SHIFT] then [1].
- Select [5: Reg].
- To calculate the correlation coefficient, select [3: r].
- To calculate the equation of the least squares regression line, select [1: A], then repeat the process and select [2: B]. Then write the equation of the line y = A + Bx.

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You can use your CASIO scientific calculator to interpolate or extrapolate values



Method 1: Use the equation of the least squares regression line. Substitute the given x-value into the equation, and calculate the y-value from there.

Method 2: Use the calculator function.

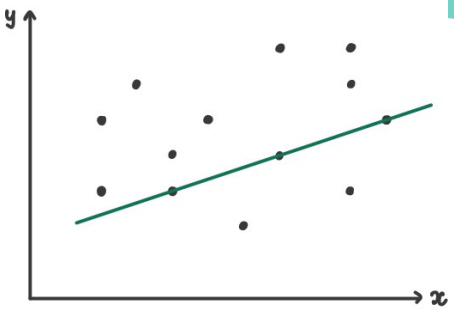
- First type the given x-value, then press [=].
- Then press [SHIFT], then [1: STAT].
- Then press [5: Reg], followed by [5: y].
- Then press [=] and the answer will be on the screen.

Make sure that you can draw a scatter plot AND interpret one.

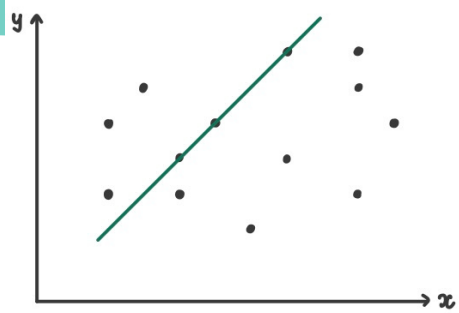
All bivariate data sets consist of pairs of values that can be written as points.
These points have x- and y-coordinates, and, when plotted on a Cartesian plane, form a scatter plot.
The question will state which variable is x and which variable is y.
Note that you do NOT have to label any of the points.

Once the points have been plotted, you should be able to draw a **line of best fit**.
It is a common misconception that a line of best fit should pass through as many points as possible. Rather draw a line that has the same number of points above and below it.

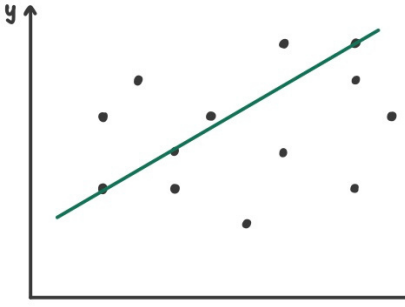
Look at the examples below:



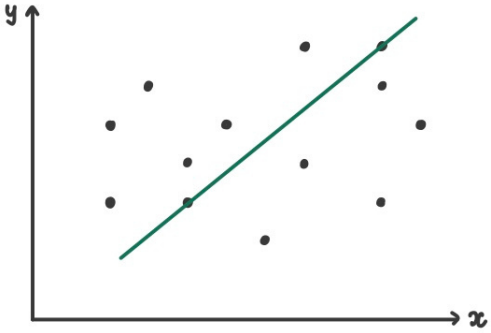
There are 8 points above and 2 points below this line, so it is NOT a line of best fit.



There are 3 points above and 7 points below this line, so it is NOT a line of best fit.



There are 4 points above and 6 points below this line, so it is NOT a line of best fit.

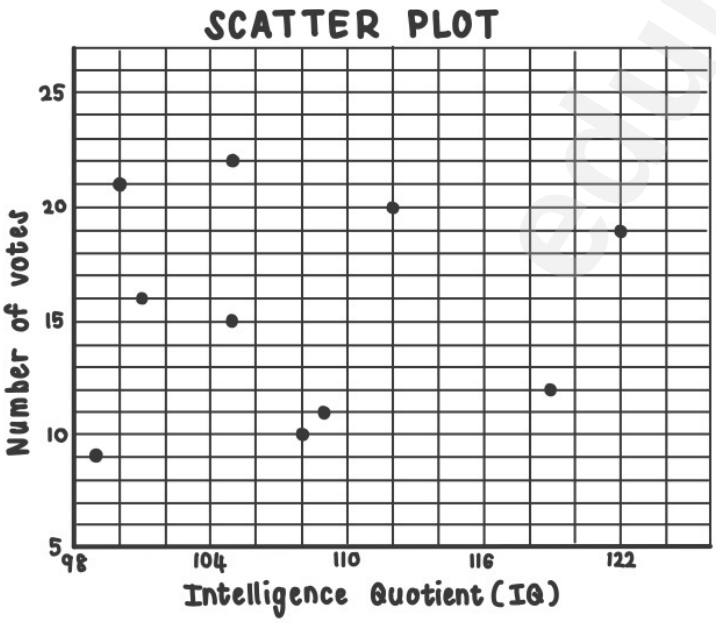


There are 6 points above and 5 points below this line, so it can be classified as a line of best fit.

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Worked example (K + R)

The matric class of a high school had to vote for the chairperson of the RCL (representative council of learners). The scatter plot below shows the IQ (intelligence quotient) of the 10 learners who received the most votes and the number of votes that they received.



Before the election, the popularity of each of these ten learners was established and a popularity score (out of a 100) was assigned to each. The popularity scores and the number of votes of the same 10 learners who received the most votes are shown in the table below.

Popularity score (x)	32	89	35	82	50	59	81	40	79	65
Number of votes (y)	9	22	10	21	11	15	20	12	19	16

Worked example (K)

1. Calculate the:

1.1. Mean number of votes that these 10 learners received

$\bar{x} = \frac{155}{10}$

$= 15,5$ votes

$\bar{x} = \frac{\text{sum of all votes}}{\text{number of learners}}$

or use the STAT function of the calculator.

1.2. Standard deviation of the number of votes that these 10 learners received

$\sigma = 4,59$ votes

Use the STAT function of the calculator.

Worked example (R)

2. The learners who received fewer votes than one standard deviation below the mean were not invited for an interview. How many learners were interviewed?

$\bar{x} - \sigma = 15,5 - 4,59$

$= 10,91$ votes

$\therefore$ 2 below 10,91 votes

$\therefore$ 8 learners were invited

One standard deviation below the mean $= \bar{x} - \sigma$

Count the number of learners with less than 10,91 votes

Out of 10 learners, 2 were not invited for an interview.

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Worked example (K)

3. Determine the equation of the least squares regression line for the data given in the table.

$A = 1,77$
$B = 0,22$
$\therefore \hat{y} = 1,77 + 0,22x$

Use the STAT function of the calculator.

Worked example (R)

4. Predict the number of votes that a learner with a popularity score of 72 will receive.

$x = 72 : y = 1,77 + 0,22(72)$
$= 17,61$ votes
or $x = 72 : y = 17,92$ votes

Substitute the number of votes given into the equation of the least squares regression line.
or use the STAT function of the calculator.

Worked example (C)

5. Using the scatter plot and table above, provide a reason why:

5.1. IQ is not a good indicator of the number of votes that a learner could receive.

There is no correlation between IQ and number of votes ∴ IQ is not a good indicator of the number of votes a learner could receive.

A statistical prediction is only reliable if it is based on data values that have a strong correlation.
The points on the scatter plot are scattered all over ∴ there is no correlation between the two variables.

5.2. The prediction in QUESTION 4 is reliable.

$r = 0,98$
There is a very strong correlation between popularity score and number of votes
∴ the prediction in 1.4 is reliable.

Since there is no scatter plot given, the best way to determine the strength of the correlation is by calculating the correlation coefficient. Do this using the STAT function of the calculator.

PRO-TIPS

If the correlation between two variables is **strong**, the prediction based on that correlation is **reliable**.

Question source: NSC Grade 12 November 2022 Paper 2

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ANALYTICAL GEOMETRY

<u>In order to achieve maximum marks for this section, you need to be able to:</u>

- calculate the **distance** between two points.
- calculate the coordinates of the **midpoint** of a line segment.
- calculate the **gradient** of a line, and use that to find the **equation of the line.**
- calculate the coordinates of the **point of intersection** of two lines.
- calculate the equation of a **median** and **altitude** of a triangle, and **perpendicular bisector** of a line.
- determine the coordinates of the **fourth vertex of a parallelogram** if the coordinates of the other three vertices are given.
- calculate the **angle of inclination** of a line.
- calculate the **size of any angle** using angles of inclination and other geometry rules.
- calculate the **area of a figure**, often made up of multiple triangles
- write the equation of a circle.
- find the coordinates of the **centre of a circle**, and the length of the **radius**, using a given equation.
- **interpret a diagram** in order to calculate various values relating to a circle.

This section is always split up into two questions:

1. A question dealing with **lines, quadrilaterals and triangles**
2. A question dealing with **lines and circles**

Even though the sub-questions can overlap, these are the main topics of the two questions.

PRO-TIPS

This is a section where the first questions are always the same/very similar!

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Question Type 1:
Lines, Quadrilaterals and Triangles

There are some basic formulae that you need to be able to apply. These formulae were all taught in Grade 9/10 and are given on the formula sheet.

- The **distance** between points $A(x_A; y_A)$ and $B(x_B; y_B)$:
  $AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$

- The **midpoint** of the line segment AB where $A(x_A; y_A)$ and $B(x_B; y_B)$:
  $M_{AB} = \left(\frac{x_A + x_B}{2}; \frac{y_A + y_B}{2}\right)$

- The **gradient** of the line passing through $A(x_A; y_A)$ and $B(x_B; y_B)$:
  $m_{AB} = \frac{y_A - y_B}{x_A - x_B}$ or $\frac{y_B - y_A}{x_B - x_A}$

- The **equation of a straight line** with a gradient m, passing through $(x_1; y_1)$:
  $y - y_1 = m(x - x_1)$

Worked example (K)

Given: $A(3; -1)$ and $B(1; 5)$

1. Calculate the length of AB.

$AB = \sqrt{(3-1)^2 + (-1-5)^2}$
$= 2\sqrt{10}$ units or 6,32 units

2. Calculate the midpoint of AB.

$M_{AB} = \left(\frac{3+1}{2}; \frac{-1+5}{2}\right)$
$= (2; 2)$

3. Calculate the gradient of AB.

$m_{AB} = \frac{-1-5}{3-1}$
$= -3$

4. Determine the equation of the straight line passing through A and B.

$y - y_1 = -3(x - x_1)$

Sub in A(3;-1):
$y + 1 = -3(x-3)$
$= -3x + 9$
$y = -3x + 8$

or sub in B(1;5):
$y - 5 = -3(x-1)$
$= -3x + 3$
$y = -3x + 8$

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You also need to be able to answer the following types of questions:

Find the point of intersection of two lines

1. You need the **equations of both lines**. If both equations are not given, you will have to determine one or both.
2. Since the point of intersection is where the two lines are equal to each other, you have to make the two **equations equal to each other**.
3. Solve for x.
4. Substitute the x-coordinate found into either of the two equations to get the y-coordinate of the point.

Worked example (K)

Find the point of intersection of $y = 3x - 1$ and $y = -x + 5$.

$3x - 1 = -x + 5$ — Make the two equations equal to each other.

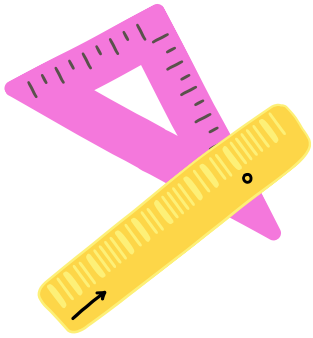
$4x = 6$

$x = \frac{3}{2}$ — Solve for x.

$\therefore y = 3\left(\frac{3}{2}\right) - 1$ — Substitute the x-coordinate into either one of the two equations.

$= \frac{7}{2}$

$\therefore \left(\frac{3}{2}; \frac{7}{2}\right)$ — Write your answer in coordinate form.



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Find the equation of a median of a triangle

A **median** of a triangle is a line from one vertex of the triangle to the midpoint of the opposite side.

When you have the coordinates of the three vertices of a triangle, you can find the equation of the median by following these steps:

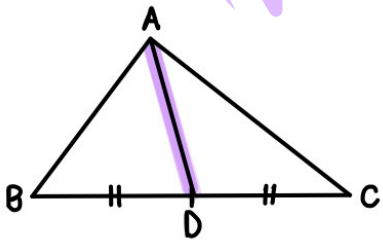
1. Determine **which side** the median is bisecting. It will always bisect the side OPPOSITE to the given vertex.

2. Calculate the **midpoint** of the side it is bisecting, using the midpoint formula.

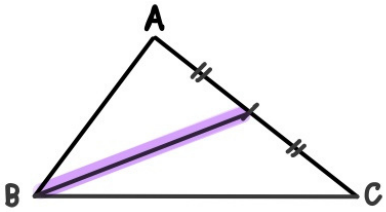
3. Now calculate the **gradient** of the median, using the coordinates of the vertex given and the midpoint calculated in step 2.

4. **Substitute** either the vertex given or the midpoint calculated in step 2 into the straight line equation and simplify.

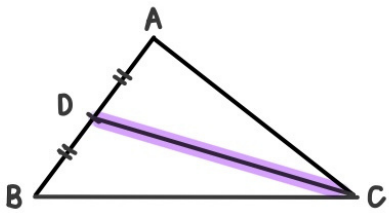
Here are some examples of medians drawn from each of the three vertices of a triangle.



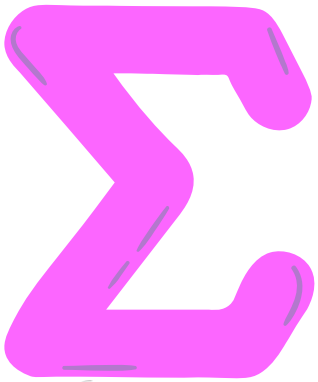
The median from vertex A which bisects the opposite side BC.



The median from vertex B which bisects the opposite side AC.



The median from vertex C which bisects the opposite side AB.



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Find the equation of an altitude/height of a triangle

An **altitude** of a triangle is a line from one vertex of the triangle, **perpendicular** to the side opposite that vertex.
It is essentially the perpendicular height of the triangle.

When you have the coordinates of the three vertices of a triangle, you can find the equation of the altitude by following these steps:

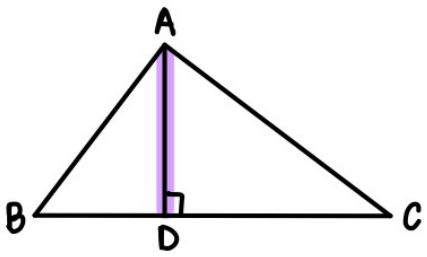
1. Determine **which side** the altitude is perpendicular to. It will always be perpendicular to the side OPPOSITE the given vertex.

2. Calculate the **gradient** of this side, using the coordinates of the two vertices given.

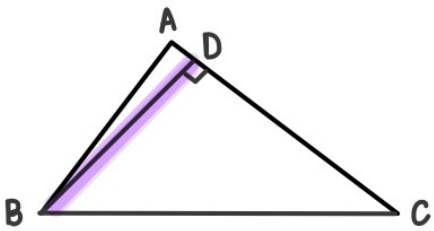
3. Now calculate the **gradient** of the altitude, using the fact that if two lines are perpendicular, the **product of their gradients is equal to -1.**

4. **Substitute** the vertex given into the straight line equation and simplify.

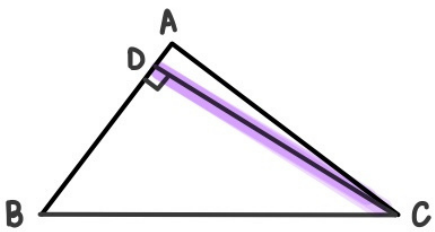
Here are some examples of altitudes drawn from all three vertices of a triangle.



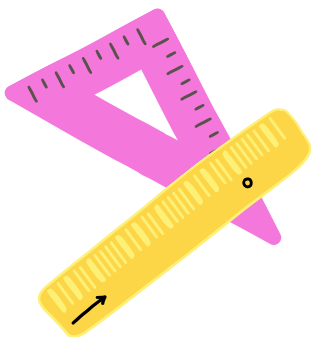
The altitude from vertex A, which is perpendicular to BC.



The altitude from vertex B, which is perpendicular to AC.



The altitude from vertex C, which is perpendicular to AB.



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Find the equation of a perpendicular bisector of a line

A perpendicular bisector of a line is a line that is **perpendicular** to another line, and also **bisects** the other line (goes through the midpoint of that line).
It does not necessarily pass through any of the vertices of the triangle.

When you have the coordinates of the three vertices of a triangle, you can find the equation of the perpendicular bisector by following these steps:

1. Determine **which side** the line is perpendicular to. This will be stated in the question.

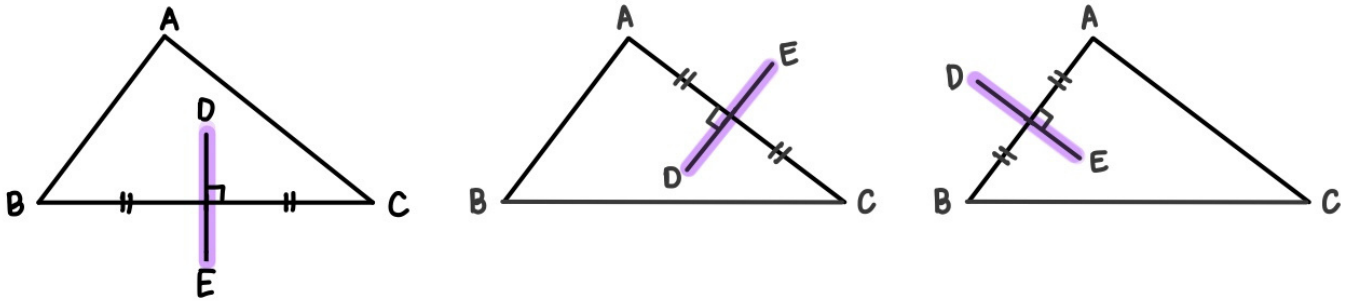
2. Calculate the **gradient** of the side it is perpendicular to, using the coordinates of the two vertices given.

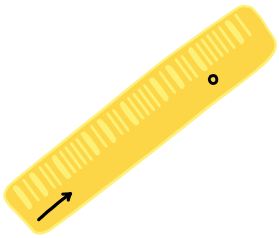
3. Now calculate the **gradient** of the perpendicular bisector, using the fact that if two lines are perpendicular, the **product of their gradients is equal to -1.**

4. Calculate the **midpoint** of the side that the perpendicular bisector is perpendicular to.

5. Now substitute both the **gradient** of the perpendicular bisector, as well as the **midpoint** found in step 4, into the straight line equation and simplify.

Here are some examples of perpendicular bisectors of all three sides of a triangle.



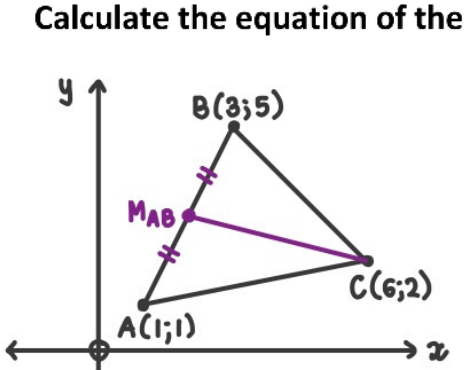


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Worked example (R)

Consider the diagram below with $A(1;1)$, $B(3;5)$ and $C(6;2)$.

Calculate the equation of the median of ΔABC from C.



$$M_{AB} = \left(\frac{1+3}{2}; \frac{1+5}{2}\right)$$
$$= (2;3)$$

(2;3) and C(6;2):

$$m = \frac{3-2}{2-6}$$
$$= -\frac{1}{4}$$

$$y - y_1 = -\frac{1}{4}(x - x_1)$$

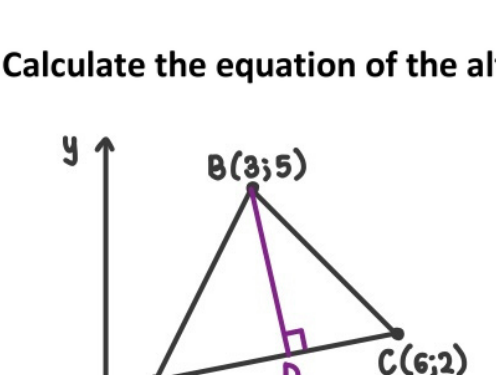
Sub in (2;3):

$$y - 3 = -\frac{1}{4}(x-2)$$
$$y = -\frac{1}{4}x + \frac{1}{2} + 3$$
$$y = -\frac{1}{4}x + \frac{7}{2}$$

or sub in C(6;2):

$$y - 2 = -\frac{1}{4}(x-6)$$
$$y = -\frac{1}{4}x + \frac{3}{2} + 2$$
$$y = -\frac{1}{4}x + \frac{7}{2}$$

Calculate the equation of the altitude of ΔABC from B.



$$m_{AC} = \frac{1-2}{1-6}$$
$$= \frac{1}{5}$$

$$m_{BD} \times m_{AC} = -1$$
$$m_{BD} \times \frac{1}{5} = -1$$
$$m = -5$$

$$\therefore y - y_1 = -5(x - x_1)$$

Sub in B(3;5):

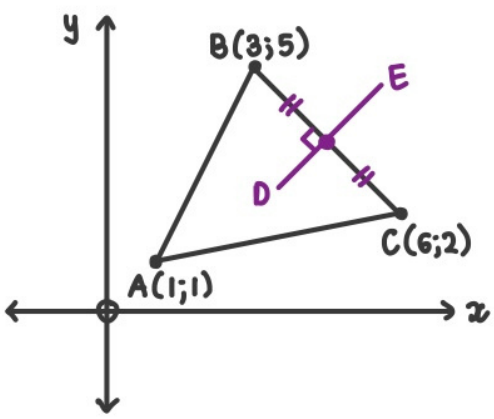
$$y - 5 = -5(x-3)$$
$$y = -5x + 15 + 5$$
$$y = -5x + 20$$

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Worked example (R) (continued)

Consider the diagram below with $A(1;1), B(3;5)$ and $C(6;2)$.

Calculate the equation of the perpendicular bisector of ΔABC of BC.



$$m_{BC} = \frac{5-2}{3-6}$$
$$= -1$$

$$m_{DE} \times m_{BC} = -1$$
$$m_{DE} \times (-1) = -1$$
$$m_{DE} = 1$$

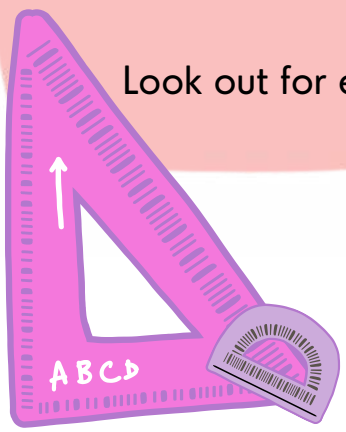
$$M_{BC} = \left(\frac{3+6}{2}; \frac{5+2}{2}\right)$$
$$= \left(\frac{9}{2}; \frac{7}{2}\right)$$

$$\therefore y - y_1 = 1(x - x_1)$$
Sub in $\left(\frac{9}{2}; \frac{7}{2}\right)$:
$$y - \frac{7}{2} = x - \frac{9}{2}$$
$$y = x - \frac{9}{2} + \frac{7}{2}$$
$$y = x - 1$$

IMPORTANT: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either **K, R, C** or **P** next to each question!



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Find the coordinates of the fourth vertex of a parallelogram

We find the coordinates of the fourth vertex of a parallelogram **by inspection.** Pay special attention to the **NAME** of the parallelogram as stated in the question. **The order of the vertices in the name is the same as the order of the vertices of the parallelogram.**

For example, if the name is **PQRS**, that means that P is connected to Q, Q is connected to R, R is connected to S and S is then connected back to P.

If the name of the parallelogram is **PQSR**, that means that P is connected to Q, Q is connected to S, S is connected to R and R is then connected back to P.

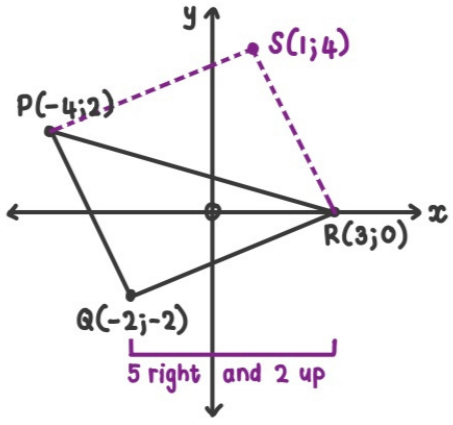
In order to find the coordinates of the fourth vertex, first **determine where the vertex will be** by looking at the name of the parallelogram.

Then look at the **change in coordinates from one vertex to another,** and apply that same change in coordinates in order to calculate the coordinates of the fourth vertex.

Worked example (R)

Consider the diagram below with points $P(-4;2)$, $Q(-2;-2)$ and $R(3;0)$.

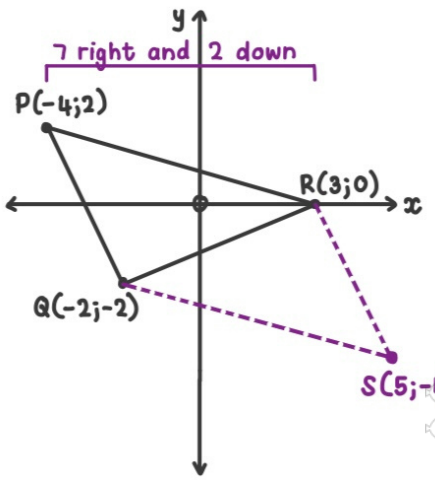
1. Determine the coordinates of S if PQRS is a parallelogram.



From Q(-2;-2) to R(3;0):
$(x;y) \rightarrow (x+5;y+2)$

∴ Apply the same translation to P:
$S = (-4+5;2+2)$
$S = (1;4)$

2. Determine the coordinates of S if PQSR is a parallelogram.



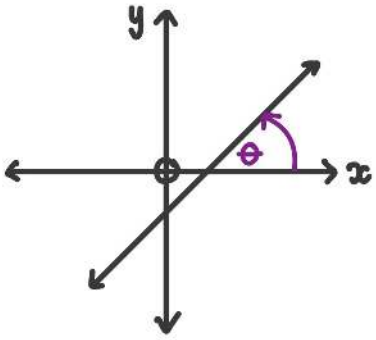
From P(-4;2) to R(3;0):
$(x;y) \rightarrow (x+7;y-2)$

∴ Apply the same translation to Q:
$S = (-2+7;-2-2)$
$S = (5;-4)$

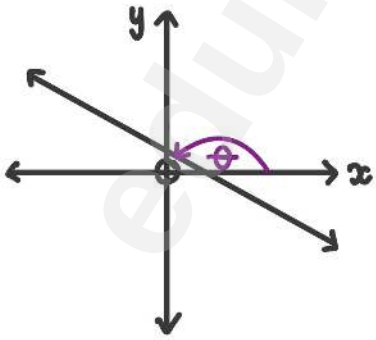
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Find the angle of inclination of a straight line

The angle of inclination of a line is the angle between that line and the positive x-axis. See the images below for examples of angles of inclination.



This line has a positive gradient, so θ is an acute angle ($0° < \theta < 90°$).



This line has a negative gradient, so θ is an obtuse angle ($90° < \theta < 180°$).

We use the following formula to calculate the angle inclination of a line:

$\tan\theta = m$, where θ is the angle of inclination and m is the gradient of the line.

$\therefore \theta = \tan^{-1}(m)$ if $m \geq 0$

or $\theta = 180° - \tan^{-1}(-m)$ if $m < 0$

Remember that in order to solve for the angle, you have to type [SHIFT] and then [tan] into the calculator.

Worked examples (R)

Calculate the angle of inclination, θ, of a straight line $y = 2x + 1$.

$m = 2$

$\therefore \tan\theta = 2$

$\theta = \tan^{-1}(2)$

$\theta = 63,43°$

The standard form of a straight line equation is $y = mx + c$.

$\tan\theta = m$

$m > 0$ (positive gradient) $\therefore \theta = \tan^{-1}(m)$

Calculate the angle of inclination, θ, of a straight line $3y + x - 1 = 0$.

$3y + x - 1 = 0$

$3y = -x + 1$

$y = -\frac{1}{3}x + \frac{1}{3}$

$m = -\frac{1}{3}$

$\therefore \tan\theta = -\frac{1}{3}$

$\theta = 180° - \tan^{-1}(\frac{1}{3})$

$\theta = 161,57°$

First write the equation in standard form: $y = mx + c$

$\tan\theta = m$

$m < 0$ (negative gradient) $\therefore \theta = 180° - \tan^{-1}(-m)$

If $m = -\frac{1}{3}$, then $-m = \frac{1}{3}$

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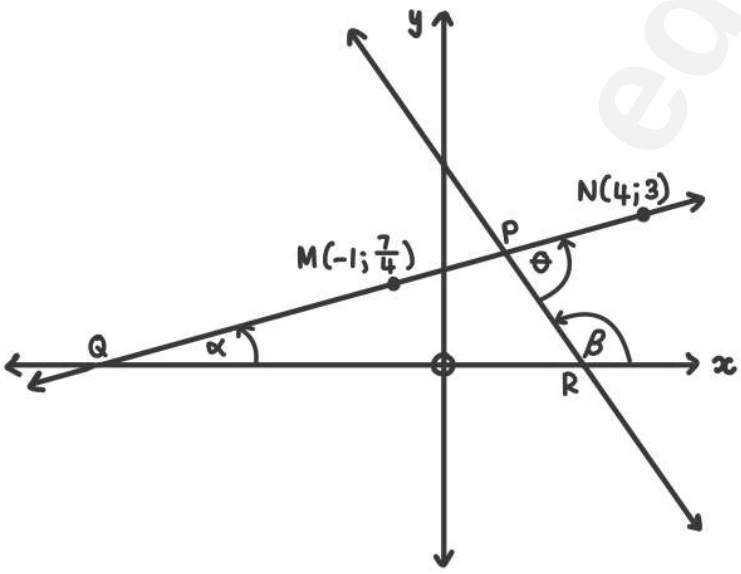
Worked example (C)

Consider the diagram below with points $M\left(-1;\frac{7}{4}\right)$, $N(4;3)$, Q, P and R.

The angle of inclination of PQ is α, the angle of inclination of PR is β and the angle between PQ and PR is θ.

The equation of PR is $y = -\frac{3}{2}x + 4$.

Calculate the value of θ, showing all working and reasons.



Calculate the angles of inclination of the lines that form the angle.

$\tan\beta = -\frac{3}{2}$

$\beta = 180° - \tan^{-1}\left(\frac{3}{2}\right)$

$= 123,69°$

The gradient of line PR is -3/2, which is used to calculate the angle of inclination of the line.

$\tan\alpha = m_{MN}$

$= \frac{\frac{7}{4} - 3}{-1 - 4}$

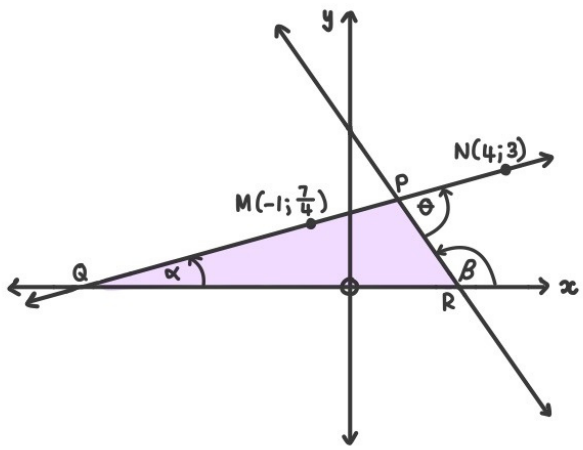
$= \frac{1}{4}$

$\alpha = \tan^{-1}\left(\frac{1}{4}\right)$

$= 14,04°$

The gradient of MN is not given and needs to be calculated first.

Now determine how these angles can be used to calculate the size of θ. Since θ is an exterior angle of the highlighted triangle, we need to calculate the other inner angle first.



$P\hat{R}Q = 180° - 123,69°$ (Ls on a straight line)

$= 56,31°$

$\therefore \theta = 56,31° + 14,04°$ (ext. $\angle$ of Δ)

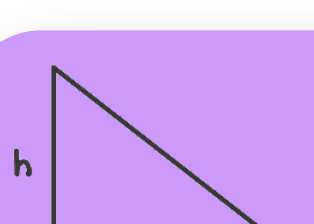
$= 70,35°$

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Calculate the area of a figure

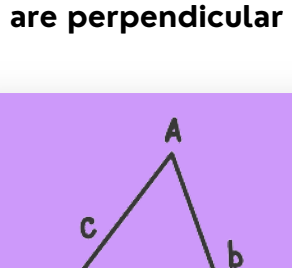
You will often be asked to calculate the area of a figure that consists of a right-angled triangle and one that is not right-angled.

In most cases, lengths of sides and sizes of angles that you have calculated earlier in the question will need to be used to calculate the respective areas, as well as the two different area formulae.



Area of a right-angled Δ
$= \frac{1}{2} \times$ base $\times$ perpendicular height
$= \frac{1}{2}bh$

The base and perpendicular height of a right-angled triangle can be any two sides that are perpendicular to each other. It can never be the hypotenuse!



Area of a non-right angled Δ:
Area $= \frac{1}{2}ab \times \sin \hat{C}$
or $\frac{1}{2}ac \times \sin \hat{B}$
or $\frac{1}{2}bc \times \sin \hat{A}$

This is the AREA RULE, which is discussed in more detail on page 75.

PRO-TIPS

Read through the given information **carefully** and indicate everything on the diagram **before** answering the questions.

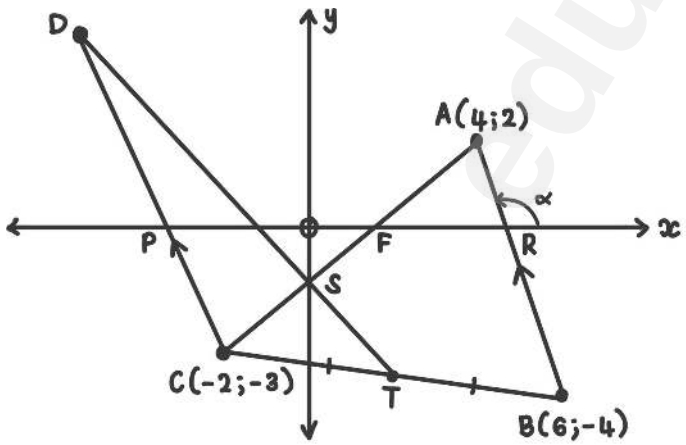
You will sometimes have to calculate some missing information before actually being able to calculate the area of the figure in the question, such as the lengths of sides or the size of angles.

Make sure that you know exactly what is required so that you can identify this missing information.

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Worked example (K + R + C)

In the diagram below, $A(4;2), B(6;-4)$ and $C(-2;-3)$ are vertices of ΔABC. T is the midpoint of BC. The equation of line AC is $5x - 6y = 8$. The angle of inclination of AB is α . ΔDCT is drawn such that $CD \parallel BA$. The lines AC and DT intersect at S, the y-intercept of AC. P, F and R are the x-intercepts of DC, AC and AB respectively.



Worked example (K)

1. Calculate the gradient of AB.

A(4;2) and B(6;-4)

$m_{AB} = \frac{2-(-4)}{4-6}$

$= -3$

Substitute the coordinates of A and B into the gradient formula.

Worked example (R)

2. Calculate the size of α

$\tan \alpha = m_{AB}$

$\tan \alpha = -3$

$\therefore \alpha = 180° - \tan^{-1}(3)$

$\alpha = 71,57°$

α is the angle between AB and the positive x-axis $\therefore \tan \alpha = m_{AB}$

3. Calculate the coordinates of T.

B(6;-4) and C(-2;-3)

$T = \left(\frac{6-2}{2}; \frac{-4-3}{2}\right)$

$T = \left(2; -\frac{7}{2}\right)$

CT = BT $\therefore$ T is the midpoint of BC. Substitute the coordinates of B and C into the midpoint formula.

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4. Calculate the coordinates of S.

$AC: 5x - 6y = 8$

$-6y = -5x + 8$

$y = \frac{5}{6}x - \frac{4}{3}$

$\therefore S(0; -\frac{4}{3})$

S is the y-intercept of AC
∴ write the given equation of AC in standard form: $y = mx + c$
c is the y-intercept

5. Determine the equation of CD in the form $y = mx + c$.

$y - y_1 = m(x - x_1)$

$m = -3$

$\therefore y - y_1 = -3(x - x_1)$

Sub in $C(-2;-3)$:

$y + 3 = -3(x + 2)$

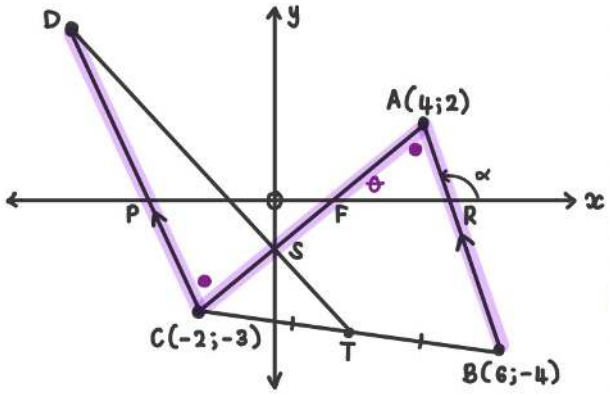
$y + 3 = -3x - 6$

$y = -3x - 9$

AB ∥ CD, and parallel lines have the same gradients ∴ $m_{CD} = m_{AB}$
Replace m with m_{CD} and then substitute the coordinates of a point on CD into the equation as well.

Worked example (C)

6. Calculate the size of $D\hat{C}A$.



First determine a strategy for calculating $D\hat{C}A$.
Notice the highlighted Z-shape.
Since AB ∥ CD, $D\hat{C}A = B\hat{A}C$.
We already know the size of α, and α is the exterior angle of ΔFAR. We can calculate $B\hat{A}C$ if we have the size of $A\hat{F}R$.
$A\hat{F}R$ is the angle of inclination of AC.

$\tan\theta = m_{AC}$

$\tan\theta = \frac{5}{6}$

$\theta = \tan^{-1}(\frac{5}{6})$

$\theta = 39,81°$

$F\hat{A}R = 71,57° - 39,81°$ (ext. ∠ of a Δ)

$F\hat{A}R = 31,76°$

$\therefore D\hat{C}A = 31,76°$ (alt. ∠s; AB ∥ CD)

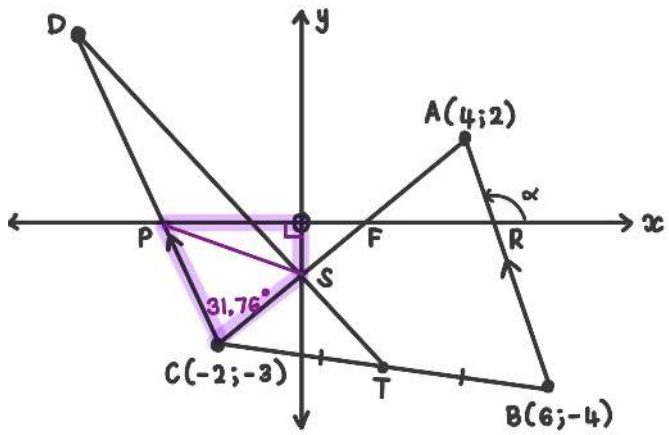
Remember to state all reasons.

PRO-TIPS

It is very useful to use **colours** to identify angles that are equal, eg. highlight the **parallel** lines in purple, as well as the angles that are equal because of those parallel lines, in the **same colour**.

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7. Calculate the area of POSC.



POSC is a quadrilateral that can be divided into right-angled ΔOPS and non-right angled ΔPCS.

To find the area of ΔOPS, we need the length of OP and OS.

P: x-intercept of CD

$\therefore$ make $y = 0$

$-3x - 9 = 0$

$-3x = 9$

$x = -3$

$\therefore P(-3;0)$

$S(0;-\frac{4}{3})$ calculated earlier

To find the area of ΔPCS, we need to calculate the lengths of PC and CS.

This will allow us to use the area rule.

Remember: we need the lengths of two sides and the size of the included angle in order to use the area rule.

$P(-3;0)$, $C(-2;-3)$ and $S(0;-\frac{4}{3})$:

$PC = \sqrt{(-3+2)^2 + (0+3)^2}$

$= \sqrt{10}$ units

$CS = \sqrt{(-2-0)^2 + (-3+\frac{4}{3})^2}$

$= \frac{\sqrt{61}}{3}$ units

$\therefore$ Area POSC = area ΔOPS + area ΔPCS

$= \frac{1}{2}(3)(\frac{4}{3}) + \frac{1}{2}(\sqrt{10})(\frac{\sqrt{61}}{3}) \cdot \sin 31{,}76°$

$= 4{,}17$ units^2

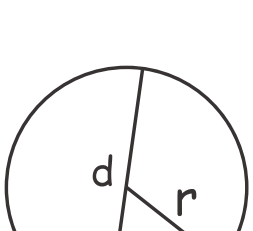
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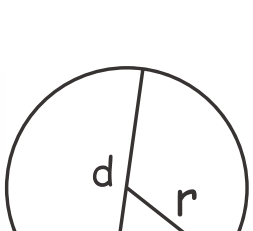
Question Type 2: Lines and Circles

The circles questions are often very similar to those asked in the lines and triangles section, but these questions usually require an understanding of the rules that apply to circles on the Cartesian plane.

We use the following formula to write the equation of a circle:



Centre-radius equation form:
$(x-a)^2 + (y-b)^2 = r^2$
where $(a;b)$ is the centre of the circle and r is the radius



If we multiply the brackets out and move all the constant terms to the right-hand side of the equation, we call the resulting equation the **standard form.**

It is important to be able to **convert** a circle equation in standard form to the centre-radius equation form. This will allow us to very easily write the coordinates of the centre of the circle, and to determine the length of the radius.

These are the steps to follow in order to do this conversion:

$x^2 + y^2 - 4x + 8y = 44$ — We have to complete the square for x and y.

$x^2 - 4x + y^2 + 8y = 44$ — Group the x- and y-terms together. The equation should now be in the form $ax^2 + bx + ay^2 + cy = d$. If a is not equal to 1, divide each term by a.

$x^2 - 4x + (2)^2 + y^2 + 8y + (4)^2 = 44 + (2)^2 + (4)^2$
$(x-2)^2 + (y+4)^2 = 64$
$\therefore$ Centre $= (2;-4)$ and radius $= 8$ units

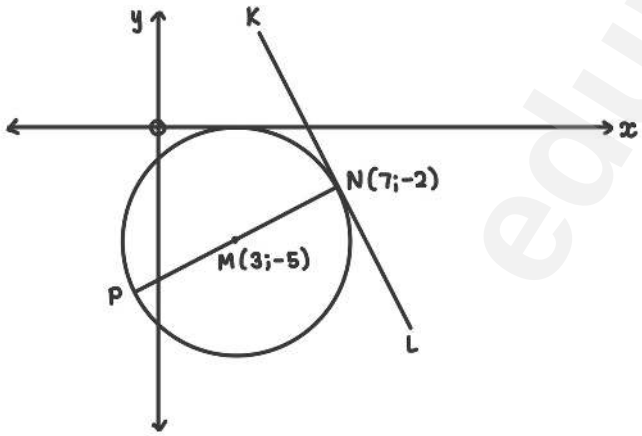
Now add $(\frac{b}{2})^2$ and $(\frac{c}{2})^2$ to both sides, and then factorise the left-hand side of the equation. Add all the values on the right-hand side together.

Now let's look at some typical exam questions so that you can see how you should be able to apply your knowledge of all analytical geometry to circle questions!

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Worked example (R + C + P)

In the diagram, $M(3;-5)$ is the centre of the circle having PN as its diameter. KL is a tangent to the circle at $N(7;-2)$.



Worked example (C)

1. Calculate the coordinates of P.

M is the midpoint of PN

$$\left(\frac{x_P+7}{2};\frac{y_P-2}{2}\right)=(3;-5)$$

$\therefore \frac{x_P+7}{2}=3$ and $\frac{y_P-2}{2}=-5$

$x_P+7=6$ $\quad$ $y_P-2=-10$

$x_P=-1$ $\quad$ $y_P=-8$

$\therefore P(-1;-8)$

Substitute the coordinates of M and N into the midpoint formula. Then split the equation up by equating the x-coordinates and y-coordinates separately. This allows you to solve for x_P and y_P.

or by inspection:

From N to M: $(x;y)\rightarrow(x-4;y-3)$

$\therefore P=(x_M-4;y_M-3)$

$P=(3-4;-5-3)$

$P=(-1;-8)$

Instead of using the midpoint formula, we can copy the translation from N to M, as it would be the same translation from M to P.

You don't need to show any working for this question.

Worked example (R)

2. Determine the equation of the circle in the form $(x-a)^2+(y-b)^2=r^2$

Centre = $(3;-5)$

$\therefore (x-3)^2+(y+5)^2=r^2$

Sub in $N(7;-2)$:

$(7-3)^2+(-2+5)^2=r^2$

$\therefore r^2=25$

$\therefore (x-3)^2+(y+5)^2=25$

First substitute the coordinates of the centre into the centre-radius equation form of a circle, $(x-a)^2+(y-b)^2=r^2$. The quickest way of calculating the value of r^2 is by substituting any point on the circumference of the circle.

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3. Determine the equation of KL in the form $y = mx + c$

KL is a tangent to the circle at N
$\therefore$ MN $\perp$ KL

A tangent to a circle is perpendicular to the radius of the circle at the point of contact.

$$m_{MN} = \frac{-5-(-2)}{3-7} = \frac{3}{4}$$

$$\therefore m_{KL} = -\frac{4}{3}$$

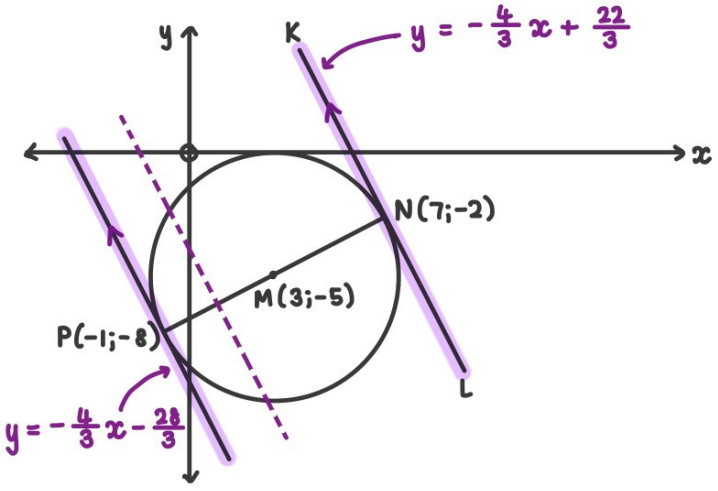
When two lines are perpendicular, the product of their gradients is -1.
$m_{MN} \times m_{KL} = -1$
$\frac{3}{4} \times m_{KL} = -1$

$$y - y_1 = -\frac{4}{3}(x - x_1)$$
Sub in N(7;-2):
$$y + 2 = -\frac{4}{3}(x-7)$$
$$y + 2 = -\frac{4}{3}x + \frac{28}{3}$$
$$y = -\frac{4}{3}x + \frac{22}{3}$$

Now substitute a point on KL into the straight line equation and simplify.

Worked example (P)

4. For which values of k will $y = -\frac{4}{3}x + k$ be a secant to the circle?



A secant is a line that cuts a circle in two points.
The two highlighted lines have gradients of $-\frac{4}{3}$.
Any other line with the given gradient that lies BETWEEN the highlighted lines will be a secant (see dotted line as an example).

We need to calculate the equation of the line passing through P.

$$y - y_1 = -\frac{4}{3}(x - x_1)$$
Sub in P(-1;-8):
$$y + 8 = -\frac{4}{3}(x+1)$$
$$y + 8 = -\frac{4}{3}x - \frac{4}{3}$$
$$y = -\frac{4}{3}x - \frac{28}{3}$$

This line has the same gradient, $-\frac{4}{3}$.

$$y = -\frac{4}{3}x + k$$
$$\therefore -\frac{28}{3} < k < \frac{22}{3}$$

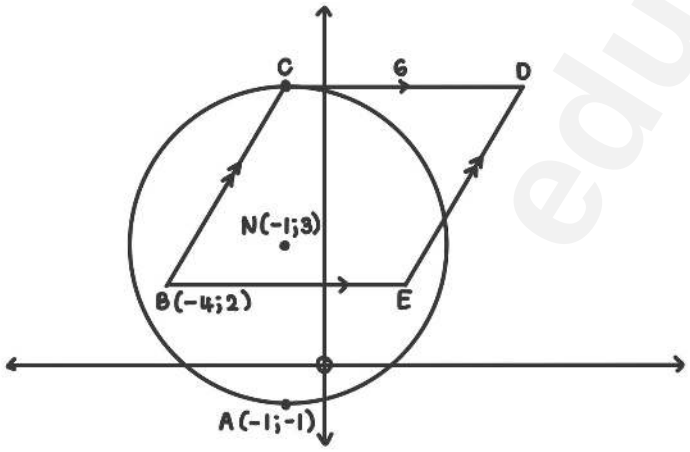
For a line to be between the two highlighted lines, the k-value must be between $-\frac{28}{3}$ and $\frac{22}{3}$.

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Worked example (R + C + P)

In the diagram, the circle centred at $N(-1;3)$ passes through $A(-1;-1)$ and C. $B(-4;2)$, C, D and E are joined to form a parallelogram such that BE is parallel to the x-axis. CD is a tangent to the circle at C and $CD = 6$ units.



Worked example (R)

1. Write down the length of the radius of the circle.

Radius = AN
= 4 units

Since A and N have the same x-value, we can simply determine the difference between the y-values of the two points.

2. Calculate the coordinates of C.

C(-1;7)

C is exactly above A and N, so it has the same x-value, and since it is also on the circumference on the circle, it is 4 units above point N ∴ add 4 to the y-value of N.

3. Calculate the coordinates of D.

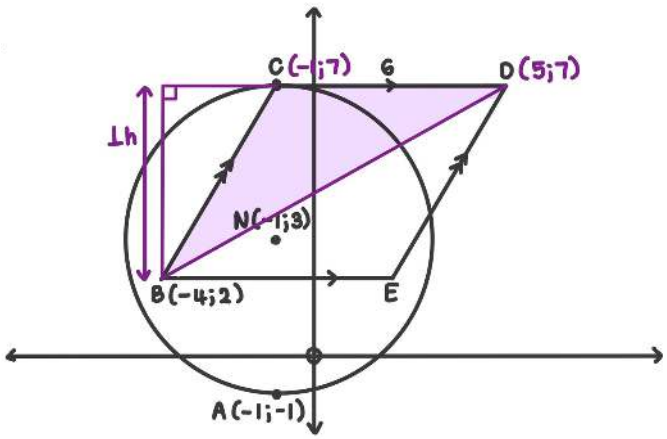
D(5;7)

BE is parallel to the x-axis, and BE ∥ CD ∴ CD ∥ x-axis
That means that C and D have the same y-value.
CD = 6 units, so add 6 to the x-value of C to get the x-value of D.

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Worked example (C)

4. Calculate the area of ΔBCD.



ΔBCD is not a right-angled Δ, but we can still use the area-formula if we can find the perpendicular height of the triangle.
⊥h = 5 units, as it's the difference between 2 and 7.

$$\text{Area } \Delta BCD = \frac{1}{2}(CD) \times \perp\text{height}$$
$$= \frac{1}{2}(6) \times 5$$
$$= 15 \text{ units}^2$$

Worked example (P)

5. The circle, centred at N, is reflected about the line $y = x$. M is the centre of the new circle which is formed. The two circles intersect at A and F.

Calculate the length of NM.

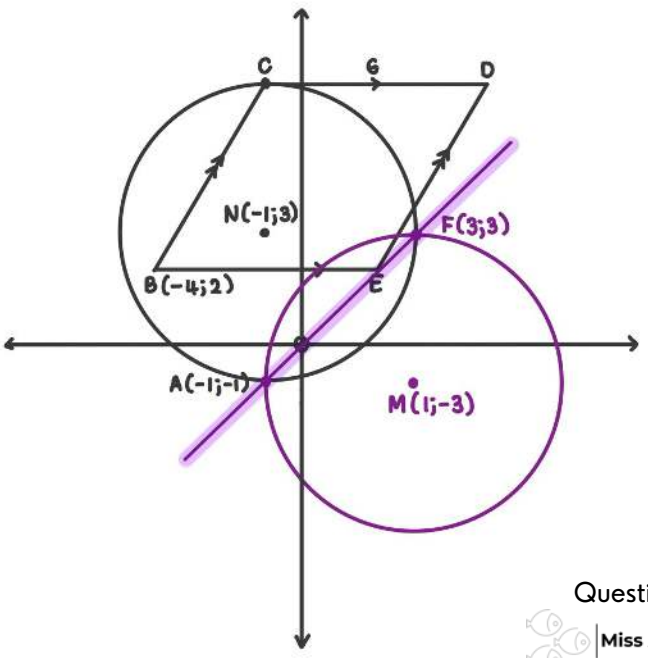
$M = (3;-1)$

When a point is reflected in the line $y = x$, the x- and y-coordinates of the point swap around.
$N(-1;3) \rightarrow M(3;-1)$

$$NM = \sqrt{(-1-3)^2 + (3+1)^2}$$
$$NM = 4\sqrt{2} \text{ units}$$
$$NM = 5,66 \text{ units}$$

Now use the distance formula to calculate the length of NM.

6. Calculate the midpoint of AF.



Apply the reflection in the line $y = x$ to the entire circle in order to find the coordinates of F.

$A = (-1;-1)$ and $F = (3;3)$

$$\therefore M_{AF} = \left(\frac{-1+3}{2}; \frac{-1+3}{2}\right)$$
$$= (1;1)$$

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TRIGONOMETRY

<u>In order to achieve full marks for this section, you need to be able to:</u>

- simplify a trigonometric expression using **reduction formulae**
- **draw a right-angled triangle** using given information
- use a right-angled triangle to write **expressions for trigonometric ratios**
- prove trigonometric **identities**
- solve trigonometric **equations**
- work with **special triangles** to simplify expressions without using a calculator

There are 5 types of questions that get tested every year:

1. Simplifying a trigonometric expression
2. Proving a trigonometric identity
3. Using a diagram on the Cartesian plane
4. Solving a trigonometric equation
5. Problem-solving questions

We will look at each of these types of questions separately, but let's first look at some basic information that you should know.

<u>The definitions of the trigonometric ratios</u>



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{AC} \text{ or } \frac{c}{b}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{BC}{AC} \text{ or } \frac{a}{b}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{BC} \text{ or } \frac{c}{a}$$

Use this trick to remember the ratios:

SOH CAH TOA

Sine is **O**pposite over **H**ypotenuse
Cos is **A**djacent over **H**ypotenuse
Tan is **O**pposite over **A**djacent

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TRIG IDENTITIES that you need to know and be able to apply:

The square identity:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\therefore \sin^2\theta = 1 - \cos^2\theta$$
$$\text{and} \quad \cos^2\theta = 1 - \sin^2\theta$$

The tan identity:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\therefore \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

The double angle identities:

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$

How to derive the cosine double angle identities:

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - \sin^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= \cos^2 x - 1 + \cos^2 x$$
$$= 2\cos^2 x - 1$$

The compound angle identities:

$$\cos(x-y) = \cos x.\cos y + \sin x.\sin y$$
$$\cos(x+y) = \cos x.\cos y - \sin x.\sin y$$
$$\sin(x-y) = \sin x.\cos y - \cos x.\sin y$$
$$\sin(x+y) = \sin x.\cos y + \cos x.\sin y$$

How to derive the compound angle identities:

$$\cos(x+y) = \cos[x-(-y)]$$
$$= \cos x.\cos(-y) + \sin x.\sin(-y)$$
$$= \cos x.\cos y + \sin x.(-\sin y)$$
$$= \cos x.\cos y - \sin x.\sin y$$

$$\sin(x-y) = \cos[90° - (x-y)]$$
$$= \cos[(90° - x) + y]$$
$$= \cos[(90° - x) - (-y)]$$
$$= \cos(90° - x).\cos(-y) + \sin(90° - x).\sin(-y)$$
$$= \sin x.\cos y + \cos x.(-\sin y)$$
$$= \sin x.\cos y - \cos x.\sin y$$

$$\sin(x+y) = \cos[90° - (x+y)]$$
$$= \cos[(90° - x) - y]$$
$$= \cos(90° - x).\cos y + \sin(90° - x).\sin y$$
$$= \sin x.\cos y + \cos x.\sin y$$

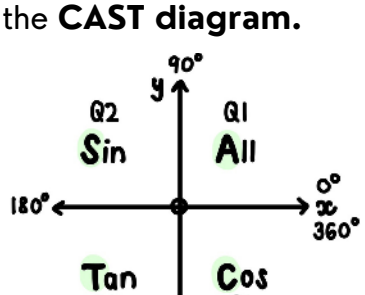
PRO-TIPS

- You have to be able to **SHOW** how two of the cos double angle identities are derived, using the first cos-identity.
- You have to be able to **SHOW** how three of the compound angle identities are derived, using the first cos-identity.

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The CAST diagram and reduction formulae:

Certain trigonometric ratios are positive only in some of the four quadrants of the Cartesian plane. We summarise which trig ratios are positive in which quadrants using the **CAST diagram.**



Quadrant 1: All of the trig ratios are positive.
Quadrant 2: Only sin of an angle in Q2 is positive.
Quadrant 3: Only tan of an angle in Q3 is positive.
Quadrant 4: Only cos of an angle in Q4 is positive.

There are fixed expressions for angles in each quadrant that we use as reduction formulae:

$180° - \theta$: Quadrant 2

$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

$180° + \theta$: Quadrant 3

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

$360° - \theta$ or $-\theta$: Quadrant 4

$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

PRO-TIPS

Take **note** that these reduction formulae are also used when solving **trig equations**!

These are, unfortunately, not the only reduction formulae that you need to work with. It is useful to have a strategy for determining which quadrant an angle lies in. Here are some examples:

θ is an acute angle, so is in the first quadrant.
$\therefore$ start in Q1 and then apply any change to that



$\theta + 180°$: Q3
$\theta - 270°$: Q2
$\theta + 360°$: Q4

When **ADDING** 180 degrees, we move in an anticlockwise direction from the first quadrant.
When **SUBTRACTING** 270 degrees, we move in a clockwise direction.

Angles that add up to 90 degrees are called **complementary angles**. We therefore call reduction formulae with 90 degrees **CO-RATIOS:**

$90° - \theta$: Quadrant 1

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$

$90° + \theta$: Quadrant 2

$\sin(90° + \theta) = \cos\theta$
$\cos(90° + \theta) = -\sin\theta$

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Question Type 1:
Simplifying a trigonometric expression

These types of questions rely on your knowledge of **reduction formulae**, **co-ratios**, **special triangles** and sometimes the use of **identities**, to simplify a complicated expression to a much simpler one.

Worked example (C)

Simplify the following trigonometric expression as far as possible:

$$\frac{\sin 2x \cdot \cos(-x) + \cos 2x \cdot \sin(360° - x)}{\sin(180° + x)}$$

$$\frac{\sin 2x \,.\, \cos(-x) + \cos 2x \,.\, \sin(360° - x)}{\sin(180° + x)}$$

(Q4, Q4, Q3)

First identify the quadrants in which each of the angles are.

$$= \frac{\sin 2x \,.\, \cos x + \cos 2x \,.\, (-\sin x)}{-\sin x}$$

$$= \frac{\sin 2x \,.\, \cos x - \cos 2x \,.\, \sin x}{-\sin x}$$

$$= \frac{\sin(2x - x)}{-\sin x}$$

Now apply the reduction formulae for each of these angles. Notice that the numerator is the expansion of the negative sine compound-angle identity.

$$= \frac{\sin x}{-\sin x}$$

Now that the numerator is simplified, we can cancel out the $\sin x$ in the numerator and the denominator.

$$= -1$$

Question source: Grade 12 NSC November 2020 Paper 2

PRO-TIPS

SHOW every sign in the reduction formula step! You get a mark for every reduction shown.

Worked example (C)

Simplify the following trigonometric expression as far as possible:

$$\frac{\sin 140° \cdot \sin(360° - x)}{\cos 50° \cdot \tan(-x)}$$

$$\frac{\sin 140° \,.\, \sin(360° - x)}{\cos 50° \,.\, \tan(-x)}$$

(Q2, Q4, Q4)

Identify the quadrants in which each of the angles lie, and then apply the reduction formulae.

The question has to be done without a calculator, so we need to change $\sin 140°$ to $\cos 50°$ in order to cancel those ratios out. We also need to apply the tan identity.

$$= \frac{\sin(90° + 50°) \,.\, (-\sin x)}{\cos 50° \,.\, (-\tan x)}$$

$$= \frac{\cos 50° \,.\, \sin x}{\cos 50° \,.\, \frac{\sin x}{\cos x}}$$

$$= \sin x \times \frac{\cos x}{\sin x}$$

Instead of dividing by a fraction, we can "tip and times". This allows us to simplify the answer.

$$= \cos x$$

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Worked example (C)

Simplify the following trigonometric expression as far as possible:

$$\frac{\cos(90^\circ+\theta)}{\sin(\theta-180^\circ)+3\sin(-\theta)}$$

Q2 Q3 Q4

$$\frac{\cos(90^\circ+\theta)}{\sin(\theta-180^\circ)+3\sin(-\theta)}$$

Identify the quadrants in which each of the angles lie, and then apply the reduction formulae.

$$= \frac{-\sin\theta}{-\sin\theta-3\sin\theta}$$

$$= \frac{-\sin\theta}{-4\sin\theta}$$

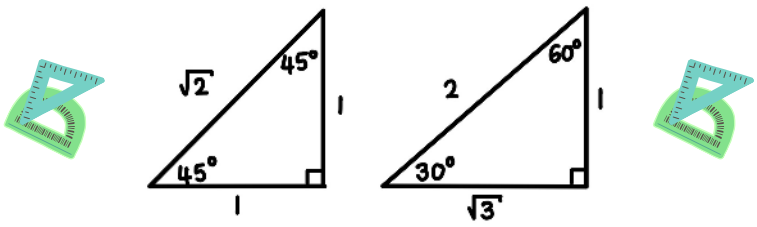
$$= \frac{1}{4}$$

Question source: Grade 12 NSC November 2020 Paper 2

Before we look at the next example, let's recap the SPECIAL TRIANGLES.

WHEN should you use these triangles?
If the question states that you **may not use a calculator** and there are **30, 45 and/or 60 degree angles,** you have to **DRAW** the triangles and **USE** them when calculating the trigonometric ratios.

Memorise these triangles!



Worked example (C)

Simplify the following trigonometric expression without the use of a calculator:

$$\tan^2 210^\circ - (1+\cos 120^\circ)\cdot\sin^2 405^\circ$$

$\tan^2 210^\circ - (1+\cos 120^\circ).\sin^2 405^\circ$

Each of these angles can be expressed in terms of special angles.

Q3 Q2 Q1

$= \tan^2(180^\circ+30^\circ) - [1+\cos(180^\circ-60^\circ)].\sin^2(360^\circ+45^\circ)$
$= \tan^2 30^\circ - (1-\cos 60^\circ).\sin^2 45^\circ$

Now apply the sign in each of these quadrants.

$= \left(\frac{1}{\sqrt{3}}\right)^2 - \left(1-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$
$= \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
$= \frac{1}{3}\times\frac{4}{4} - \frac{1}{4}\times\frac{3}{3}$
$= \frac{4}{12} - \frac{3}{12}$
$= \frac{1}{12}$

We use the special triangles to calculate the values of each of the trig ratios.

Question source: Grade 12 NSC November 2022 Paper 2

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Question Type 2:
Proving a trigonometric identity

A trigonometric identity is a trigonometric statement that is always true.
You have learnt some trigonometric identities that appear on the formula sheet (the square and tan identities), but there are much more complicated identities that we don't have to know or use, but must be able to prove.

Take note of the following hints when proving an identity:

- **You have to use the correct layout.** Choose a side of the identity to work on and label your first line with that chosen side. Use "LHS" for the left-hand side of the identity, and "RHS" for the right-hand side of the identity. **You MAY NOT work on both sides of the identity at the same time.** You may, however, work on one side of the identity, and then work on the other side separately, and prove both of these sides equal to each other.

- The purpose is to show that the two sides of the identity are equal to each other, and then state that at the end as **"LHS = RHS"**.

- Look out for opportunities to use the **square and tan identities**. Simply applying these identities should earn you a mark.

- Look out for opportunities to use the **compound and double angle identities**, especially the sine double angle identity, as there is only one version of that identity.

- When having to use a cos double angle identity, **try to identify the best one** to use depending on what you are trying to prove.

- If you are trying to get to an expression with only sine, use the cos double angle identity that **only contains sine**.

- If you are trying to get to an expression with only cosine, use the cos double angle identity that **only contains cos**.

- If you are trying to get to an expression with sine and cosine, use the cos double angle identity that **contains both**.

- Look out for opportunities to **factorise** expressions, either by taking out a common factor, factorising a difference of two squares or factorising trinomials.

- If an expression consists of multiple separate fractions, and the other side of the identity consists of **only one fraction**, make sure to write the separate fractions as one by finding the lowest common denominator of the separate fractions.

- There are often multiple ways of proving an identity, so just **try something!** Even if you are unable to prove the identity, you could get some marks for applying any of the hints listed above.

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Now let's look at some typical exam type questions!

Worked example (C)

Prove the following trigonometric identity:

$$\frac{-2\sin^2 x + \cos x + 1}{1 - \cos(540° - x)} = 2\cos x - 1$$

$$LHS = \frac{-2\sin^2 x + \cos x + 1}{1 - \cos(540° - x)}$$

$540° = 360° + 180°$
$\therefore \cos(540° - x) = \cos(180° - x)$

Q2

$$= \frac{-2(1 - \cos^2 x) + \cos x + 1}{1 - (-\cos x)}$$

Use the square identity to replace $\sin^2 x$ with $1 - \cos^2 x$. We do this because there is already a $\cos x$ in the numerator.

$$= \frac{-2 + 2\cos^2 x + \cos x + 1}{1 + \cos x}$$

$$= \frac{2\cos^2 x + \cos x - 1}{1 + \cos x}$$

After simplifying the numerator, we are left with a trinomial that we can factorise. This allows us to cancel out one of the brackets in the numerator with the denominator.

$$= \frac{(2\cos x - 1)(\cos x + 1)}{1 + \cos x}$$

$$= 2\cos x - 1$$

$$= RHS$$

Remember to start with "LHS = " and end with " = RHS".

Question source: Grade 12 NSC November 2021 Paper 2

Worked example (C)

Prove the following trigonometric identity:

$$\cos(x + y) \cdot \cos(x - y) = 1 - \sin^2 x - \sin^2 y$$

The LHS of this identity is the product of two compound angle identities, so we have to expand the identities and then multiply the expansions.

$$LHS = \cos(x+y).\cos(x-y)$$
$$= (\cos x.\cos y - \sin x.\sin y)(\cos x.\cos y + \sin x.\sin y)$$
$$= \cos^2 x.\cos^2 y + \cos x.\cos y.\sin x.\sin y - \cos x.\cos y.\sin x.\sin y - \sin^2 x.\sin^2 y$$
$$= \cos^2 x.\cos^2 y - \sin^2 x.\sin^2 y$$

We can get rid of $\cos^2 x$ and $\cos^2 y$ by using the square identity.

$$= (1 - \sin^2 x)(1 - \sin^2 y) - \sin^2 x.\sin^2 y$$

Multiply the brackets out.

$$= 1 - \sin^2 y - \sin^2 x + \sin^2 x.\sin^2 y - \sin^2 x.\sin^2 y$$

Add / subtract like terms.

$$= 1 - \sin^2 x - \sin^2 y$$
$$= RHS$$

Question source: Grade 12 NSC November 2022 Paper 2

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Question Type 3:
Using a diagram on the Cartesian plane

When a question states that it should be done without using a calculator, that often points to one of two methods:

1. Working with **special triangles** to determine the values of special angles (see pg. 44)
2. Drawing a **right-angled triangle** on the Cartesian plane and using that triangle to determine the values of various trigonometric ratios and expressions.

When a question relies on you drawing a triangle:

Enough information will be given for you to draw a triangle with the lengths of all three sides and the sizes of all three angles, or expressions for these.

The question will only state the size of one angle.

The triangle that you will draw will always be right-angled, so that means that one of the other angles will be equal to **90 degrees**.

The size of the third angle can then be **calculated** by subtracting the given angle and the right angle from 180 degrees.

In these questions, **an expression for the value of one trigonometric ratio is always given.**

Remember the definitions of the trig ratios:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

If the given value/expression is a fraction, then the **numerator** is equal to the side of the triangle that is in the numerator of the formula alongside.

The **denominator** is equal to the side of the triangle that is in the denominator of the formula alongside.

If the value/expression is not given as a fraction, **rewrite it as a fraction by writing it over 1**, eg. 6 is equal to 6 over 1.
The one side of the triangle is then equal to 6 units, and the other side is equal to 1 unit.

PRO-TIPS

Once you have determined which two sides of the triangle are given, you can calculate the length of the third side using the Theorem of Pythagoras.
Remember to write the reason **(Pythagoras)** when doing this calculation.

Now let's look at some typical exam questions!

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Worked example (R)

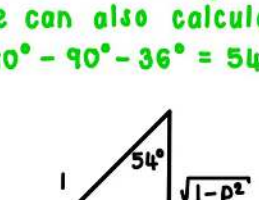
Given: $\sin 36° = \sqrt{1-p^2}$

Without using a calculator, determine each of the following in terms of p.

$\sin 36° = \dfrac{\sqrt{1-p^2}}{1}$ ← opposite, ← hypotenuse

We need to rewrite the expression given for sin 36° as a fraction so that we can apply the definition of sine.

Now that we have an angle (36°), an expression for the length of the side opposite that angle ($\sqrt{1-p^2}$) and the length of the hypotenuse (1 unit), we can draw a right-angled triangle.
We can then calculate an expression for the length of the third side using the theorem of Pythagoras.
We can also calculate the third angle in the triangle.
$180° - 90° - 36° = 54°$



$x^2 = 1^2 - (\sqrt{1-p^2})^2$ (Pythagoras)
$= 1 - (1-p^2)$
$= p^2$
$x = p$

1. $\tan 36°$

$\tan 36° = \dfrac{\sqrt{1-p^2}}{p}$

Now that we have a right-angled Δ with a 36° angle, we can use the definition of tan to determine the answer of tan 36°.
Remember that $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

Now that we have a Δ with a 36° and 54° angle, we need to rewrite 108° in terms of 36° or 54°. This will then allow us to use the Δ.

2. $\cos 108°$

$\cos 108° = \cos 2(54°)$

There are three double-angle identities for cos, so we can answer this question using any one of these.

$\cos 2(54°) = \cos^2 54° - \sin^2 54°$
$= \left(\dfrac{\sqrt{1-p^2}}{1}\right)^2 - \left(\dfrac{p}{1}\right)^2$
$= 1 - p^2 - p^2$
$= 1 - 2p^2$

or $1 - 2\sin^2 54°$
$= 1 - 2\left(\dfrac{p}{1}\right)^2$
$= 1 - 2p^2$

or $2\cos^2 54° - 1$
$= 2\left(\dfrac{\sqrt{1-p^2}}{1}\right)^2 - 1$
$= 2(1-p^2) - 1$
$= 2 - 2p^2 - 1$
$= 1 - 2p^2$

Question source: Grade 12 NSC November 2021 Paper 2

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Worked example (R)

Given that $\sqrt{13}\sin x + 3 = 0$, where $x \in (90°; 270°)$.

Without using a calculator, determine the value of:

1. $\tan x$

$\sqrt{13}\sin x + 3 = 0 \quad ; \quad x \in (90°; 270°)$ ← This Δ must be drawn on the Cartesian plane.

$\sqrt{13}\sin x = -3$

$\sin x = -\frac{3}{\sqrt{13}}$ ← opposite, hypotenuse

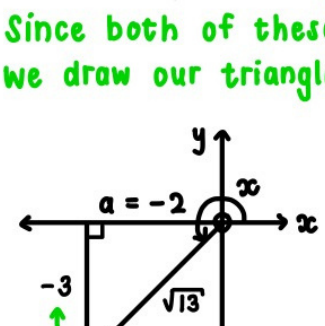
Before we can draw a Δ, we first need to make sin x the subject of the equation.

To determine which quadrant the Δ is in, we must consider the two pieces of given information. We choose the quadrant in which BOTH of these statements are true.

$\sin x = -\frac{3}{\sqrt{13}}$ sin x is negative in Q3 and Q4

$x \in (90°; 270°)$ Angles between 90° and 270° are in Q2 and Q3

Since both of these statements are true in the third quadrant, we draw our triangle in Q3.



$a^2 = (\sqrt{13})^2 - (-3)^2$ (Pythagoras)

$= 4$

$a = \pm 2$

but Q3 $\therefore a = -2$ ← on the negative side of the x-axis

$\therefore \tan x = \frac{-3}{-2}$ ← opposite, adjacent

$= \frac{3}{2}$

2. $\cos(180° + x)$

Q3

$\cos(180° + x) = -\cos x$

$= -\left(-\frac{2}{\sqrt{13}}\right)$

$= \frac{2}{\sqrt{13}}$

180° + x is an angle in the third quadrant, where cos is negative.

Question source: Grade 12 NSC November 2022 Paper 2

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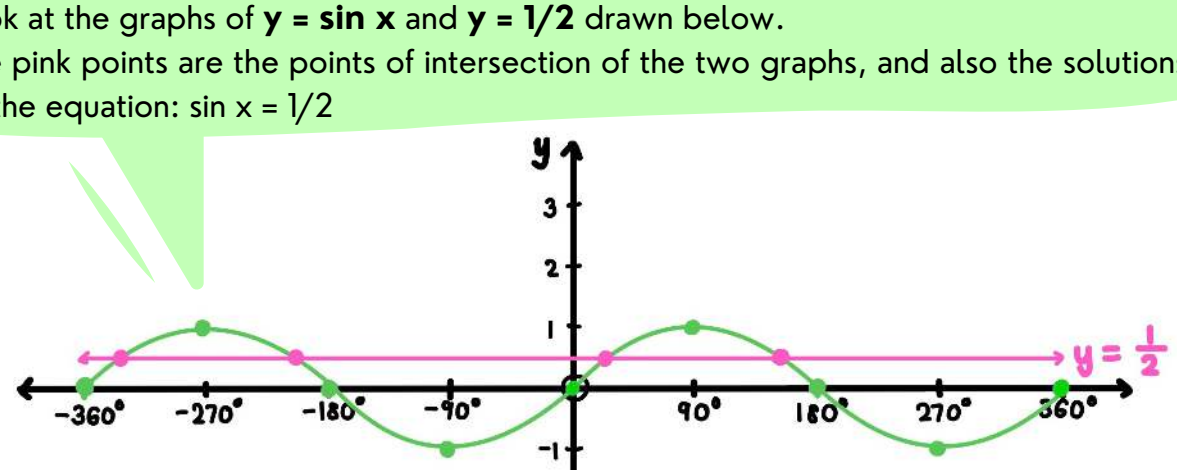
Question Type 4:
Solving a trigonometric equation

When solving a trigonometric equation, we are finding the value(s) of angles that make the equation true.
When a specific interval is given in the question, then we need to list the x-values that satisfy the equation in that interval.
When no specific interval is given, then we need to write a general solution.

But what is a general solution?
It is a formula for **all possible x-values that satisfy an equation**, written in terms of a variable of your choice.
In this study guide, we will use "k" as the variable in our general solutions. You may, however, use any variable that you want, except for the variable that you are solving.

A trigonometric equation is different from other algebraic equations, because a trig function is **repetitive**. If we think of trig graphs, the same shape repeats over and over again, so **there are many different x-values that have the same y-value**.

Look at the graphs of **y = sin x** and **y = 1/2** drawn below.
The pink points are the points of intersection of the two graphs, and also the solutions to the equation: sin x = 1/2



- Notice that there are **four solutions** to this equation just in the domain drawn above.
- These two graphs can be drawn for **any domain**, and will have the **same shape**, so will have more points of intersection.
- We therefore use the **general solution** to write a formula that will yield these solutions, as well as all other solutions in any given domain.

Now let's look at some examples of the many different types of trig equations that you need to be able to solve!

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Worked example (R): A basic trig equation

A basic trig equation is an equation where the reference angle can be found immediately. **NO** simplification of the equation is needed before it can be solved.

No interval is given in this question, so we simply have to write the general solutions.

Solve for x:

$\sin x = 0,32$

$\sin x = 0,32$ — 0,32 is a positive number, so the solution(s) for x have to be angles that are in quadrants 1 or 2. Remember that sine of angles in these quadrants are positive. This means that we are going to write a GENERAL solution for quadrant 1 and 2.

$RA = \sin^{-1}(0,32)$ — Calculate the reference angle.
$= 18,66°$

$Q1: x = 18,66° + k.360°; k \in \mathbb{Z}$ — Now apply the quadrant formulae for quadrants 1 and 2, and simplify.
$Q2: x = 180° - 18,66° + k.360°$
$= 161,34° + k.360°$

But what would we do if a specific interval was given? Let's take a look.

Solve for x where $x \in [-180°; 180°]$:

$\sin x = 0,32$

The general solutions found for this equation above are:
1. $x = 18,66° + k.360°; k \in \mathbb{Z}$
2. $x = 161,34° + k.360°$

"$k \in \mathbb{Z}$" means that these two formulae will yield specific solutions for x when we substitute integer values for k.

In formula 1: $x = 18,66° + k.360°$
Let $k = 0$: $x = 18,66°$ — This value is in the given interval.
Let $k = 1$: $x = 198,66°$ — This value is NOT in the given interval.
Let $k = -1$: $x = -341,34°$ — This value is NOT in the given interval.

In formula 2: $x = 161,34° + k.360°$
Let $k = 0$: $x = 161,34°$ — This value is in the given interval.
Let $k = 1$: $x = 521,34°$ — This value is NOT in the given interval.
Let $k = -1$: $x = -198,66°$ — This value is NOT in the given interval.

$\therefore x = 18,66°$ or $x = 161,34°$ — List the valid solutions found.

You DO NOT have to show the substitution step. You can simply list the final solutions.

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Worked example (R): A basic trig equation with complex angles

Solve for x:

$3\cos(2x + 10°) = -0,8$

First make the trig ratio the subject of the equation, and then find the reference angle.
Make sure to use the **FULL** angle in the general solution, and then solve for x.

$3\cos(2x + 10°) = -0,8$
$\cos(2x + 10°) = -\frac{0,8}{3}$

First make the trig ratio the subject of the equation.

$RA = \cos^{-1}\left(\frac{0,8}{3}\right)$
$= 74,53°$

A reference angle is an acute angle, so we leave out the negative when calculating the reference angle.

Q2 : $2x + 10° = 180° - 74,53° + k.360°;\ k \in \mathbb{Z}$
$2x = 105,47° + k.360°$
$x = 52,74° + k.180°$

Now apply the quadrant formulae for quadrants 2 and 3, and solve for x in each of these formulae.

Q3 : $2x + 10° = 180° + 74,53° + k.360°$
$2x = 244,53° + k.360°$
$x = 122,27° + k.180°$

Worked example (R): A trig equation that can be done without the use of a calculator

Solve for x:

$\sin 2x = -\sin 72°$

If a trig equation has the **same** trig ratio on both sides, we can **identify** the **reference** angle by looking at the angle on the right-hand side of the equation.

$\sin 2x = -\sin 72°$
$RA = 72°$

We have the same trig ratio on both sides of the equation, so the reference angle is given.
Since the expression on the right-hand side is negative, we have to find general solutions for the quadrants in which sine is negative.

Q3 : $2x = 180° + 72° + k.360°;\ k \in \mathbb{Z}$
$= 252° + k.360°$
$x = 126° + k.180°$

Now apply the quadrant formulae for quadrants 3 and 4, and solve for x in each of these formulae.

Q4 : $2x = 360° - 72° + k.360°$
$= 288° + k.360°$
$x = 144° + k.180°$

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Worked example (C): A trig equation that can be done without a calculator after one of the trig ratios are changed

DO NOT expand the trig ratio on the left-hand side with a compound angle identity! It will severely complicate the calculations.

Solve for x:

$\cos(x + 45°) = \sin 3x$

$\cos(x+45°) = \sin 3x$
$= \cos(90° - 3x)$

This equation has variables on both sides, and the angles are different, so we need to make sure that we have the same trig ratios on both sides.
To change sine to cos, we use co-ratios.

$RA = 90° - 3x$

Now that we have the same trig ratio on both sides of the equation, the angle on the right-hand side is the reference angle.

Q1: $x + 45° = 90° - 3x + k.360°$; $k \in \mathbb{Z}$
$4x = 45° + k.360°$
$x = 11{,}25° + k.90°$

Q4: $x + 45° = 360° - (90° - 3x) + k.360°$
$= 360° - 90° + 3x + k.360°$
$-2x = 225° + k.360°$
$x = -112{,}5° - k.180°$
or $-112{,}5° + k.180°$

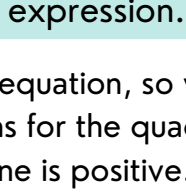
Since the expression on the right-hand side of the equation is positive, we need to write general solutions for the quadrants in which cos is positive ∴ Q1 and Q4.

Since k can be any integer, $k.360°$ can be positive or negative. This means that the sign in front of $k.360°$ does not matter.

This equation also could have been solved by changing the trig ratio on the left-hand side of the equation to a sine expression.

$\cos(x+45°) = \sin 3x$
$\sin[90° - (x+45°)] = \sin 3x$
$\sin(90° - x - 45°) = \sin 3x$
$\sin(45° - x) = \sin 3x$
$RA = 3x$

This is now a sine equation, so we have to write general solutions for the quadrants in which sine is positive.



Q1: $45° - x = 3x + k.360°$; $k \in \mathbb{Z}$
$-4x = -45° + k.360°$
$x = 11{,}25° - k.90°$
Q2: $45° - x = 180° - 3x + k.360°$
$2x = 135° + k.360°$
$x = 67{,}5° + k.180°$

Even though we are finding general solutions that are slightly different from the solutions above, these general solutions will yield the same specific solutions. Both of these sets of general solutions are correct.

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Worked example (C): A trig equation requiring the use of identities and factorising

PRO-TIPS

- When any equation has a degree of 2 we classify it as a **quadratic equation**, in which case it is very likely that we will have to factorise the equation in order to solve it.
- **First** determine how the equation needs to be written in order to be factorised.

Solve for x:

$3\cos^2 x - 5\sin x = 1$

$3\cos^2 x - 5\sin x = 1$

The "$\cos^2 x$" tells us that this is a quadratic equation, so we have to have zero on the one side and then factorise the quadratic expression on the other side.

$3\cos^2 x - 5\sin x - 1 = 0$

We can't factorise this because there are two different trig ratios, so we use the square identity to replace $\cos^2 x$ with $1 - \sin^2 x$.

$3(1 - \sin^2 x) - 5\sin x - 1 = 0$
$3 - 3\sin^2 x - 5\sin x - 1 = 0$
$-3\sin^2 x - 5\sin x + 2 = 0$
$3\sin^2 x + 5\sin x - 2 = 0$
$(3\sin x - 1)(\sin x + 2) = 0$

Now simplify the equation so that we can write it in standard form:
$a\sin^2 x + b\sin x + c = 0$
We can now either factorise the trinomial or use the quadratic formula to solve for $\sin x$.

$3\sin x - 1 = 0$ or $\sin x + 2 = 0$
$3\sin x = 1$ $\quad\quad$ $\sin x = -2$ ②
$\sin x = \frac{1}{3}$ ① $\quad\quad$ no solution
RA = 19,47°

Equation ① is a type A trig equation.
Equation ② is a trig equation that cannot be solved.
The graph of $y = \sin x$ never has a y-value of -2 ∴ there is no x-value that will make $\sin x = -2$.

Q1: $x = 19{,}47° + k.360°;\ k \in \mathbb{Z}$
Q2: $x = 180° - 19{,}47° + k.360°$
$= 160{,}53° + k.360°$

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Worked example (C): A trig equation requiring the use of double angle identities

This equation requires the use of the tan identity, and the sine double angle identity. After applying these identities we can simplify the equation so that it can be solved.

Solve for x:

$\tan x = 2\sin 2x \ \ where \ \cos x < 0$

$\tan x = 2\sin 2x$ where $\cos x < 0$

This equation contains multiple different trig ratios, so let's use identities to simplify the equation.

$\tan x = \frac{\sin x}{\cos x}$ and $\sin 2x = 2\sin x \cos x$

$$\frac{\sin x}{\cos x} = 2(2\sin x \cos x)$$

$$\frac{\sin x}{\cos x} = 4\sin x \cos x$$

Multiply both sides of the equation by $\cos x$ so that we no longer have a fraction on the left-hand side.

$$\sin x = 4\sin x \cos^2 x$$
$$\sin x - 4\sin x \cos^2 x = 0$$
$$\sin x(1 - 4\cos^2 x) = 0$$

We now have a quadratic equation, so we have to move all terms to one side of the equation (make it equal to zero), factorise the expression on the left, and then solve for x.

$\sin x = 0$ ①

$x = 0° + k.360°; k \in \mathbb{Z}$

$x = 180° + k.360°$

For equation ①, we have to think of the graph of $y = \sin x$.

Where does the graph have a y-value of zero?

In other words, where is this graph's x-intercepts?

At $x = 0°$ and $x = 180°$, and every $360°$ before and after those values.

or $1 - 4\cos^2 x = 0$ ②

$(1 + 2\cos x)(1 - 2\cos x) = 0$

$1 + 2\cos x = 0$ or $1 - 2\cos x = 0$

$\cos x = -\frac{1}{2}$ $\quad\quad$ $\cos x = \frac{1}{2}$

but $\cos x < 0$

$\therefore \cos x = -\frac{1}{2}$

$RA = 60°$

Equation ② yields two solutions for $\cos x$.

The question stated that $\cos x < 0$ (negative), so we can disregard the solution that is equal to $\frac{1}{2}$.

Q2: $x = 180° - 60° + k.360°$

$= 120° + k.360°$

Q3: $x = 180° + 60° + k.360°$

$= 240° + k.360°$

We now find general solutions for the two quadrants in which cos is negative.

Question source: Grade 12 NSC November 2021 Paper 2

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Worked example (C): A trig equation requiring the use of compound angle identities

Solve the following equation:

$$\sin(x+25°)\cdot\cos 15° - \cos(x+25°)\cdot\sin 15° = \tan 165°$$

$\sin(x+25°).\cos 15° - \cos(x+25°).\sin 15° = \tan 165°$
$\sin(x+25°-15°) = \tan 165°$
$\sin(x+10°) = -0{,}267...$
$RA = \sin^{-1}(0{,}267...)$
$= 15{,}54°$

The left-hand side of this equation is the expansion of a compound angle identity.
$\sin A.\cos B - \cos A.\sin B = \sin(A-B)$

$Q3: x = 180° + 15{,}54° + k.360°;\ k \in Z$
$= 195{,}54° + k.360°$
$Q4: x = 360° - 15{,}54° + k.360°$
$= 344{,}46° + k.360°$

Since the value on the right-hand side of the trig equation is negative, we have to write general solutions for the quadrants in which sine is negative, Q3 and Q4.

Question source: Grade 12 NSC November 2019 Paper 2

PRO-TIPS

You will only ever have to use a compound angle identity in an EQUATION to change an **expanded version** of the identity into the **condensed version** of the identity, like in the example done above.

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Question Type 5: Problem-solving questions

The last question in the trigonometry section is often a problem-solving question.

The only way to ensure that you are able to do these questions is to know the basics of trigonometry very well, and to practice applying these basics to different types of questions.

We will now look at two of these questions from past papers as examples.

Worked example (P)

Given $x + \frac{1}{x} = 3\cos A$ and $x^2 + \frac{1}{x^2} = 2$

Determine the value of $\cos 2A$ without a calculator.

$$x + \frac{1}{x} = 3\cos A$$
$$\left(x + \frac{1}{x}\right)^2 = (3\cos A)^2$$
$$\left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) = 9\cos^2 A$$
$$x^2 + 1 + 1 + \frac{1}{x^2} = 9\cos^2 A$$
$$x^2 + \frac{1}{x^2} + 2 = 9\cos^2 A$$
$$2 + 2 = 9\cos^2 A$$
$$4 = 9\cos^2 A$$
$$\therefore \cos^2 A = \frac{4}{9}$$

To get $x^2 + \frac{1}{x^2}$, we have to square both sides of the equation. Remember to use the FOIL-method when squaring the bracket.

$x^2 + \frac{1}{x^2} = 2$

FOIL stands for **First**, **Outer**, **Inner** and **Last**.
It is a suggestion for how to multiply two binomials.

Since we now know what $\cos^2 A$ is equal to, we can use a double angle identity to express $\cos 2A$ in terms of $\cos^2 A$.

$$\cos 2A = 2\cos^2 A - 1$$
$$= 2\left(\frac{4}{9}\right) - 1$$
$$= \frac{8}{9} - 1$$
$$= -\frac{1}{9}$$

Question source: Grade 12 NSC November 2020 Paper 2

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Worked example (P)

PRO-TIPS

This question relies on your knowledge of trig identities, reduction formulae **AND** sigma notation, even though sigma notation is a **Paper 1 topic**.

Evaluate, without using a calculator:

$$\sum_{A=38^\circ}^{52^\circ} \cos^2 A$$

$\sum_{A=38^\circ}^{52^\circ} \cos^2 A$ Sigma means "sum". We have to add all the terms with a general term of $\cos^2 A$, from where $A = 38^\circ$ to $A = 52^\circ$.

$= \cos^2 38^\circ + \cos^2 39^\circ + \cos^2 40^\circ + \cos^2 41^\circ + \cos^2 42^\circ + \cos^2 43^\circ + \cos^2 44^\circ + \cos^2 45^\circ$
$+ \cos^2 46^\circ + \cos^2 47^\circ + \cos^2 48^\circ + \cos^2 49^\circ + \cos^2 50^\circ + \cos^2 51^\circ + \cos^2 52^\circ$

We may not use a calculator, so we need to use co-ratios.
$\cos^2 38^\circ = \sin^2 52^\circ$; $\cos^2 39^\circ = \sin^2 51^\circ$; $\cos^2 40^\circ = \sin^2 50^\circ$; and so on.

$= \sin^2 52^\circ + \sin^2 51^\circ + \sin^2 50^\circ + \sin^2 49^\circ + \sin^2 48^\circ + \sin^2 47^\circ + \sin^2 46^\circ + \cos^2 45^\circ$
$+ \cos^2 46^\circ + \cos^2 47^\circ + \cos^2 48^\circ + \cos^2 49^\circ + \cos^2 50^\circ + \cos^2 51^\circ + \cos^2 52^\circ$

Each of the pairs of terms that are highlighted in the same colour add up to 1.
$\sin^2 52^\circ + \cos^2 52^\circ = 1$; $\sin^2 51^\circ + \cos^2 51^\circ = 1$; $\sin^2 50^\circ + \cos^2 50^\circ = 1$; and so on.
There are seven of these pairs, and $\cos^2 45^\circ$ is left over.

$= 7(1) + \left(\frac{1}{\sqrt{2}}\right)^2$ We can find the value of $\cos 45^\circ$ using a special triangle.
$= 7\frac{1}{2}$ or $\frac{15}{2}$

Question source: Grade 12 NSC November 2018 Paper 2

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TRIGONOMETRIC FUNCTIONS

In order to get maximum marks for this topic, you need to be able to do the following:

- **Draw** a sine, cos or tan graph for any given domain.
- **Interpret** a trig function when a graph is given, by answering questions relating to **period, amplitude, range and maximum/minimum value.**
- Apply and/or interpret **translations** (shifts) and **reflections** of trig functions.
- **Find the equation** of a trig function when the graph is given.

The standard forms of the three trigonometric functions:

$y = a \sin k(x-p) + q$

$y = a \cos k(x-p) + q$

a = amplitude

Period = $\frac{360°}{k}$

$y = a \tan k(x-p) + q$

A tan graph does not have an amplitude.

Period = $\frac{180°}{k}$

p: shifts the graph to the left or right

q: shifts the graph up or down

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Now let's discuss the properties of trigonometric functions:

- Trig functions are **periodic**, which means that they repeat themselves after a certain interval. This interval is known as the **period**.

- **Amplitude:** the height of the graph, measured from the central axis. This can also be described as the distance between the minimum/maximum value and the central axis.
  Note that an amplitude can never be negative.
  A tan graph **does not have an amplitude**, as it does not have a minimum or maximum value.

- **Asymptotes**: a tan graph has asymptote(s) at x-values that make the function undefined. These asymptotes also repeat themselves every period.
  For the basic tan graph, the asymptotes are at $x = 90 + k.180$, where k is an integer (refer to Trigonometric equations on page 50 for an explanation of this notation).

- **Maximum value:** the greatest y-value on a sine or cos graph.

- **Minimum value:** the smallest y-value on a sine or cos graph.

- **Domain:** the set of all x-values of a graph. The domain of a trig graph is the interval that is specified in the question.

- **Range:** the set of all y-values of a graph. The range of a trig graph is all y-values between the minimum value and the maximum value. You are allowed to write the range in interval notation or inequality notation.

- **Step:** the step of a trig graph is calculated by dividing the period by 4.
  This indicates how often you need to plot a point or draw an asymptote.
  For example, if the period of a sine graph is 360 degrees, the step is 90 degrees.
  This means that you have to plot a point every 90 degrees.

Drawing a trigonometric graph

When asked to draw a trig graph, there are two methods that you can use:

1. Know the **basic graph** and apply all parameter changes to it.
2. Use the **TABLE function** on the calculator.

Both of these methods are described on the next pages.

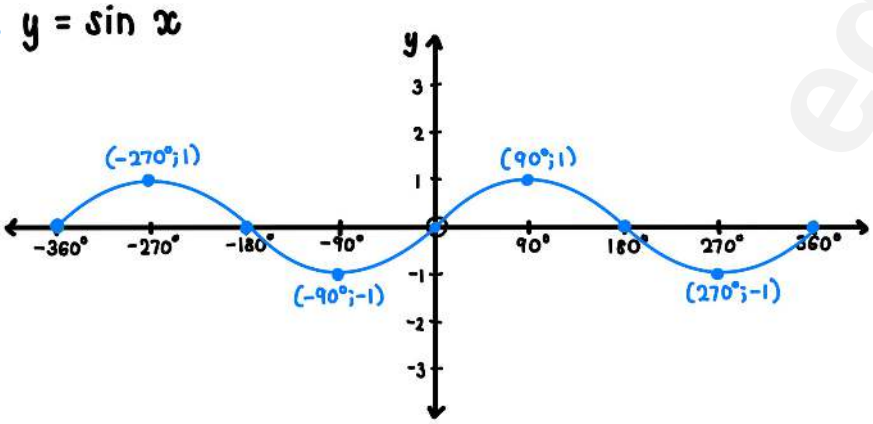
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The basic trigonometric graphs

The graphs below are the basic trig graphs.

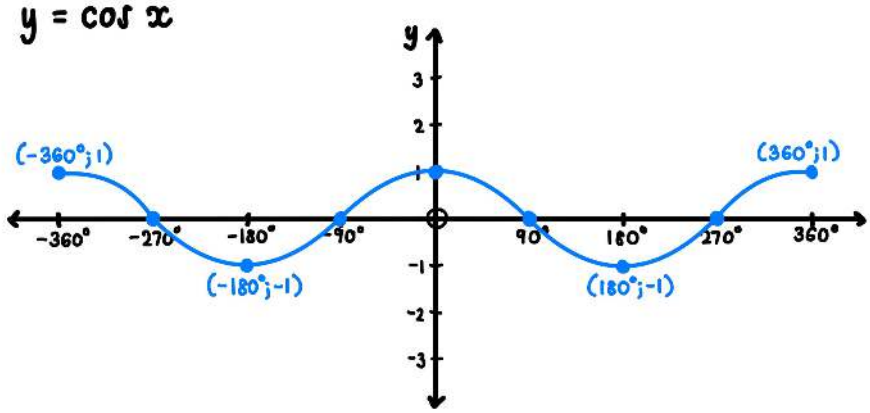
The best method for drawing trig graphs is to know the shapes and properties of these graphs, and then apply any parameter changes given in a question to these.

$y = \sin x$



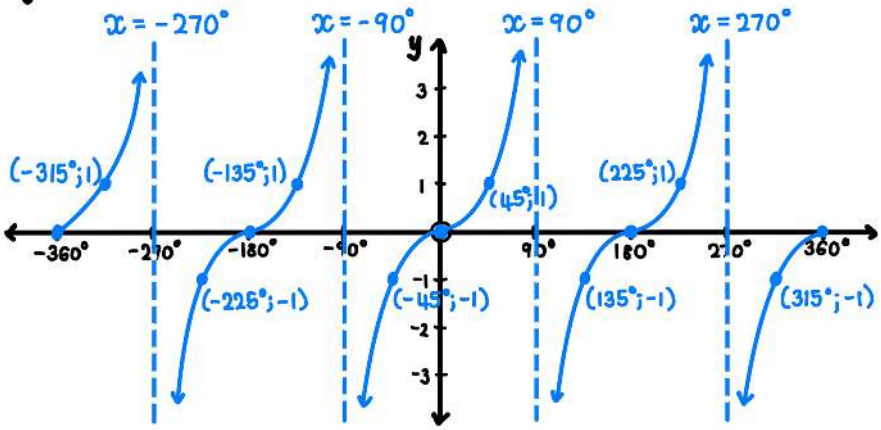
Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

$y = \cos x$



Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

$y = \tan x$



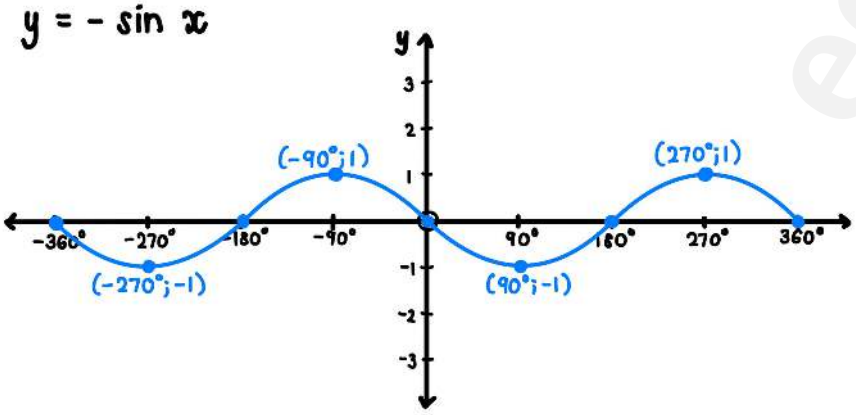
Amplitude = none
Period = 180 degrees
Step = 45 degrees
Maximum value = none
Minimum value = none

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What happens when the a-value is negative?

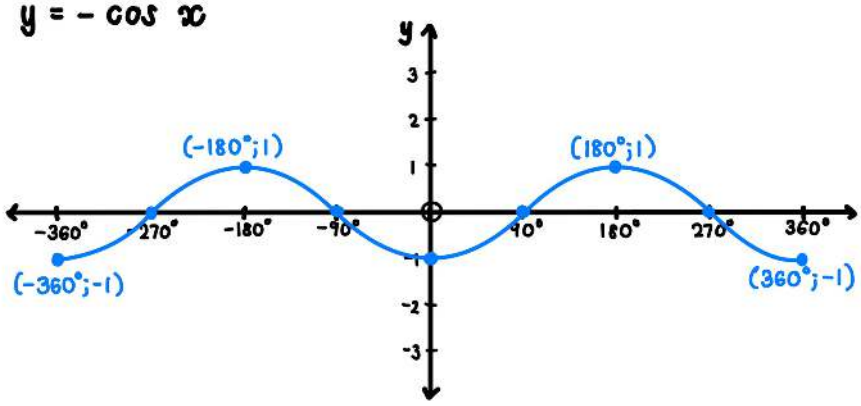
A negative a-value indicates that the basic graphs have been **reflected in the x-axis.** This means that the **signs of the y-coordinates** of the original points have to **change**. **Remember that since the amplitude is a distance, it does not change when the a-value is negative.**

$y = -\sin x$



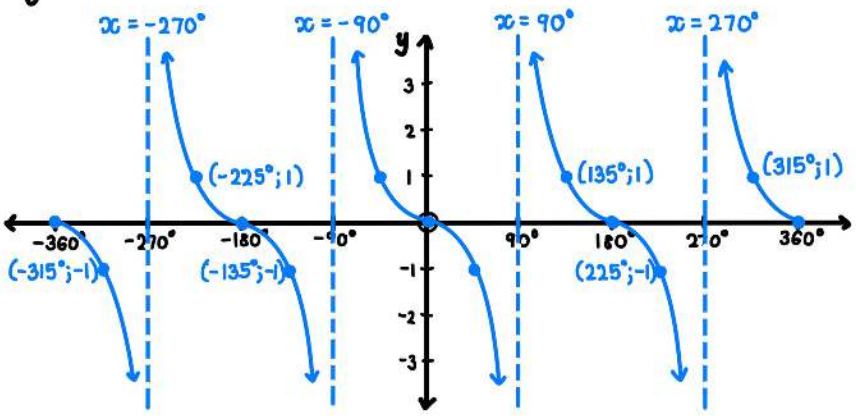
Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

$y = -\cos x$



Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

$y = -\tan x$



Amplitude = none
Period = 180 degrees
Step = 45 degrees
Maximum value = none
Minimum value = none

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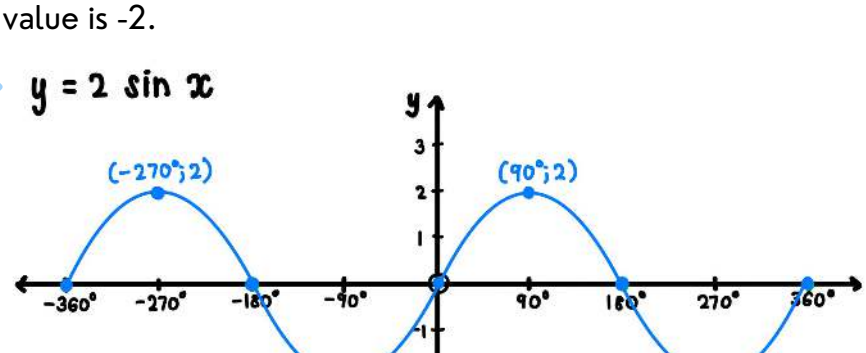
What happens when the amplitude is not 1?

The **a-value** is the amplitude of a sine and cosine graph. This causes a vertical stretch of the graph.
The amplitude is **half the distance between the maximum and minimum values** of the graph.

Since a tan graph doesn't have a maximum or minimum value, it also does not have an amplitude.
When the a-value of a tan graph is not equal to 1, it affects the slope of the graph.

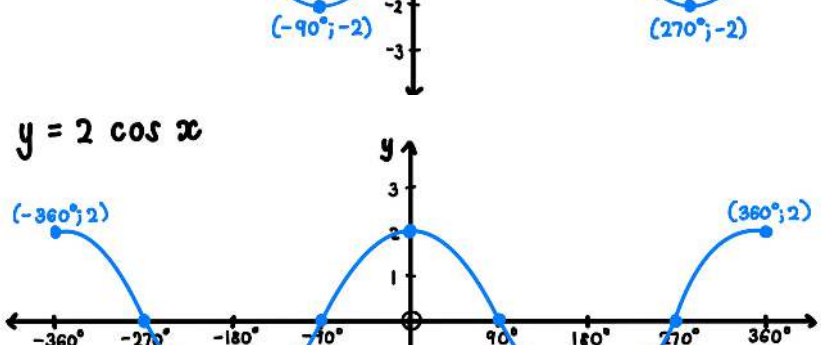
Since the amplitude is a distance, it is **not affected** when the a-value is negative.
The graph of y = -2 sin x, for example, still has an amplitude of 2, even though the a-value is -2.

$y = 2 \sin x$



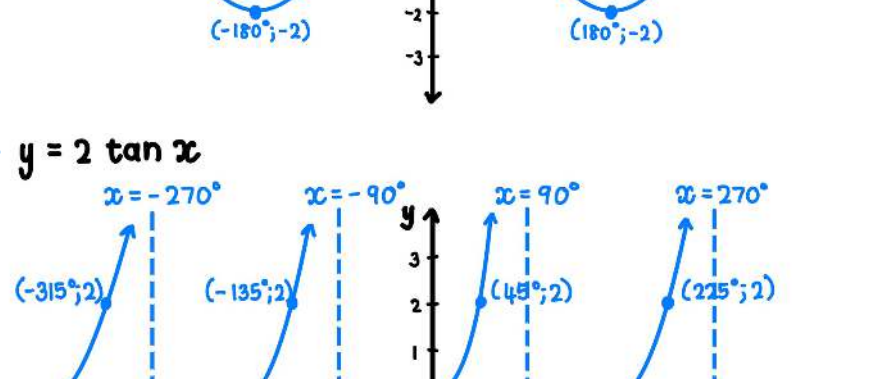
Amplitude = 2
Period = 360 degrees
Step = 90 degrees
Maximum value = 2
Minimum value = -2

$y = 2 \cos x$



Amplitude = 2
Period = 360 degrees
Step = 90 degrees
Maximum value = 2
Minimum value = -2

$y = 2 \tan x$



Amplitude = none
Period = 180 degrees
Step = 45 degrees
Maximum value = none
Minimum value = none

The graph of y = tan x has points (-45 ; -1) and (45 ; 1), etc.
Since the a-value of this graph is 2, the graph has points (-45 ; -2) and (45 ; 2), etc.

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What happens when the q-value is not zero?

When the q-value of a trig graph is changed, that causes the graph to **shift up or down.**

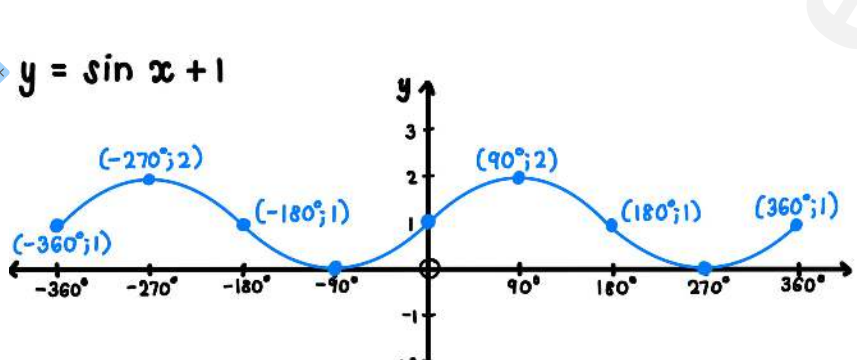
If the q-value if **positive**, that shifts every point on the graph **up**.

If the q-value is **negative**, that shifts every point on the graph **down**.

The **number of units** by which the graph is shifted is equal to the **q-value.**

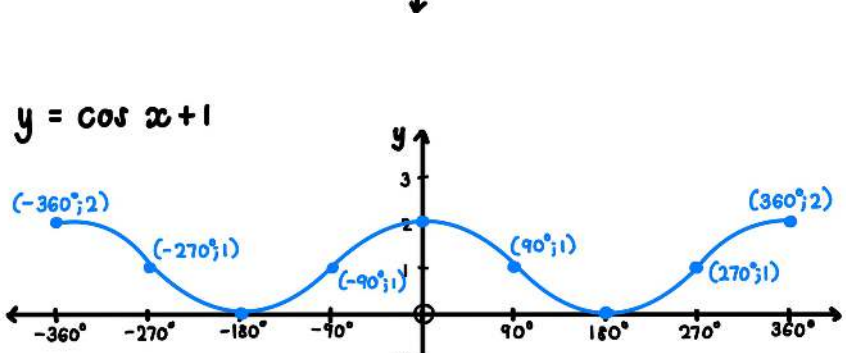
Since the range is the set of all y-values on the graph, the **range** is also affected by a change in q-value.

$y = \sin x + 1$



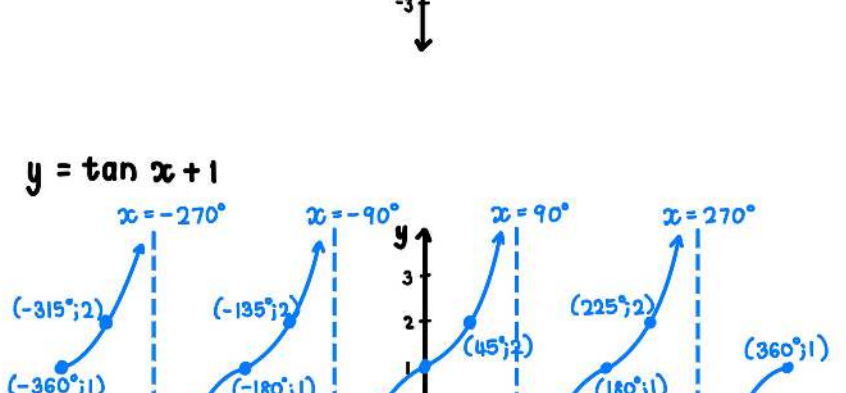
Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 2
Minimum value = 0
Graph of y = sin x has shifted 1 unit up.

$y = \cos x + 1$



Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 2
Minimum value = 0
Graph of y = cos x has shifted 1 unit up.

$y = \tan x + 1$



Amplitude = none
Period = 180 degrees
Step = 45 degrees
Maximum value = none
Minimum value = none
Graph of y = tan x has shifted 1 unit up.

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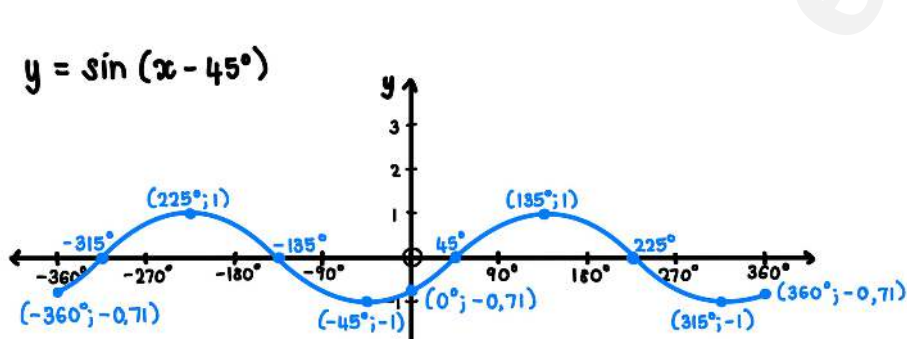
What happens when the p-value is not zero?

When the p-value of a trig graph is changed, that causes the graph to **shift to the left or right.**

If the bracket has a **positive** value, every point on the graph shifts to the **left, eg. (x + 15) = 15 degrees to the left.**

If the bracket has a negative value, every point on the graph shifts to the right, **eg. (x - 15) = 15 degrees to the right.**

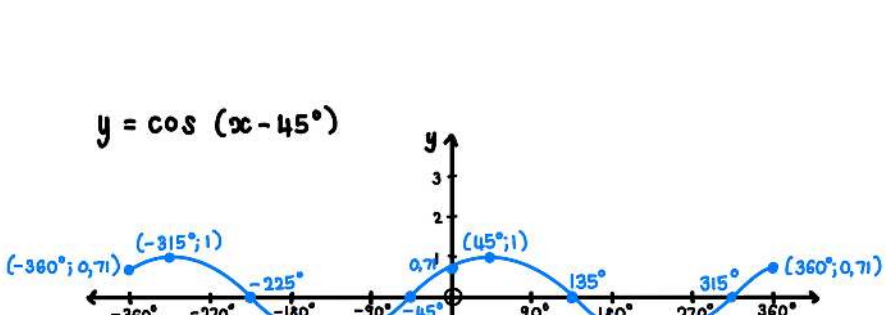
$y = \sin(x - 45°)$



Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

Graph of y = sin x has shifted 45 degrees to the right.

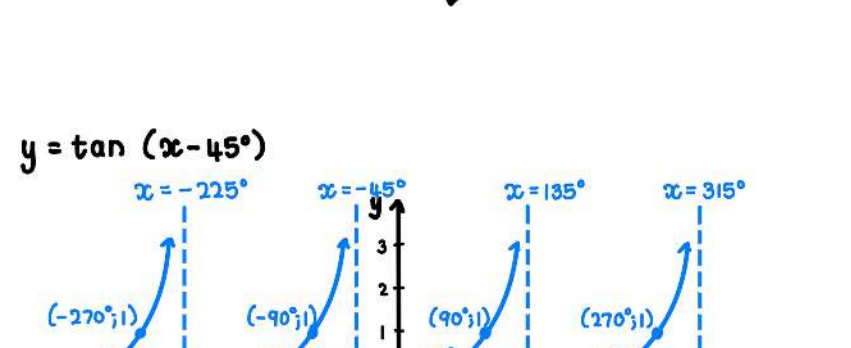
$y = \cos(x - 45°)$



Amplitude = 1
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -1

Graph of y = cos x has shifted 45 degrees to the right.

$y = \tan(x - 45°)$



Amplitude = none
Period = 180 degrees
Step = 45 degrees
Maximum value = none
Minimum value = none

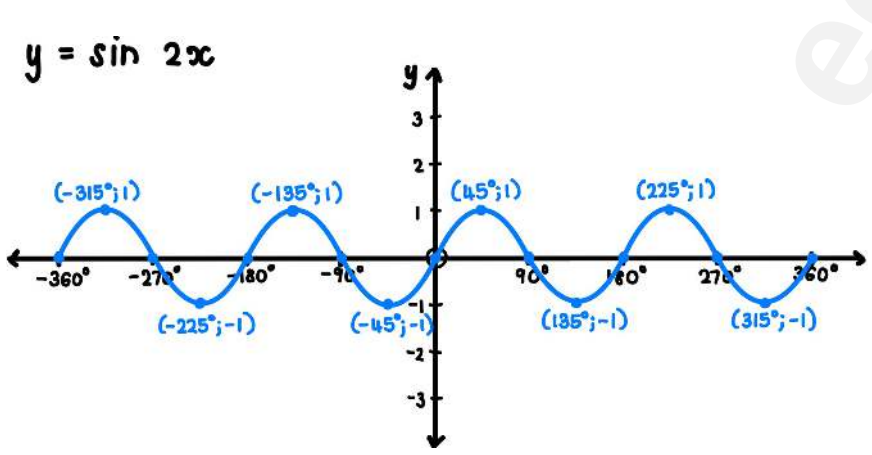
Graph of y = tan x has shifted 45 degrees to the right.

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What happens when the k-value is not 1?

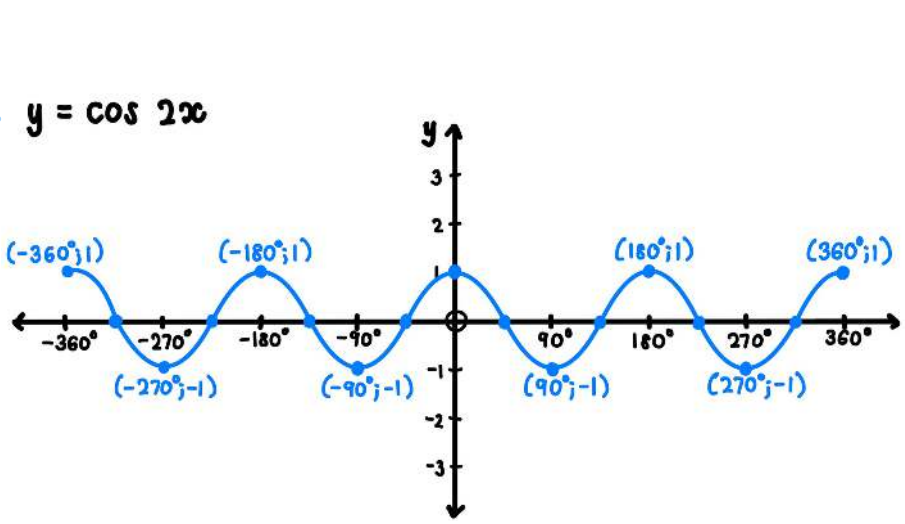
The **period** of a trig graph is affected by the k-value.
The period of a **sine and cosine** graph is **360 degrees DIVIDED by k**.
The period of a **tan** graph is **180 degrees DIVIDED by k.**
The **step** of the graph is then calculated by **dividing the new period by 4.**

$y = \sin 2x$



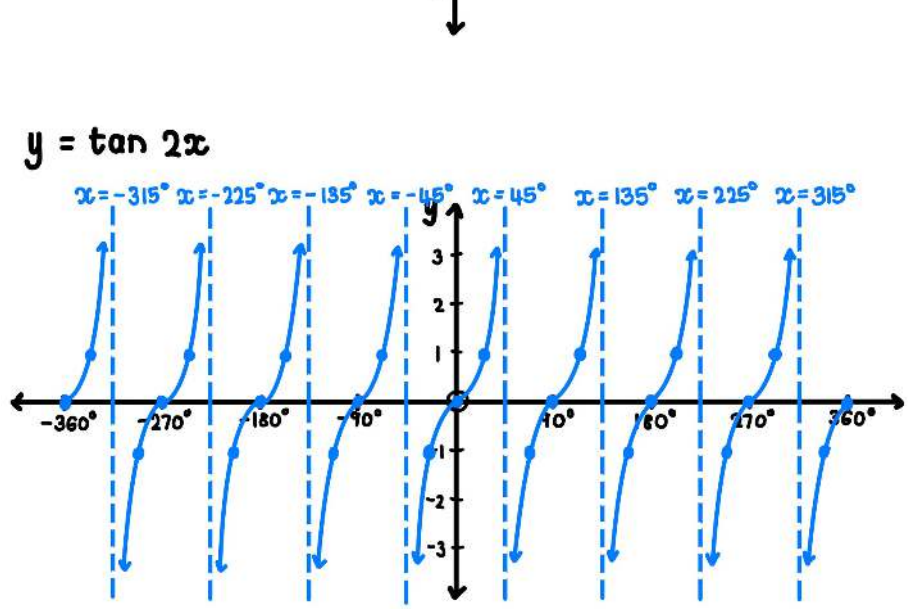
Amplitude = 1
Period = 180 degrees
Step = 45 degrees
Maximum value = 1
Minimum value = -1
There are 2 full sine curves in the original period of 360 degrees.

$y = \cos 2x$



Amplitude = 1
Period = 180 degrees
Step = 45 degrees
Maximum value = 1
Minimum value = -1
There are 2 full cosine curves in the original period of 360 degrees.

$y = \tan 2x$



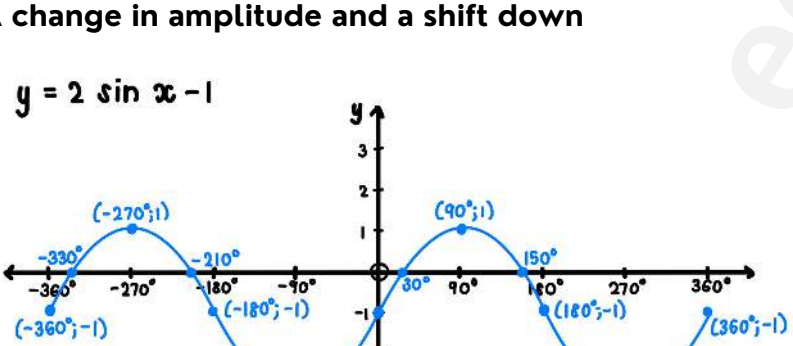
Amplitude = none
Period = 90 degrees
Step = 22,5 degrees
Maximum value = none
Minimum value = none
There are 2 full tan curves in the original period of 180 degrees.

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Drawing graphs with multiple parameter changes

When there are multiple parameter changes, determine how each of these changes would affect the basic graph, and then apply them together.

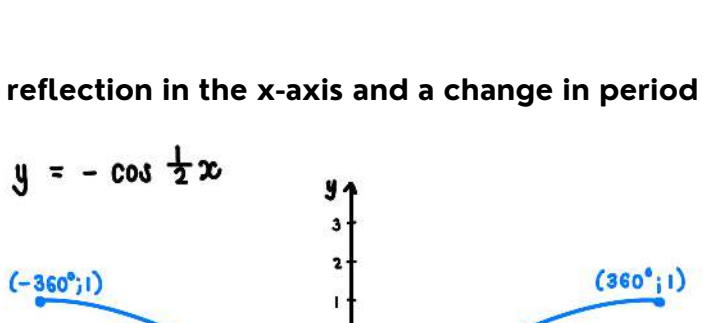
1. A change in amplitude and a shift down



Amplitude = 2
Period = 360 degrees
Step = 90 degrees
Maximum value = 1
Minimum value = -3

1. When a = 2, that means that the graph should have a minimum value of -2 and a maximum value of 2.
2. Now the graph is shifted 1 unit down. This means that it will now have a minimum value of -3 and a maximum value of 1.

2. A reflection in the x-axis and a change in period



Amplitude = 1
Period = 720 degrees
Step = 180 degrees
Maximum value = 1
Minimum value = -1

1. When a = -1, that means that the graph should still have an amplitude of 1, but the graph is reflected in the x-axis.
2. The original period of a cos graph is 360 degrees. This now changes to 720 degrees, because 360 divided by 1/2 is 720 degrees. This means that the step is 180 degrees, so each of our critical points have to be 180 degrees apart.

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Using the TABLE function on the calculator

You are able to use the **TABLE function** on a CASIO scientific calculator to determine the points that you have to plot when sketching a graph.

Please note that this is a **LAST RESORT** method and it is highly recommended that you use your knowledge of the basic trig graphs, as well as how each parameter affects each of these graphs, to draw them.

Follow these steps when using the TABLE function:

1. Set up the correct mode by typing [MODE] and then selecting the option that reads [TABLE]. This option might have different numbers depending on the model of calculator that you are using.

2. The screen should now show **"f(x) ="**. You now **type the equation of the graph**, using **[ALPHA] and [ )]** when having to type x. Make sure to close the bracket in the equation. **Press [=].**

3. You now have the option to type a second equation, g(x). If you are only drawing one graph, you can simply press [=] again to skip this step.

4. The screen should now show **"Start?"** You now type the **starting value of the given domain, and press [=]**.

5. The screen should now show **"End?"** You now type the **end value of the given domain, and press [=].**

6. The screen should now show **"Step?"** You now type the step for the graph that you calculate by **dividing the period by 4**, and **press [=].**

7. The screen should now show a **table of values**. This table has two columns, one for x, and one for f(x). The x-values are all of the x-coordinates of the points that you have to plot, and the f(x)-values are all of the y-coordinates of the points that you have to plot. Press the down arrow to see all of the points in the table.

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Worked example (R)

Draw the graph of $y = -3\sin x$ for $x \in [0°; 360°]$.

You first have to calculate the **PERIOD** of the graph, and then use that to calculate the **STEP** of the graph.

Since this is a sine graph, we calculate the period by **dividing 360 by the k-value**, which in this case is equal to 1. We then **divide the period**, 360 degrees, **by 4** in order to calculate the step.

$y = -3\sin x$ for $x \in [0°; 360°]$

Period = $360°$

Step = $360° \div 4 = 90°$

$f(x) = -3\sin x$

Start? $0°$

End? $360°$

Step? $90°$

	x	$f(x)$
1	0	0
2	90	-3
3	180	0
4	270	3
5	360	0

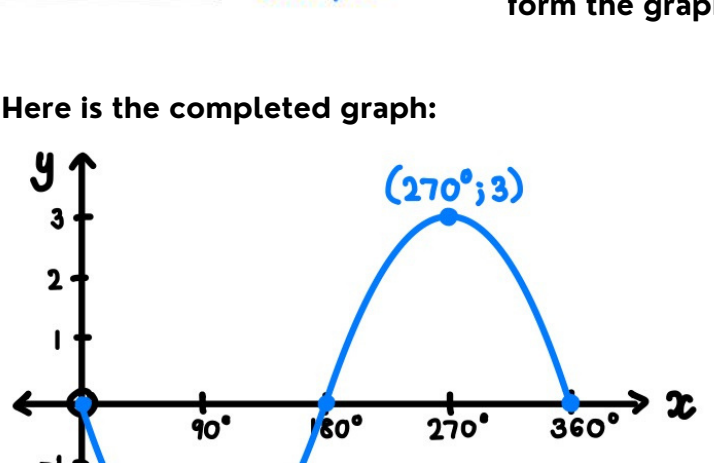
Points to plot
(0°; 0)
(90°; -3)
(180°; 0)
(270°; 3)
(360°; 0)

PRO-TIPS

Now that we know the period and the step of the graph, we can press **[MODE]** and **[TABLE]** to get into the correct mode. We now follow the prompts **(in black)** and type the correct values **(in blue)** in order to get a table of coordinates.

These points, when plotted and connected. form the graph of y = -3 sin x, as shown below.

Here is the completed graph:



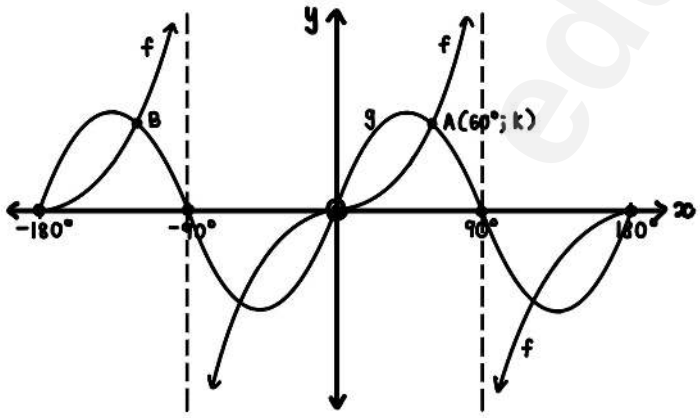
Now let's look at some typical exam questions!

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Worked example (K + R + C)

In the diagram below, the graphs of $f(x) = \tan x$ and $g(x) = 2\sin 2x$ are drawn for the interval $x \in [-180°; 180°]$.

$A(60°; k)$ and B are two points of intersection of f and g.



1. Write down the period of g.
2. Calculate the value of k.
3. Calculate the coordinates of B.
4. Write down the range of $2g(x)$.
5. For which values of x will $g(x + 5°) - f(x + 5°) \leq 0$ in the interval $x \in [-90°; 0°]$?

Solutions:

Worked example (K)

1. Period of $g = \frac{360°}{2}$
   $= 180°$

The period of a cos graph is $\frac{360°}{k}$, where k is the coefficient of x.

Worked example (R)

2. $A(60°; k)$: $k = \tan 60°$
   $= \sqrt{3}$

   or $k = 2\sin 2(60°)$
   $= \sqrt{3}$

k is the y-coordinate of point A, which lies on both f and g. We can calculate k by substituting $x = 60°$ into the equations of either f or g.

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Worked example (K)

3. $B(-120°; \sqrt{3})$

Point B is exactly 180° before point A, as it is the same point of intersection of the two graphs, but one period (180°) earlier.

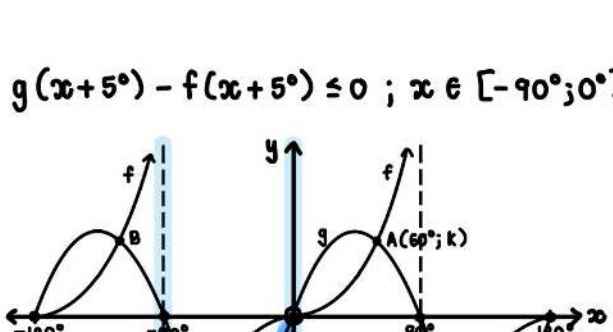
Worked example (C)

4. Range of $g(x)$: $y \in [-2;2]$
   or $-2 \leq y \leq 2$
   $\therefore$ Range of $2g(x)$: $y \in [-4;4]$
   or $-4 \leq y \leq 4$

The minimum value of g is $y=-2$ and the maximum value of g is $y=2$. $2g(x) = 4\sin 2x$, which has an amplitude of 4.

Worked example (C)

5. $g(x+5°) - f(x+5°) \leq 0$; $x \in [-90°;0°]$:



We can rewrite the question in a different way:
$g(x+5°) \leq f(x+5°)$

We can now identify where the graph of $g(x)$ is smaller than (BELOW) or equal to $f(x)$.

This is the highlighted section, which is between the point of intersection of the two graphs (labelled as C) and the origin.
Since A is the point $(60°; \sqrt{3})$, we know that C is $(-60°; -\sqrt{3})$.
$g(x) \leq f(x)$: $-60° \leq x \leq 0°$
$\therefore g(x+5°) \leq f(x+5°)$: $-65° \leq x \leq -5°$

$(x+5°)$ is a shift of 5° to the left, so the entire interval shifts 5° to the left as well.

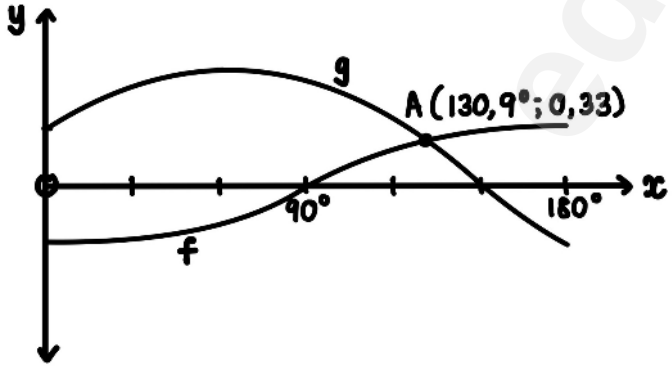
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Worked example (K + R + C + P)

The graphs of $f(x) = -\frac{1}{2}\cos x$ and $g(x) = \sin(x + 30°)$ for the interval $x \in [0°; 180°]$, are drawn below.

$A(130,9°; 0,33)$ is the approximate point of intersection of the two graphs.



1. Write down the period of g.
2. Write down the amplitude of f.
3. Determine the value of $f(180°) - g(180°)$.
4. Use the graphs to determine the values of x, in the interval $x \in [0°; 180°]$, for which:
   - 4.1. $f(x - 10°) = g(x - 10°)$
   - 4.2. $\sqrt{3}\sin x + \cos x \geq 1$

Solutions:

Worked example (K)

1. Period of g = 360° The k-value of g is 1, so the period of g is $\frac{360°}{1}$.

2. Amplitude of f = $\frac{1}{2}$ The a-value of f is $-\frac{1}{2}$, but the amplitude is a distance, so it is positive.

Worked example (R)

3. $f(180°) - g(180°) = \frac{1}{2} - (-\frac{1}{2})$
   $= 1$

$f(180°) = -\frac{1}{2}\cos 180°$
$= \frac{1}{2}$

$g(180°) = \sin(180° + 30°)$
$= -\frac{1}{2}$

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Worked example (C)

4.1.

$f(x-10°) = g(x-10°)$:
$x = 130,9° + 10°$
$= 140,9°$

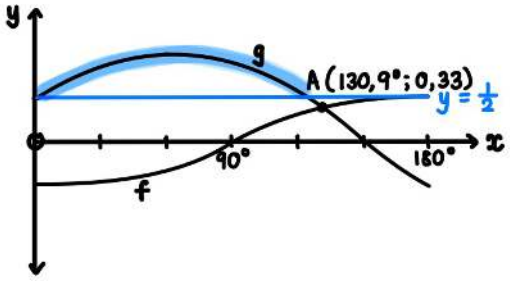
$f(x) = g(x)$ at point A, which is at $x = 130,9°$
$(x-10°)$ is a shift of $10°$ to the right, so we add $10°$ to $130,9°$

Worked example (P)

4.2. In order to use the graph(s), we need to rewrite the question so that it actually contains one of the equations.

$\sqrt{3}\sin x + \cos x \geq 1$
$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x \geq \frac{1}{2}$
$\sin x \cos 30° + \cos x \sin 30° \geq \frac{1}{2}$
$\sin(x+30°) \geq \frac{1}{2}$

Divide each term by 2.
$\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$
The left-hand side is the expansion of a sine compound angle identity.



We now identify where the graph of g is above or equal to $y = \frac{1}{2}$.
It's useful to draw in a line at $y = \frac{1}{2}$ and highlight the section above that line.
$\sin(x+30°) = \frac{1}{2}$ at $x = 0°$ and $x = 120°$

$\therefore 0° \leq x \leq 120°$ or $x \in [0°; 120°]$

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SOLUTION OF TRIANGLES

In order to achieve full marks for this section, you need to be able to:

- Use the **trigonometric ratios** to calculate the lengths of sides or sizes of angles of a right-angled triangle.
- Use the **sine rule** to calculate the length of a side or the size of an angle of a non-right angled triangle.
- Use the **cosine rule** to calculate the length of a side or the size of an angle of a non-right angled triangle.
- Use the **area rule** to calculate the area of a non-right angled triangle.
- **Prove** the three triangle rules.

We can use **trigonometry** to solve both right-angled and non-right angled triangles.

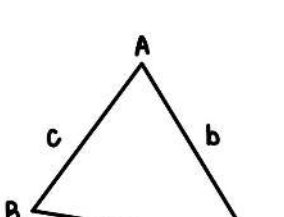
"Solving triangles" means to calculate the lengths of sides, and the sizes of angles, of triangles.

To solve **right-angled triangles**, we use the **trigonometric ratios**, as explained in the previous section.

To solve **non-right angled triangles**, we use the **triangle rules:** the sine rule, cosine rule and area rule.

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THE SINE RULE:



To calculate a side:

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

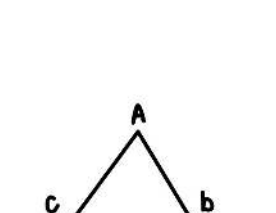
Use the sine rule to calculate the length of a side if you have the size of the opposite angle, and one other pair of an opposite side and angle.

To calculate an angle:

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

Use the sine rule to calculate the size of an angle if you have the size of the opposite angle, and one other pair of an opposite side and angle.

THE COSINE RULE:



To calculate a side:

$a^2 = b^2 + c^2 - 2bc.\cos \hat{A}$

or $b^2 = a^2 + c^2 - 2ac.\cos \hat{B}$

or $c^2 = a^2 + b^2 - 2ab.\cos \hat{C}$

Use the cosine rule to calculate the length of a side if you have the lengths of the other two sides, and the size of the included angle (the angle opposite the unknown side).

To calculate an angle:

$$\cos \hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

or $$\cos \hat{B} = \frac{a^2 + c^2 - b^2}{2ac}$$

or $$\cos \hat{C} = \frac{a^2 + b^2 - c^2}{2ab}$$

Use the cosine rule to calculate the size of ANY angle in a triangle if you have the lengths of all three sides.

THE AREA RULE:



To calculate the area of a triangle that is not right-angled:

Area $\Delta ABC = \frac{1}{2}ab.\sin \hat{C}$

or Area $\Delta ABC = \frac{1}{2}ac.\sin \hat{B}$

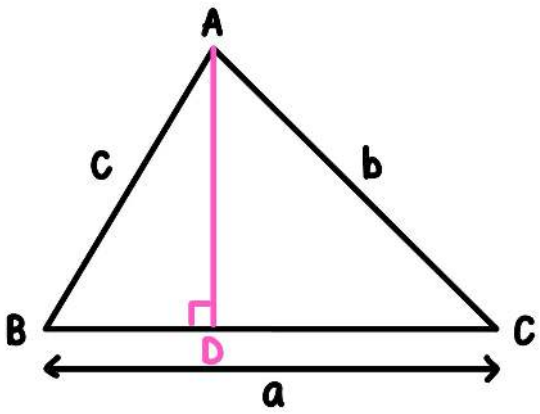
or Area $\Delta ABC = \frac{1}{2}bc.\sin \hat{A}$

In order to use the area rule, you need the lengths of any two sides and the size of the included angle (the angle between the two sides).

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PROVING THE SINE RULE:

You also need to be able to prove all three of the rules for your final exam!
Make sure that you understand the three proofs, and are able to adjust the proofs in case the given triangles have different letters as the vertices.



- This **proof** uses a construction that divides the given triangle into two right-angled triangles.
- We then use the **common** side AD to write an expression for the sine trig ratio in each of these triangles, and then we work with the two ratios at the same time to set up the **sine rule**.



Given: ΔABC, an acute triangle.

Required to prove: $\frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$

Construction: AD, the perpendicular height of ΔABC

The construction divides ΔABC into two right-angled triangles.

Proof: We use the definition of $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an expression for $\sin \hat{B}$ in ΔABD and for $\sin \hat{C}$ in ΔACD.

$\sin \hat{B} = \frac{AD}{c}$ and $\sin \hat{C} = \frac{AD}{b}$

Both of these expressions include AD, so we make AD the subject of both.

$AD = c \sin \hat{B}$ $\quad$ $AD = b \sin \hat{C}$

We then make the two expressions for AD equal to each other, and then manipulate the equation.

$\therefore c \sin \hat{B} = b \sin \hat{C}$

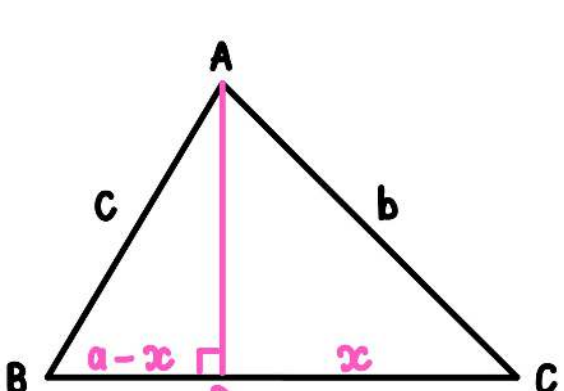
$\frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$

PRO-TIPS

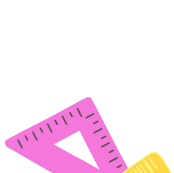
In this proof, you will **NEVER** be asked to prove **all three** fractions of the sine rule. They will only ever ask you to prove the **first two** fractions, as done in this example.

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PROVING THE COSINE RULE:



- This **proof** uses a construction that divides the given triangle into two right-angled triangles.
- We then use the **theorem** of **Pythagoras** in both of these triangles and finally, since this is the **cosine rule**, we also use the trig **ratio** of **cosine** in order to set up the cosine rule.



Given: ΔABC, an acute triangle.

Required to prove: $c^2 = a^2 + b^2 - 2ab.\cos\hat{C}$

Construction: AD, the perpendicular height of ΔABC

The construction divides ΔABC into two right-angled triangles.

Proof: Let $DC = x$
$\therefore BD = a - x$

For this proof, we have to introduce another variable, x.

In ΔABD:
$c^2 = AD^2 + (a-x)^2$ (Pythagoras)
$= AD^2 + a^2 - 2ax + x^2$

We can apply the theorem of Pythagoras in ΔACD to get an expression for c^2. Multiply the brackets out.

In ΔACD:
$b^2 = AD^2 + x^2$ (Pythagoras)
$\therefore c^2 = a^2 + b^2 - 2ax$

We now apply the theorem of Pythagoras in the other right-angled Δ, ΔACD, so that we can get rid of AD^2 and x^2 in the expression found for c^2 earlier in the proof.

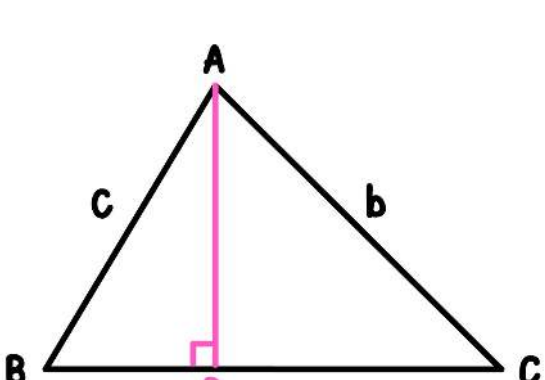
$\cos\hat{C} = \frac{x}{b}$
$x = b\cos\hat{C}$

Now we need to get rid of the x in the expression for c^2, so we express $\cos\hat{C}$ in terms of x in ΔACD, and then make x the subject.

$\therefore c^2 = a^2 + b^2 - 2ab.\cos\hat{C}$

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PROVING THE AREA RULE:



- This **proof** uses a construction that divides the given triangle into two right-angled triangles.
- We then use the formula for calculating the area of a right-angled triangle for the given triangle, and since the **area rule** uses **sine** of an angle.
- We also use the **trig ratio** of sine in one of the right-angled triangles.

Given: ΔABC, an acute triangle.

Required to prove: Area $\Delta ABC = \frac{1}{2}ab.\sin\hat{C}$



Construction: AD, the perpendicular height of ΔABC

The construction divides ΔABC into two right-angled triangles.

Proof: Area $\Delta ABC = \frac{1}{2} a \times AD$

To find the area of a right-angled Δ, we use the formula:
Area = $\frac{1}{2}$ base x perpendicular height

$\sin\hat{C} = \frac{AD}{b}$

$\therefore AD = b\sin\hat{C}$

We now use the definition of $\sin\hat{C} = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an expression for AD.
This allows us to replace AD in the formula for the area of ΔABC.

$\therefore$ Area $\Delta ABC = \frac{1}{2}ab\sin\hat{C}$

Now let's look at some typical exam questions!

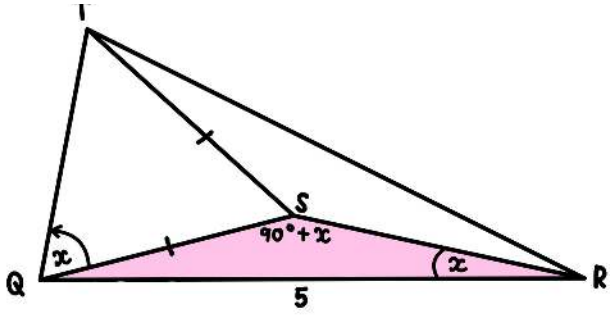
It is important to note that there is only ONE question covering this topic in your final exam, and the question usually counts around 10 marks.

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Worked example (C)

In the diagram below, T is a hook on the ceiling of an art gallery. Points Q, S and R are on the same horizontal plane from where three people are observing the hook T. The angle of elevation from Q to T is x.

$\hat{QSR} = 90° + x$, $\hat{QRS} = x$, QR = 5 units and TS = SQ.



PRO-TIPS

Remember that you need to be able to use reduction formulae, co-ratios and trig identities in the simplification of these expressions.

1. Prove that $QS = 5 \tan x$

$$\frac{QS}{\sin x} = \frac{5}{\sin(90° - x)}$$

$$\frac{QS}{\sin x} = \frac{5}{\cos x}$$

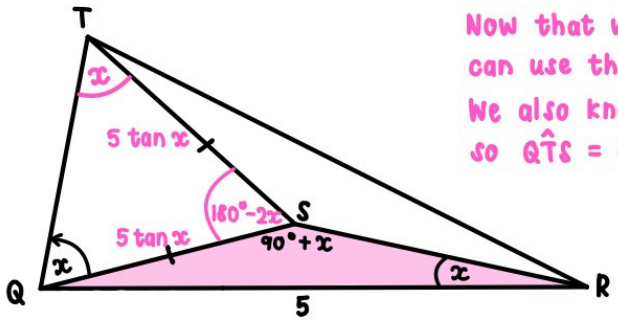
$$QS = \frac{5 \sin x}{\cos x}$$

$$QS = 5 \tan x$$

ΔQRS is not a right-angled Δ. Since we have the angle opposite QS, and another side and opposite angle (QR and $\hat{QSR}$), we should use the sine rule.
$\sin(90° - x) = \cos x$

We make QS the subject by multiplying both sides by $\cos x$.
$\frac{\sin x}{\cos x} = \tan x$

2. Prove that the length of $QT = 10 \sin x$



Now that we know that $QS = 5 \tan x$, we can use that in the rest of our calculations. We also know that ΔQTS is an isosceles Δ, so $\hat{QTS} = \hat{SQT}$.

$$\frac{QT}{\sin(180° - 2x)} = \frac{QS}{\sin x}$$

$$\frac{QT}{\sin 2x} = \frac{QS}{\sin x}$$

$$QT = \frac{QS \sin 2x}{\sin x}$$

$$QT = \frac{QS \cdot 2 \sin x \cos x}{\sin x}$$

$$QT = 5 \tan x \cdot 2 \cos x$$

$$QT = \frac{5 \sin x}{\cos x} \cdot 2 \cos x$$

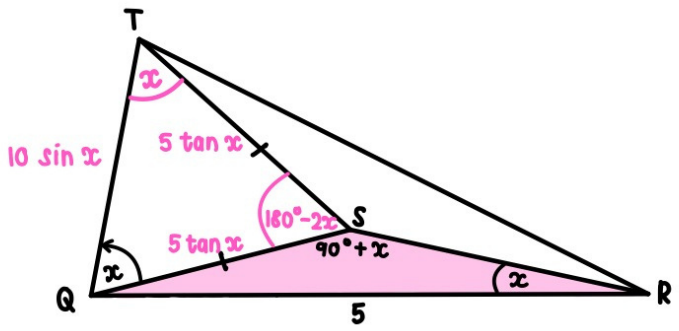
$$\therefore QT = 10 \sin x$$

Now that we have the angle opposite side QT, we can use the sine rule.
$\sin(180° - 2x) = \sin 2x$

We can make QT the subject of the equation by multiplying both sides by $\sin 2x$.
$\sin 2x = 2 \sin x \cos x$ (this is a double angle identity)
We now replace QS with $5 \tan x$, from the previous question.
$\tan x = \frac{\sin x}{\cos x}$

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3. Calculate the area of ΔTQR if $T\hat{Q}R = 70°$ and $x = 25°$



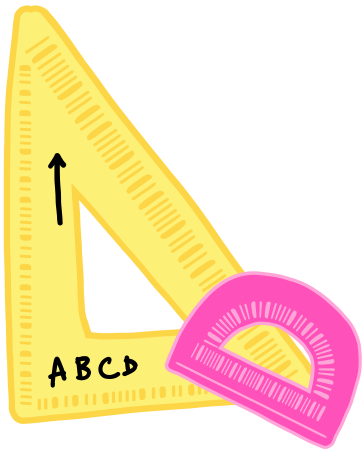
$QT = 10 \sin x$
$QT = 10 \sin 25°$

We use the expression for QT from the previous question. We now replace x with 25°. The value of this is 4,226182617. Either use the full expression for QT in calculations, or store this value in your calculator.

$QR = 5$ units
$T\hat{Q}R = 70°$

ΔTQR is not a right-angled Δ, so we have to use the area rule. We need to use the lengths of two sides and the size of the included angle.

$$\therefore \text{Area } \Delta TQR = \frac{1}{2} QT \cdot QR \cdot \sin T\hat{Q}R$$
$$= \frac{1}{2}(10 \sin 25°)(5)(\sin 70°)$$
$$= 9{,}93 \text{ units}^2$$



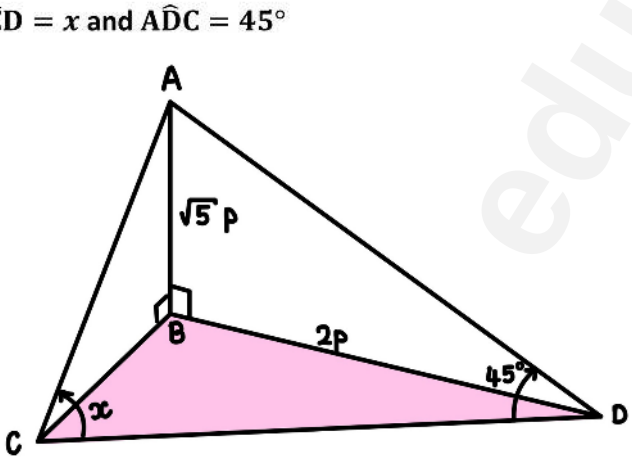
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Worked example (C)

AB is a vertical flagpole that is $\sqrt{5}p$ metres long. AC and AD are two cables anchoring the flagpole. B, C and D are in the same horizontal place.

$BD = 2p$ metres, $A\hat{C}D = x$ and $A\hat{D}C = 45°$



1. Determine the length of AD in terms of p.

$AD^2 = (2p)^2 + (\sqrt{5}p)^2$ (Pythagoras)

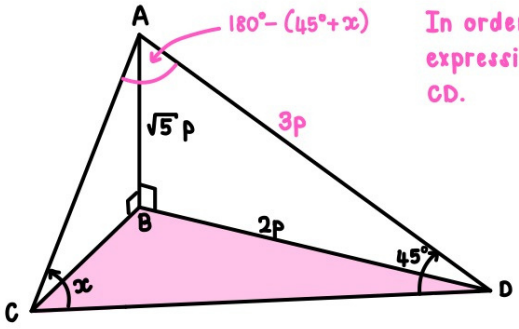
$= 4p^2 + 5p^2$

$= 9p^2$

$AD = 3p$

ΔABD is a right-angled Δ, so we can use Pythagoras to calculate the length of AD, the hypotenuse.

2. Show that the length of $CD = \dfrac{3p(\sin x + \cos x)}{\sqrt{2}\sin x}$



In order to calculate CD, we need an expression for $C\hat{A}D$, the angle opposite CD.

$C\hat{A}D = 180° - (45° + x)$ (∠ sum in Δ)

$\dfrac{CD}{\sin[180° - (45° + x)]} = \dfrac{3p}{\sin x}$

Apply the sine rule.

$\dfrac{CD}{\sin(45° + x)} = \dfrac{3p}{\sin x}$

$\sin[180° - (45° + x)] = \sin(45° + x)$

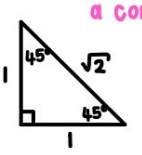
$CD = \dfrac{3p\sin(45° + x)}{\sin x}$

Multiply both sides by $\sin(45° + x)$.

$CD = \dfrac{3p(\sin 45°.\cos x + \cos 45°.\sin x)}{\sin x}$

We now expand $\sin(45° + x)$ using a compound angle identity.

$CD = \dfrac{3p\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)}{\sin x}$



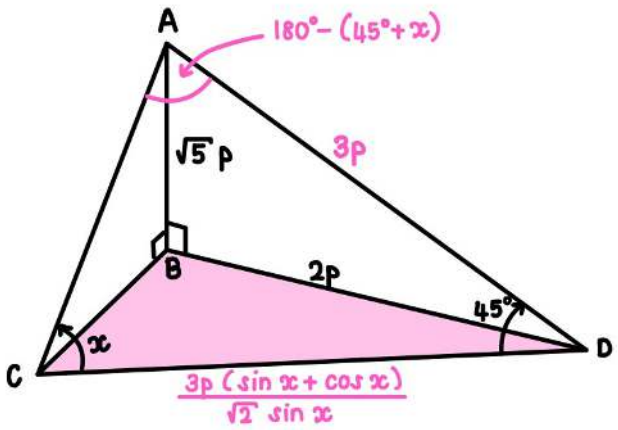
Use the 45° special Δ to determine the values of $\sin 45°$ and $\cos 45°$. Then take out $\frac{1}{\sqrt{2}}$ as a common factor.

$CD = \dfrac{3p \cdot \frac{1}{\sqrt{2}}(\sin x + \cos x)}{\sin x}$ $\times\sqrt{2}$ $\times\sqrt{2}$

$\therefore CD = \dfrac{3p(\sin x + \cos x)}{\sqrt{2}\sin x}$

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3. If it is further given that $p = 10$ and $x = 110°$, calculate the area of ΔADC.



$AD = 3(10)$

$AD = 30$ units

$CD = \frac{3(10)(\sin 110° + \cos 110°)}{\sqrt{2}\sin 110°}$

$CD = 13,492...$ units

Store the value of CD in your calculator.

$\text{Area } \Delta ACD = \frac{1}{2} AD \cdot CD \cdot \sin 45°$

$= \frac{1}{2}(30)(13,492...)(\sin 45°)$

$= 143,11 \text{ units}^2$

Since ΔACD is not a right-angled Δ, we use the area rule.

IMPORTANT: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either **K, R, C** or **P** next to each question!

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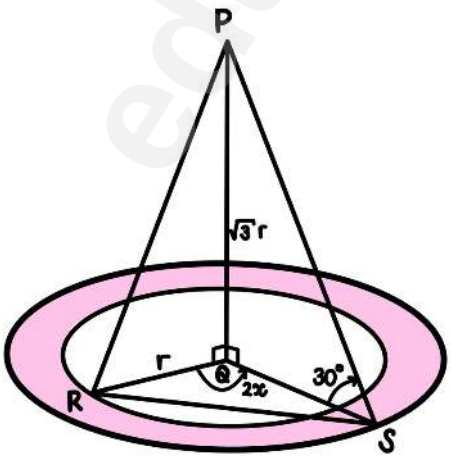
Worked example (C)

A landscape artist plans to plant flowers within two concentric circles around a vertical light pole PQ. R is a point of the inner circle and S is a point on the outer circle. R, Q and S lie in the same horizontal plane. RS is a pipe used for the irrigation system in the garden.

The radius of the inner circle is r units and the radius of the outer circle is QS.

The angle of elevation from S to P is $30°$.

$R\hat{Q}S = 2x$ and $PQ = \sqrt{3}r$



1. Show that $QS = 3r$

$\tan 30° = \frac{\sqrt{3}r}{QS}$

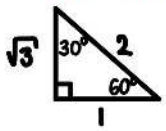
$QS \tan 30° = \sqrt{3}r$

$QS = \frac{\sqrt{3}r}{\tan 30°}$

$QS = \frac{\sqrt{3}r}{\frac{1}{\sqrt{3}}}$ $\times \sqrt{3}$ $\times \sqrt{3}$

$\therefore QS = 3r$

ΔPQS is a right-angled Δ, so we can use the definition of $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.



Replace tan 30° with the value of the ratio.

Now multiply the numerator and denominator by $\sqrt{3}$ so that we can show that $QS = 3r$.

2. Determine, in terms of r, the area of the flower garden.

Area flower garden
= area outer circle − area inner circle
= $\pi(3r)^2 - \pi(r)^2$
= $9\pi r^2 - \pi r^2$
= $8\pi r^2$ units^2

The formula for the area of a circle is πr^2, where r is the length of the radius.

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3. Show that $RS = r\sqrt{10 - 6\cos 2x}$

$RS^2 = r^2 + (3r)^2 - 2(r)(3r).\cos 2x$

$= r^2 + 9r^2 - 6r^2\cos 2x$

$= 10r^2 - 6r^2\cos 2x$

$= r^2(10 - 6\cos 2x)$

$\therefore RS = r\sqrt{10 - 6\cos 2x}$

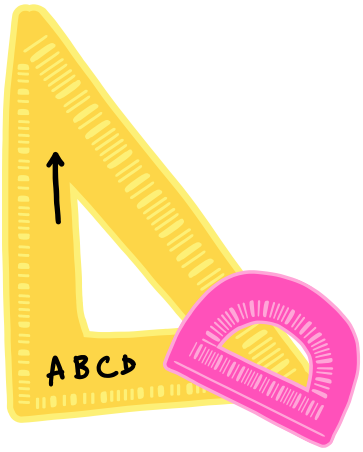
We have expressions for the lengths of the other two sides of ΔGRS and the angle opposite RS, so we can use the cosine rule, and then take out r^2 as a common factor.

4. If $r = 10$ metres and $x = 56°$, calculate RS.

$RS = 10\sqrt{10 - 6\cos 2(56°)}$

$RS = 35$ m

Substitute the given values into the expression for RS.



Question source: NSC Grade 12 November 2020 Paper 2

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EUCLIDEAN GEOMETRY

Euclidean Geometry is a section that you have actually been doing since Grade 8!

Do you remember **FUZYX**? Working with parallel lines and classifying triangles? That was your first taste of Euclidean Geometry, and even though this section is primarily tested using circles and similarity of triangles, you have to know all the basics in order to answer the more complicated questions.

<u>The following strategy can be used when you are solving geometry questions:</u>

1. **Read through the given information very carefully.**
Make sure that everything that is mentioned in writing is indicated on the diagram as well. If it isn't, add it into the diagram in **BLACK** or in pencil, so that you know that it was given information.

2. **Now analyse the diagram very carefully.**
Use different pencils, pens or highlighters to mark sides and/or angles that are equal, identify various parts of the diagrams, such as parallel lines, cyclic quads, etc. and indicate all values that you can determine right at the start on the diagram as well. Make sure to use a different colour for every value that was **NOT GIVEN**. You will have to state that value as well as a reason for it if you use it in a calculation.

3. **Analyse what you are asked to calculate/prove in the question.**
Ask yourself how you could do that, and which values or what other information you might need to be able to answer the question.

4. When writing the solution, **make sure that you write down all angles or sides that you are using in order to get to the answer.**
You also have to write a reason for every statement.

PRO-TIPS

Marks are awarded for values shown on the diagram, even if you didn't write the values down, so make sure to indicate **EVERY** value that you possibly can!

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Summary of ALL Euclidean Geometry knowledge that you should have:

Note that all reasons are highlighted.
You may only use the reasons that are specified in the Grade 12 NSC Exam Guidelines.
All reasons in this chapter are allowed to be used.

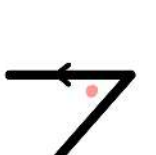
1. FUZYX:



corresponding angles are equal
(corresp. Ls; ll lines)



co-interior angles add up to 180°
(co-int. Ls; ll lines)



alternate angles are equal
(alt. Ls; ll lines)



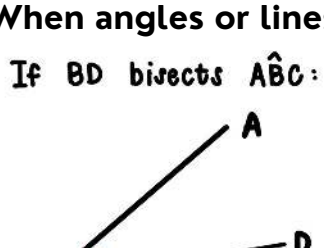
angles on a straight line add up to 180°
(Ls on a str line)



vertically opposite angles are equal
(vert opp Ls =)

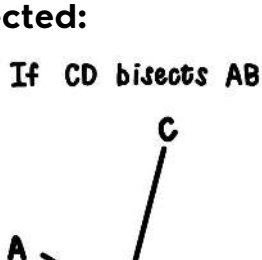
2. When angles or lines are bisected:

If BD bisects $A\hat{B}C$:



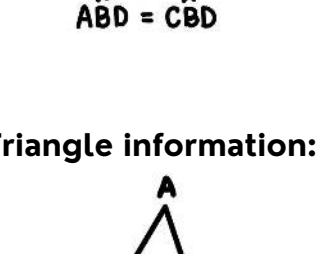
$A\hat{B}D = C\hat{B}D$

If CD bisects AB:



AE = BE and E is the midpoint of AB.

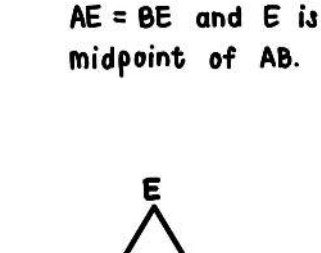
3. Triangle information:



$\hat{A} + \hat{B} + A\hat{C}B = 180°$
(∠ sum in Δ) or
(sum of Ls in Δ)

$A\hat{C}D = \hat{A} + \hat{B}$ (ext. ∠ of Δ)

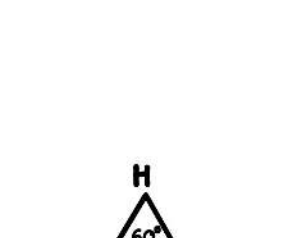
ΔABC is a scalene Δ.



If EF = EG : $\hat{F} = \hat{G}$
(Ls opp equal sides)

If $\hat{F} = \hat{G}$: EF = EG
(sides opp equal Ls)

ΔEFG is an isosceles Δ.

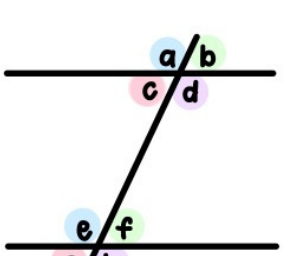


If HI = HJ = IJ :
$\hat{H} = \hat{I} = \hat{J} = 60°$

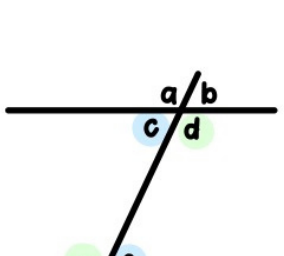
ΔHIJ is an equilateral Δ.

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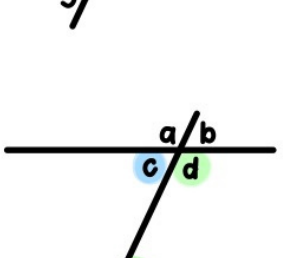
4. To prove that lines are parallel:



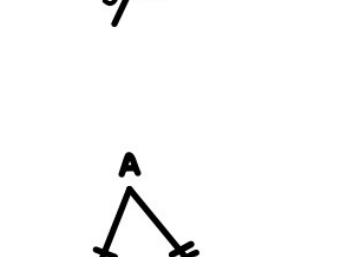
Prove that:
$a = e$ or $b = f$ or $c = g$ or $d = h$
(corresp. Ls equal)
because corresponding angles are only equal if they are formed by parallel lines.



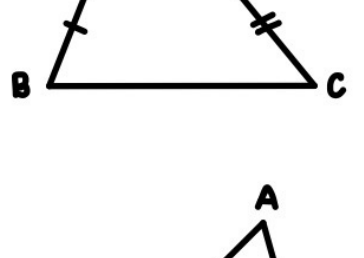
Prove that:
$c = f$ or $d = e$
(alt. Ls equal)
because alternate angles are only equal if they are formed by parallel lines.



Prove that:
$c + e = 180°$ or $d + f = 180°$
(co-int. Ls supp)
because co-interior angles only add up to 180° if they are formed by parallel lines.



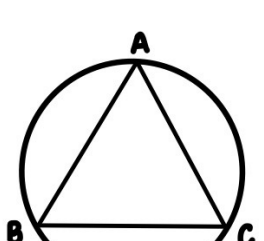
Prove that AD = BD and AE = CE
(Midpt theorem)
because a line from the midpoint of one side of a triangle to the midpoint of the second side, is parallel to the third side.



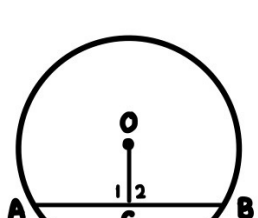
Prove that AB and AC are in proportion
eg. $\frac{AD}{BD} = \frac{AE}{CE}$ or $\frac{AD}{AB} = \frac{AE}{AC}$
(line divides 2 sides in prop)
because a line only divides two sides of a triangle in the same proportion if that line is parallel to the third side of the triangle.

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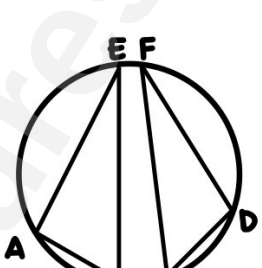
5. Ways of proving that two sides are equal:



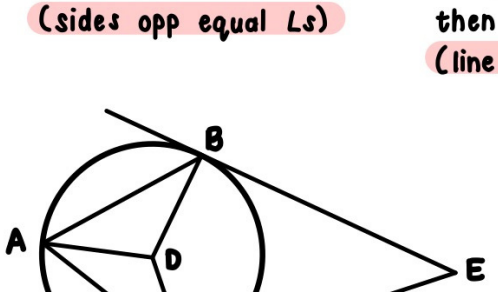
To prove AB = AC:
Prove that $A\hat{B}C = A\hat{C}B$
then AB = AC
(sides opp equal Ls)



To prove that AC = BC:
Prove that $\hat{C}_1 = 90°$
or $\hat{C}_2 = 90°$
then AC = BC
(line from centre ⊥ to chord)



To prove that AB = CD:
Prove that $\hat{E} = \hat{F}$
then AB = CD
(equal Ls; equal chords)



To prove that AD = CD:
Prove that ΔABD ≡ ΔACD
then AD = CD (≡ Δs)

If BE and CE are tangents,
then BE = CE (tans from common pt)

6. When the centre of a circle is given:



(∠ at centre = 2 × ∠ at circumference)

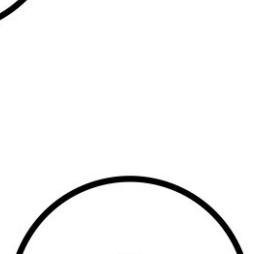


If OM ⊥ AB: AM = BM
(line from centre ⊥ to chord)

If AM = BM: OM ⊥ AB
(line from centre to midpt of chord)



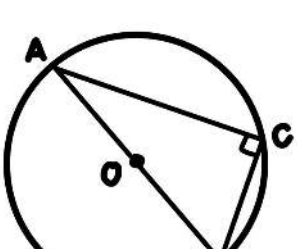
(radius ⊥ tangent)



OA = OB (radii)
$O\hat{A}B = O\hat{B}A$ (radii; Ls opp equal sides)

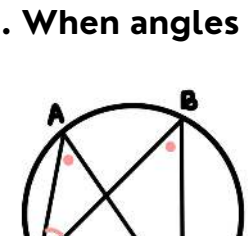
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7. When the diameter of a circle is given:

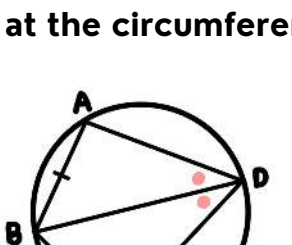


$A\hat{C}B = 90°$ (L in semicircle)
or (diameter subtends right L)

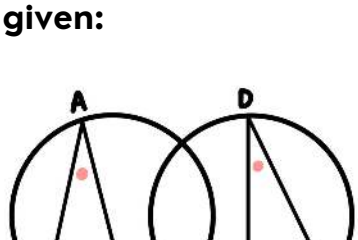
8. When angles formed at the circumference are given:



$\hat{A} = \hat{B}$ and $\hat{C} = \hat{D}$
(Ls in same segment)

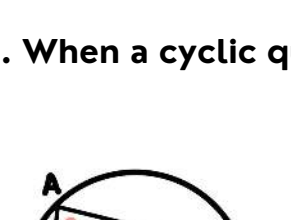


If AB = BC : $A\hat{D}B = C\hat{D}B$
(equal chords; equal Ls)

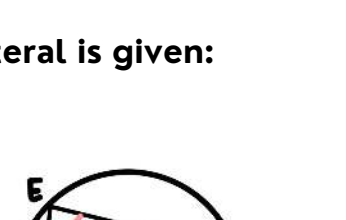


If ABC and DEF are equal circles
and BC = EF :
$\hat{A} = \hat{D}$ (equal chords; equal Ls)

9. When a cyclic quadrilateral is given:



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$
(opp Ls of cyclic quad)



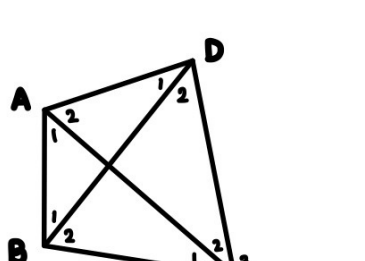
$H\hat{G}I = \hat{E}$
(ext. Ls of cyclic quad)



$K\hat{J}L = K\hat{M}L$ and $L\hat{J}M = L\hat{K}M$
and $J\hat{K}M = J\hat{L}M$ and $J\hat{N}K = J\hat{L}K$
(Ls in same segment)

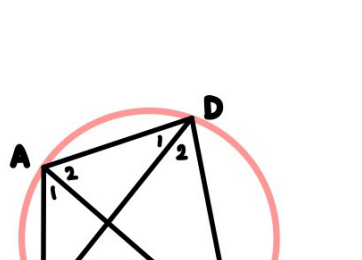
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10. Ways of proving that a quadrilateral is cyclic:



ABCD is a cyclic quadrilateral if any of the following conditions can be proved:

1. $\hat{A} + B\hat{C}D = 180°$ or $\hat{B} + \hat{D} = 180°$
   (converse opp $\angle$s of cyclic quad)
2. $D\hat{C}E = \hat{A}$ (converse ext. $\angle$ of cyclic quad)
3. $\hat{A}_1 = \hat{D}_2$ or $\hat{A}_2 = \hat{B}_2$ or $\hat{C}_1 = \hat{D}_1$ or $\hat{C}_2 = \hat{B}_1$
   (converse $\angle$s in same segment)



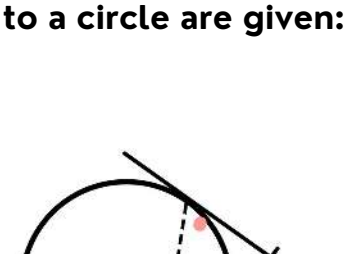
If it has been proven that ABCD is a cyclic quadrilateral, you can draw a circle passing through the vertices.

All properties of a cyclic quadrilateral can then be used.

11. When tangents to a circle are given:



(tangent ⊥ radius)



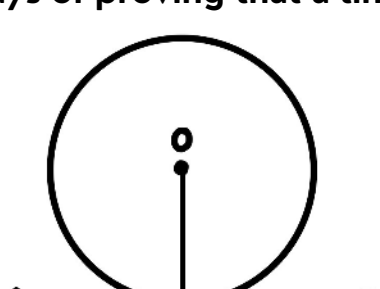
(tangents from common pt)



(tan chord theorem)

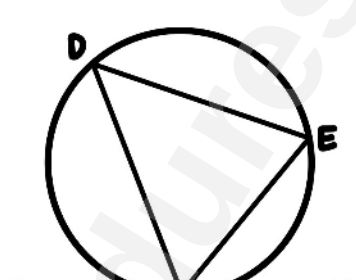
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12. Ways of proving that a line is a tangent:



Prove that $O\hat{B}A = 90°$ or $O\hat{B}C = 90°$

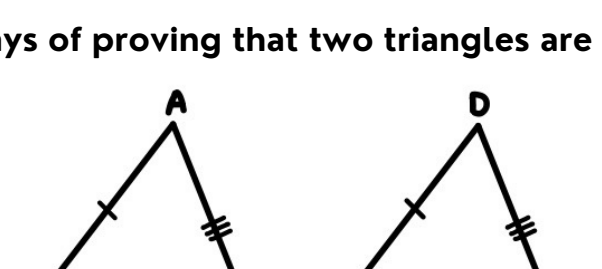
(converse tangent ⊥ radius)



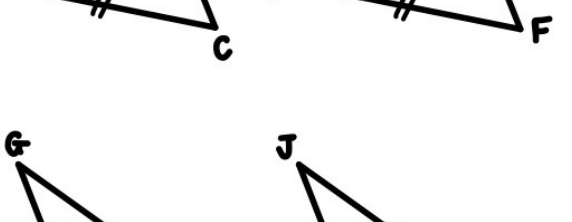
Prove $D\hat{F}G = \hat{E}$ or $E\hat{F}H = \hat{D}$

(converse tan chord theorem)

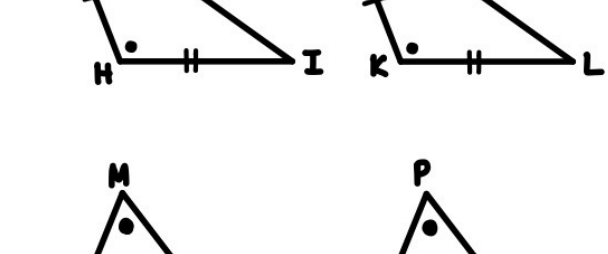
13. Ways of proving that two triangles are congruent:



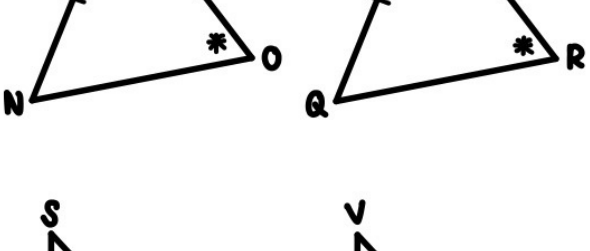
In ΔABC and ΔDEF:
1. AB = DE (given)
2. BC = EF (given)
3. AC = DF (given)
∴ ΔABC ≡ ΔDEF (SSS)



In ΔGHI and ΔJKL:
1. GH = JK (given)
2. $\hat{H} = \hat{K}$ (given)
3. HI = KL (given)
∴ ΔGHI ≡ ΔJKL (SAS)



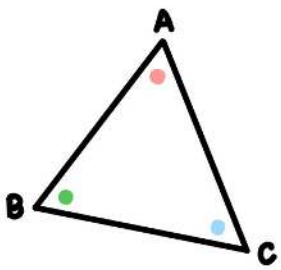
In ΔMNO and ΔPQR:
1. MN = PQ (given)
2. $\hat{M} = \hat{P}$ (given)
3. $\hat{O} = \hat{R}$ (given)
∴ ΔMNO ≡ ΔPQR (AAS)

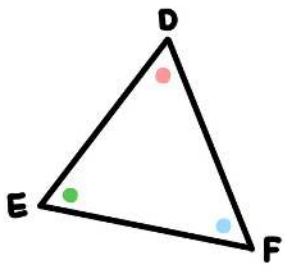


In ΔSTU and ΔVWX:
1. ST = VW (given)
2. $\hat{T} = \hat{W} = 90°$ (given)
3. SU = VX (given)
∴ ΔSTU ≡ ΔVWX (RHS)

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14. Ways of proving that two triangles are similar:

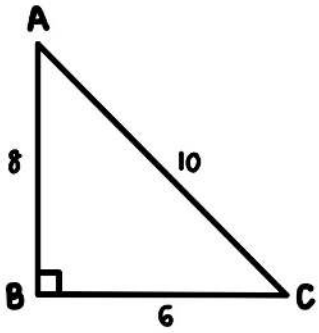


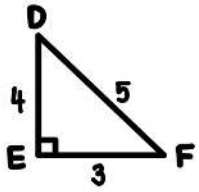


In ΔABC and ΔDEF:

1. $\hat{A} = \hat{D}$ (given)
2. $\hat{B} = \hat{E}$ (given)
3. $\hat{C} = \hat{F}$ (given)

$\therefore \Delta ABC \,|||\, \Delta DEF$ (equiangular)



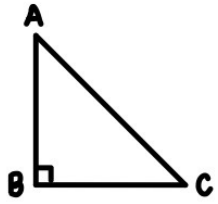


In ΔABC and ΔDEF

1. $\frac{AB}{DE} = \frac{8}{4} = 2$
2. $\frac{AC}{DF} = \frac{10}{5} = 2$
3. $\frac{BC}{EF} = \frac{6}{3} = 2$

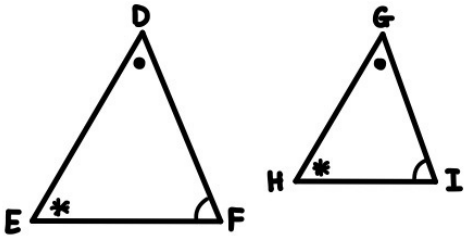
$\therefore \Delta ABC \,|||\, \Delta DEF$ (sides in prop)

15. Ways of calculating the lengths of the sides of a triangle:



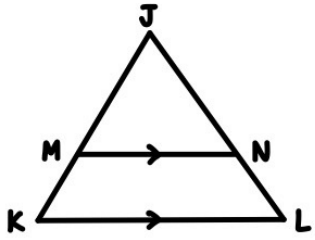
$AC^2 = AB^2 + BC^2$

(Pythagoras)



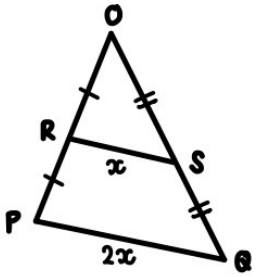
If $\Delta DEF \,|||\, \Delta GHI$:

$\frac{DE}{GH} = \frac{EF}{HI} = \frac{DF}{GI}$ (||| Δs)

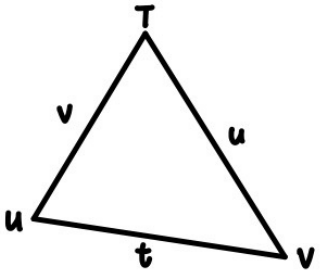


$\frac{JM}{MK} = \frac{JN}{NL}$

(Prop theorem; MN || KL)



If R is the midpoint of OP (OR = RP) and S is the midpoint of OQ (OS = SQ), then PQ = 2 × RS (Midpt theorem)



$$\frac{t}{\sin \hat{T}} = \frac{u}{\sin \hat{U}} = \frac{v}{\sin \hat{V}}$$

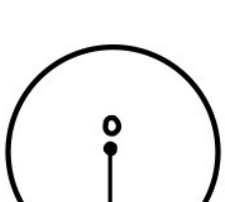
or $t^2 = u^2 + v^2 - 2uv.\cos \hat{T}$

$u^2 = t^2 + v^2 - 2tv.\cos \hat{U}$

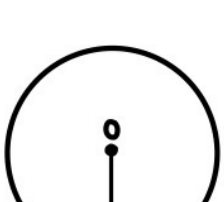
$v^2 = t^2 + u^2 - 2tu.\cos \hat{V}$

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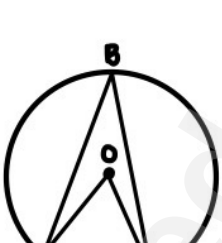
Summary of all Euclidean Geometry CIRCLE theorems:



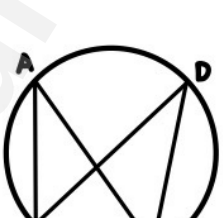
AC = BC
(line from centre ⊥ to chord)



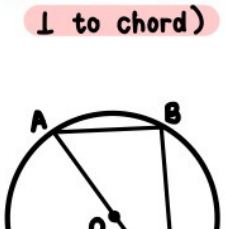
OC ⊥ AB
(line from centre to midpt of chord)



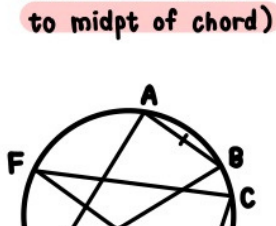
$A\hat{O}C = 2 \times A\hat{B}C$
(∠ at centre = 2 × ∠ at circumference)



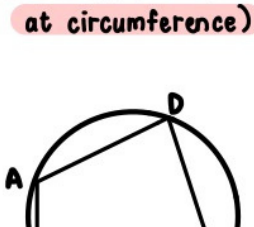
$\hat{A} = \hat{D}$ and $\hat{B} = \hat{C}$
(∠s in same segment)



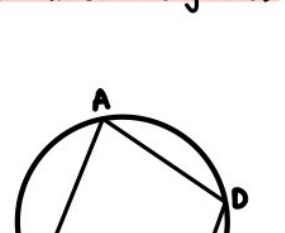
$A\hat{B}C = 90°$
(∠ in semicircle)



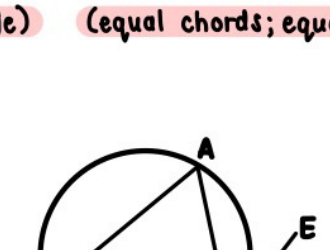
$\hat{E} = \hat{F}$
(equal chords; equal ∠s)



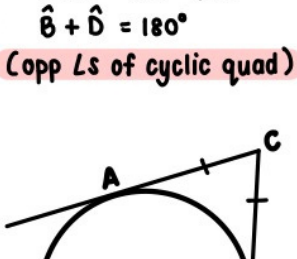
$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$
(opp ∠s of cyclic quad)



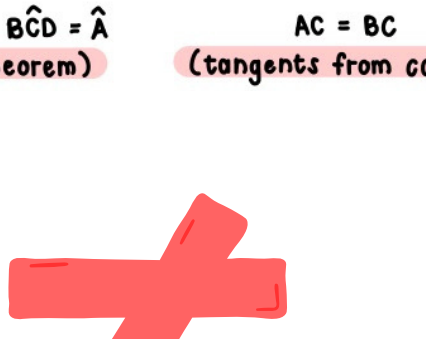
$D\hat{C}E = \hat{A}$
(ext.∠ of cyclic quad)



$A\hat{C}E = \hat{B}$ and $B\hat{C}D = \hat{A}$
(tan chord theorem)



AC = BC
(tangents from common pt)



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Euclidean Geometry CIRCLE theorem proofs

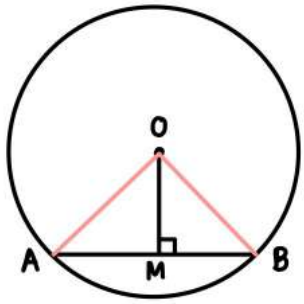
You can expect at least one proof in your final exam.

Make sure that you are able to write these proofs out in full, without leaving out any steps or reasons.

You also have to describe the construction, and draw it on the diagram.

Theorem 1: Line from centre perpendicular to chord

A line from the centre of a circle, perpendicular to a chord, bisects the chord.



Given: A circle with centre O, line OM which is perpendicular to chord AB.

Required to prove: AM = MB

Construction: radii OA and OB

Proof:

In ΔAOM and ΔBOM

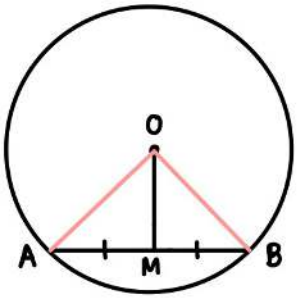
1. AO = BO (radii)
2. OM is common.
3. $O\hat{M}A = O\hat{M}B$ ($OM \perp AB$)

$\therefore \Delta AOM \equiv \Delta BOM$ (RHS)

$\therefore$ AM = BM (congruent Δs)

Converse of Theorem 1: Line from centre to midpoint of chord

A line from the centre of a circle to the midpoint of a chord, is perpendicular to that chord.



Given: A circle with centre O and line OM which bisects chord AB.

Required to prove: $OM \perp AB$

Construction: radii OA and OB

Proof:

In ΔAOM and ΔBOM:

1. AO = BO (radii)
2. AM = BM (given)
3. OM is common.

$\therefore \Delta AOM \equiv \Delta BOM$ (SSS)

$\therefore A\hat{M}O = B\hat{M}O$ (congruent Δs)

and $A\hat{M}O + B\hat{M}O = 180°$ ($\angle$s on a straight line)

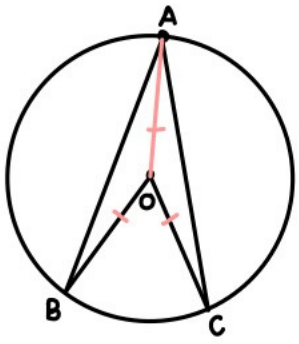
$\therefore A\hat{M}O = B\hat{M}O = 90°$

$\therefore OM \perp AB$

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Theorem 2: Angle at centre = 2 x angle at circumference

The angle subtended by an arc or chord at the centre of a circle is double the size of an angle subtended by the same arc or chord at the circumference of the circle.



Given: A circle with centre O and points A, B and C on the circumference.

Required to prove: $B\hat{O}C = 2B\hat{A}C$

Construction: radius OA

Proof:

Let $O\hat{A}B = x$ and $O\hat{A}C = y$

so $B\hat{A}C = x + y$

$O\hat{B}A = x$ and $O\hat{B}C = y$ (Ls opp equal sides; radii)

$B\hat{O}A = 180° - 2x$ and $C\hat{O}A = 180° - 2y$ (∠ sum in Δ)

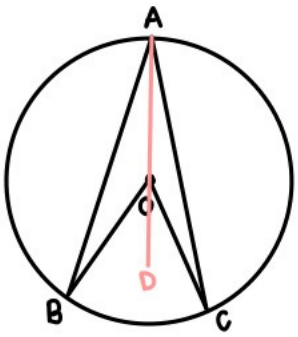
$\therefore B\hat{O}C = 360° - (180° - 2x) - (180° - 2y)$ (Ls around a point)

$= 360° - 180° + 2x - 180° + 2y$

$= 2x + 2y$

$= 2(x + y)$

$= 2B\hat{A}C$



Alternate proof:

Construction: Join AO and produce to D.

Proof:

Let $O\hat{A}B = x$ and $O\hat{A}C = y$

so $B\hat{A}C = x + y$

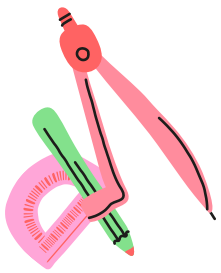
$O\hat{B}A = x$ and $O\hat{B}C = y$ (Ls opp equal sides; radii)

$B\hat{O}D = 2x$ and $C\hat{O}D = 2y$ (ext. ∠ of a Δ)

$\therefore B\hat{O}C = 2x + 2y$

$= 2(x + y)$

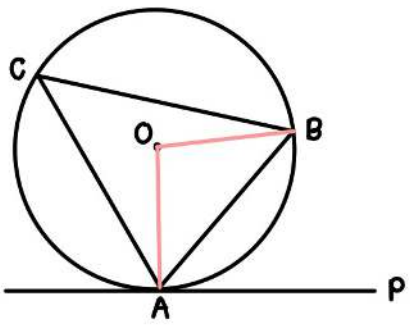
$= 2B\hat{A}C$



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Theorem 3: Tan chord theorem

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle subtended by the same chord in the alternate segment.



Given: Circle with centre O. AP is a tangent to the circle at A and chord AB subtends $\hat{C}$ at the circumference.

Required to prove: $B\hat{A}P = A\hat{C}B$

Construction: radii OA and OB

Proof:

Let $A\hat{C}B = x$

$A\hat{O}B = 2x$ ($\angle$ at centre = 2 × $\angle$ at circumference)

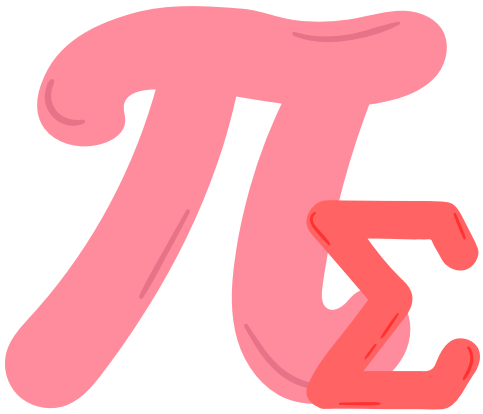
$O\hat{A}B = 90° - x$ ($\angle$ sum in Δ; $\angle$s opp equal sides; radii)

and $O\hat{A}P = 90°$ (tangent $\perp$ radius)

$\therefore B\hat{A}P = 90° - (90° - x)$

$= x$

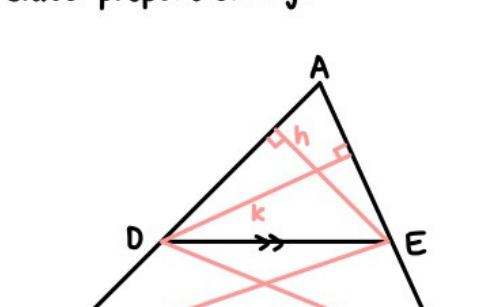
$\therefore B\hat{A}P = A\hat{C}B$



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Theorem 4: Proportional Division Theorem

A line parallel to one side of a triangle will divide the other two sides proportionally.

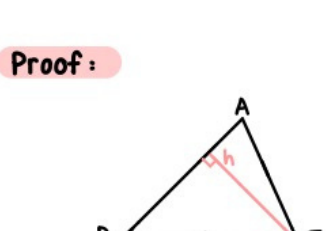


Given: In ΔABC, D lies on AB and E lies on AC such that $DE \parallel BC$.

Required to prove: $\frac{AD}{DB} = \frac{AE}{EC}$

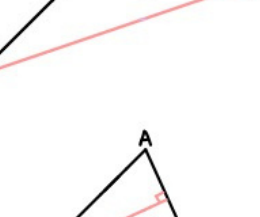
Construction: Draw altitudes h and k of ΔADE relative to bases AD and AE. Join DC and BE.

Proof:



$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\frac{1}{2} AD \times h}{\frac{1}{2} DB \times h} = \frac{AD}{DB}$$

(ΔADE and ΔBDE have the same height, h)



$$\frac{\text{Area } \Delta ADE}{\text{Area } \Delta CDE} = \frac{\frac{1}{2} AE \times k}{\frac{1}{2} CE \times k} = \frac{AE}{CE}$$

(ΔADE and ΔCDE have the same height, k)

but area ΔBDE = area ΔCDE
(same base DE and height between parallel lines DE and BC)

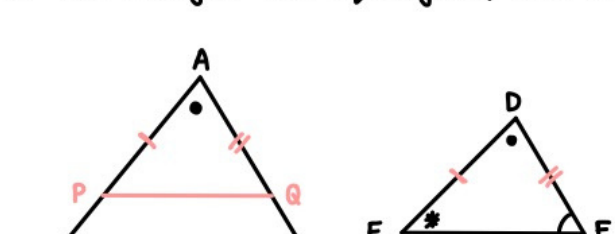
$$\therefore \frac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \frac{\text{Area } \Delta ADE}{\text{Area } \Delta CDE}$$

$$\frac{AD}{DB} = \frac{AE}{CE}$$

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Theorem 5: Equiangular triangles have sides in proportion

If two triangles are equiangular, then the triangles are similar.



Given: ΔABC and ΔDEF, with $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$.

Required to prove: $\Delta ABC \,|||\, \Delta DEF$ $\left(\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}\right)$

Construction: On AB, mark off P such that AP = DE and on AC, mark off Q such that AQ = DF.
Join PQ.

Proof:

In ΔAPQ and ΔDEF:

1. $AP = DE$ (construction)
2. $\hat{A} = \hat{D}$ (given)
3. $AQ = DF$ (construction)

$\therefore \Delta APQ \equiv \Delta DEF$ (SAS)

$\therefore A\hat{P}Q = \hat{E}$ ($\equiv \Delta s$) and $\hat{E} = \hat{B}$ (given)

$\therefore A\hat{P}Q = \hat{B}$

$\therefore PQ \,||\, BC$ (corresponding $\angle s$ equal)

$\therefore \frac{AB}{AP} = \frac{AC}{AQ}$ (Prop theorem; $PQ \,||\, BC$)

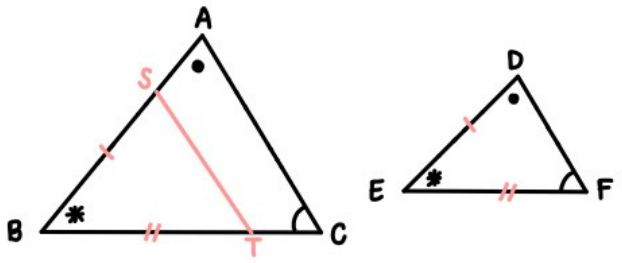
but $AP = DE$ and $AQ = DF$ (construction)

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

Similarly, by constructing S on AB such that SB = DE and T on BC such that BT = ET, we can prove that:

$\frac{AC}{DF} = \frac{BC}{EF}$

$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$



Only do the "similarly" part of this proof if the question requires you to prove that all three pairs of fractions are equal.

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HOW IS THIS SECTION TESTED?

Now that we've summarised everything you need to know for this section, we need to know how to apply all of this knowledge.

There are 4 question types that you can expect in your final exams:

1. A basic question involving finding the sizes of angles or lengths of sides using Euclidean geometry theorems.
2. A more complex question requiring the use of variables to prove tangents, diameters, cyclic quadrilaterals, etc.
3. A question that requires working with proportion.
4. A question that combines all of the above.

A method that works very well for all of these types of questions is to do a **pre-question analysis.**

<u>This requires reading through the given information very carefully.</u>

- At the end of each sentence/statement, fill in the information that you can **identify** from that statement, on your diagram.
- Then, using the other information on the diagram, see if you can fill in any other **angles** or **sides**.
- From here, make sure to fill in everything that you are **calculating** in each question on the diagram as well.
- The questions are usually **ordered** in a way so that you need to use the answer from the **previous question** in your solution to the following question.

Note that even if you are unable to prove what is asked in the question, you can still use that statement in the following question.

For example: if question 1 is asking you to prove that ABCD is a cyclic quadrilateral, but you are unable to do so, you are allowed to use the fact that ABCD is, in fact, a cyclic quadrilateral, in the next questions.

PRO-TIPS

When **marking** your final exam, the markers look at what you've filled in on your diagrams. They will sometimes **award marks** for values that you've shown there, even if you haven't managed to write those values down as part of your solution!

For your pre-question analysis, it's useful to think of the abbreviation <u>DR CPT</u>.

- Is there a **diameter**?
- Is there a **radius** or **radii** that create isosceles triangles?
- Are there any **cyclic quadrilaterals**?
- Are there any **parallel lines**?
- Are there any **tangents**?

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Question Type 1:
Calculating angles using Euclidean Geometry theorems

Worked example (R)

In the diagram, PQRS is a cyclic quadrilateral.

Chord RS is produced to T.

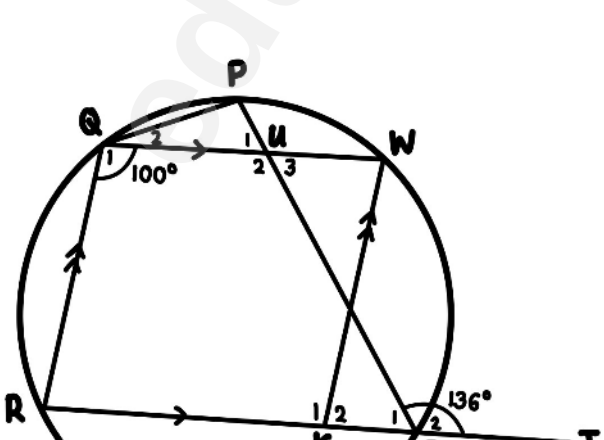
K is a point on RS and W is a point on the circle such that QRKW is a parallelogram.

PS and QW intersect at U.

$P\hat{S}T = 136°$ and $\hat{Q}_1 = 100°$

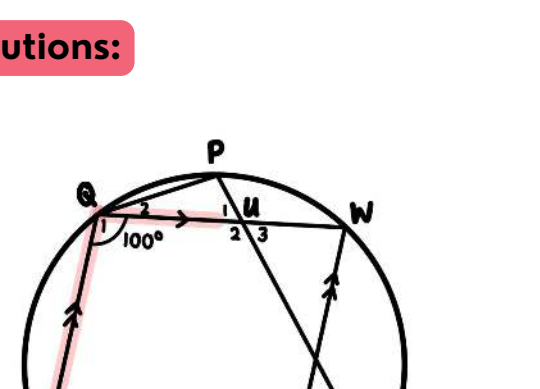
Determine, with reasons, the size of:

1. $\hat{R}$
2. $\hat{P}$
3. $P\hat{Q}W$
4. $\hat{U}_2$



Solutions:

1.

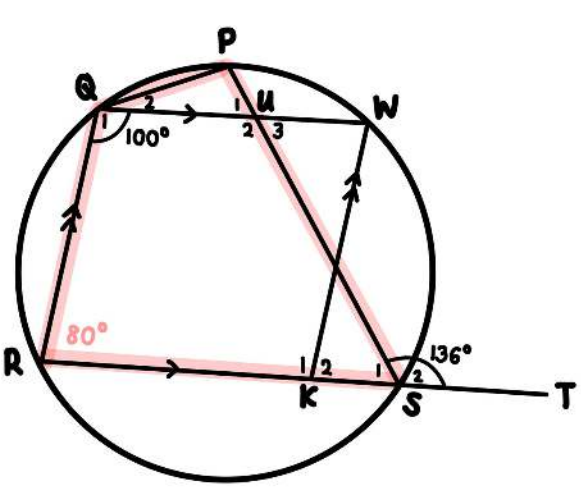


$\hat{Q}_1$ and $\hat{R}$ are co-interior angles that are formed by parallel lines ∴ they are supplementary (add up to 180°).

$\hat{R} = 80°$ (co-int. Ls; QW ∥ RK)

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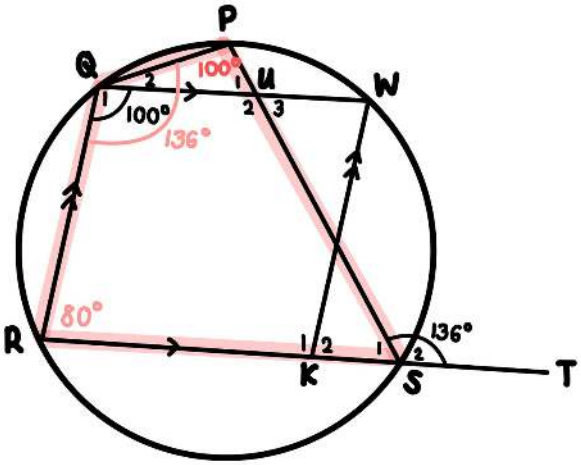
2.



PQRS is a cyclic quad because all 4 vertices lie on the circumference of the circle $\therefore \hat{R} + \hat{P} = 180°$

$\hat{P} = 100°$ (opp Ls of cyclic quad)

3.



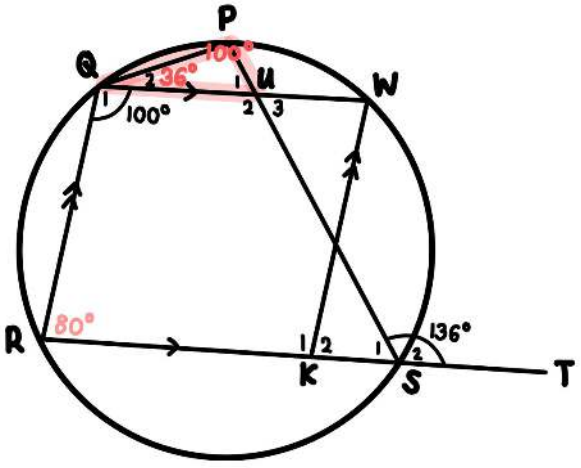
$\hat{S}_2$ is an exterior angle of cyclic quad PQRS and is therefore equal to $P\hat{Q}R$, the opposite interior angle.

$P\hat{Q}R = 136°$ (ext. L of cyclic quad)

$\therefore P\hat{Q}W = 136° - 100°$

$= 36°$

4.



$\hat{U}_2$ is an exterior angle of ΔPQU, so it is equal to the sum of the opposite interior angles.

$\hat{U}_2 = 100° + 36°$ (ext. L of Δ)

$= 136°$

Question source: NSC Grade 12 November 2019 Paper 2

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Question Type 2:
Complex questions that require working with variables

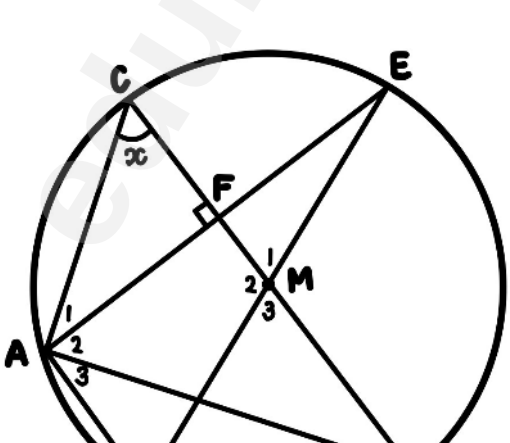
Worked example (C)

In the diagram, BE and CD are diameters of a circle having M as centre.

Chord AE is drawn to cut CD at F. AE ⊥ CD.

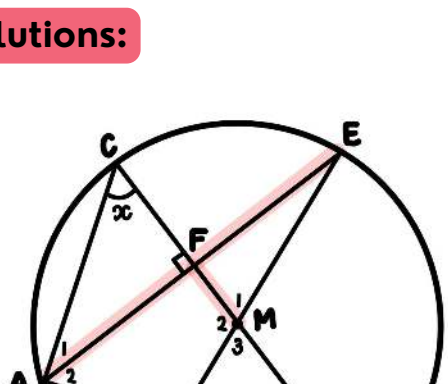
Let $\hat{C} = x$.

1. Give a reason why $AF = FE$.
2. Determine, giving reasons, the size of $\hat{M}_1$ in terms of x.
3. Prove, giving reasons, that AD is a tangent to the circle passing through A, C and F.
4. Given that CF = 6 units and AB = 24 units, calculate, giving reasons, the length of AE.



Solutions:

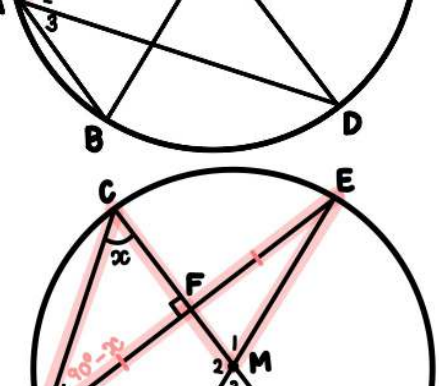
1.



It is given that MF ⊥ AE, so AF is equal to FE.

AF = FE (line from centre ⊥ to chord)

2.

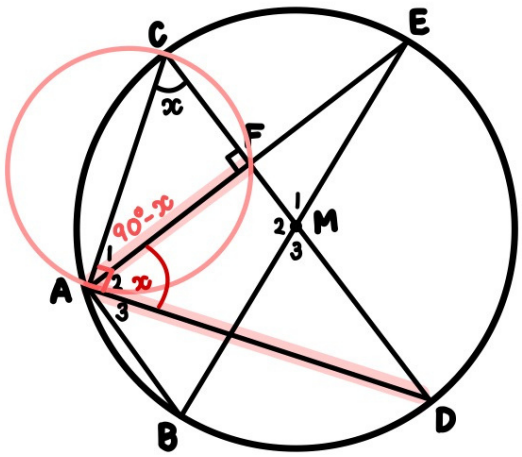


We can determine the size of $\hat{A}_1$ by subtracting $\hat{C}$ and $A\hat{F}C$ from 180°. $\hat{A}_1$ is subtended by chord CE at the circumference, and $\hat{M}_1$ is subtended by the same chord at the centre. This means that $\hat{M}_1$ is double the size of $\hat{A}_1$.

$\hat{A}_1 = 180° - 90° - x$ (∠ sum in △)
$= 90° - x$
$\therefore \hat{M}_1 = 2(90° - x)$ (∠ at centre = 2 × ∠ at circumference)
$= 180° - 2x$

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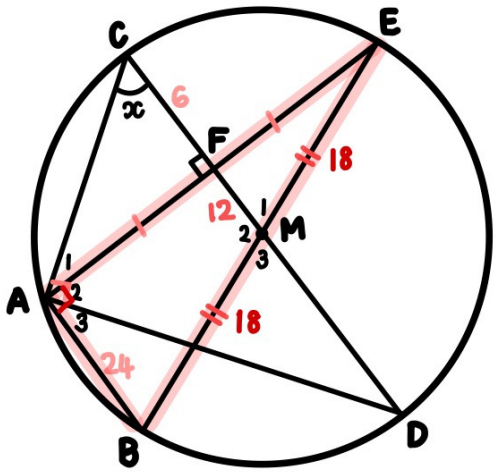
3.



Draw a circle through points A, C and F to get an idea of how to prove that AD is a tangent.
If AD is a tangent and AF is a chord of a circle, then $\hat{A}_2$ would be equal to $\hat{C}$, because $\hat{A}_2$ would be the angle between the tangent and the chord, and $\hat{C}$ would be subtended by chord AF.

$C\hat{A}D = 90°$ (∠ in semicircle)
$\therefore \hat{A}_2 = 90° - (90° - x)$
$= 90° - 90° + x$
$= x$
$\therefore \hat{A}_2 = \hat{C}$
∴ AD is a tangent to the circle passing through A, C and F. (converse tan chord theorem)

4.



F is the midpoint of AE and M is the midpoint of BE (BE is a diameter and M is the centre of the circle).

FM = 12 units (Midpoint theorem)
∴ CM = 18 units
∴ BE = 18 + 18 (radii)
= 36 units

Since ΔABE is right-angled and we have the lengths of two of the three sides of the triangle, we can use the theorem of Pythagoras to calculate AE.

$B\hat{A}E = 90°$ (∠ in semicircle)
$AE^2 = 36^2 - 24^2$ (Pythagoras)
$= 720$
$AE = 12\sqrt{5}$ units

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Question Type 3: Working with proportion

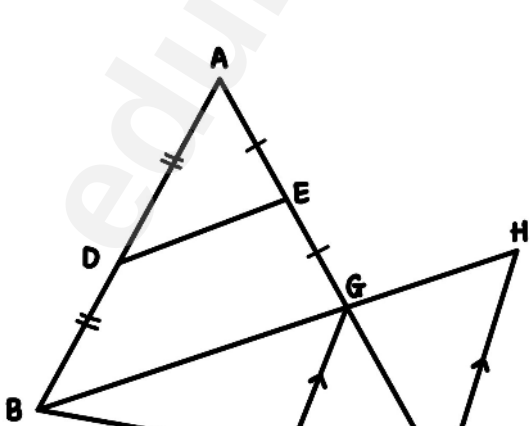
Worked example (R)

In the diagram, ΔABG is drawn.

D and E are midpoints of AB and AG respectively.

AG and BG are produced to C and H respectively.

F is a point on BC such that FG ∥ CH.

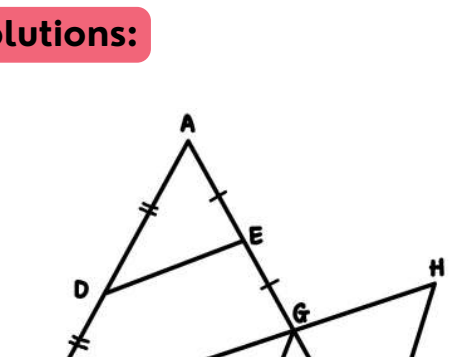


1. Give a reason why DE ∥ BH.

2. If it is further given that $\frac{FC}{BF} = \frac{1}{4}$,
DE = $3x - 1$ and GH = $x + 1$,
calculate, giving reasons, the value of x.

Solutions:

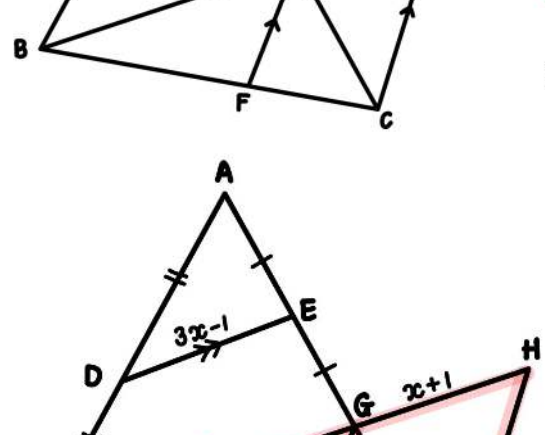
1.



D is the midpoint of AB and E is the midpoint of AG.
Midpoint theorem states that a line joining the midpoints of two sides of a triangle is parallel to the third side.

DE ∥ BH (Midpoint theorem)

2.



Midpoint theorem also states that the line joining the midpoints of two sides of a triangle is half the length of the third side.

BG = 2(3x-1) (Midpoint theorem)
= 6x-2

We can now use Prop theorem in ΔBCH to solve for x.

$\frac{BG}{GH} = \frac{BF}{FC}$ (Prop theorem; FG ∥ CH)

$\frac{6x-2}{x+1} = \frac{4}{1}$

$6x - 2 = 4(x+1)$
$6x - 2 = 4x + 4$
$2x = 6$
$x = 3$

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Question Type 4:
Combining angle and proportionality theorems

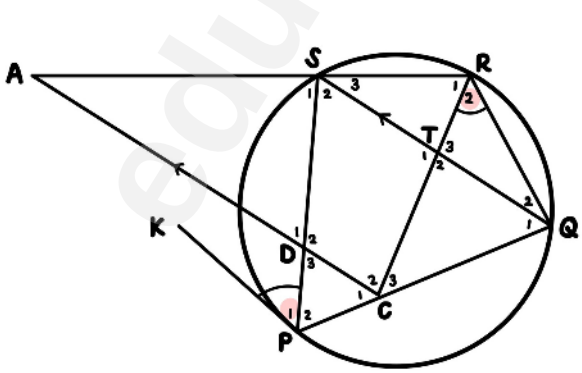
Worked example (C)

In the diagram, PQRS is a cyclic quadrilateral.

KP is a tangent to the circle at P.

C and D are points on chords PQ and PS respectively and CD produced meets RS produced at A.

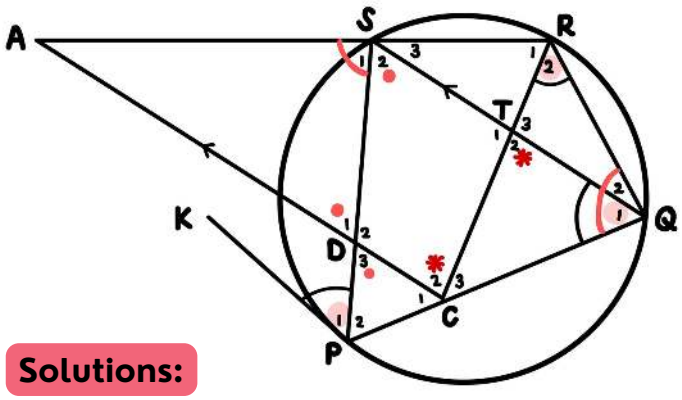
CA ∥ QS. RC is drawn. $\hat{P}_1 = \hat{R}_2$.



Prove, giving reasons, that:

1. $\hat{S}_1 = \hat{T}_2$
2. $\frac{AD}{AR} = \frac{AS}{AC}$
3. $AC \times SD = AR \times TC$

Pre-question analysis:

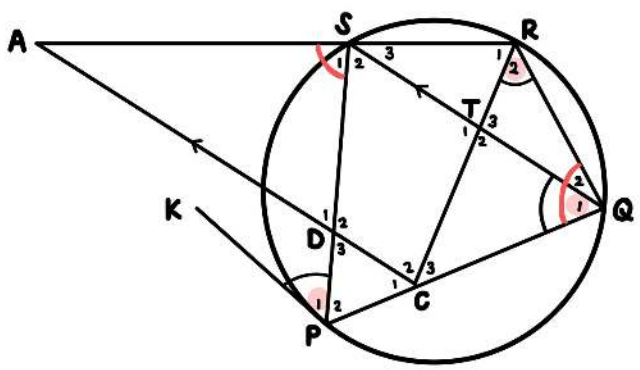


Upon reading through the paragraph, we can identify the following things:

- $\hat{S}_1 = \hat{Q}_1 + \hat{Q}_2$ (ext.∠ of cyclic quad)
- $\hat{P}_1 = \hat{Q}_1$ (tan chord theorem)
  ∴ $\hat{Q}_1$ is also equal to $\hat{R}_2$ (given that $\hat{P}_1 = \hat{R}_2$)
- $\hat{S}_2 = \hat{D}_1$ (alt.∠s; ST ∥ AD)
  and $\hat{S}_2 = \hat{D}_3$ (corresp.∠s; ST ∥ AD)
- $\hat{T}_2 = \hat{C}_2$ (alt.∠s; ST ∥ AD)

Solutions:

1.



First write an expression for $\hat{S}_1$.

$\hat{S}_1 = \hat{Q}_1 + \hat{Q}_2$ (ext.∠ of cyclic quad)
$\hat{P}_1 = \hat{Q}_1$ (tan chord theorem)
∴ $\hat{S}_1 = \hat{P}_1 + \hat{Q}_2$

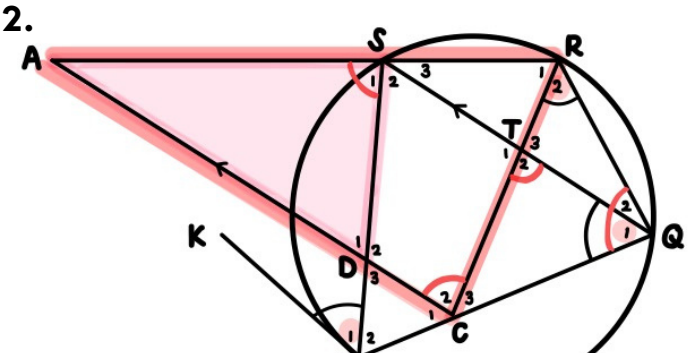
Now we need to show that $\hat{T}_2$ can also be written as $\hat{P}_1 + \hat{Q}_2$.

$\hat{T}_2 = \hat{R}_2 + \hat{Q}_2$ (ext.∠ of Δ)
$\hat{P}_1 = \hat{R}_2$ (given)
∴ $\hat{T}_2 = \hat{P}_1 + \hat{Q}_2$

∴ $\hat{S}_1 = \hat{T}_2$

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2.



RTP: $\frac{AD}{AR} = \frac{AS}{AC}$

All of the sides mentioned are sides of two triangles: ΔADS and ΔACR. We need to prove that these two triangles are similar so that we can state that the sides are in the same proportion.

List the angles in the two triangles so that it is easier to identify the angles that are equal.

ΔADS:	ΔARC:	
$\hat{A}$	$\hat{A}$	common
$\hat{D}_1$	$\hat{R}_1$	3rd pair of Ls
$\hat{S}_1$ (via $\hat{T}_2$)	$\hat{C}_2$	

Now write the full similarity proof with all reasons.

In ΔADS and ΔARC:

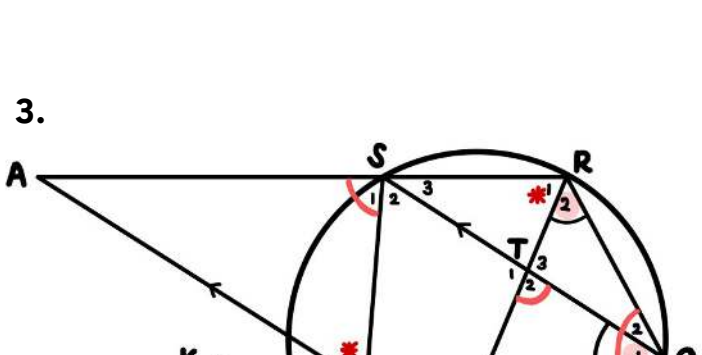
1. $\hat{A}$ is common.
2. $\hat{S}_1 = \hat{T}_2$ (proven above)
   $\hat{T}_2 = \hat{C}_2$ (alt. Ls; SQ ∥ AC)
   $\therefore \hat{S}_1 = \hat{C}_2$
3. $\hat{D}_1 = \hat{R}_1$ (∠ sum in Δ)

$\therefore$ ΔASD ||| ΔACR (equiangular)

$\therefore \frac{AS}{AC} = \frac{SD}{CR} = \frac{AD}{AR}$ (||| Δs)

You don't have to write this middle fraction.

3.



RTP: AC × SD = AR × TC

There are no two triangles which AC, SD, AR and TC are sides of. We can, however, use a combination of the fractions found in the previous question, as well as Prop theorem, to prove this statement.

$\frac{AS}{AC} = \frac{SD}{CR}$ (||| Δs) ...

and $\frac{AS}{AR} = \frac{CT}{CR}$ (Prop theorem; ST ∥ AC)

AS and CR are in both of these statements, but not in the statement that we have to prove.

$\therefore$ AS × CR = AC × SD

and AS × CR = AR × TC

$\therefore$ AC × SD = AR × TC since both sides are equal to AS × CR

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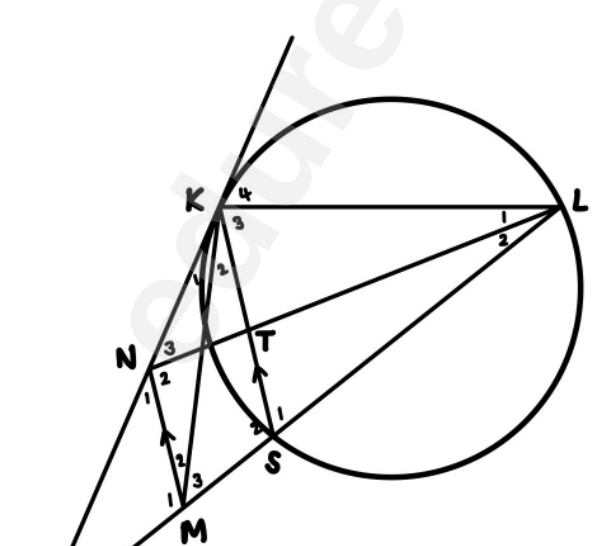
Worked example (C)

In the diagram, PK is a tangent to the circle at K.

Chord LS is produced to P.

N and M are points on KP and SP respectively such that MN ∥ SK.

Chords KS and LN intersect at T.



1. Prove, giving reasons, that:

   1.1. $\hat{K}_4 = N\hat{M}L$

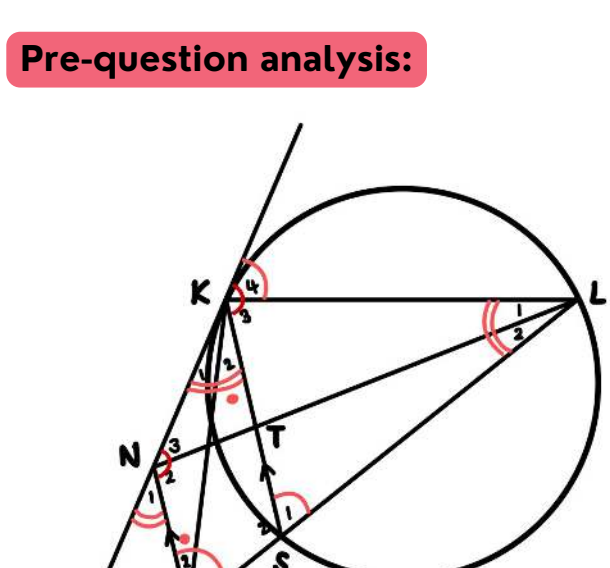
   1.2. KLMN is a cyclic quadrilateral

2. Prove, giving reasons, that $\Delta LKN \| \| \Delta KSM$.

3. If LK = 12 units and 3KN = 4SM, determine the length of KS.

4. If it is further given that NL = 16 units, LS = 13 units and KN = 8 units, determine, with reasons, the length of LT.

Pre-question analysis:



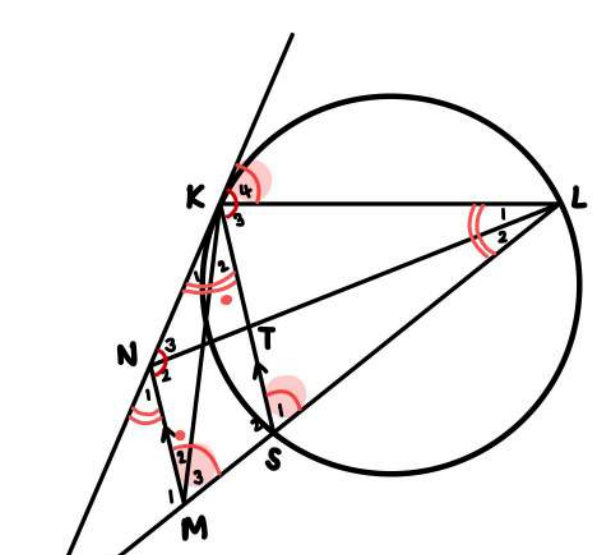
Upon reading through the paragraph, we can identify the following things:

- $\hat{K}_4 = \hat{S}_1$ (tan chord theorem)
- $\hat{K}_1 + \hat{K}_2 = \hat{L}_1 + \hat{L}_2$ (tan chord theorem)
- $\hat{S}_1 = \hat{M}_2 + \hat{M}_3$ (corresp. Ls; NM ∥ SK)
- $\hat{M}_2 = \hat{K}_2$ (alt. Ls; NM ∥ SK)
- $\hat{N}_1 = \hat{K}_1 + \hat{K}_2$ (corresp. Ls; NM ∥ SK)
- $\hat{K}_3 + \hat{K}_4 = \hat{N}_2 + \hat{N}_3$ (corresp. Ls; NM ∥ SK)

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Solutions:

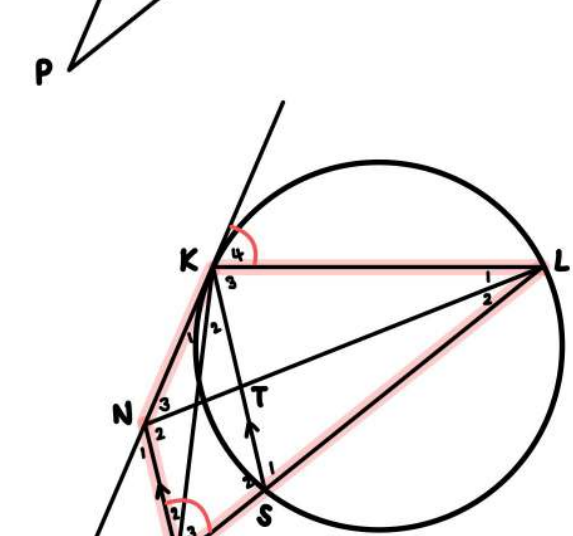
1.



We have already identified how these angles are equal to each other in our pre-question analysis.

$\hat{K}_4 = \hat{S}_1$ (tan chord theorem)
$\hat{S}_1 = N\hat{M}L$ (corresp. Ls; NM ∥ SK)
$\therefore \hat{K}_4 = N\hat{M}L$

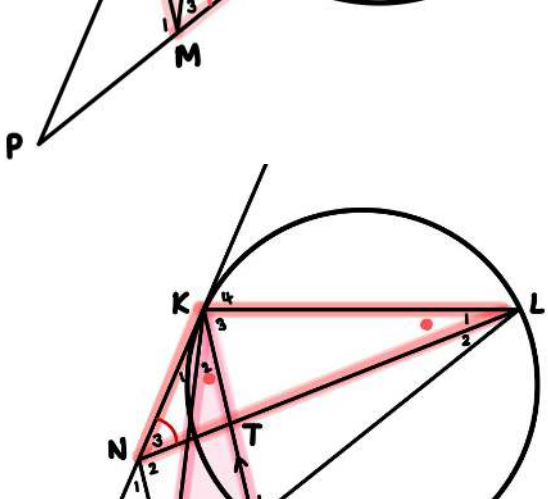
2.



In the previous question, we proved that $\hat{K}_4 = N\hat{M}L$.
$\hat{K}_4$ is an exterior angle of KLMN, and it is equal to the opposite interior angle, $N\hat{M}L$.

$\hat{K}_4 = N\hat{M}L$ (proven above)
$\therefore$ KLMN is a cyclic quad
(converse ext. L of cyclic quad)

3.



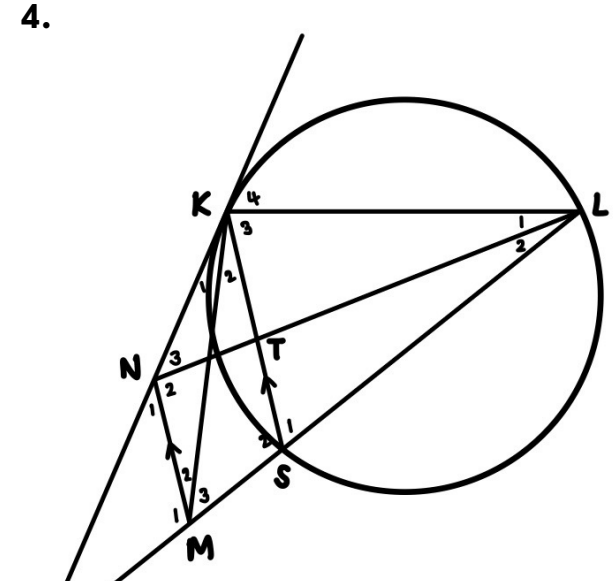
List the angles in the two triangles so that it is easier to identify the angles that are equal.

ΔLKN:		ΔKSM:	
$\hat{L}_1$	via $\hat{M}_2$	$\hat{K}_2$	
$L\hat{K}N$		$\hat{S}_2$	3rd pair of Ls
$\hat{N}_3$		$\hat{M}_3$	Ls in same segment (both subtended by chord KL)

In ΔLKN and ΔKSM:
1. $\hat{L}_1 = \hat{M}_2$ (Ls in same segment)
   $\hat{M}_2 = \hat{K}_2$ (alt. Ls; NM ∥ KS)
   $\therefore \hat{L}_1 = \hat{K}_2$
2. $\hat{N}_3 = \hat{M}_3$ (Ls in same segment)
3. $L\hat{K}N = \hat{S}_2$ (L sum in Δ)
$\therefore \Delta LKN \,|||\, \Delta KSM$ (equiangular)

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4.



Proven above:
$\Delta LKN \text{ III } \Delta KSM$
$\therefore \frac{LK}{KS} = \frac{KN}{SM} = \frac{LN}{KM}$ (III Δs)

Given: $3KN = 4SM$

$\frac{KN}{SM} = \frac{4}{3}$

Since $\frac{KN}{SM} = \frac{3}{4}$, and $\frac{LK}{KS} = \frac{KN}{SM}$, we can state the following:

$\frac{LK}{KS} = \frac{4}{3}$

Since we're solving for KS, it's easier to flip the fractions so that KS is the numerator.

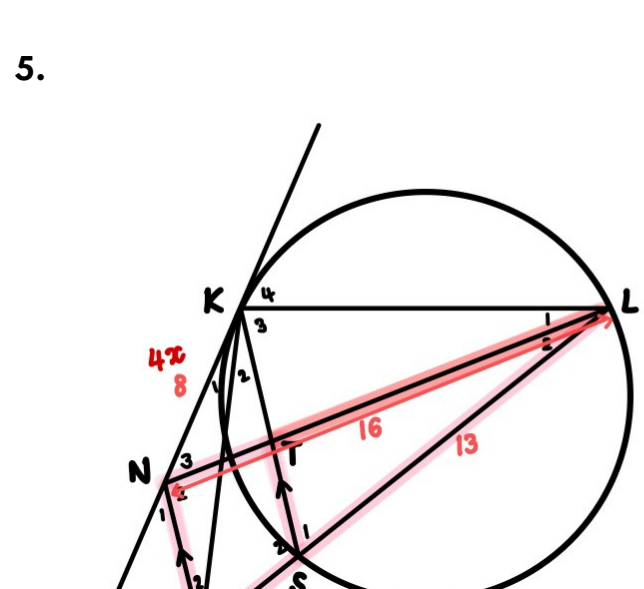
so $\frac{KS}{LK} = \frac{3}{4}$

$\frac{KS}{12} = \frac{3}{4}$

$KS = \frac{3}{4} \times 12$

$KS = 9$ units

5.



Since TS II NM, we could use Prop theorem to calculate LT if we have the length of SM.
We can use the ratio given in the previous question, as well as the length of KN, to calculate the length of SM.

$\frac{SM}{KN} = \frac{3}{4}$

$\frac{SM}{8} = \frac{3}{4}$

$SM = \frac{3}{4} \times 8$

$= 6$ units

$\therefore \frac{LT}{LN} = \frac{LS}{LM}$ (Prop theorem; TS II NM)

$\frac{LT}{16} = \frac{13}{13+6}$

$LT = \frac{13}{19} \times 16$

$LT = 10,95$ units

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INFORMATION SHEET

Familiarise yourself with the placement of the formulae that are given on the information/formula sheet by using this labelled formula sheet.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ quadratic formula

$A = P(1 + ni)$ simple growth

$A = P(1 - ni)$ simple decay/straight line depreciation

$A = P(1 - i)^n$ compound decay/reducing balance depreciation

$A = P(1 + i)^n$ compound growth

$T_n = a + (n-1)d$ arithmetic: term formula

$S_n = \frac{n}{2}[2a + (n-1)d]$ arithmetic: sum formula

$T_n = ar^{n-1}$ geometric: term formula

$S_n = \frac{a(r^n - 1)}{r - 1}$; $r \neq 1$ geometric: sum formula

$S_\infty = \frac{a}{1 - r}$; $-1 < r < 1$ sum to infinity

$F = \frac{x[(1+i)^n - 1]}{i}$ future value annuities

$P = \frac{x[1 - (1+i)^{-n}]}{i}$ present value annuities

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ derivative by first principles

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ distance formula

$M\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$ midpoint formula

$y = mx + c$ straight line equations

$y - y_1 = m(x - x_1)$ straight line equations

$m = \frac{y_2 - y_1}{x_2 - x_1}$ gradient formula

$m = \tan\theta$ angle of inclination

$(x-a)^2 + (y-b)^2 = r^2$ equation of a circle

In ΔABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ sine rule

$a^2 = b^2 + c^2 - 2bc.\cos A$ cosine rule

area $\Delta ABC = \frac{1}{2}ab.\sin C$ area rule

$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$
$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$
$\cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$
trig compound angle identities

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ trig double angle identities

$\bar{x} = \frac{\Sigma x}{n}$ mean of a set of data values

$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$ variance of a set of data values

$P(A) = \frac{n(A)}{n(S)}$ probability of event A

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ probability identity

$\hat{y} = a + bx$ equation of least squares regression line

$b = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2}$ gradient of least squares regression line

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