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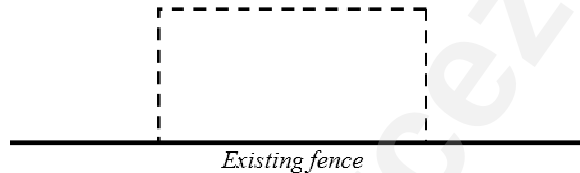
NoteSheet on Optimisation



Optimisation involves finding the optimum (best value) for a real world problem.

Consider the problem of a farmer who wishes to fence a **rectangular** field, using a straight existing fence for one side. He has 200m of fencing. What dimensions should he make the field in order that it will be as big as possible?

A good first step is to draw a picture. In the picture the fence the farmer will erect is shown with a dotted line and the existing fence is shown with a solid line.



A next useful step is to play with some numbers to check we understand that fields of different areas are possible.

Complete the following table of some possible field sizes:

(3)

The first row has been completed for you.

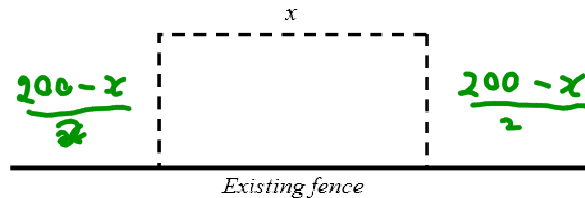
Remember that he only has 200m of fencing and that he only has to fence three sides (one parallel to the existing fence and two perpendicular)

Length of the side parallel to the existing fence	Length of each of the 2 sides which is perpendicular to the existing fence	Area of field ($length \times breadth$)
120m	40m	4800m ²
80m	60m	4800 m ²
110m	45	4950 m ²

Looking at the table above it is apparent that we can get different field sizes. We are interested in the optimum (in this case the biggest) possible field

Let's see if we can find this using calculus!

Suppose we let the side parallel to the river be x then our sketch will look like this:



To find the area of this field we need to know what each of the sides perpendicular to the river would be:

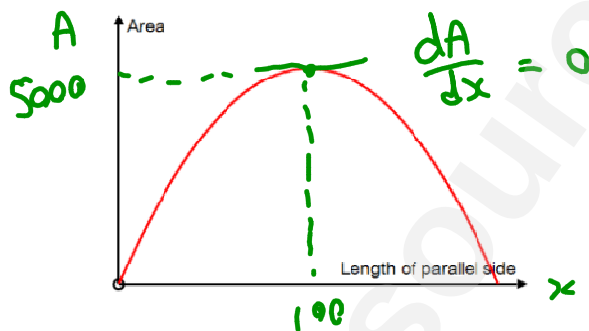
So far we have used x meters so we have $200 - x$ left. (1)

However, this needs to fence two equal sides so each side will be $\frac{200-x}{2}$ (1)

And the area of our field will be

$$A = \left(\frac{200-x}{2} \right) x$$

If we draw a graph of the area against the length of the parallel side it looks like this:



To find the optimum (in this case the biggest) area we obviously wish to find the point at which the curve turns. We do this using calculus!

$$A = \left(\frac{200-x}{2} \right) x$$

$$A = \frac{200x - x^2}{2}$$

$$A = \frac{200x}{2} - \frac{1}{2}x^2$$

$$A = 100x - \frac{1}{2}x^2$$

$$\frac{dA}{dx} = 100 - x = 0$$

$$x = 100$$

Biggest field =
100m x 50m

$$\underline{5000 \text{ m}^2}$$

Differentiate the expression for area (call it $\frac{dA}{dx}$) and equate it to zero (the value of the gradient where the graph turns!) (2)

See prev. pg.

Solve for x

See prev. pg.

You should get:

Each of the perpendicular sides would be 50 (1)

This means that the field will have an optimum (in this case biggest) area of 5000m² (1)

when the dimensions are 100 x 50 (2)

In summary:

We solve an optimisation problem as follows:

- 1) Make sure that you understand the problem. A sketch is often useful.
- 2) Introduce a variable (suppose it is x) and get the other variables in terms of it.
- 3) Determine an expression for the commodity that we are attempting to optimize. Suppose it is M
- 4) Differentiate the expression $\frac{dM}{dx}$ with respect to the relevant variable.
- 5) Equate the derivative to zero and solve for the variable.
- 6) Determine whether we have found a maximum or a minimum. We use the second derivative for this!
- 7) Calculate the actual optimum if asked for. This is done by substituting back into the original equation.

Let's try some more examples:

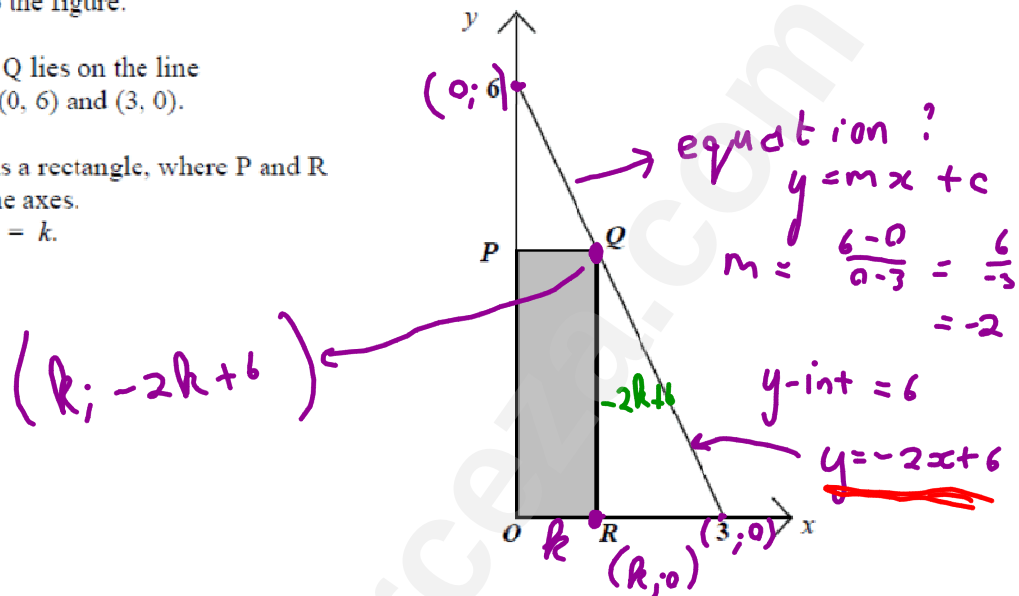
Example 1

Refer to the figure.

A point Q lies on the line joining (0, 6) and (3, 0).

OPQR is a rectangle, where P and R lie on the axes.

Let $OR = k$.



- ✓ (1) Write an expression for QR in terms of k . (2)
- (2) Find the coordinates of Q for which the rectangle has a maximum area. (5)
- (3) Hence determine this maximum. (2)

$$A = k(-2k+6)$$

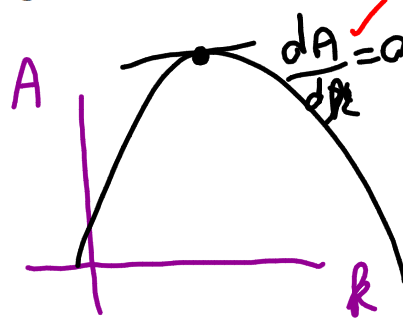
$$A = -2k^2 + 6k$$

$$\frac{dA}{dk} = -4k + 6 = 0$$

$$-4k = -6$$

$$k = \frac{6}{4} \text{ or } 1.5$$

$$Q(1.5; 3)$$



$$\text{Area}_{\max} = (1.5)(3) = \underline{4.5 \text{ units}^2}$$

Example 2

A garden has 200 kg of watermelons growing in it. Every day, the total mass of watermelon increases by 5 kg. However, every day the price per kg of watermelon goes down by 1c.

- (1) If the current price is 90c per kg then what is the crop worth if it is harvested today?

$$90 \times 200$$

$$\underline{\underline{R\ 180,00}}$$

- (2) What will I earn if I sell tomorrow?

$$89 \times 205$$

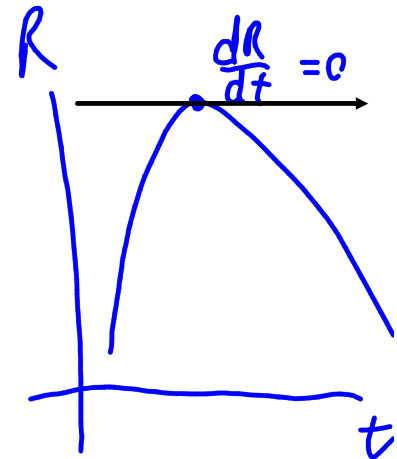
$$\underline{\underline{R\ 182,45}}$$

(3) What will the crop be worth in t days?

revenue $\swarrow$

$$R = (200 + 5t)(90 - t)$$

$$\underline{\underline{R = 18000 + 250t - 5t^2}}$$



- (4) How much longer should the watermelons be left to grow in order to maximize the income?

$$\frac{dR}{dt} = 250 - 10t = 0$$

$$250 = 10t$$

$$\underline{\underline{25 = t}}$$

25 days

(5) What is max income?

$$18000 + 250(25) - 5(25)^2$$

$$\underline{\underline{= R\ 211,25}}$$

Example 3

The sum of 2 positive numbers is 12. Find the numbers when their product is a maximum.

one is 3 other is 9 Product = 27
 one is 5 other is 7 Product = 35
 one is x other is $12-x$ Product = $x(12-x)$

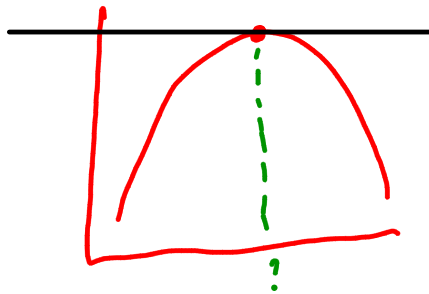
$$P = 12x - x^2$$

$$\frac{dP}{dx} = 12 - 2x = 0$$

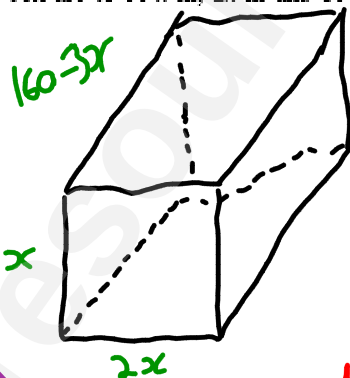
$$12 = 2x$$

$$x = 6$$

Numbers are 6 and 6

**Example 4**

The edges of a rectangular box are to be x m, $2x$ m and $160-3x$ m respectively. Determine the size of x to maximise the volume.



$$V = x(2x)(160-3x)$$

$$V = 2x^2(160-3x)$$

$$V = 320x^2 - 6x^3$$

$$\frac{dV}{dx} = 640x - 18x^2 = 0$$

$$x(640 - 18x) = 0$$

$$x = 0 \quad \text{or} \quad 640 - 18x = 0$$

$$640 = 18x$$

$$x = 35,6 \text{ m}$$

$$\frac{d^2V}{dx^2} = 640 - 36x$$

$$\text{if } x = 0, \frac{d^2V}{dx^2} > 0 \therefore \cup$$

$$\text{if } x = 35, \frac{d^2V}{dx^2} < 0$$

$\therefore \cap$

$$\frac{x}{1} \times \frac{9}{x} = 9$$

Example 5

The area of a rectangular rug is to be 9m^2 . Determine the dimensions which minimize the perimeter.

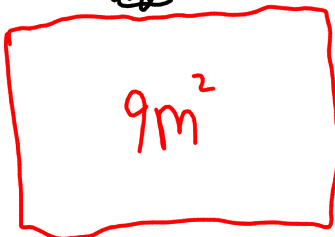


Diagram of a rectangle with area 9m^2 and sides x and $\frac{9}{x}$.

Handwritten notes:

$$\frac{dP}{dx} = 2 - 18x^{-2} = 0$$

$$2 - \frac{18}{x^2} = 0 \rightarrow 2x^2 - 18 = 0$$

$$2x^2 = 18 \rightarrow x^2 = 9 \rightarrow x = 3$$

Perimeter calculations:

$$P = x + x + \frac{9}{x} + \frac{9}{x}$$

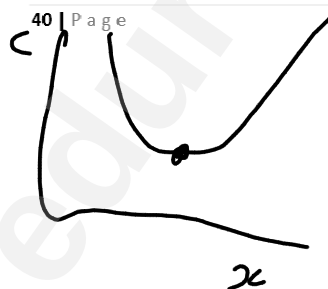
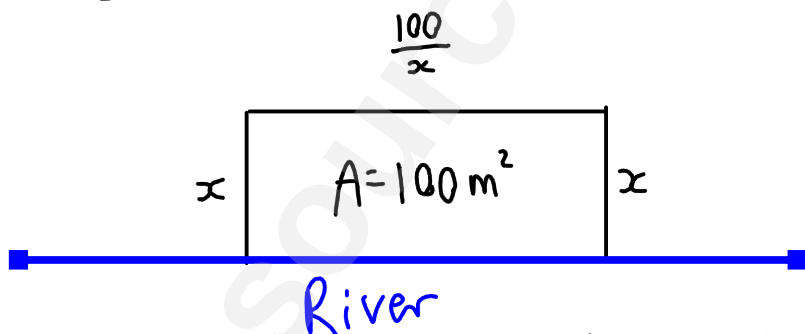
$$P = 2x + \frac{18}{x}$$

$$P = 2x + 18x^{-1}$$

Result: $x = 3$

Example 6

A rectangular paddock is to be fenced in on 3 sides, the fourth side being formed by a river. The cost of fencing at right angles to the river is R20 per metre, while that parallel to the river costs R15 per metre. If the paddock is to be 100m^2 , find its dimensions so that the cost of fencing will be a minimum.



$$C = 20x + 20x + \frac{15}{1} \left(\frac{100}{x} \right)$$

$$C = 40x + \frac{1500}{x}$$

$$C = 40x + 1500x^{-1}$$

$$\frac{dC}{dx} = 40 - 1500x^{-2} = 0$$

$$40 - \frac{1500}{x^2} = 0$$

$$40x^2 - 1500 = 0$$

$$40x^2 = 1500$$

$$x^2 = \frac{1500}{40}$$

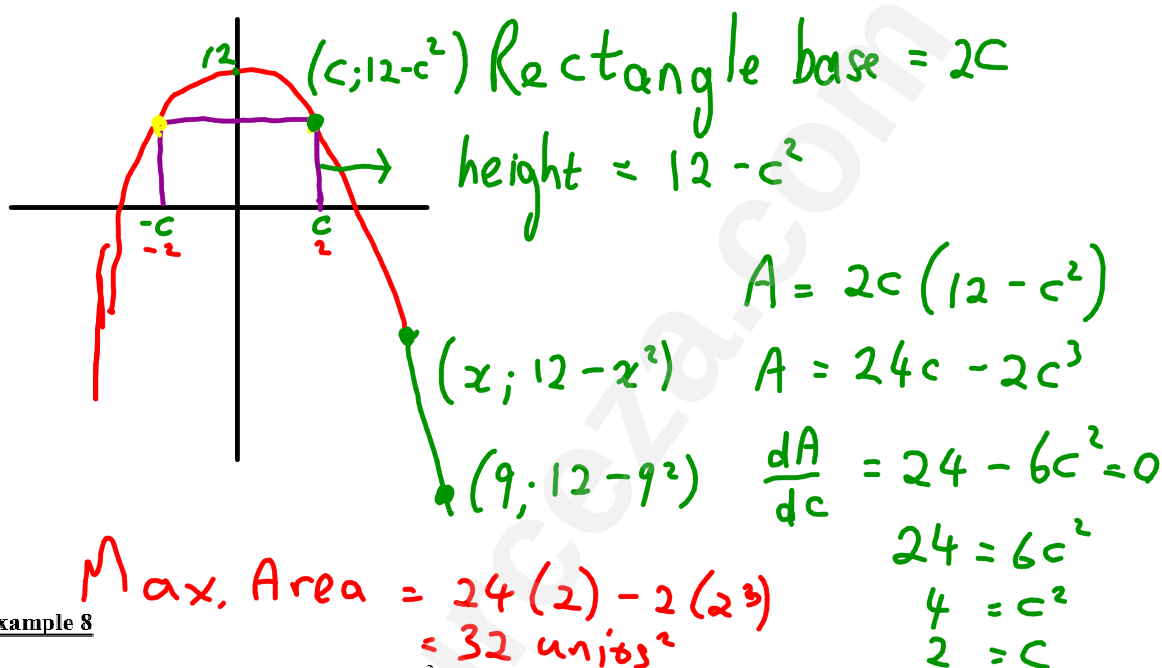
$$x = 6.1$$

$$6.1 \times 16.4$$

are best dimensions

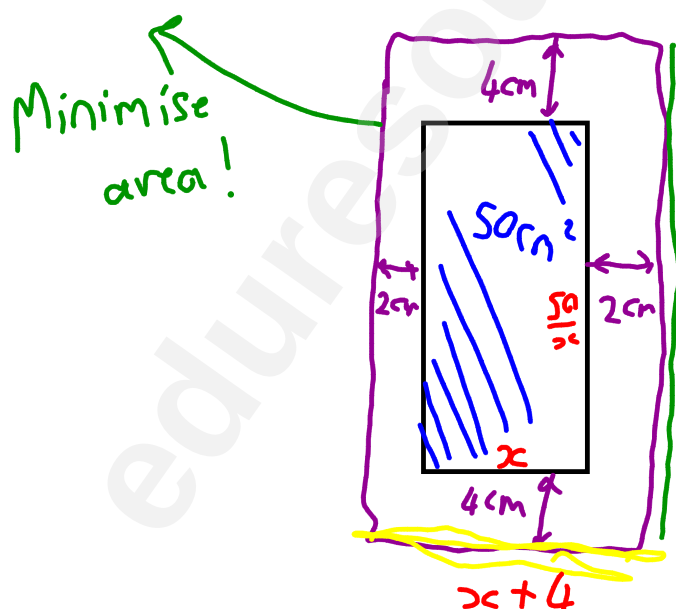
Example 7

A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area it can have?



Example 8

You are designing a poster to contain 50 cm^2 of print with margins of 4 cm each at top and bottom and 2 cm at each side. What overall dimensions will minimise the amount of paper used.



$$A = (x+4)\left(\frac{50}{x} + 8\right)$$

$$A = 50 + 8x + \frac{200}{x} + 32$$

$$\frac{50}{x} + 8 \quad A = 82 + 8x + 200x^{-1}$$

$$\frac{dA}{dx} = 8 - 200x^{-2} = 0$$

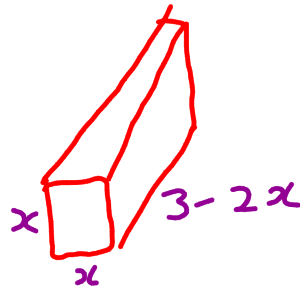
$$8 - \frac{200}{x^2} = 0$$

$$8x^2 = 200 \quad x^2 = 25$$

$$x = 5$$

Example 9

The post office will accept a box for posting only if the sum of the length and the girth (distance around) does not exceed 3m. Find the dimensions of the largest acceptable box with a square end.



$$L + B + H = 3m$$

$$x + x + ? = 3$$

$$V = x \cdot x \cdot (3 - 2x)$$

$$V = 3x^2 - 2x^3$$

$$\frac{dV}{dx} = 6x - 6x^2 = 0$$

$$x - x^2 = 0$$

$$x(1 - x) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

Min

Max

Biggest possible box = $1 \times 1 \times 1$
(cube)