



eduresourceza.com

Calculus

Calculus – Sketching the graph of a cubic function

In this lesson we are going to demonstrate one of the applications of differential calculus by answering a typical question that relies on Calculus. We are going to sketch a graph of a cubic function and we are going to determine the equation of a tangent to the graph at some point on the graph.

Lesson Outcomes

By the end of this lesson, you should be able to:

- sketch the graphs of a cubic function
- determine the equation of a tangent to a graph.

Curriculum Links

LO 2: Functions and Algebra

12.2.7 (d) Generate sketch graphs of cubic functions using differentiation to determine the stationary points (maxima, minima and points of inflection) and the factor theorem and other techniques to determine the intercepts with the x -axis.

Lesson Notes

The question addressed by this lesson is:
Sketch the graph of the function:

- $f(x) = -x^3 + 14x^2 - 49x + 36$; and
- Determine the equation of the tangent to $f(x)$ at the point $(3; f(3))$

In order to draw the graph of the cubic function there are three sets of critical values that we need to determine. We want to know:

- the co-ordinates of the turning points of the graph, if they exist;
- the intersections with the x -axis; and
- the intersection with the y axis.

Determining the turning points of the graph:

Given: $f(x) = -x^3 + 14x^2 - 49x + 36$

Determine the derivative:

$$\begin{aligned} f'(x) &= -3x^{3-1} + 14 \times 2x^{2-1} - 49 \times 1x^{1-1} \\ &= -3x^2 + 28x - 49 \end{aligned}$$

Stationary points

At the stationary points: $f'(x) = 0$

$$-3x^2 + 28x - 49 = 0$$

$$3x^2 - 28x + 49 = 0 \quad (\div -1)$$

$$(3x - 7)(x - 7) = 0$$

$$\text{either } 3x - 7 = 0 \quad \text{or} \quad x - 7 = 0$$

$$3x = 7$$

$$x = 7$$

$$x = \frac{7}{3}$$

y -values of stationary points:

$$\begin{aligned} x = \frac{7}{3} \quad f\left(\frac{7}{3}\right) &= -\left(\frac{7}{3}\right)^3 + 14\left(\frac{7}{3}\right)^2 - 49 \times \frac{7}{3} + 36 \\ &\approx -14,8 \end{aligned}$$

$$\begin{aligned} x = 7 \quad f(7) &= -(7)^3 + 14(7)^2 - 49 \times 7 + 36 \\ &= 36 \end{aligned}$$

Stationary points are:

$$\left(\frac{7}{3}; -14,8\right) \text{ and } (7; 36)$$

Determining the points of intersection with the axes:

y -intercept:

The y -intercept is given by $(0; f(0))$

$$f(0) = -0^3 + 14 \times 0^2 - 49 \times 0 + 36 = 36$$

$\therefore y$ -intercept: $(0; 36)$

Calculus

x-intercept(s):

The x -values of the x -intercept(s) are given by

$$f(x) = 0$$

i.e. we must solve $-x^3 + 14x^2 - 49x + 36 = 0$

$$x^3 - 14x^2 + 49x - 36 = 0 \quad (\div -1)$$

$$\text{since } f(1) = -1^3 + 14 \times 1^2 - 49 \times 1 + 36$$

$$= -1 + 14 - 49 + 36$$

$$= 0$$

$(x - 1)$ is a factor of $f(x)$

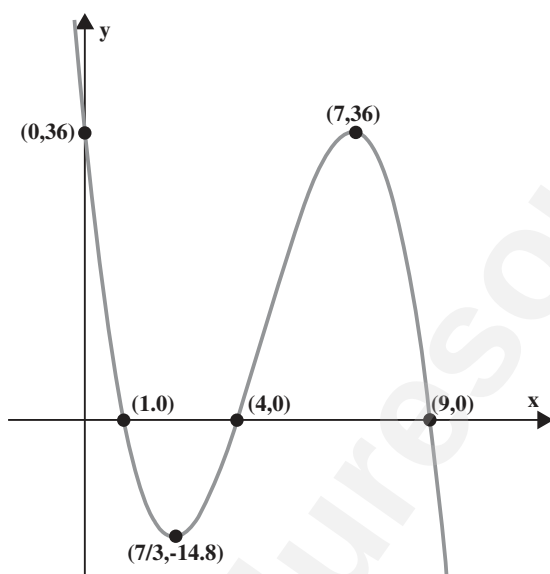
$$\text{it follows that } x^3 - 14x^2 + 49x - 36 = 0$$

$$(x - 1)(x^2 - 13x + 36) = 0$$

$$(x - 1)(x - 4)(x - 9) = 0$$

$\therefore$ x -intercepts: $(1;0);(4;0);(9;0)$

Next we can draw the graph of the function.



The second part to this question was to determine the equation of the tangent to $f(x)$ at the point $(3; f(3))$.

To determine the equation of a straight line (tangent) we need at least one point on the line and the gradient of the line.

A point:

x -co-ordinate: $x = 3$

$$\begin{aligned} y\text{-co-ordinate: } f(3) &= -3^3 + 14 \times 3^2 - 49 \times 3 + 36 \\ &= -27 + 126 - 147 + 36 \\ &= -12 \end{aligned}$$

$\therefore$ point: $(3, -12)$

The gradient using calculus:

$$f'(x) = -3x^2 + 28x - 49$$

$$f'(3) = -3 \times 3^2 + 28 \times 3 - 49$$

$$= -27 + 84 - 49$$

$$= 8$$

The line (tangent): $y = g(x) = 8x - 36$:

This line is a tangent at the point $(3; -12)$ but will cut the curve at another point. You can determine the point of intersection with the curve as follows:

$$\text{Curve: } f(x) = -x^3 + 14x^2 - 49x + 36$$

$$\text{Tangent: } g(x) = 8x - 36$$

At intersection:

$$g(x) = f(x)$$

$$8x - 36 = -x^3 + 14x^2 - 49x + 36$$

$$x^3 - 14x^2 + 57x - 72 = 0$$

$$(x - 3)(x - 3)(x - 8) = 0$$

$$\therefore x = 3 \text{ or } x = 8$$

Points of intersection:

$$x = 3: (3; -12)$$

$$\begin{aligned} x = 8: f(8) &= -8^3 + 14 \times 8^2 - 49 \times 8 + 36 = 20 \\ &(8; 20) \end{aligned}$$

Task

Draw sketch graphs of the following functions:

1. $f(x) = -x^3 + 4x^2 - 4x$ and determine the equation of the tangent to the curve at $x = 3$.
2. $f(x) = x^3 - 3x - 2$
3. $f(x) = x^3 - 9x^2 + 24x - 20$ and determine the equation of the tangent to the curve at $x = 1$
4. $f(x) = (x - 2)^3 + 8$

