



GRADE 12



CALCULUS

It is the branch of mathematics that is concerned with limits and with the differentiation and integration of functions. The two basic forms of calculus are differential calculus and integral calculus. In Grade 12 we focus on differential calculus.

DIFFERENTIAL CALCULUS

It is a description, in a precise manner, of the ways in which related quantities change. It is about finding the slope (gradient) of a tangent to the graph of a function, or finding the rate of change of one quantity with respect to another quantity.

Differential calculus is the collective study of all forms of differentiation. A derivative is obtained through the process of differentiation.

CONCEPTS TO BE TAUGHT

1. LIMIT

2. DERIVATIVE

- First principle
- Rules for differentiation

3. TANGENTS TO THE CURVE

- Equation of a tangent

4. CUBIC FUNCTIONS

- Sketching
- Finding the equation of the graph
- Inflection



- Concavity
- Increasing & Decreasing

5. APPLICATION OF CALCULUS

- Rates of change
- The calculus of motion

CONCEPTS FROM EARLIER GRADES

1. Algebraic Expressions

Factorization by:

- i. Taking out the common factor
- ii. Factorizing a trinomial of a second degree
- iii. Difference of two squares
- iv. The sum and difference of two cubes

Classroom Exercise 1

Factorise the following expressions

- (a) $x^2 + 5x - 24$
- (b) $3a^3 - 9a^2 + 6a$
- (c) $9x^2 - y^2$
- (d) $x^3 + 8$
- (e) $x^3 - 27$

2. Surds and Exponents

- i. Laws of exponents
- ii. Converting surds to exponential form
- iii. Algebraic Manipulation

Classroom Exercise 2

1. Express the following terms in exponential form

- a) $\sqrt{x}$
- b) $\sqrt[3]{x^2}$
- c) $\sqrt[5]{p^3}$

2. Simplify the following expressions

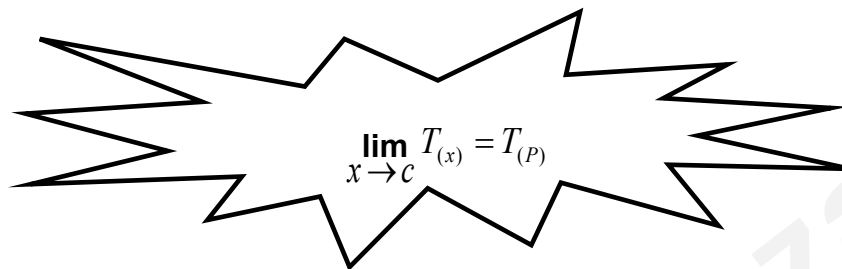
- a) $(x - 3)(2x + 5)$
- b) $(3p - 5)^2$
- c) $(1 - \sqrt{x})(1 - 2\sqrt{x})$
- d) $\frac{p^3+1}{p+1}$

3. Given: $f(x) = 2x + 5$, Determine the following

- (a) $f(2)$
- (b) $f(-1)$
- (c) $f(x + 2)$
- (d) $f(a + h)$

1. LIMIT

The limit of a polynomial functions, $T_{(x)}$ approaches p is $T_{(p)}$. It is expressed as:

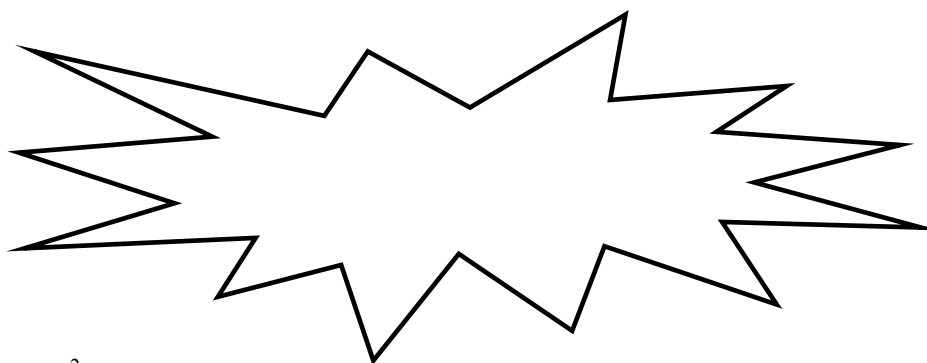


$$\lim_{x \rightarrow c} T_{(x)} = T_{(p)}$$

Limit: What happens to a function $f(x)$ when x tends to a specific value. A specific value can be approached from both sides so we have a limit from left and a limit from the right

THEOREMS ON LIMIT

1. $\lim_{x \rightarrow c} k = k$
i.e. the limit of a constant is that constant
2. $\lim_{x \rightarrow c} x = c$
i.e. where there is x we put c
3. $\lim_{x \rightarrow c} x^n = c^n$



Example: $\lim_{x \rightarrow 2} x^2 + 3x = 2^2 + 3(2) = 10$

 CLASSROOM EXERCISE 1

Evaluate the following limits (if they exist)

a) $\lim_{x \rightarrow 1} 8x$

b) $\lim_{x \rightarrow 3} (x^2 + 4x - 5)$

c) $\lim_{x \rightarrow 1} \frac{x-2}{x+2}$

d) $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$

e) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{2x + 4}$

Gradient (Slope) of a curve at a point

The gradient of a point on a curve is given by: $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

Example: Find the gradient of $f(x) = x^2$ at the point where $x = 3$

$$\text{gradient} = \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \rightarrow 0} 6 + h = 6 + 0 = 6$$

 CLASSROOM EXERCISE 2

Find the gradient at the given point

- a) $f(x) = 3x^2$ at $x = -1$
- b) $f(x) = 2x - 1$ at $(3;9)$
- c) $f(x) = 2x^2 + 1$ at $x = 2$

2. DERIVATIVE OF A FUNCTION

The slope of a tangent at the point $[x; f(x)]$ on the graph of f is called the derivative of f at x and is written as $f'(x)$.

The derivative of a function at x is given by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(Provided the limit exist)

It is called the first principle

TIPS ON HOW TO APPLY THE FIRST PRINCIPLE

- ✓ Find $f(x+h)$ separately
 - ✓ Write down the formula
- $$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
- ✓ Substitute $f(x)$ and $f(x+h)$ into the formula
 - ✓ Be careful with the exponent of $(x+h)$ as well as the signs
 - ✓ Simplify and remember to cancel out h from the denominator
 - ✓ Substitute h by zero (at this stage we no longer write $\lim_{h \rightarrow 0}$)

Example: Find $f'(x)$ from first principle if:

$$f(x) = 2x^2$$

$$f(x+h) = 2(x+h)^2 = 2x^2 + 4xh + 2h^2$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

$$f'(x) = \lim_{h \rightarrow 0} 4x + 2h$$

$$\therefore f'(x) = 4x + 2(0) = 4x$$

CLASSROOM EXERCISE 3

Determine $f'(x)$ from the first principle if:

a) $f(x) = x^2$

b) $f(x) = x^2 - 2$

c) $f(x) = 2x^2$

d) $f(x) = \frac{1}{x}$

e) $f(x) = x^2 + 3x$

f) $f(x) = x^3$

g) $f(x) = 2 - x^2$

RULES FOR DIFFERENTIATION

A function is differentiable if it has a derivative everywhere in its domain. It must be continuous and smooth.

1. **Constant Rule:** if c is a constant then,

$$\frac{d}{dx}[c] = 0$$

i.e. the derivative of a constant is zero

Example: $y = 7$
 $y' = 0$

2. **Power Rule:** if n is any number then,

$$\frac{d}{dx}[x^n] = n.x^{n-1}$$

For any given function $y = x$; **the derivative:** $y' = nx^{n-1}$

Example: if $f(x) = x^5$ then $f'(x) = 5x^{5-1} = 5x^4$

Functions that do not at first appear to be a power of a variable

The first thing is to express them as powers i.e. $\sqrt{x} = x^{\frac{1}{2}}$, then using Rule 2,

Ways to write the derivative of $y = f(x)$

- $f'(x) \rightarrow f$ prime x or the derivative of f with respect to x
- $y' \rightarrow y$ prime

- $\frac{dy}{dx}$ → dee why dee ecks or the derivative of y with respect to x
- $\frac{df}{dx}$ → dee eff dee ecks or derivative of f with respect to x
- D_x → dee ecks the derivative with respect to x

NOTE $\frac{dy}{dx}$ **does** not mean dy divide by dx

CLASSROOM EXERCISE 4

Differentiate the following functions:

(a) $f(x) = (4x^3 + 2)(1 - 3x)2$

(b) $g(x) = (x^2 + x + 2)(x^2 + 1)$

(c) $f(x) = 5x^2 - 2\sqrt{x}$

(d) $y = 2x^{-7} + x\frac{3}{x^2}$

(e) $g(x) = \frac{x^3 + 7x^2 - 2x + 4}{x}$

(f) $y = 5t^8 + \frac{t}{\sqrt{t}}$

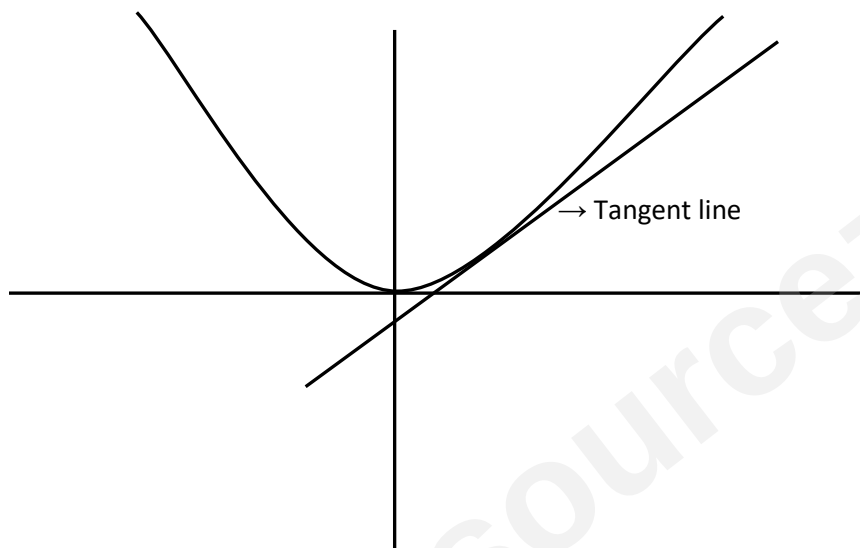
(g) $f(x)(4x^3 + 2)(3x - 1)$

(h) $g(t) = \left(\sqrt{t} + \frac{1}{t}\right)(2t + 1)$

(i) $y = (3x^2 - 5)^2$

3. THE EQUATIONS OF A TANGENT TO THE CURVE

A tangent is a straight line that touches a curve at one point.



The equation of a tangent is determined by two conditions. The two conditions are a gradient and a point of contact.

The equation of a tangent is in the form $y = mx + c$. The gradient (m) is determined from the derivative i.e.

The derivative of f at $x = c$ is equivalent to the gradient of the tangent line to the curve of f at

$$x = c$$

$$m = \frac{dy}{dx} \text{ or } f'(x)$$

The curve and its tangent line have the same gradient at the point of contact. The two – point form for finding the equation of a tangent is given by:

$$y - f(a) = f'(a)(x - a)$$

where a is the value of x at a point of contact, $f(a)$ is the value of y at a point of contact and $f'(a)$ is the slope/gradient at the point of contact

Example: For $f(x) = x^2 + x$, find the equation of a tangent to $f(x)$ at the point (1; 2)

$$m = f'(x) = 2x + 1$$

$$m = 2(1) + 1 = 3$$

$$y - f(x) = f'(x)(x - 1)$$

$$y - 2 = 3(x - 1)$$

$$y = 3x - 1$$

CLASSROOM EXERCISE 5

Find the equation of the tangent to the following curves at a given value of x

a) $y = x^2 - x - 6$; $x = 1$

b) $y = 3x^2 + 2x + 1$; $x = 0$

c) $y = -\frac{6}{x}$; $x = -2$

d) $y = x^2 + 2$; $x = -1$

e) $y = x^2 - 6x + 5$; $x = 2$

MIXED EXERCISES

QUESTION 1

1.1 If $f(x) = x^2 + 3x$, determine the derivative, $f'(x)$ from first principles.

1.2 The equation of a curve is given by

$$f(x) = x + \frac{4}{x^2}$$

Find the equation of the tangent to the curve at the point (1; 5).

1.3 Determine $Dx \left[\frac{x^2 + 4x - 12}{x - 2} \right]; x \neq 2$

1.4 Find $\frac{dy}{dx}$ if $y = \frac{x^2 - 9}{x - 3}$

1.5 If $y = \frac{p^3 + 1}{p + 1}$ find $\frac{dy}{dx}$

1.6 Evaluate $Dx \left[\frac{2x^3 - 16}{x^2 + 2x + 4} \right]$

1.7 Given $f(x) = \frac{x^4 - x}{x - 1}$ determine f'

1.8 $g(x) = \frac{x^3 + 8}{x + 2}$ find g'

1.9 $y = \frac{4x^2 - 12x + 9}{2x - 3}$ find $\frac{dy}{dx}$

1.1 Determine $f'(x)$ by first principles if $f(x) = 3x^2 - 5$

1.2 Differentiate the following functions expressing your answers with positive exponents where necessary

1.1.1 $y = (5x - 2)(2x + 3)$

1.1.2 $y = (x^2 - x)^2$

1.1.3 $y = (1 - t)(1 - 2t)$

1.1.4 $y = 3x(2x - \frac{2}{5x})$

1.1.5 $y = \sqrt{p}(\sqrt{p^3} - 2p)$

1.1.6 $y = \sqrt{x}(\sqrt{x} - a) + a$

1.1.7 $y = (4 - 2x)^2$

1.3 For which value of x is the function $y = x^2 + 3x - 5$ parallel to the line $3y + 2x = 6$?

QUESTION 2

2.1 Given the function $f(x) = 2 - 3x^2$,

determine $f'(x)$ from first principles

2.2 Determine the derivatives of each of the following functions:

$$2.2.1 \quad f(x) = \frac{\sqrt{x} + 5}{x}$$

$$2.2.2 \quad y = \frac{x^4}{8} + \sqrt[4]{x}$$

$$2.2.3 \quad y = \frac{3x}{5x^2} - \frac{1}{2\sqrt{x}}$$

$$2.2.4 \quad f(x) = \frac{2\sqrt{x} - 1}{x\sqrt{x}}$$

2.3 If $g(x) = \frac{1}{3}x^3 - x^2 - 3x$ find the equation of

the tangent to the curve at $x = 3$

2.1 Given $f(x) = \frac{-2}{x}$, differentiate f from first principles

2.2 Evaluate the following expressions

$$2.2.1 \quad Dx[\theta x^2 + x^\theta]$$

$$2.2.2 \quad \frac{dy}{dx} \text{ if } y(x+1) = 3x^2 + 4x + 1$$

$$2.2.3 \quad \frac{dy}{dx} \text{ if } yx + y = 3x^2 + 4x + 1$$

2.3 Given $g(x) = -x^3 + 3x - 2$. Calculate the value(s) of x where the gradient of g is equal to -2

4. CUBIC GRAPHS

It is given by an expression of the third degree (x^3). It has three (3) possible roots and two (2) possible turning points. Two or even three of them may be repeated. Unlike a quadratic function, a cubic function has at least one real root.

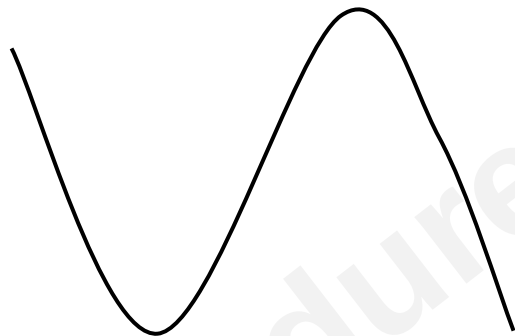
A cubic graph is of the form:

$$f(x) = ax^3 + bx^2 + cx + d$$

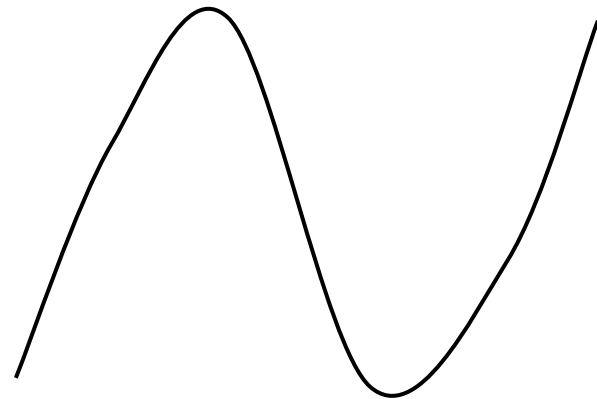
Where $a \neq 0$

Properties of a cubic function

1. **Shape:** If the coefficient of x^3 is negative



If the coefficient of x^3 is positive



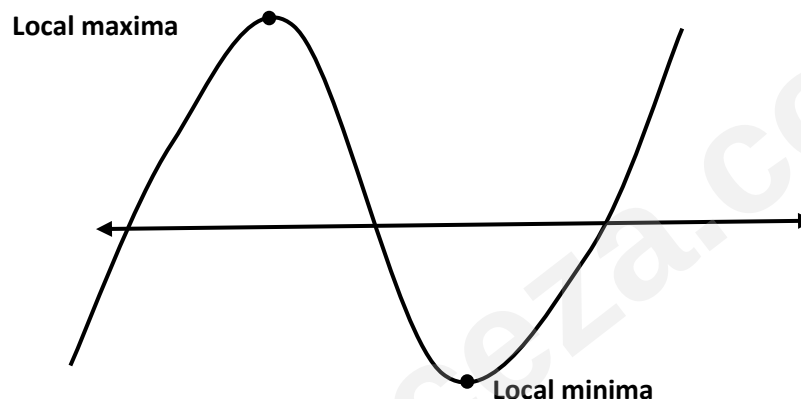
2. **y-intercept:** It is represented by the constant from the given function

3. **Intercepts with the x-axis:** It is determined by solving for the roots of the function. At this point $f(x) = 0$

4. Types of Stationary points (Minima & Maxima)

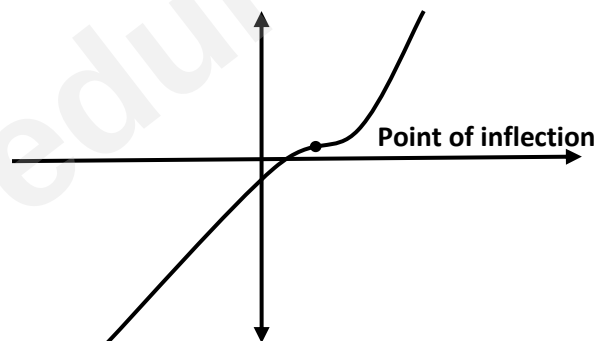
(a) Local minima – the bottom of the valley

(b) Local maxima – the top of the hill



The term “local” simply means that the top of the hill is not necessarily the highest point in the world, but merely the highest point in the local vicinity. Sometimes we use the word relative instead of local. The first derivative is used to calculate the values of x at these points. At this point the first derivative is equal to zero.

(c) Point of inflection – Where the graph changes its concavity. It is neither local minimum nor local maximum. The value of x at this point is calculated by using the second derivative. The second derivative at this point is equal to zero.



Example: Sketch the following graph: $f(x) = x^3 - 12x^2 + 36x$

1. Intercepts

a) $y = 0$

b) x - intercepts

$$x^3 - 12x^2 + 36x = 0$$

$$x(x^2 - 12x + 36) = 0$$

$$x(x - 6)(x - 6) = 0$$

$$\therefore x = 0 \text{ or } x = 6$$

➤ Stationery Points

$$f'(x) = 3x^2 - 24x + 36 = 0$$

$$3(x^2 - 8x + 12) = 0$$

$$(x - 6)(x - 2) = 0$$

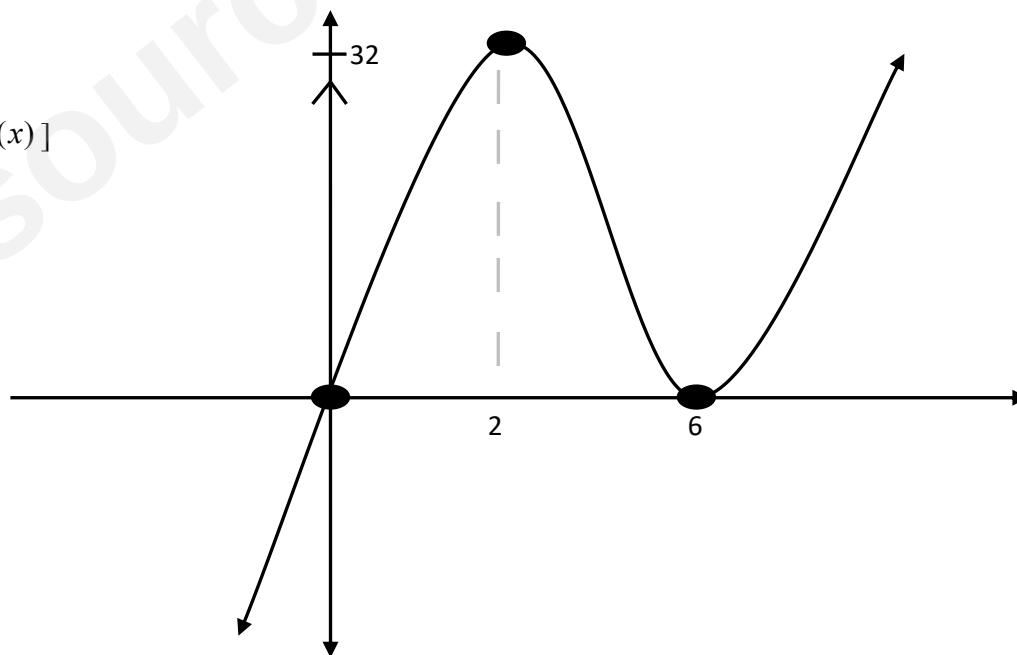
$$\therefore x = 6 \text{ or } x = 2$$

Substitute for x in $f(x)$ [NOT $f'(x)$]

$$y = 6^3 - 12(6)^2 + 36(6) = 0$$

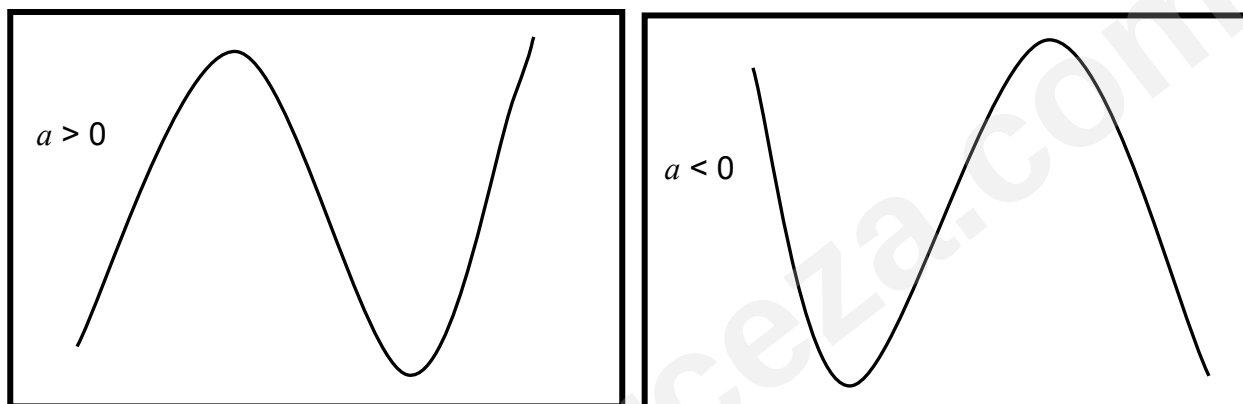
or

$$y = 2^3 - 12(2)^2 + 36(2)$$

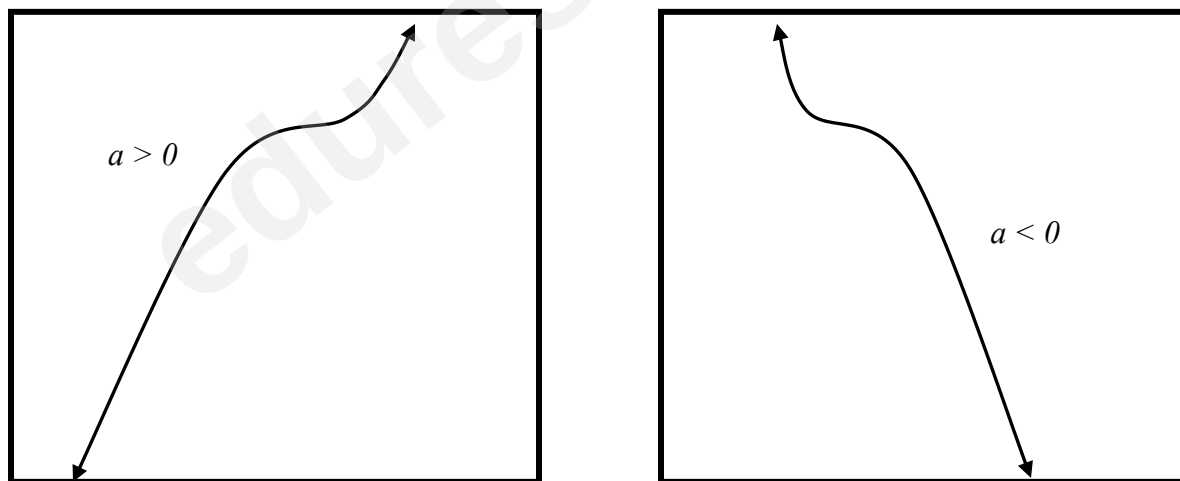


SUMMARY OF A CUBIC GRAPH

1. For 2 stationery points



2. For 1 stationery point



 PRACTISE EXERCISE 6

1. **Given:** $f(x) = -x^3 + 3x + 2$

- a) Show that $x - 2$ is a factor of $f(x)$ and factorise $f(x)$ completely
- b) Find all the zeros of f
- c) Determine the stationary points of f
- d) Sketch the graph of f

2. **Sketch the following curves, using calculus method**

- a) $y = x^3 - 12x$
- b) $y = x^3 + 3x^2 - 9x$
- c) $y = x^3 + 2x$
- d) $y = x^3 - 6x^2 + 12x - 4$
- e) $y = x^2 + 3x + 2$

POINT OF INFLECTIONS

The point at which the function changes its concavity. At this point $f''(x) = 0$ i.e. the second derivative is equal to zero.

The direction of bending changes an inflection point. The graph is concave down on one side of an inflection and concave up on the other side of an inflection. The second derivative tells about change in slope.

We can use the fact that the x -coordinate of the point of inflection is half way between the two critical values of the graph of f . So

$x = \frac{x_A + x_B}{2}$, if A and B are turning points of $f(x)$. So we first find the x -values of the turning point (critical values), then use the midpoint formula to find the value of x of inflection, then substitute this in $f(x)$ and find the corresponding y value.

Example: Given $f(x) = -x^3 + 5x^2 + 8x - 12$. Find the co-ordinates of the point of inflection of f

Answer: $f'(x) = -3x^2 + 10x + 8$

$$f''(x) = -6x + 10$$

$$-6x + 10 = 0$$

$$\therefore x = -\frac{2}{3}$$

Substitute for x in f

$$f\left(-\frac{2}{3}\right) = -\left(-\frac{2}{3}\right)^3 + 5\left(-\frac{2}{3}\right)^2 + 8\left(-\frac{2}{3}\right) - 12$$

$$\therefore f\left(-\frac{2}{3}\right) = \frac{-400}{27}$$

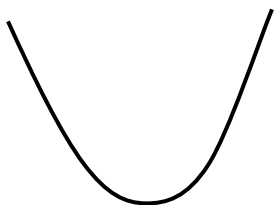
$$\therefore \text{The point of inflection is: } k\left(-\frac{2}{3}; \frac{-400}{27}\right)$$

Concavity of a cubic function

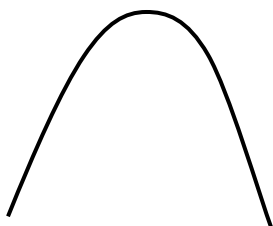
All cubic functions undergo a single change in concavity and this change always occurs at the cubic's centre (point of inflection)

Intervals for which concavity occurs

- Concave up ($f''(x) > 0$)



- Concave down ($f''(x) < 0$)



PRACTICE EXERCISE 7

Find the inflection point for the following functions

- (a) $f(x) = 3x^3 + 7x^2 + 2x - 3$
- (b) $f(x) = x^3 + 3x^2 - 4$
- (c) $f(x) = x^2 - 2x + 4$

MIXED EXERCICES

Given the function $f(x) = x^3 + kx^2 - 8x - 12$ 1.1 Show that $k = 5$ if $(x + 1)$ is a factor of f 1.2 Hence factorize f fully 1.3 Find the co-ordinates of the turning point of f 1.4 Find the co-ordinates of the point of inflection of f 1.5 Sketch the graph of f, showing ALL the intercepts with the axes, the co-ordinates of the turning point as well as point of inflection	Consider the function: $f(x) = x^3 - 6x^2 + 9x - 4$ 1.1 Determine: 1.1.1 The intercepts with both axes 1.1.2 The coordinates of any stationary points 1.1.3 The x-coordinate of the point of inflection 1.2 Sketch the graph of f on the axes provided 1.3 Using your graph determine the value(s) of x if: 1.3.1 $f(x) \leq 0$ 1.3.2 $f(x) f'(x) > 0$ 1.3.3 $f' > 0$
Given; $f(x) = x^3 - 5x^2 - 8x + 12$ 2.1 Calculate the x-intercepts of $f(x)$ 2.2 Calculate the co-ordinates of the stationery points of $f(x)$ 2.3 Sketch the graph of $f(x)$, clearly indicating the co-ordinates of the turning points and ALL the intercepts with the axes 2.4 For which value(s) of x is $f(x)$ concave up? 2.5 $y = -11x + 3$ is a tangent to $f(x)$ at $(a;b)$. Determine the values of a and b if a is a whole number	Given $g(x) = x^3 - 3x^2 + kx + 8$ where k is a constant. The graph has a turning point at $x = 1$. Find the value of k.

MIXED EXERCISES

1. Given $p(x) = ax^3 + bx^2 + cx + d$ as well as the following

- $a < 0$
- $p(-3) = p(8) = p(0) = 0$
- $p'(-1) = p'(5) = 0$

Draw a neatly labeled sketch of $p(x)$. Show all information. You are not required to label the y -coordinate of the turning point

2. A cubic function f has the following properties

- $f\left(\frac{1}{2}\right) = f(3) = f(-1) = 0$
- $f'(2) = f'\left(-\frac{1}{3}\right) = 0$
- f decreases for $x \in \left(-\frac{1}{3}; 2\right)$ only

Draw a possible sketch graph of f , clearly indicating the x -coordinate of the turning point and ALL the x -intercepts

3. The function $y = f(x)$ has the following properties

- $f'(x) > 0$ when $x < 3$
- $f'(3) = f'(5) = 0$
- $f''(5) > 0$
- $f(1) = 0$
- $f(5) = 3$

Motivate by means of a sketch whether the y -intercept is positive or negative

4. The following information about a cubic polynomial, $y = f(x)$ is given:

- $f(-1) = 0$
- $f(2) = 0$
- $f(1) = -4$
- $f(0) = -2$
- $f'(1) = f'(-1) = 0$
- if $x < -1$ then $f'(x) > 0$
- if $x > 1$ then $f'(x) > 0$

4.1 Use the information to draw a neat sketch of f

4.2 For which values of x is f strictly increasing?

4.3 Use your graph to determine the x -coordinate of the point of inflection of f

4.4 For which values of x is f concave up?

5A cubic graph f is defined by the following properties:

- $f(-2) = f(1) = f(6) = 0$
- $f(0) = 12$
- $a > 0$

5.1 By using the given information above, sketch a basic graph clearly indicating the intercepts with the axes

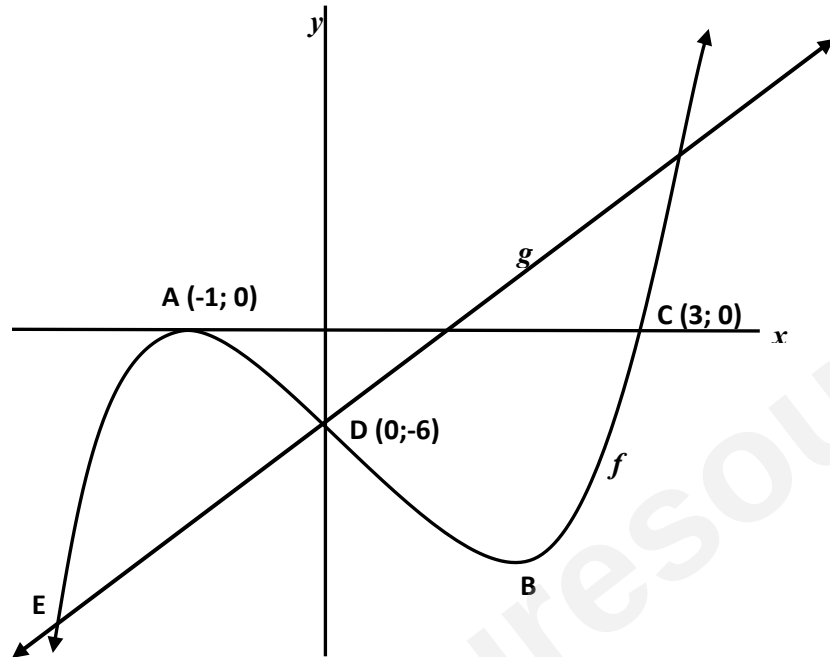
5.2 Show that $f(x) = x^3 - 5x^2 - 8x + 12$

5.3 Determine the values of x for which $f'(x) < 0$

5.4 Calculate the value of x for which f is concave up

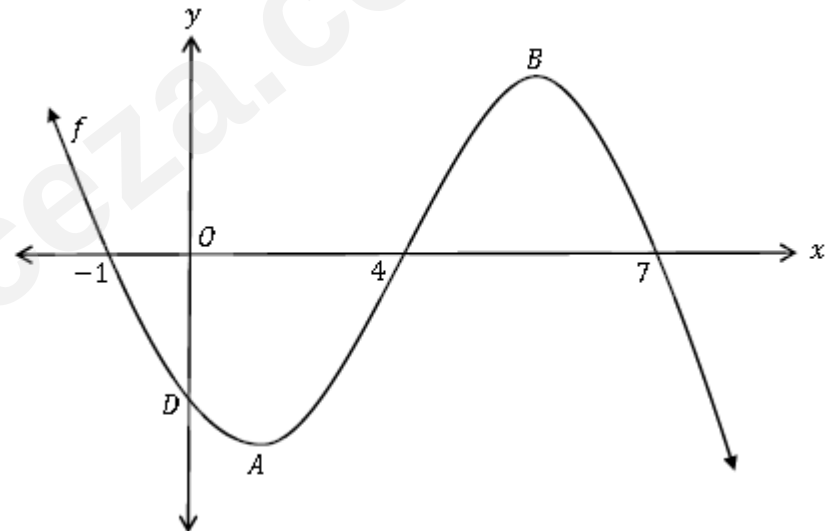
MIXED EXERCISES

The graphs of $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = 6x - 6$ are sketched below. A $(-1; 0)$ and C $(3; 0)$ are the x -intercepts of f . The graph of f has turning points at A and B. D $(0; -6)$ is the y -intercept of f . E and D are points of intersection of the graphs of f and g .



- 1.1 Show that $a = 2$; $b = -2$; $c = -10$ and $d = -6$.
- 1.2 Calculate the x -coordinate of the turning point B.
- 1.3 $h(x)$ is the vertical distance between $f(x)$ and $g(x)$, that is $h(x) = f(x) - g(x)$. Calculate x such that $h(x)$ is a maximum, where $x < 0$.

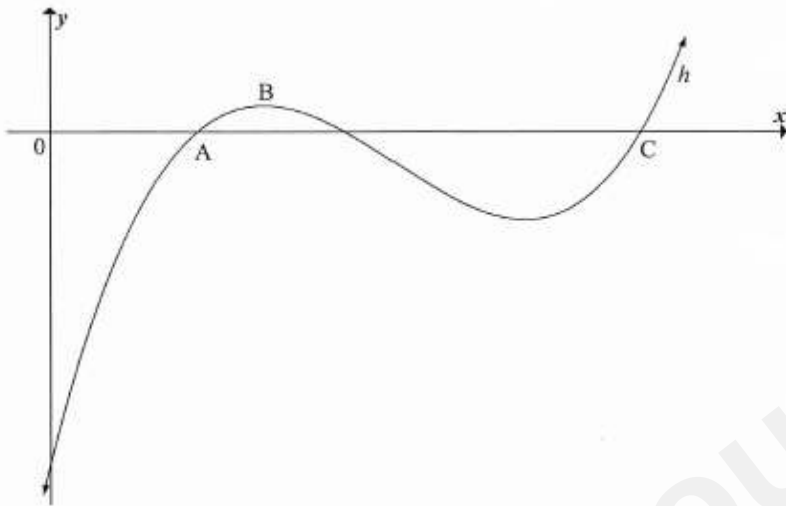
The sketch below shows the graph of $f(x) = -x^3 + 10x^2 - 17x + d$. The x -intercepts of f are $(-1; 0)$, $(4; 0)$ and $(7; 0)$. A and B are turning points of f and D is the y -intercept of f . The sketch is not drawn to scale.



- 1.1 Write down the value of d
- 1.2 Determine the coordinates of A and B
- 1.3 Determine the value of x where the concavity of f changes
- 1.4 Determine the coordinates of the point on f with a maximum gradient
- 1.5 Determine for which value(s) of x is $f(x) \cdot f'(x) > 0$

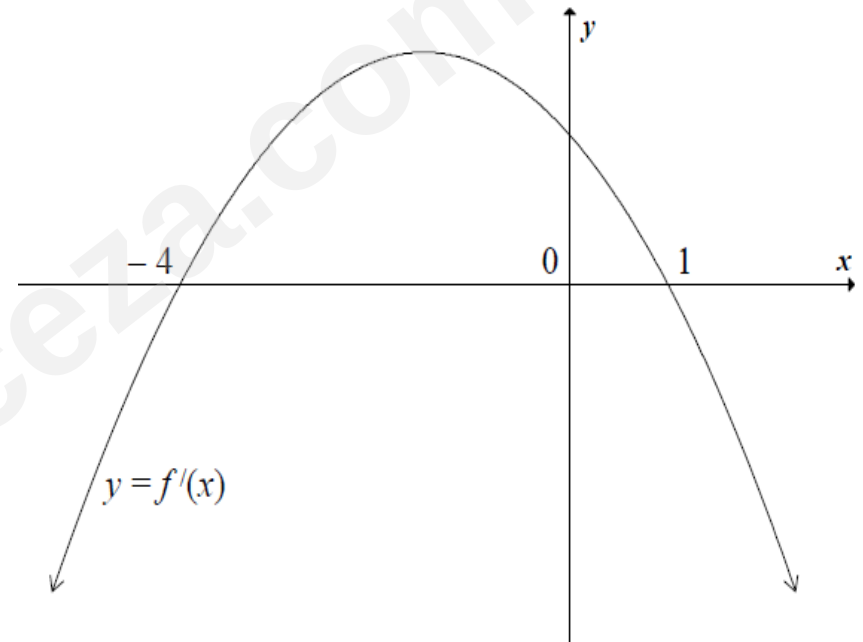
MIXED EXERCISES

The sketch below shows the graph of $h(x) = x^3 - 7x^2 + 14x - 8$. The x -coordinator of point A is 1. C is another x -intercept of h .



- 2.1 Determine $h'(x)$
- 2.2 Determine the x -coordinator of the turning point
- 2.3 Calculate the coordinator of C
- 2.4 The graph of h is concave down for $x < k$. Calculate the value of k

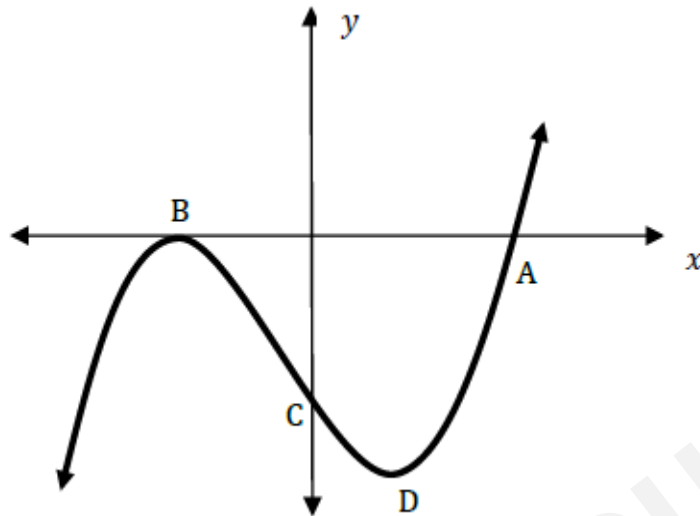
The graph of $y = f'(x)$, where f is a cubic function, is sketched below. Use the graph to answer the following questions:



- 2.1 For which values of x is the graph of $y = f'(x)$ decreasing?
- 2.2 At which value of x does the graph of f have a local minimum? Give reasons for your answer.

MIXED EXERCISES

The diagram below shows the graph of: $f'(x) = x^3 + x^2 - x - 1$

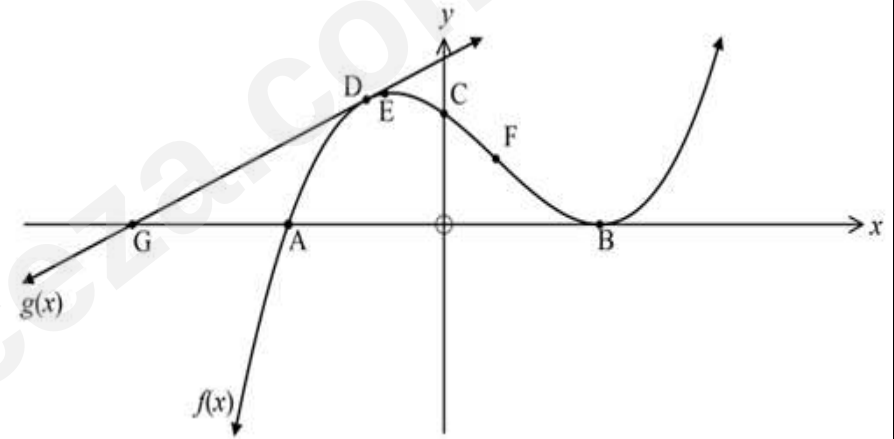


- 3.1 Calculate the distance between A and B, the x -intercepts
- 3.2 Calculate the coordinates of D, a turning point of f
- 3.3 Show that the concavity of f changes at $x = -\frac{1}{3}$
- 3.4 For which values of x is:
 - 3.4.1 $f(x) > 0$
 - 3.4.2 $f(x) \cdot f'(x) < 0$

The sketch given below represents the function f with equation:

$$f(x) = x^3 - 2x^2 - 4x + 8$$

The curve cuts the x -axis at A and B. B and E are the turning points.



- 3.1 Determine the coordinates of A, B and E.
- 3.2 Determine the x -coordinate of the point F, the point of inflection of $f(x)$
- 3.3 The straight line g with equation $g(x) = 3x + c$ is a tangent to f at D. Determine the coordinates of G.

MIXED EXERCISES

For certain function f , the first derivative is given as $f'(x) = 3x^2 + 8x - 3$

- 1.1 Calculate the x - coordinates of the stationary points of f
- 1.2 For which values of x is f concave up
- 1.3 Determine the values of x for which f is strictly increasing?
- 1.4 If it is further given that $f(x) = ax^3 + bx^2 + cx + d$ and $f(0) = -18$, determine the equation of f

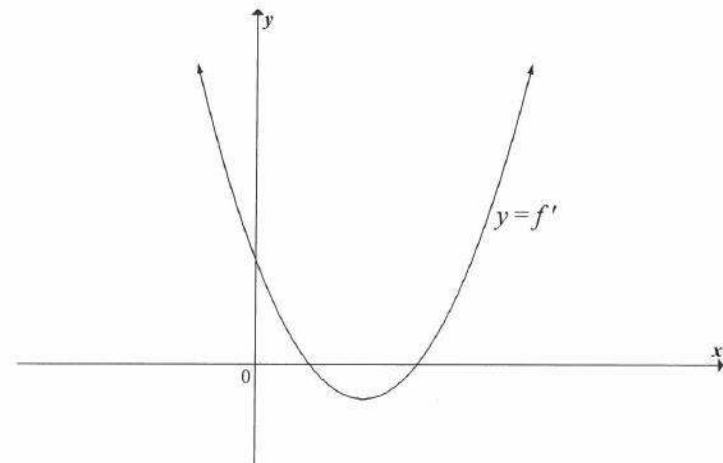
The turning points of a cubic function $y = g(x)$ are $(-1; 6)$ and $(5; 4)$. Draw a sketch graph of y' .

Label the x -intercepts clearly.

QUESTION 3

The graph of $y = ax^2 + bx + c$ below represents the derivative of f . It is given that $f'(1) = 0$, $f'(3) = 0$ and $f'(0) = 6$.

- 3.1 Write down the x -coordinates of the stationary points of f
- 3.2 For which value(s) of x is strictly decreasing?
- 3.3 Explain at which value of x the stationary point of f will be a local minimum
- 3.4 Determine the x -coordinate of the point of f
- 3.5 For which value (s) of x is concave up?



5. OPTIMIZATION

Optimization is the action of making the best or most effective use of a situation or resource by maximizing desired factors and minimizing undesired factors. Optimal means the best. The mathematical procedures involved is by finding the largest or the smallest value of a function subject to one or more constraints. The constraints are usually described by an equation (objective function) that must absolutely be true no matter what our solution is.

Optimization consists of maximizing and minimizing a real function.

1. RATES OF CHANGE

It is the extent to which one quantity affects a change in another related quantity. There are three scenarios where rates of change is applied.

Scenario 1: Practical Real – Life Problems.

The objective function is given and learners must identify the variables represented in the function and solve the problem.

Example 1: The data collected in a certain town over many years shows that the number of people affected by a disease increases according to the function $N(t) = 3t^3 + 20t$, where t is the number of years and N the number of people affected by a disease.

Determine the rate at which the disease will spread in the tenth year.

Solution: Write the objective function

$$N(t) = 3t^3 + 20t$$

N (number of people) is what we wish to optimize

t (number of years) is the constraint

Find the 1st derivative

$$N'(t) = 9t^2 + 20$$

Substitute t with 10 and calculate the rate at which the disease will spread in the 10th year.

$$N'(10) = 9(10)^2 + 20 = 920$$

Example 2: In a small factory the profit (P) per week is $P = -x^3 + 108x + 1\,600$, where x is the number of workers. Calculate

- How many workers are necessary to assure a maximum profit?
- The maximum profit

Solution: a) Write the objective function

$$P = -x^3 + 108x + 1\,600$$

P(profit) is what we wish to optimize

x (workers) is the constraint

Find the 1st derivative and set it equal to zero (maximum profit) - $3x^2 + 108 = 0$

$$\text{Solve for } x: -x^2 + 36 = 0$$

$$(x + 6)(x - 6) = 0$$

$x = -6$ or $x = 6$ therefore 6 workers are necessary

- For maximum profit, substitute x by 6 from the given function and solve for P: $P = -6^3 + 108(6) + 1\,600 = 2\,032$

PRACTICE EXERCISE 1

- A motor car uses P litres of petrol on a journey where

$$P = 32x^2 + \frac{2}{x^2} \quad \text{and } x \text{ is the average speed in kilometres per hour. Determine}$$

- the average speed for which the petrol consumption will be lowest.
- P, the litres of petrol for this consumption

- It is estimated that t years from now, the circulation of a soccer magazine will be given by the function:

$$n(t) = 50t^2 - 200t + 3000$$

- At the time of first publication, what was the circulation of the magazine?
- Derive an expression for the rate at which the circulation will be changing t years from now .
- At what rate will the circulation be changing 3 years from now?
- Determine by how much the circulation will change during the second year. Will it be increasing or decreasing?

- 3) The volume V in cm^3 of water remaining in a leaking tank after t seconds is:

$$V = 2000 - 40t + 0,2t^2$$

3.1 Find the average rate of change of volume from $t = 30$ seconds to $t = 40$ seconds

3.2 How fast is the volume decreasing when $t = 30$ seconds

- 4) A biologist states that when a certain type of antibacterial is introduced into a culture of bacteria, the number of bacteria present is given by the formula: $b(t) = -4t^2 + 60t + 1\,500$ where $b(t)$ in millions is the number of bacteria present at time t , measured in hours

4.1 How many bacteria were present at the beginning?

4.2 At what moment was the maximum number of bacteria present?

- 5) A profit yielded on a taxi is dependent on the average speed at which it is driven. The profit (P) in rand per hour is calculated from the formula: $P = \frac{3}{80}x^2 + 6x - 80$ where x is the average speed in kilometres per hour, and $x \geq 30$

5.1 Calculate the speed at which neither profit nor loss is yielded.

5.2 Determine the most economical speed and the corresponding profit per hour.

5.3 At which speed will the owner show a loss?

- 6) A water tank has both an inlet and outlet which regulates the depth of water it contains. The depth (D) in metres is given by the function: $D = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$, where D is measured in metres and t is measured in hours starting from 8H00.

6.1 Calculate the depth of the water at 8H00

6.2 Calculate the rate at which the depth is changing at 11H00

6.3 What happens to the depth of water at 11H00?

6.4 At what time will the inflow of water be the same as the outflow?

- 7) The number of molecules of a certain drug in the bloodstream t hours after it has been taken is represented by the function:

$$M(t) = -t^3 + 3t^2 + 7t, \quad 0 < t < 10$$

7.1 Determine the number of molecules of the drug in the bloodstream 3 hours after the drug was taken.

7.2 Determine the rate at which the number of molecules of the drug in the bloodstream is changing at exactly 2 hours after the drug was taken.

7.3 How many hours after taking the drug will the rate at which the number of molecules of the drug in the bloodstream is changing, be maximum?

- 8) The City of Cape Town wanted to determine the amount of water used every week during the recent drought. They measured the amount of water drawn from a tank and used by residents.

They determined the volume of water, in litres, t weeks after they started measuring was: $V = -100t^2 + 200t + 2\,400$

8.1 After how many weeks was the volume at maximum?

8.2 After how many weeks will the tank be empty?

8.3 Determine the rate at which the volume changed according to the time if $t = 3$.

Scenario 2: Knowledge of formulae of surface area and volume of right prism is required.

Example 3: A farmer has 200 meters of fencing with which he wishes to enclose a rectangular plot of land in which he wishes to plant vegetables.

1. If the length is x meters, determine the breadth in terms of x
2. Calculate the dimensions of the plot in order for him to obtain the largest number of vegetables from the land he wishes to enclose.

Solution: 1. $2x + 2b = 200$ (this is a constraints equation) $b = 100 - x$

2. Area of a rectangle = $l \times b$ (area is what we wish to optimize)

Objective function in one variable: $A = x(100 - x)$

$$A = 100x - x^2$$

$$A' = 100 - 2x = 0$$

$x = 50$ so the length is 50 metres

breadth = $200 - 50 = 150$ metres

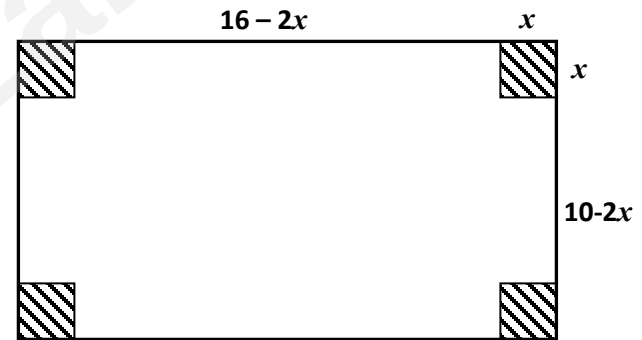
PRACTISE EXERCISE 2

MIXED EXERCISES

The length of a rectangular basket with a lid that closes is three times the width, x . The total surface area of the basket is 450 squared units.

- 1.1 Express the length in terms of x
- 1.2 Show that the height is given by $h = \frac{225}{4x} - \frac{3x}{4}$
- 1.3 Prove that the volume of the basket is given by $V = \frac{1}{4}(675x - 9x^3)$
- 1.4 Hence, determine the value of x for which the volume is a maximum

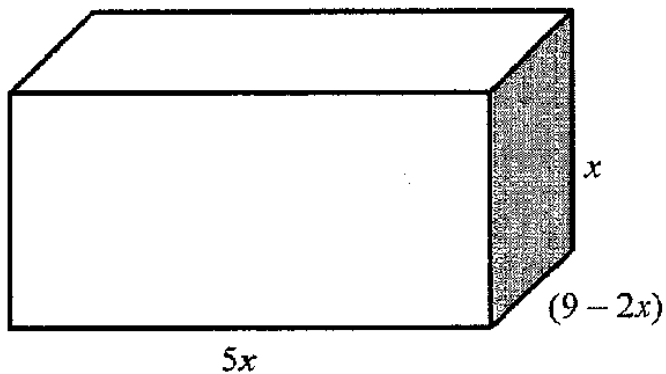
An open box with a rectangular base is to be constructed from a rectangular piece of card measuring 16cm by 10cm. A square is to be cut out from each corner and the sides folded up.



- 1.1 Show that the volume of the box is given by: $V = 4x^3 - 52x^2 + 160x$
- 1.2 Find the size of the cut out squares so that the resulting box has the largest possible volume.

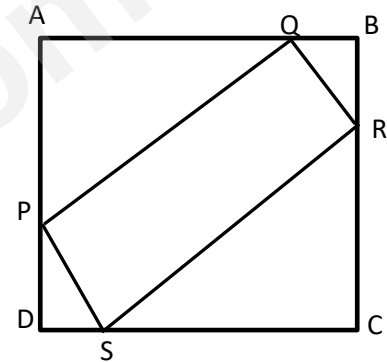
MIXED EXERCISES

A rectangular box has a length of $5x$ units, breadth of $(9 - 2x)$ units and its height of x units.



- 2.1 Show that the volume (V) of the box is given by $V = 45x^2 - 10x^3$
- 2.2 Determine the value of x for which the box will have maximum volume.

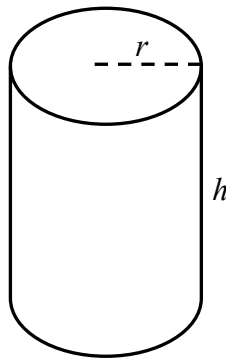
ABCD is a square with sides 20 mm each. PQRS is a rectangle that fits inside the square such that $QB = BR = DS = DP = k$ mm



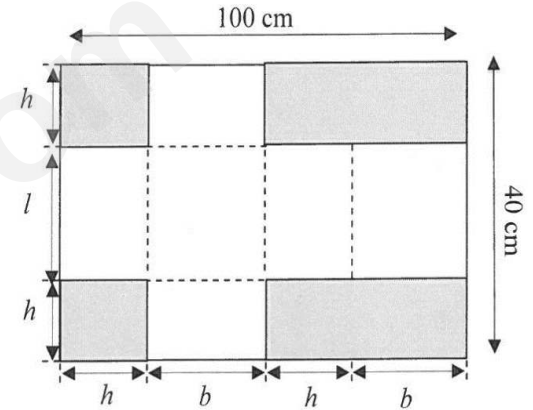
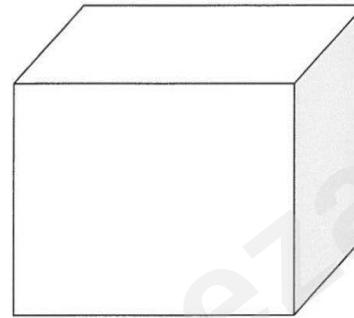
- 2.1 Prove that the area of PQRS $= -2k(k - 20) = 40k - 2k^2$
- 2.2 Determine the value of k for which the area of PQRS is a maximum

MIXED EXERCISES

A soft drink can has a volume of 340 cm^3 , a height of $h \text{ cm}$ and a radius of $r \text{ cm}$.



- 3.1 Express h in terms of r
- 3.2 Show that the surface area of the can is given by $A(r) = 2\pi r^2 + 680r^{-1}$
- 3.3 Determine the radius of the can that will ensure that the surface area is a minimum



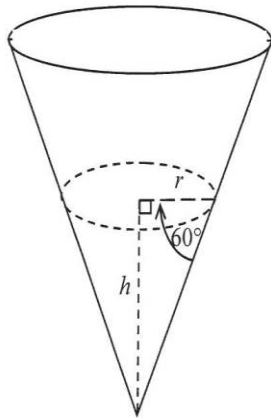
A box is made from a rectangular piece of cardboard, 100 cm by 40cm, by cutting out the shaded areas and folding along the dotted lines as shown in the diagram above.

- 3.1 Express the length l in terms of the height h
- 3.2 Hence prove that the volume of the box is given by $V = h(50 - h)(40 - 2h)$
- 3.3 For which value of h will the volume of the box be a maximum?

Scenario 3: A list of formulae will be provided for the surface area and volume of cones, spheres and pyramids. Learners must select the correct one.

MIXED EXERCISES

A rain gauge is in the shape of a cone. Water flows into the gauge. The height of the water is h cm when the radius is r cm. The angle between the cone edge and the radius is 60° , as shown in the diagram below.



Formulae for volume:

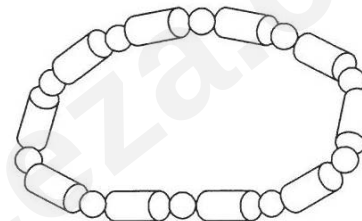
$$V = \pi r^2 h \quad V = \frac{1}{3} \pi r^2 h$$

$$V = lbh \quad V = \frac{4}{3} \pi r^3$$

1.1 Determine r in terms of h . Leave your answer in surd form

1.2 Determine the derivative of the volume of water with respect to h when h is equal to 9 cm

A necklace is made by using 10 wooden spheres and 10 wooden cylinders. The radii, r , of the spheres and the cylinders are exactly the same. The height of each cylinder is h . The wooden spheres and cylinders are to be painted. (Ignore the holes in the spheres and cylinders)



$$V = \pi r^2 h \quad S = 2\pi r^2 + 2\pi r h$$

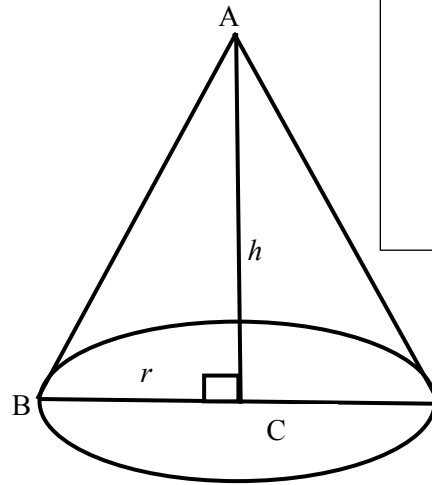
$$V = \frac{4}{3} \pi r^3 \quad S = 4\pi r^2$$

- 1.1 If the volume of a cylinder is 6 cm^3 , write h in terms of r
- 1.2 Show that the total surface area (S) of all the painted surfaces of the necklace is equal to $S = 60\pi r^2 + \frac{120}{r}$
- 1.3 Determine the value of r so that the least amount of paint will be used

MIXED EXERCISES

Refer to the figure. A mine is discharging waste on a dump which forms a cone. The radius of the circular base is r meters, the height is h meters, and the semi-vertical angle is a constant 40° (i.e. $\hat{BAC} = 40^\circ$).

The volume of the cone given by:



$$V = \pi r^2 h$$

$$V = \frac{4}{3} \pi r^3$$

$$V = lbh$$

$$V = \frac{1}{3} \pi r^2 h$$

2.1 Show that $V = \frac{\pi r^3}{3 \tan 40^\circ}$

2.2 Find the radius of the base when the volume is $72\,000\text{m}^3$, correct to the nearest meter.

2.3 Find the rate of increase of volume for each meter increase in radius, when the radius has the value found in 3.2 above

2. CALCULUS OF MOTION

Displacement, Velocity and Acceleration

If $f(x)$ is the equation of motion which gives position (s) at time t , then:

- velocity = $\frac{ds}{dt}$ where s represents displacement(position) and t represents time
- acceleration = $\frac{dv}{dt}$ OR $\frac{ds^2}{dt^2}$ where v represents velocity and t is time

Example 5: A particle travels along the x -axis such that $x(t) = 4t^3 - 2t + 5$, where x represents its displacement in metres from the origin t seconds after observation began

1. How far from the origin is the particle at the start of observation?
2. Calculate the velocity of the particle after 3 seconds
3. Calculate the acceleration of the particle after 3 seconds

Solution:

1. No need to use the derivative because we are looking for the displacement. At the start (beginning) $t = 0$

$$x(0) = 4(0)^3 - 2(0) + 5 = 5$$

the particle is 5 metres away

2. velocity is the derivative of displacement . therefore we are going to use the derivative

$$v = \frac{ds}{dt} = 12t^2 - 2$$

$$v = 12(3)^2 - 2 = 106\text{m.s}^{-1}$$

3. acceleration = $\frac{dv}{dt} = 24t = 24(3) = 72\text{m.s}^{-2}$

PRACTICE EXERCISE 4

1. An object moves in a straight line so that at the end of t seconds its distance from a fixed point on the line is S cm where: $S(t) = 25 + 100t - 2t^2$. Calculate
 - 1.1 the distance from a fixed point after 5 seconds
 - 1.2 the velocity after 4 seconds
 - 1.3 how long does it take before the object comes momentarily to rest?
2. The displacement, S , of a car moving towards a fixed point after t seconds, is given by: $S = t^3 - 9t^2 + 24t + 8$
 - 2.1 Determine the expression for the velocity of the car
 - 2.2 Calculate the time t , for which the car was reversing
3. In the formula $S = 44t - 6t^2$, S is the distance in metres that is travelled by the bus in t seconds after the brakes were applied
 - 3.1. Show that the speed at the time where the brakes were first applied was 158,4 kilometres per hour
 - 3.2 Solve for t if $\frac{ds}{dt} = 0$
 - 3.3 How many metres did the bus travel until it stops?
 - 3.4 If there was a stationary vehicle 60 metres in front of the bus at the time of braking, would the bus have collided with the stationary vehicle? Explain
4. A ball is thrown vertically upwards. The height of the ball in metres above the thrower, t seconds after being thrown, is given by: $h(t) = 20t - 5t^2$
 - 4.1 Calculate the height of the ball after 3 seconds
 - 4.2 After how many seconds does the ball reach its maximum height?
 - 4.3 Find the velocity of the ball after 1,5 seconds
 - 4.4 After how many seconds is the velocity of the ball 12m. s^{-1} ?

5. A stone thrown vertically into the air reaches a height of h metres after t seconds, and

$$h = 120t - 16t^2. \text{ Find:}$$

- 5.1 The time taken by the stone to reach its maximum height
- 5.2 The maximum height reached by the stone
- 5.3 The speed attained after 2 seconds
- 5.4 When the speed will be 24 metres per second