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Calculus

Calculus – Solving a problem

In this lesson we will be using all of the knowledge and skills that we have developed over the previous eight lessons on Calculus to solve a problem.

Lesson Outcomes

By the end of this lesson, you should be able to:

- interpret the derivative as a measure of the rate of change of a function

Curriculum Links

LO 2: Functions and Algebra

- 12.2.7 (d) Generate sketch graphs of cubic functions using differentiation to determine the stationary points (maxima, minima and points of inflection) and the factor theorem and other techniques to determine the intercepts with the x -axis.
- (e) Solve practical problems involving optimisation and rates of change

Lesson Notes

The problem that we are going to consider involves predicting the population of a bug. The problem is as follows:

The population size (in 100s of bugs per square metre) of a particular bug is given by:

$$f(x) = \frac{x^3 - 18x^2 + 60x + 592}{8}; 0 \leq x \leq 14,5$$

Where x represents the time in months.

Questions:

- After how many months does the model predict that the population will be at its maximum?
- After how many months does it predict that the population be at its minimum?
- What are the maximum and minimum values?

In order to find out after how many months the population reaches its maximum and minimum values, we have to find the turning points of the graph. We do this by determining when the gradient of the tangent to the curve is zero. In other words we need to find the derivative of the function. This is shown below:

$$\begin{aligned} f(x) &= \frac{x^3 - 18x^2 + 60x + 592}{8} \\ &= \frac{x^3}{8} - \frac{18x^2}{8} + \frac{60x}{8} + \frac{592}{8} \\ f'(x) &= \frac{3 \times x^{3-1}}{8} - \frac{2 \times 18x^{2-1}}{8} + \frac{1 \times 60x^{1-1}}{8} + 0 \\ &= \frac{3 \times x^2}{8} - \frac{9x}{2} + \frac{15}{2} \end{aligned}$$

At the turning (stationary) points $f'(x) = 0$

$$\therefore \frac{3 \times x^2}{8} - \frac{9x}{2} + \frac{15}{2} = 0$$

$$(\times 8) \quad 3x^2 - 36x + 60 = 0$$

$$(\div 3) \quad x^2 - 12x + 20 = 0$$

$$(x - 2)(x - 10) = 0$$

$$\Rightarrow (x - 2) = 0 \text{ or } (x - 10) = 0$$

$$x = 2 \text{ or } x = 10$$

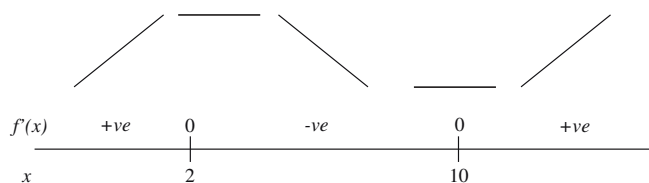
To determine which is the maximum and which is the minimum we could substitute the x -values of the stationary points into the original function as shown below:

$$\begin{aligned} f(2) &= \frac{2^3 - 18 \times 2^2 + 60 \times 2 + 592}{8} \\ &= \frac{8 - 72 + 120 + 592}{8} \\ &= \frac{648}{8} \\ &= 81 \Rightarrow (2; 81) \end{aligned}$$

$$\begin{aligned} f(10) &= \frac{10^3 - 18 \times 10^2 + 60 \times 10 + 592}{8} \\ &= \frac{1000 - 1800 + 600 + 592}{8} \\ &= \frac{392}{8} \\ &= 49 \Rightarrow (10; 49) \end{aligned}$$

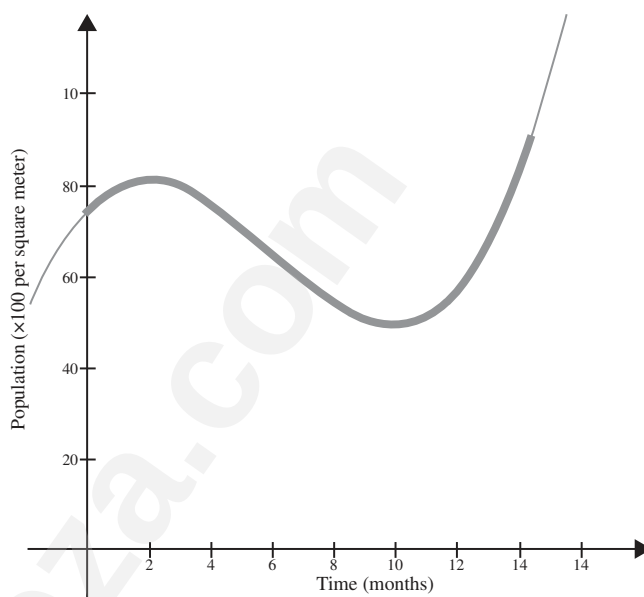
Calculus

Another method of determining which x -value gives the maximum and which one gives the minimum is to draw a sketch like the one below:



The sketch shows that if the gradient (derivative) is positive then the function is increasing and if the gradient (derivative) is negative then the function is decreasing. By drawing in the gradient lines you can see that a local maximum will occur at 2 months and a local minimum will occur at 10 months.

When we look at the actual graph of the function (shown below) we notice that the graph has a turning point at (2; 81) and that this turning point is a local maximum – that is, the population at this turning point is greater than the population at the turning point (10; 49). However, we also notice from the graph that in the interval $0 \leq \text{time} \leq 14,5$ months the population gets larger than it does at the turning point. For 14,5 months the population will be $f(14,5) \approx 90,77$ – this is the largest that the population becomes in the interval $0 \leq \text{time} \leq 14,5$. We refer to the population value at 14,5 months as the absolute maximum population value for the given interval and we refer to the population value at 2 months as the local maximum. The important message is that we should always investigate the end points of the domain for which the function that we are studying is defined to ensure that the maximum/minimum turning point (or local maximum/minimum) is indeed the optimal value for our problem.



Task

The population size (in 100s of bugs per square metre) of a particular bug is given by:

$$f(x) = -x^3 + 9x^2 - 15x + 17; 0 \leq x \leq 6$$

Where x represents the time in months.

After how many months does the model predict that the population will be at its maximum and after how many months does it predict that the population be at its minimum?

What are the maximum and minimum values?

