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Module: 1

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ALGEBRA, EQUATIONS & LOGARITHMS

In this chapter we shall be solving linear, quadratic, and simultaneous equations.

1. Determine all real solutions for each of the following:

(a) $(x + 3)(x - 7) = (x + 3)(4x - 5)$

(b) $2x^2 - 5x = 8$

(c) $\sqrt{5 - x} + x = -1$

(d) $\frac{(x + 3)(x + 1)}{3 - 2x} = \frac{x + 1}{2}$

2. If $f(x) = \sqrt{\frac{2x^2 - x}{x + 5}}$, for which values of x is:

2.1 $f(x) = 0$?

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2.2 $f(x)$ undefined?

2.3 $f(x)$ real?

3. If we define $\sqrt{-1} = k$ write the roots of $x^2 + 2x + 5 = 0$ in terms of k in simplest form.

4. Using the factor theorem, or otherwise, factorise:

(a) $3x^3 + 2x^2 - 19x + 6$

(b) $-X^3 + 3x - 2$

5. Solve for x:

(a) $-3x^2 + 6x < -9$

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(b) $X^2 - 4x + 7 > 0$

(c) $x \leq \frac{3}{x-2}$

6. Solve for x:

(a) $3x^3 + 14x^2 - 23x + 6 = 0$

(b) $X^3 + 2x^2 - 17x - 18 = 0$

(c) $x^3 - x^2 + 12 = 0$

7. Solve for x and y in the following simultaneous equations:

$3x^2 - 2xy + y^2 = 3x - 2y = 1$

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8. Solve for x:

(a) $3^{x+2} + 3^{x} = 30$

(b) $3^{x} = 34$

(c) $\log_2 (x + 6) = 5$

9. Solve for x:

(a) $\log_2 (x + 1) + \log_2 (3x - 2) = 3$

(b) Hence or otherwise solve $\log_2 (x + 1) + \log_2 (3x - 2) \leq 3$.

10. Solve for a: $2^{2a+1} - 2^{a} = 3$

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Marks: 50

Question 1:

1.1 Simplify without a calculator:

1.1.1 $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$ (3)

1.1.2 $\frac{2\sqrt{8}-\sqrt{50}}{(1-\sqrt{2})(1+\sqrt{2})}$ (4)

1.2 Simplify:

$\frac{9^{n-1}.27^{3-2n}}{81^{2-n}}$ (4)

Question 2:

2.1 Solve for x:

2.1.1 $3x^2+17x-6=0$ (3)

2.1.2 $3x^2-4x=5$ *(Correct to TWO decimal places.)* (5)

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2.1.3 $x^2 - 2x \leq 15$ (3)

2.1.4 $2.3^x + 3^x.2^{-1} = 7\frac{1}{2}$ (4)

2.1.5 $3\sqrt{x-1} = 12$ (3)

2.2 Solve for x and y simultaneously:

$3x^2 + 4xy + 2y^2 = 9$

$x + 2y = 3$ (7)

Question 3:

3.1 Give the nature of roots for a quadratic equation if:

3.1.1 $\Delta = 16$ (1)

3.1.2 $\Delta = 27$ (1)

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3.1.3 $\Delta = 0$ (1)

3.1.4 $\Delta = -5$ (1)

3.2 For what value of k will the equation $kx^2 = 3(3 - x)$ have equal roots? (5)

Question 4:

I have 12 metres of fencing. What are the dimensions of the largest rectangular area that I can enclose with this fencing by using an existing wall as one side? Show all your calculations.

(5)

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Exercise: 2

Date:_____

Marks: 66

Question 1:

1. Solve for x in each of the following:

1.1 $\frac{4x}{x-3} \leq 2x$ (5)

1.2 $x^2 + 7x = 1$ (3)

1.3 $2x^2 + 6 = px$ (by completing the square). (5)

Question 2:

2. Solve the following equations:

2.1 $12 = 8x^2 - 8x - (x^2 - x)^2$ (7)

2.2 $4^x + 2^3 = 3.2^{x+1}$ (4)

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Question 3:

3. Solve the following:

3.1 $\frac{3-4x}{x-2} > x$ (5)

3.2 $2x-\sqrt{25x-1}=-3$ (5)

3.3 $3=3x^2+x$ (3)

Question 4:

4. Solve the following equations:

4.1 $x(x+7)+16+\frac{60}{x^2+7x}=0$ (4)

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4.2 $4.3^{-x} - 3^{1-2x} = 1$ (5)

4.3 $2x^2 - ax + 1 = 0$ (by completing the square). (5)

Question 5:

5.1 Simplify the following without a calculator: $\sqrt[4]{(\sqrt{19}+\sqrt{3})}.\sqrt[4]{(\sqrt{19}-\sqrt{3})}$ (3)

5.2 Simplify: $\frac{16^x.81^{2x-1}.6^{-4x}}{9^{x-0,5}}$ (3)

5.3 Solve the following: $x - 3 - \sqrt{x-1} = 0$ (4)

5.4 Solve for x: $1 - 2x - \sqrt{9x-2} = 0$ (5)

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Exercise: 3 Date:_____

Marks: 36

Question 1:

Simplify without the use of calculator. (Leave answer with positive exponents.)

1.1 $\dfrac{\sqrt{20} + \sqrt{5}}{4\sqrt{5} - 5^{\frac{1}{2}}}$ (3)

1.2 $\dfrac{3^{t+4} - 5.3^{t} + 1}{2.\ 3^{t+1}}$ (4)

Question 2:

2.1 Solve for x and leave answer in ***SURD FORM.***

$2x^2 = 3x + 1$ (3)

2.2 Hence describe the roots for the equation in 2.1. (2)

2.3 Determine the value of P if: $2x^2 + px + 2$ *has non − real roots.* (3)

2.4 Without solving the equation $(10x - 1)x = 2$; determine the nature of roots. (3)

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Question 3:

Solve without the use of calculator.

3.1.1 $6x^2 \leq 1 - x$ (4)

3.1.2 $x - 12 = \sqrt{x}$ (4)

3.1.3 $\frac{x+2}{x+1} - \frac{3}{x-2} = \frac{1}{x+1}$ (5)

3.2 Solve for x and y in the following system of equations.

$y = -x + 2$

$x^2 + y^2 = 2$ (5)

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Exercise: 4 Date:_____

Marks: 56

Question 1:

1.1 Solve for x

1.1.1 $x^2 = 12x$ (3)

1.1.2 $2x^2 + 4x - 7 = 0$ **(Leave your answer in surd form)** (4)

1.1.3 $(x + 2)(x - 5) \geq -6$ (5)

1.2 Solve for x and y if they satisfy the equations
$x - 2y + 7 = 0$ and $x^2 + xy + y^2 - 21 = 0$ (7)

Question 2:

2.1 Solve for x:

2.1.1 $X(x+1) = 6$ (3)

2.1.2 $3x^2 - 4x = 8$ (4)

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2.1.3 $4x^2 + 1 \geq 5x$ (5)

2.2 Consider the equation: $x^2 + 5xy + 6y^2 = 0$

2.2.1 Calculate the values of the ratio $\frac{y}{x}$ (4)

2.2.2 Hence, calculate the values of x and y if $x + y = 8$. (4)

Question 3:

3.1 Solve for x

3.1.1 $x(x-1) = 0$ (2)

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3.1.2 $2x^2 - 5x - 7 = 0$ (Round off to one decimal place) (4)

3.1.3 $7x^2 + 18x - 9 \leq 0$ (5)

3.2 Solve for x and y simultaneously

$2x - y = 4$

$x^2 + xy = 19 - y^2$ (6)

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Module: 2
PATTERNS AND SEQUENCES

Key Concepts

In this session we will focus on summarizing what you need to know about:

- How to find the nth term of a quadratic sequence. (constant second difference)
- Finding a term of an arithmetic sequence.
- Finding a term of a geometric sequence.
- Using simultaneous equations to find a sequence.
- Develop a formula for the sum of a arithmetic and geometric series.

Terminology & definitions

Sequence: Listing of terms. i.e. **T_1; T_2;T_3……**

Series: Adding terms of a sequence i.e **$T_1 + T_2 + T_3 +$……..**

Arithmetic Sequence: Has a constant first difference **$d = T_2 - T_1$**

Quadratic Sequence: Has a constant second difference.

Geometric Sequence: Has a constant ratio. $r = \frac{T_2}{T_1}$

nth term of an A.S $= a + (n-1)d$

nth term of an G.S $= ar^{n-1}$

For a quadratic sequence,

The nth term is given by $T_n = an^2 + bn + c, where\ \ a, b\ and\ c\ are\ integers.$ To calculate the values of $a, b\ and\ c$, the equations below can be used

- $2a = 2^{nd}\ difference$
- $3a + b = T_2 - T_1$
- $a + b + c = T_1$

Symbols, Units & Equations

Common difference = d

First term = a Common ratio =r

1. Given the following sequence 7; 2; – 3; – 8 ……

Determine:

(a) The n th term, T $_n$

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(b) The sum of the first 20 terms.

2. Given the following sequence 6; 3; 1½; ¾; ……
Determine:

(a) The n th term, T_n

(b) The sum of the first 10 terms

(c) The sum to infinity

3. Determine n if, $\sum_{K=1}^{n} 2(\frac{1}{2})^K = \frac{63}{32}$

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4. Find the n th term of the sequence 3; 6; 13; 24; ….

5. Calculate $\sum_{i=1}^{7}(-1)^{i}(4i-3)$

6. A particular arithmetic series has $S_n = \frac{3n^2 - 55n}{2}$.

Determine

6.1 The 24 th term.

6.2 The sum of the first 25 terms

7. How many terms of the arithmetic series 40 + 36 + 32 ……
Must be taken for the sum to be less than 180?

8. Find the first three numbers which form an infinite geometric series, such that the sum of the first three terms is 26 and their product is 216.

9. The first term of an arithmetic sequence is 110 and the last term is – 6. Determine the sum of the first 18 terms.

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10. Prove that $\left(\sum_{r=1}^{n} r\right)^2 - \left(\sum_{r=1}^{n-1} r\right)^2 = n^3$

11. Calculate $\sum_{i=1}^{\infty}(8)\left(\frac{1}{\sqrt{2}}\right)^{i-1} + \sum_{i=1}^{8}(3)(2)^{i-1} + \sum_{i=1}^{20}\{6 + 4(i-1)\}$

Leave your answer in simplified surd form.

12. If $x > 1$, find the sum to infinity of $\frac{x^2}{x-1} + x + (x-1)$ …… and also the value of x for the given series to have a sum to infinity.

13. In an arithmetic series, the 3 **rd** term is twice the 8 **th** term. Find the sum of the first 25 terms in terms of d.

14. The sum of the first three terms of a geometric sequence Is 42 and the sum of the second three terms is 5¼. Find the 8 **th** term.

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Exercise: 1 Date:_____

Marks: 60

Question 1:

1.1 In an arithmetic series, the second term equals -8 and the eight term equals 58. Calculate:

1.1.1 the first term and the common difference. (5)

1.1.2 the sum of the 1st 301 terms of the series. (3)

1.2 Simplify:

$$\sum_{k=-3}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^k$$ (4)

Question 2:

2.1 Prove that $a + ar + ar^2 + \ldots$ (to n terms) $= \frac{a(1-r^n)}{1-r}$ for $r \neq 1$ (5)

2.2 Given the geometric series $15 + 5 + \frac{5}{3} + \ldots$

2.2.1 Explain why the series converges. (2)

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2.2.2 Evaluate: $\sum_{n=1}^{\infty} 5(3^{2-n})$ (3)

2.3 The sequence 4; 9; x; 37; . . . .is a quadratic sequence. Calculate x. (3)

Question 3:

The arithmetic series 3 + 9 +15 + ... has n terms.

3.1 Show that the sum of the first n terms is given by: $S_{n=}3n^2$ (4)

3.2 Show also that the sum of the first (n-4) terms is $3n^2 - 24n + 48$ (3)

3.3 If the sum of the last four terms is 600, calculate the value of n (3)

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Question 4:

A larger quantity of waste material contains metal. At the successive passes through a recovery Process, the mass of the of the metal recovered is;
32Kg for the first pass,
24Kg for the second pass,
18Kg for the third pass,
13.5Kg for the fourth pass, and so on to form a geometric sequence.

4.1 How much metal would be recovered at the eighth pass? (3)

4.2 Find the total mass of metal recovered in the first eight pass? (4)

4.3 In order to be economical, at least 2.4Kg of metal must be recovered in any pass. Show that the eleventh pass would be uneconomical. (3)

Question 5:

5.1 The 1st, 3rd and 11th terms of an A.S are also the first three terms of a G.S. If the first term of the A.S is 2; determine the 7th term of the G.S. (5)

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5.2 Given that $3 + 3(x + 4) + 3(x+4)^2 + \ldots..$ is a convergent series,

5.2.1 Calculate the sum to infinity in terms of x for the series (3)

5.2.2 Hence, find the sum to infinity if $x = -4$ (2)

5.3 Given the mixed sequence 3, 2, 3, -1, 3, $\frac{1}{2}$, . . .

5.3.1 Write down the next two terms of the sequence (2)

5.3.2 Find the sum of the first 20 terms of the sequence (3)

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Exercise: 2 Date:_____

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Question 1:

1.1 The first 4 terms of an arithmetic sequence are: 3; p; q; 21. Determine the values of p and q. (3)

1.2 Prove that the sum of n terms of an arithmetic series is given by the following formula:

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$ (4)

1.3 The sum of the first 3 terms of an arithmetic sequence is 21, while their product is 315. Determine these three terms. (5)

1.4 The sum of the first 16 terms of an arithmetic sequence is 552. The sum of the next 4 terms is 258. Find the first three terms of the series. (5)

1.5 If the second term of an arithmetic sequence is 15 and the fifth term is 24, determine the third and fourth terms of the sequence. (5)

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1.6 The sum of the first 53 terms of an arithmetic series is 4 240, while the seventh term is equal to 20. Find the first term and the common difference. (6)

1.7 If the sum of n terms of an arithmetic sequence is given by $S_n = 4n + 3n^2$, determine:

1.7.1 The first 3 terms. (4)

1.7.2 The general term. (2)

1.8 The sum of the fifth and the ninth terms of an arithmetic sequence is 6, while the twenty-fifth term is –24. Calculate the sum of the first 50 terms. (7)

Question 2:

2. Given the sequence 3; 6; 9; …; 60.

2.1 Determine the number of terms in the sequence. (3)

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2.2 Determine the sum of the terms in the sequence. (3)

2.3 Determine the sum of all the natural numbers from 1 to 60 which are not multiples of 3. (4)

2.4 Prove that the sum of n terms of a geometric series is given by the following formula:

$$S_n = \frac{a(1-r^n)}{1-r}$$ (4)

Question 3:

3.1 A tennis ball is dropped from a height of 10 metres above the ground and loses 5% of its previous height with each bounce. Determine the total distance the ball travels before it stops bouncing. (4)

3.1.1 Hence write down the formula for the sum of a geometric series above. (1)

3.1.2 Deduce a formula for the sum to infinity of a convergent geometric series. (4)

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Question 4:

4. A radio station has a competition where prize money is awarded over a period of 5 days. On the first day, R5 000 is given to the winner. On the second day, 80% of this amount is awarded to the winner and on each of the next 3 days, 80% of the amount awarded on the previous day is given to the winner of the day.

4.1 How much money is given to the winner on the fourth day? (3)

4.2 How much money is given out by the radio station over the 5 days? (5)

4.3 If the radio station has R22 000 to give out for this competition, over how many days could this competition last before there is no more money to be awarded? (7)

4.4 Write the series $3+6x+9x^2+12x^3+\ldots$ with 45 terms in $\sum$ – notation. (4)

4.1 Calculate the value of $\sum_{k=1}^{\infty} 7.4^{2-k}$ (4)

4.6 Write the following series in $\sum$ – notation: $4x-8x^3+16x^5-32x^7+\ldots$ with p terms. (4)

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4.7 Determine the value of: $\sum_{k=0}^{5} \frac{-27}{(-3)^k}$ (5)

4.8 Calculate the value of n if $\sum_{t=1}^{n} \frac{25}{4}\left[\frac{4}{5}\right]^t = \frac{11529}{625}$ (6)

4.9 If $\sum_{m=2}^{18} \log x^{1-m} = 612$, find the value of x. (6)

Question 5:

5. Given: $\sum_{t=2}^{22} (2t+5)$

5.1 Find the first 3 terms. (3)

5.2 How many terms are there in this series? (1)

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5.3 Find the sum of the series. (3)

Question 6:

6.1 Determine the least number of terms for which $\sum_{p=1}^{n}\left(\frac{2}{3}\right)^{1-p}$ will have a value of more than 9 972. (7)

6.2 If the sum of a convergent geometric series is 1 and the second term is $\frac{1}{4}$, determine the first three terms of the series. (5)

6.3 $2+\frac{2}{x}+\frac{2}{x^2}+\frac{2}{x^3}+\ldots$ Is a converging geometrical series.

6.3.1 Determine the possible values of x. (2)

6.3.2 Determine the solution to the equation: $2+\frac{2}{x}+\frac{2}{x^2}+\frac{2}{x^3}+\ldots=8$ (3)

6.4 The nth term of a sequence is 6^{1-n}.

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6.4.1 Determine the fourth term. (1)

6.4.2 Determine the sum to infinity. (4)

6.5 If the sum of an infinite convergent geometric series is 18 and the second term is 4, determine the third term of the series. (5)

6.6 For a particular geometric progression, the term formula is given by: $T_n = \frac{12}{2^n}$

Determine the sum to infinity. (5)

Question 7:

7. Given the following infinite series: $3 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{3x^3}{8} + \ldots$

7.1 For which values of x will the series converge? (2)

7.2 If $x = \frac{3}{4}$, find the sum of the series. (4)

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7.3 Given the series: $(1+2x)+(1+2x)^2+(1+2x)^3+...$

7.3.1 For what values of x will the series converge? (3)

7.3.2 If $(1+2x)+(1+2x)^2+(1+2x)^3+...=1$, determine the value of x. (3)

7.4 A certain plant reaches a height of 0,8 m during its first year. It then grows 2,5 cm during the following year. In each of the following years, it grows $\frac{3}{4}$ of the previous years' increase in height.

7.4.1 Determine the height of the plant after five years. (3)

7.4.2 Show that the height of the plant will never reach 0,9 m. (4)

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Exercise: 3 Date:_____

Marks: 92

Question 1:

1.1 The first three terms of the sequence -4, $x, y, 36$ form an A.S while the last three terms of this Sequence form a G.S. Find the possible values of x *and* y (6)

1.2 The 10th term of an A.S is 28 and the 7th term is 19. Calculate;

1.2.1 The common difference and the first term of the sequence (4)

1.2.2 The 80th term (2)

1.2.3 The sum of the first 80 terms (3)

Question 2:

The marks scored in weekly tests by a Grade 12 learner are given by a Quadratic sequence 8, 18, 30, 44, . . .

2.1 Calculate the nth term of the sequence (4)

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2.2 Hence find the marks scored by the learner in the 7th test if this trend is maintained. (2)

Question 3:

3.1 Thando and Joy's teacher asked that they use their own rule to construct a sequence of numbers, starting with 4. The sequences that they have constructed are given below. Thando's sequence: 4; 9; 14; 19; 24...Joy's sequence: 4; 64; 1024; 16384; 262144…

Write down the nth term (or the rule in term of n) of:

3.1.1 Thando's sequence (3)

3.1.2 Joy's sequence (3)

3.1.3 $5; x; y;$ Is an arithmetic sequence and $x; y; 81$ is a geometric sequence. ALL terms in the sequences are integers. Calculate the values of x and y. (8)

Question 4:

4.1 The nth term of a sequence is given by $T_{n=}\ 3(m-1)^{n+1}$ where m is a Real number, $m \neq 1$.

4.1.1 Show that the sequence is geometric. (3)

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4.1.2 Determine the values of m for which the series $f(m) = \sum_{n=1}^{\infty}(m-1)^{n+1}$ converges. (3)

4.1.3 Calculate (in terms of$m$) the sum to infinity of this series. (3)

Question 5:

The following sequence of numbers forms a quadratic sequence: 9; 19; 33; 51; 73;

5.1 The first difference of the above sequence also forms a sequence. Determine an expression for the general term of the first difference. (3)

5.2 Determine the expression for the nth term of the quadratic sequence. (4)

5.3 Calculate the first difference between 66th and 67th terms of the quadratic sequence. (2)

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5.4 Explain why the sequence of number will never contain a negative term. (2)

Question 6:

6.1 The first term of a geometric sequence is 3 and the sum of the first 4 terms is 5 times the sum of the first 2 terms. The common ratio is greater than 1. Calculate:

6.1.1 The first three terms of the sequence, and (4)

6.1.2 The value of n for which the sum to n terms will be greater than 765 (5)

6.2 Consider the sequence: 12; 4; 14; 7; 16; 10;...

6.2.1 If the pattern continues in the same way; write down the next TWO terms in the sequence. (2)

6.2.2 Calculate the sum of the first 50 terms of the sequence. (5)

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Question 7:

Consider the sequence: 5; 18; 37; 62; 93; …

7.1 If the sequence behaves consistently, determine the next **TWO** terms of the sequence. (2)

7.2 Calculate a formula for the nth term of the sequence. (4)

7.3 Use your formula to calculate n if the nth term in the sequence is 1 278. (5)

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Module: 3
GRAPHS AND FUNCTIONS

In this chapter we shall deal with turning points (**local minimum point/ Maximum Turning Point**) of the parabola.
Straight line graphs whose **general functional equation** is of the form

$$y = mx + c \ ,$$

Were studied at grades 10 and 11, and thus we assume familiarity with them.

EXAMPLE 1

Given: $p(x) = -x^2 + 8x + 20$ and $g(x) = 4x + 8$

1. Draw sketch graphs of p and g on the same system of axes.

   Indicate all intercepts with the axes and the turning point.

2. Let $A(x; y)$ be a point on the parabola, $x > 0$, and B on the straight line, so that

   AB is parallel to the $y-$axis.

   Show that the length of AB can be expressed as

   $-x^2 + 4x + 12$

3. Calculate the **maximum length** of AB.

SOLUTION:

Consider the parabola given by $p(x) = -x^2 + 8x + 20$. In order to sketch we proceed as follows: $x-$intercepts, let $p(x) = 0$, so that:

Step 1:

$$-x^2 + 8x + 20 = 0$$

The two numbers whose product is $-20 = -1 \times 20$ and whose sum is -8

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are -10 and 2.

$$-x^2+8x+20=0$$

$$\Rightarrow x^2-10x+2x-20=0$$

$$\Rightarrow x(x-10)+2(x-10)=0$$

$$\Rightarrow (x-10)(x+2)=0$$

$$\therefore x=10 \text{ or } x=-2$$

Therefore $(10;0)$ and $(-2;0)$ are $x-$intercepts of p.

Step 2: Given $p(x)=-x^2+8x+20$, the **abscissa** and the **ordinate** of the turning point are given by:

$x=-\frac{b}{2a}$, where $a=-1$, $b=8$ and $c=20$ from $p(x)=-x^2+8x+20$

$$\therefore x=-\frac{8}{2(-1)}=4$$

And $y=\frac{4ac-b^2}{4a}$

$$y=\frac{4(-1)(20)-8^2}{4(-1)}$$

$$=\frac{-80-64}{-4}$$

$$=36$$

OR

Alternatively it is common practice to substitute the value of $x=-\frac{b}{2a}$ into the original equation.

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Thus, $x = 4$ (as obtained above) yields: $p(x) = -x^2 + 8x + 20$

$$p(4) = -4^2 + 8(4) + 20$$

$$\therefore p(4) = 36$$

Step 3: For the $y-$intercept of $p(x) = -x^2 + 8x + 20$, let $x = 0$, so that:

$$p(x) = -x^2 + 8x + 20$$

$$\Rightarrow p(0) = -(0)^2 + 8(0) + 20$$

$$\therefore p(0) = 20$$

$(0; 20)$ is the $y-$intercept of p.

Consider a straight line given by

Given $g(x) = 4x + 8$, we can sketch if we find the $x-$intercept and $y-$intercept.

For $x-$intercept, let $g(x) = 0 \Rightarrow 4x + 8 = 0$

$$\therefore x = -2$$

Thus, $(-2; 0)$ is the $x-$intercept.

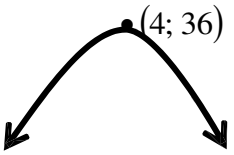
$y-$intercept : Let $x = 0$, $g(x) = 4x + 8$

$$\Rightarrow g(0) = 4(0) + 8$$

$$\therefore g(0) = 8$$

Thus $(0; 8)$, is the $y-$intercept.

Remark: Just before we draw the **parabola**, we must note that, since the co-efficient of x^2 in $p(x) = -x^2 + 8x + 20$ is $a = -1 < 0$, it follows that our graph will have a maximum turning point.

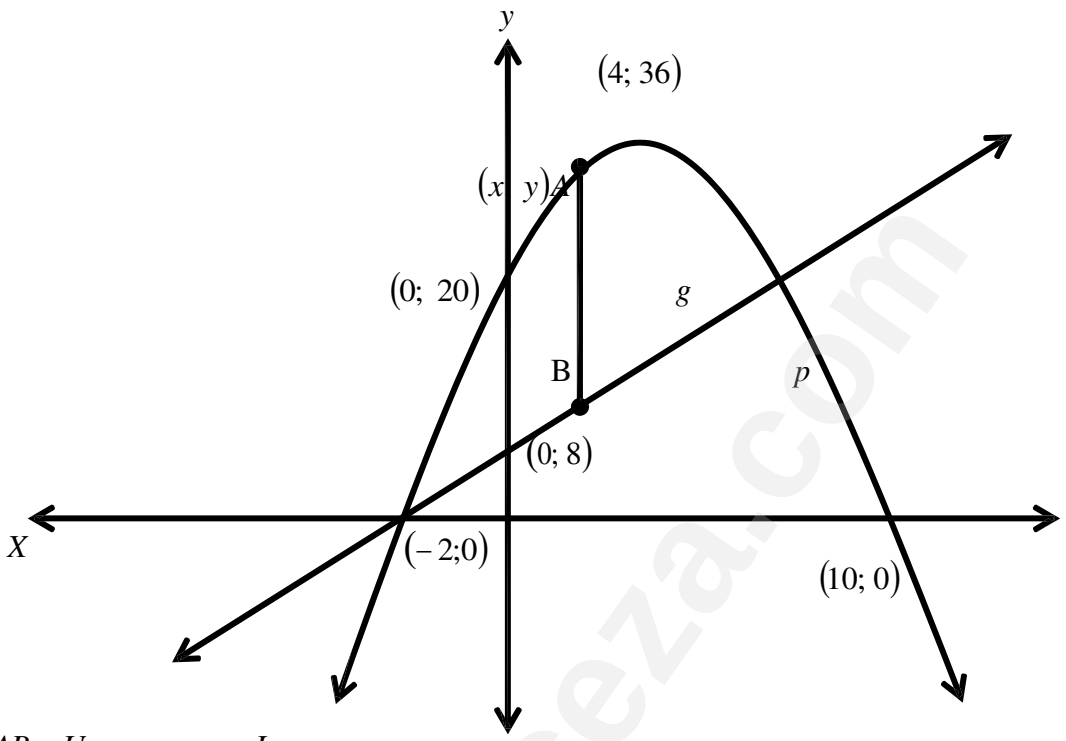


Expected shape.

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Thus we are ready to sketch



1.2 $AB = Upper\ curve - Lower\ curve$

$$AB = -x^2 + 8x + 20 - (4x + 8)$$

$$= -x^2 + 4x + 12$$

1.3 If $AB = -x^2 + 4x + 12$ in 1.2 above we have that since co-efficient of x^2 is $a = -1 < 0$, the parabola has a maximum turning point, so that AB has a maximum value as required.
Now, maximum length of AB is given by:

$$y = \frac{4ac - b^2}{4a}$$

$$= \frac{4(-1)(12) - 4^2}{4(-1)}$$

$$= \frac{-48 - 16}{-4}$$

$$= \frac{-64}{-4}$$

$$= 16 \text{ units}$$

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OR/ Alternatively

If $AB = -x^2 + 4x + 12$ we have that $x = -\frac{b}{2a}$

$$= -\frac{4}{2(-1)}$$

$$= 2$$

, so that

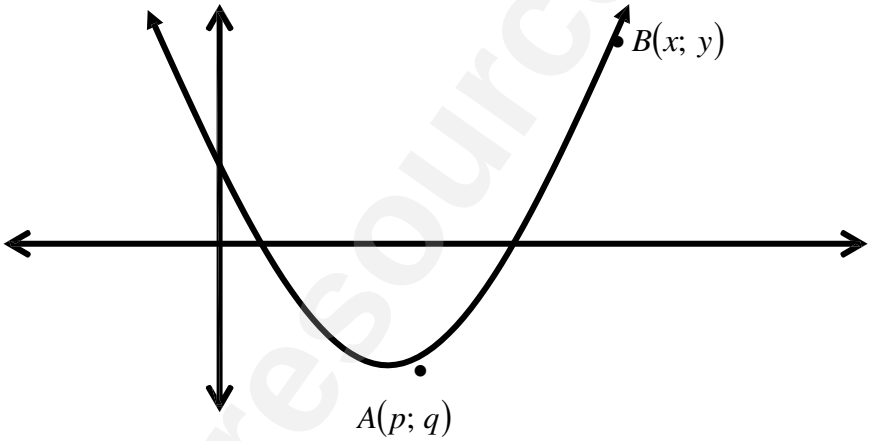
$AB = -2^2 + 4(2) + 12$

$= -4 + 8 + 12$

$= 16$ units

FINDING THE EQUATION OF A DRAWN GRAPH

1. Given a turning point and one other point on a graph. For instance:



1. **We use the following formula**.

$y = a(x - p)^2 + q$

- By first substituting $(p; q)$
- Determine a, by substituting the co-ordinates of the extra point.

2. **Given the roots α and β plus another point.**

Use the formula

$$y = a(x - \alpha)(x - \beta)$$

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And substitute the co-ordinates of the extra point to determine a .

3. Given the co-ordinates of three points on the parabola, one of which is the $y-$intercept.

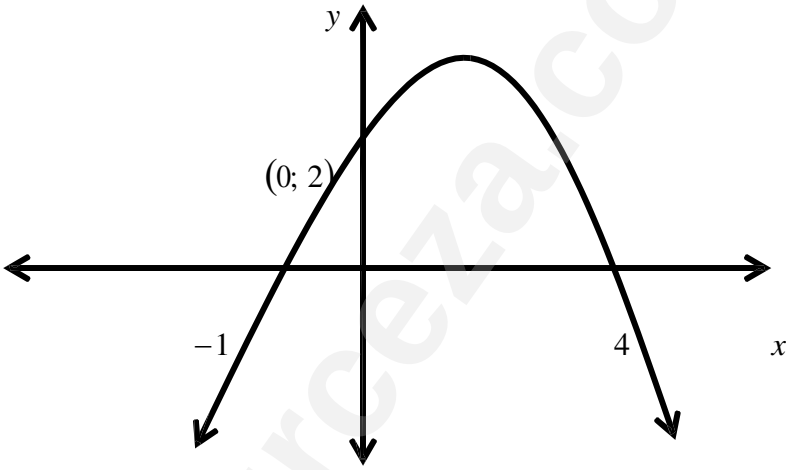
Substitute the co-ordinates of the other two points into $y = ax^2 + bx + c$ (given)

And solve equations simultaneously for a and b .

EXAMPLES

Determine the equations of each of the following parabola:

1.



SOLUTION:

$$y = a(x+1)(x-4)$$

Substitute $(0;\ 2)$ in:

$$2 = a(0+1)(0-4)$$

$$2 = -4a$$

$$-\frac{1}{2} = a$$

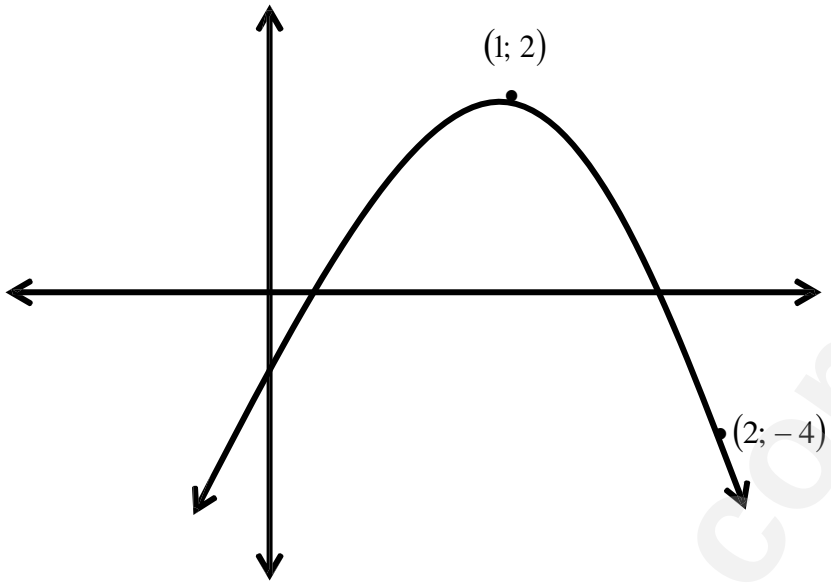
$$\therefore y = -\frac{1}{2}(x+1)(x-4)$$

$$= -\frac{1}{2}x^2 + \frac{3}{2}x + 2$$

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2.



Solution:

$y = a(x-p)^2 + q$

Substitute $(2;-4)$ in:

$-4 = a(2-1)^2 + 2$

$-4 = a + 2$

$\therefore a = -6$

$y = -6(x^2 - 2x + 1) + 2$

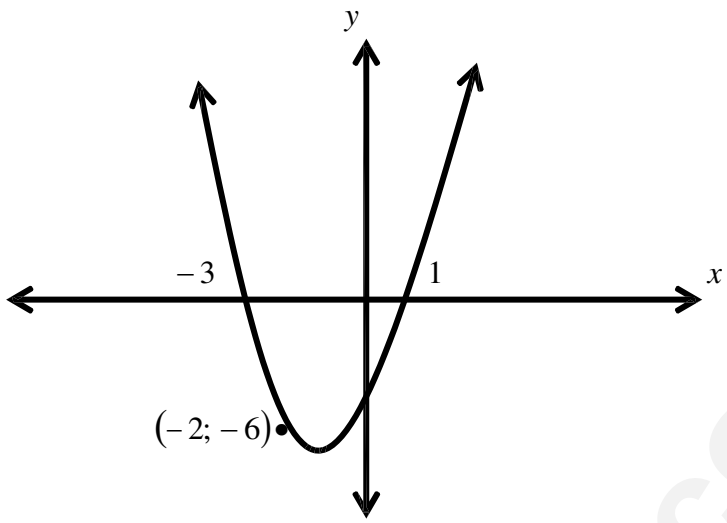
$\therefore y = -6x^2 + 12x - 4$

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3.



Solution:

$y = a(x-\alpha)(x-\beta)$

Substitute $(-2;-6)$ in:

$-6 = a(-2+3)(-2+1)$

$-6 = -3a$

$2 = a$

Thus, $y = 2(x+3)(x-1)$

$\therefore y = 2x^2 + 4x - 6$

EXAMPLE

$g = \{(x;\ y): y = +2(x+1)^2 + 4\}$

1.1 What is the max./ min. value of y the and the corresponding $x-$value?

1.2 What is the equation of the axis of symmetry?

1.3 What are the co-ordinates of the turning point?

1.4 Where does the graph intersect the $y-$ axis?

1.5 Determine the intercepts with the $x-$ axis.

1.6 What is the nature of the roots?

1.7 Draw a neat sketch graph and show all the important co-ordinates.

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1.8 What is the domain of g ?

1.9 What is the range of g ?

1.10 Determine the value of $g(-2)$.

Use the graph to determine:

1.11 For which values of x is:

1.11.1 $2x^2+4x+6\geq 0$

1.11.2 $2x^2+4x\leq 0$

1.12 For which values of p will the roots of $2x^2+4x+p=0$ be

1.12.1 equal

1.12.2 real

1.13 For which values of m will $2(x+1)^2+m=0$ have:

1.13.1 equal roots

1.13.2 non-real roots.

1.14 If $x-$ increases from -1 to ∞, will the function values increase of decrease?

SOLUTION:

1.1 Min. $y=4$ if $x+1=0$

$x=-1$

1.2 $x=-1$ Note that it is an equation we cannot only write -1 .

1.3 $(-1;\ 4)$

1.4 Let $x=0$: $y=2(0+1)^2+4$

$=6$

1.5 Let $y=0$: $2(x+1)^2+4=0$

$\therefore 2(x+1)^2=-4$ but $(x+1)^2\geq 0$

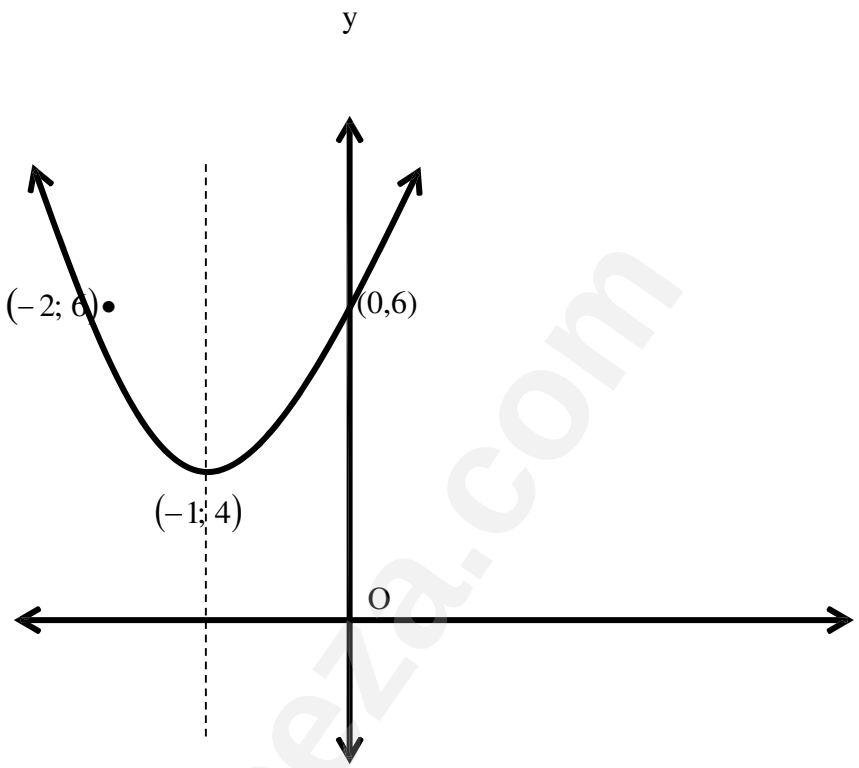
$\therefore$ no real solution

1.6 Roots are non-real

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1.7



1.8 Domain $\equiv \{x : x \in R\}$

1.9 Range $\equiv \{y : y \geq 4,\ y \in R\}$

1.10 $g(-2) = 2(-2+1)^2 + 4$

$= 6$

1.11.1 $y = 2(x+1)^2 + 4$

$= 2x^2 + 4x + 2 + 4$

$= 2x^2 + 4x + 6$

$\therefore 2x^2 + 4x + 6 \geq 0$ if $g(x) \geq 0$

$\therefore$ All values of x.

1.11.2 $2x^2 + 4x \leq 0$

$\Rightarrow 2x^2 + 4x + 6 \leq 6$

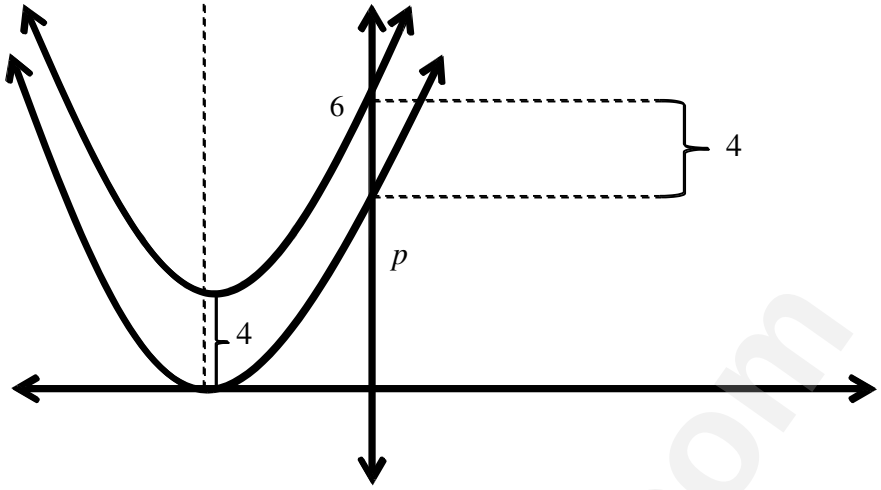
$\Rightarrow g(x) \leq 6$

$\therefore -2 \leq x \leq 0;\ \ x \in$

1.12.1 $2x^2 + 4x + p = 0$

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NOTE
***P* is the *y*-intercept if the parabola just touches the *x*-axis. The whole parabola has to**
move down 4 units and each function value decreases by 4.

$p = 6 - 4$

$p = 2$

1.12.2 $p \leq 2$ The parabola must touch or intersect the *x*-xis.

1.13.1 $m = 0$ m is the minimum value of y when the parabola touches the x-axis

(for equal roots) or has no intersection with the x-axis (for non-real roots).

1.13.2 $m > 0$

1.14 Increase The value of y is increasing and the graph has a positive gradient.

INVERSES

f^{-1} is used to represent the inverse of f.

The inverse f^{-1} is obtained by swopping the x and y values in the defining equation.

The graph and its inverse is always symmetrical about the line $y = x$.

To determine the inverse of a parabola, it is preferable to have it in the form
$y = a(x - p)^2 + q$

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EXAMPLE

1.1 Draw a sketch graph of f if $f(x)=2x^2$.

1.2 Determine $f^{-1}(x)$ and draw the graph of f^{-1} on the same system of axes as 1.1.

1.3 Determine algebraically the point of intersection of the two graphs.

SOLUTION:

1.1 Axis of symmetry: $x=0$ (The y-axis)

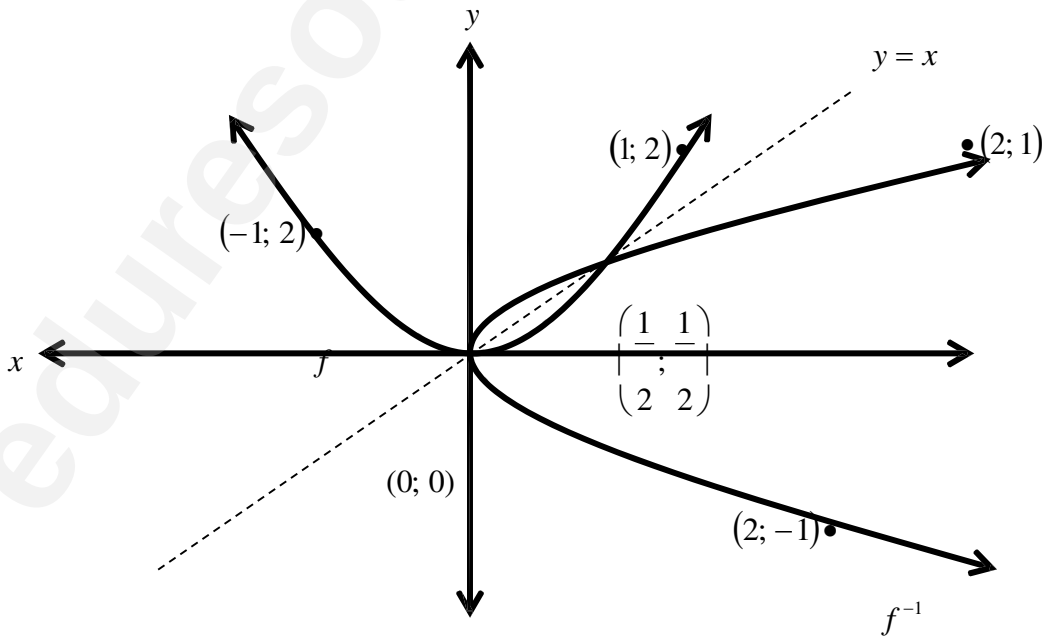
Min. $y=f(0)=0$ ($a>0$)

Show that co-ordinates of at least one more point e.g. $(1;\ 2)$.

1.2 f^{-1} : $x=2y^2$

$\therefore y=\pm\sqrt{\frac{x}{2}}$

$\therefore f^{-1}(x)=\pm\sqrt{\frac{x}{2}}$



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1.3 $y = 2x^2$ and $y = \pm\sqrt{\frac{x}{2}}$

$$\therefore \pm\sqrt{\frac{x}{2}} = 2x^2$$

$$\Rightarrow \left(\pm\sqrt{\frac{x}{2}}\right)^2 = \left(2x^2\right)^2$$

$$\Rightarrow \frac{x}{2} = 4x^4$$

$$\Rightarrow x = 8x^4$$

$$\Rightarrow x\left(1 - 8x^3\right) = 0$$

$$\Rightarrow x = 0 \text{ or } 8x^3 = 1$$

$$x^3 = \frac{1}{8}$$

$$x = \frac{1}{2}$$

Therefore points of intersection are $(0; 0)$ and $\left(\frac{1}{2}; \frac{1}{2}\right)$.

The two graphs are symmetrical about the line $y = x$ and the $x-$ and $y-$ co-ordinates of the points of intersection are equal.

EXAMPLE

Given: $f(x) = 2x + 6$

1.1 Draw sketch graphs of f and its inverse, f^{-1}, on the same system of axes.

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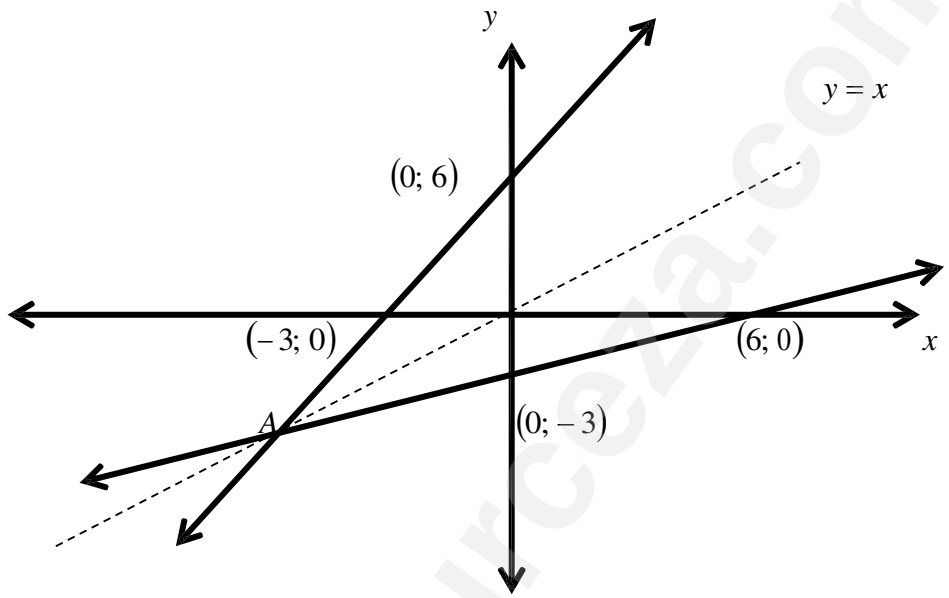
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SOLUTION:

$f : x = 2x + 6$

$f^{-1} : x = 2y + 6$

$\therefore y = \frac{1}{2}x - 3$



NOTE

The two graphs above are symmetrical about the line $y = x$.

If $(0; 6)$ lies on the graph of f , then $(6; 0)$ will lie on the graph of f^{-1}.

1.2 Determine algebraically the point of intersection of the graphs of f and f^{-1}.

$y = 2x + 6$ and $y = \frac{1}{2}x - 3$, so that

$2x + 6 = \frac{1}{2}x - 3$

$\Rightarrow 4x + 12 = x - 6$

$\Rightarrow 3x = -18$

$\therefore x = -6$

$\therefore y = 2(-6) + 6$

$= -6$

$A(-6; -6)$

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The two graphs intersect on the line $y = x$; the $x-$ and $y-$ co-ordinates of the point of Intersection will therefore always be equal.

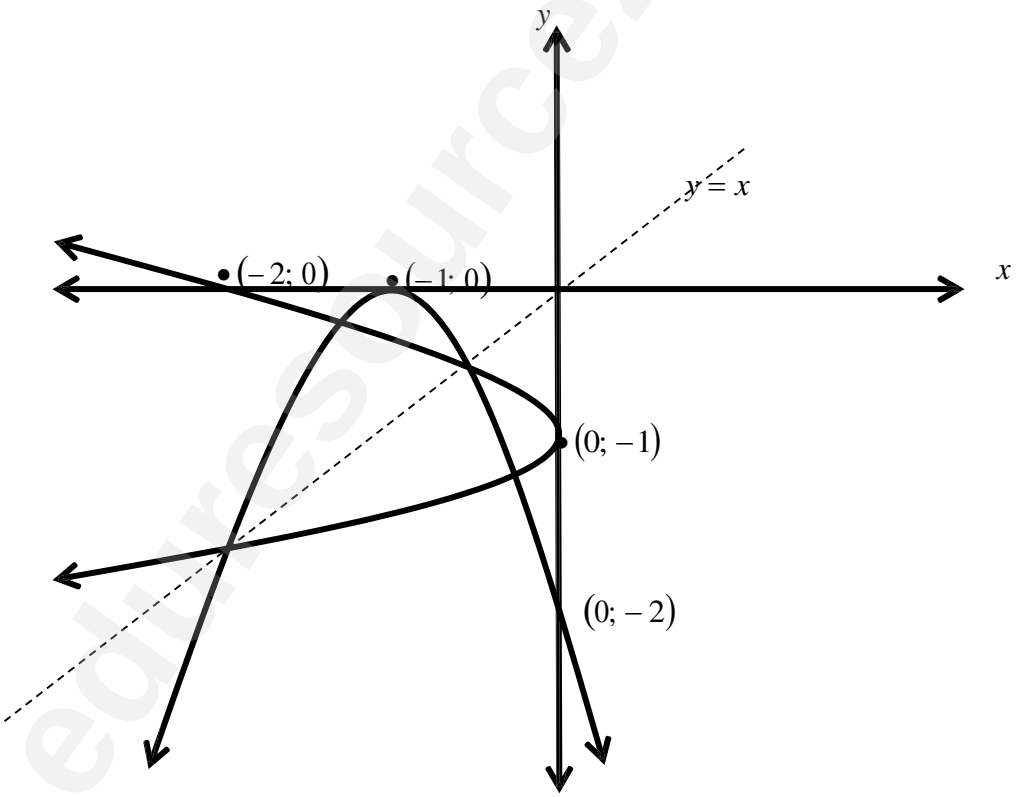
EXAMPLE

$Q = \{(x;\ y): y = -2(x+1)^2;\ y, x \in R\}$

1.1 Draw the graphs of Q and Q^{-1} on the same system of axes.
1.2 Determine the equation of Q^{-1}.
1.3 Determine the domain of Q^{-1}.

SOLUTION:

1.1 Q: has turning point $(-1; 0)$
Q^{-1} has turning point $(0; -1)$
Q : $y-$intercept: -2
Q^{-1}: $x-$intercept: -2



1.2 Q^{-1}: $x = -2(y+1)^2$

$$-\frac{x}{2} = (y+1)^2$$

$$\pm\sqrt{\frac{x}{-2}} = y+1$$

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$$\therefore y = -1 \pm \sqrt{\frac{-x}{2}}$$

$$\therefore Q^{-1} = \left\{(x;\ y):\ y = -1 \pm \sqrt{-\frac{x}{2}};\ x \le 0\right\}$$

1.3 $\{x : x \le 0;\ x \in R\}$

MOVING THE AXES (*X* OR *Y*)

1. If the $x-$axis is moved k units up or down, the graph moves k units in the opposite direction and thus determines the new $y-$ intercept.
   (The values of a and b are interchanged, because the axis of symmetry remains the same).

2. If the $y-$axis is moved p units to the left or right, the graph moves in the opposite direction. Determine the new $x-$intercepts, use $y = a(x-\alpha)(x-\beta)$ or the new turning point, $y = a(x-p)^2 + q$ (the $a-$value remains unchanged).

3. If there is a move in two directions, determine the new turning point and use the format: $y = a(x-p)^2 + q$ (the $a-$value remains the same).

EXAMPLE

1.1 Sketch the graph of $f(x) = -\frac{1}{2}x^2 + x + 4$

1.2 Determine the equation of f if the $x-$axis is moved up by 3 units.

1.3 Determine the equation of f if the graph is moved 3 units to the right.

1.4 Determine the equation of f if the graph is moved down 2 units, and one unit to the right.

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SOLUTION:

1.1 $y-$intercept

Let $x=0$, $f(x)=-\frac{1}{2}x^2+x+4$

$$f(0)=-\frac{1}{2}(0)^2+0+4=4$$

$x-$intercepts ; Let $y=0$: $-\frac{1}{2}x^2+x+4=0$

$$\Rightarrow x^2-2x-8=0$$

$$\Rightarrow (x-4)(x+2)=0$$

$$\therefore x=4 \text{ or } x=-2$$

Turning point (T.P.) : $x=-\frac{b}{2a}$

$$=-\frac{1}{2\left(-\frac{1}{2}\right)}$$

$$=1$$

Thus, $f(x)=-\frac{1}{2}x^2+x+4$

$$\Rightarrow f(1)=-\frac{1}{2}(1)^2+1+4$$

$$\Rightarrow f(1)=4\tfrac{1}{2}$$

1.2

The graph moves up by 3 units

$$y=-\frac{1}{2}x^2+x+4$$

$$\Rightarrow y=-\frac{1}{2}x^2+x+4+3$$

$$\therefore y=-\frac{1}{2}x^2+x+7$$

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1.3

New x – intercepts: 1 and 7

$$y = -\frac{1}{2}(x-1)(x-7), \quad a = -\frac{1}{2} \text{ - unchanged}$$

$$= -\frac{1}{2}x^2 + 4x - \frac{7}{2}$$

1.4 **New turning point:** $(2;\ 2\frac{1}{2})$

$$y = -\frac{1}{2}(x-2)^2 + 2\frac{1}{2}$$

$$= -\frac{1}{2}x^2 + 2x - 2 + 2\frac{1}{2}$$

$$= -\frac{1}{2}x^2 + 2x + \frac{1}{2}$$

SIMPLE TRANSFORMATIONS

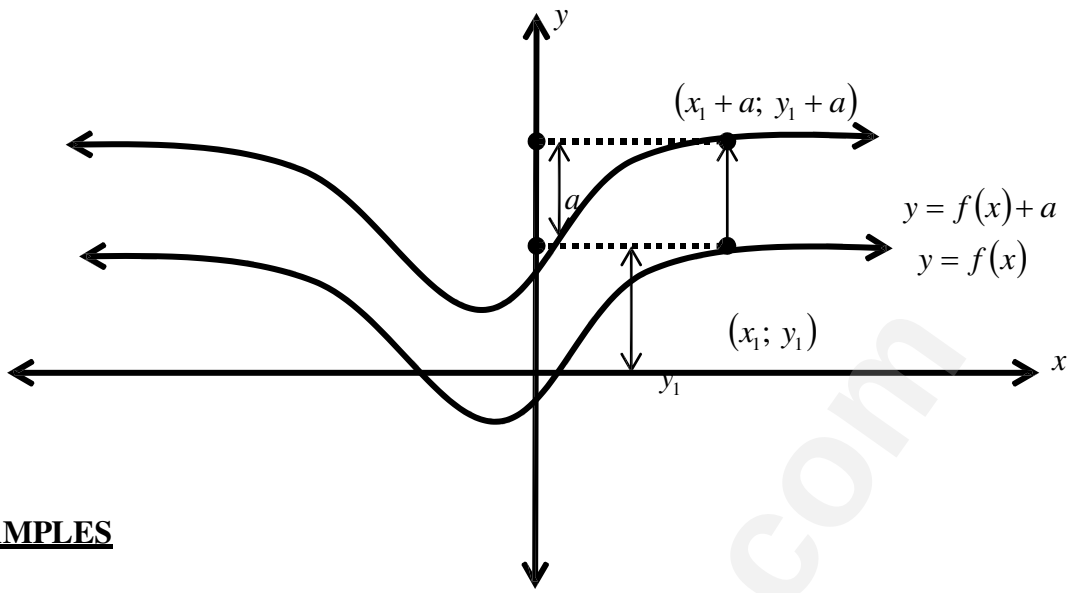
The reader will be familiar with the simple transformations, i.e. translations, reflections, rotations and enlargements. From the graph of $y = f(x)$ it is possible to deduce the graphs of other functions which are transformations of $y = f(x)$.
In this section we shall see how the graph of $y = f(x)$ can help us draw the graphs of $y = f(x) + a$, $y = f(x-a)$, $y = -f(x)$, $y = f(-x)$, $y = af(x)$ and $y = f(ax)$.

The transformation: $y = f(x) + a$

Consider a point $(x_1;\ y_1)$ on the graph of $y = f(x)$, i.e. $y_1 = f(x_1)$. The point on $y = f(x) + a$ with an $x-$ coordinate of x_1, will have a $y-$coordinate of $(y_1 + a)$. Thus for every point $(x_1;\ y_1)$ on $y = f(x)$, there exists a point $(x_1;\ y_1 + a)$ on $y = f(x) + a$.
Therefore , if we translate the graph of $y = f(x)$ by a units parallel to the $y-$axis , we obtain the graph of $y = f(x) + a$.

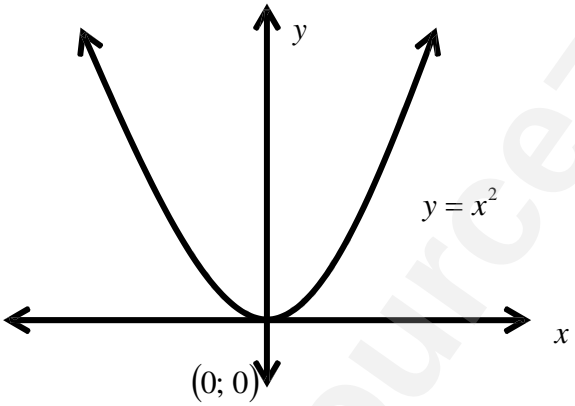
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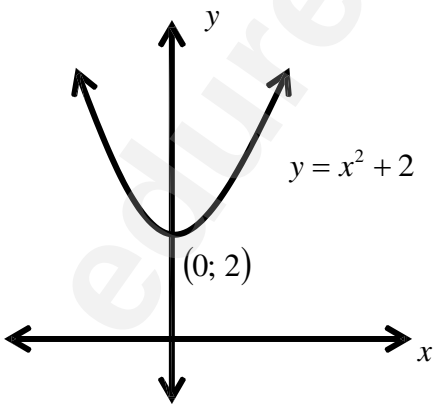


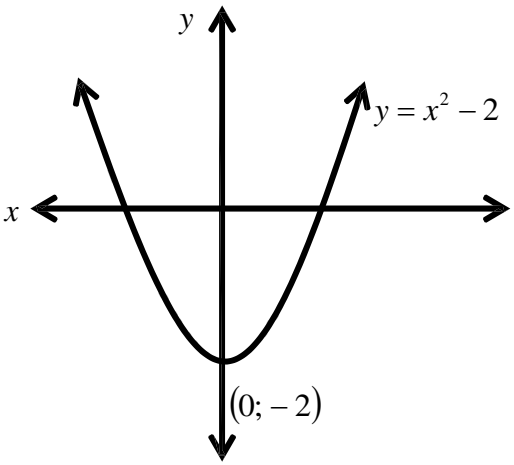
EXAMPLES

1. Given:



It follows that:





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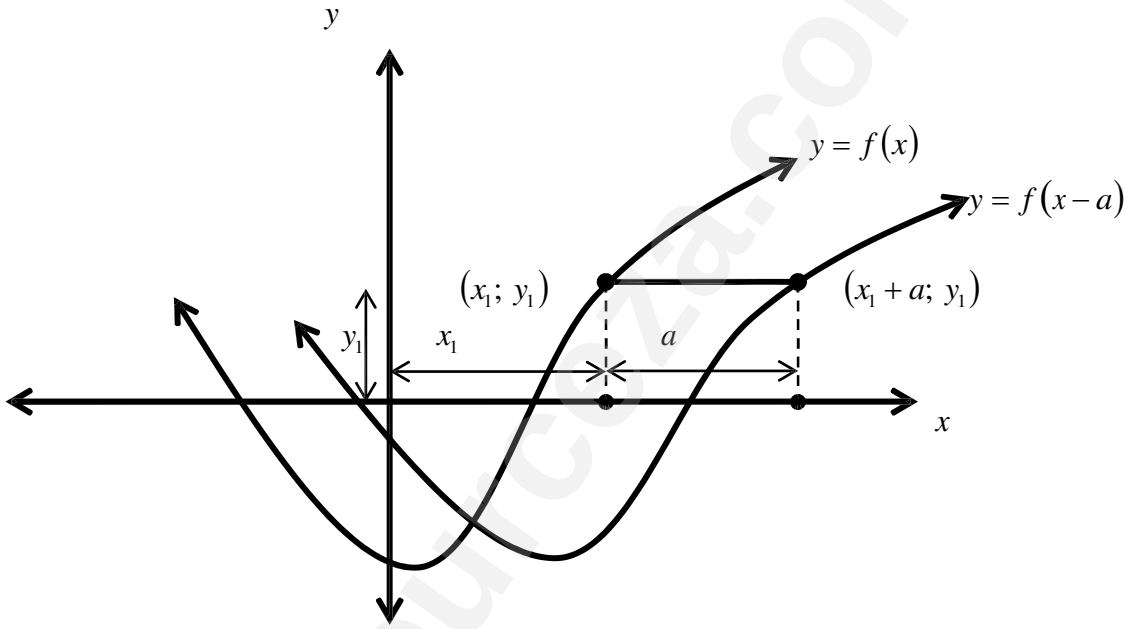
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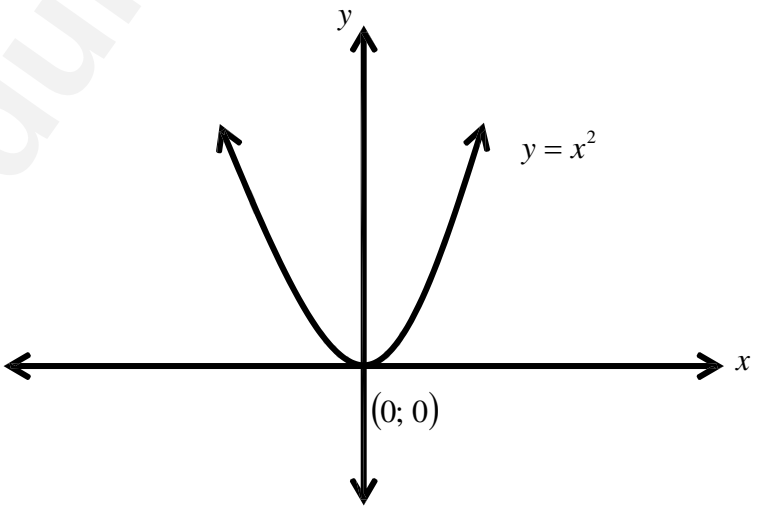
The transformation: $y = f(x-a)$

Consider a point $(x_1; y_1)$ on the graph of $y = f(x)$, i.e. $y_1 = f(x_1)$. The point on $y = f(x-a)$ with a $y-$ coordinate of y_1 will have an $x-$coordinate of $(x_1 + a)$. Thus for every point $(x_1; y_1)$ on $y = f(x)$, there exists a point $(x_1 + a; y_1)$ on $y = f(x-a)$.
Therefore, if we translate the graph of $y = f(x)$ by a units parallel to the $x-$axis, we obtain the graph of $y = f(x-a)$.



EXAMPLES

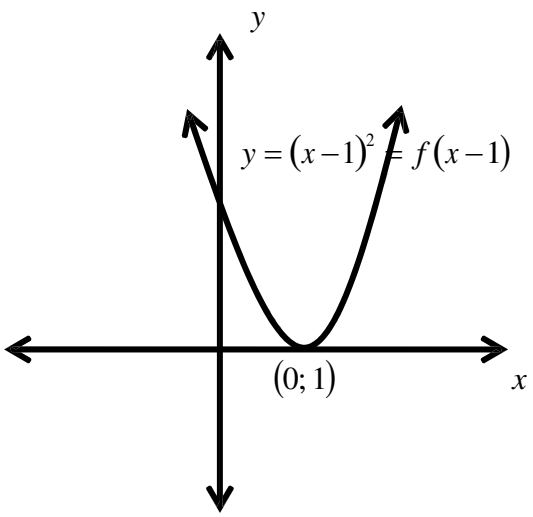
1. Given:

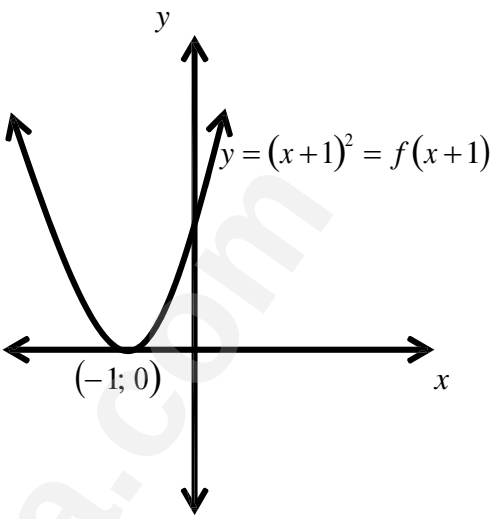


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Then the following transformations are possible:





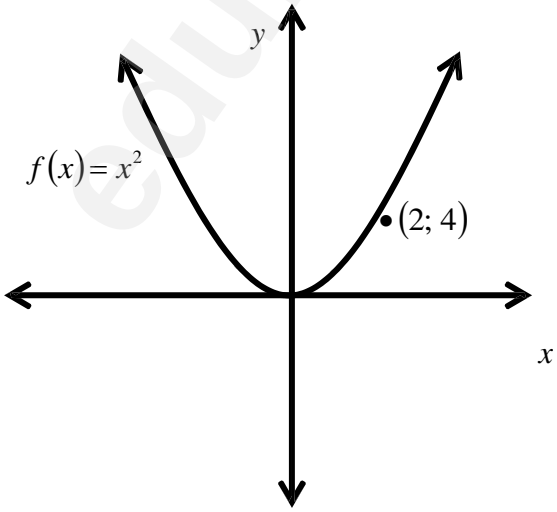
The transformation: $y=-f(x)$ and $y=f(-x)$

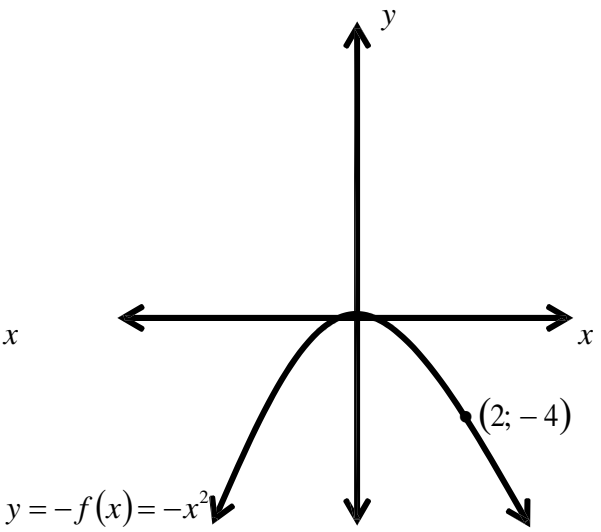
For each point $(x_1;\ y_1)$ on the graph of $y=f(x)$ there exists a point $(x_1;\ -y_1)$ on $y=-f(x)$.

Thus the graph of $y=-f(x)$ can be obtained by reflecting $y=f(x)$ in the $x-$axis.

For each point $(x_1;\ y_1)$ on the graph of $y=f(x)$, there exists a point $(-x_1;\ y_1)$ on $y=f(-x)$. Thus the graph of $y=f(-x)$ can be obtained by reflecting $y=f(x)$ in the $y-$axis.

If $f(x)=x^2$, then

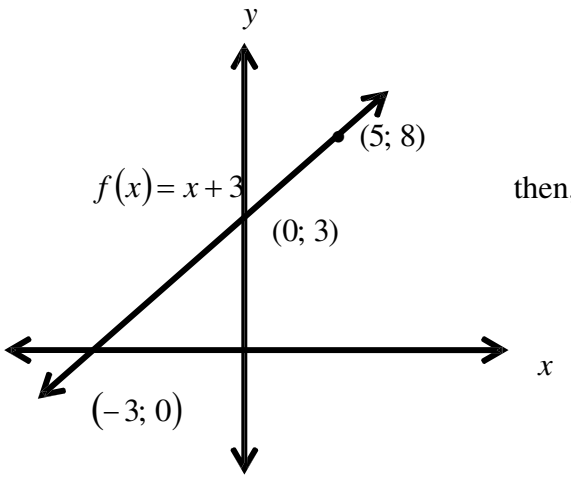




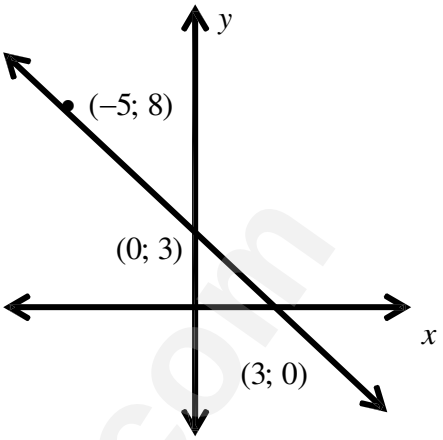
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Also if



then,



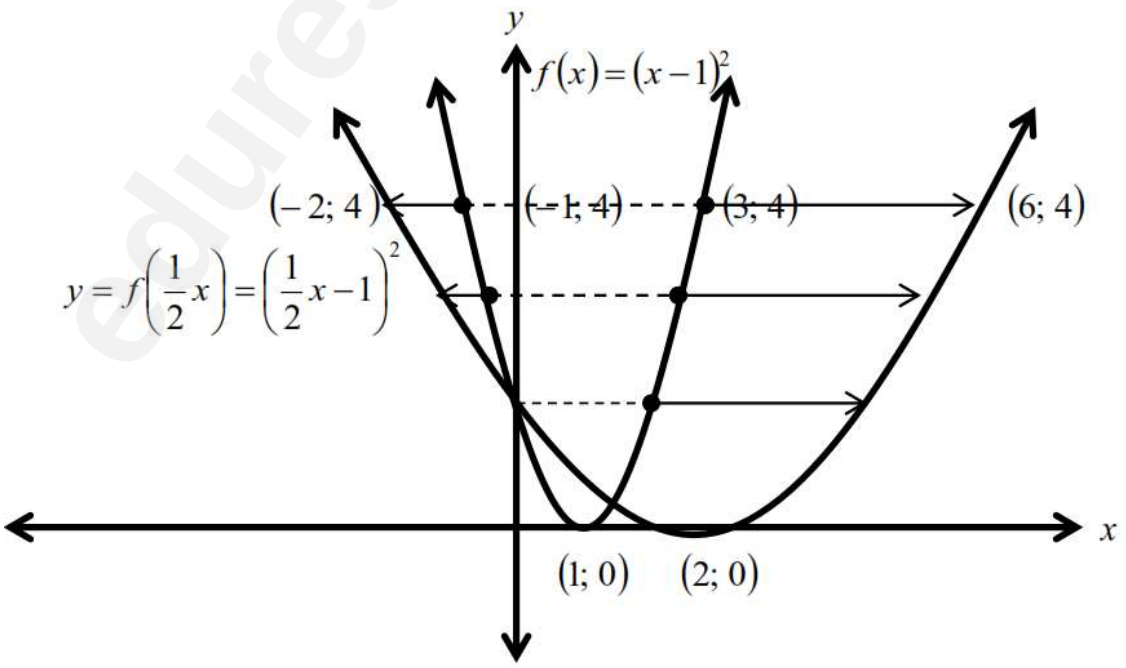
$y = f(-x) = -x + 3$

The transformation rule for reflections in the $y-$axis is $(x;\ y) \rightarrow (-x;\ y)$.

The transformation: $y = f(ax)$

For each point $(x_1;\ y_1)$ on the graph of $y = f(x)$, there exists a point $\left(\frac{x_1}{a};\ y_1\right)$ on the graph of $y = f(ax)$. Thus the graph of $y = f(ax)$ can be obtained by **stretching** $y = f(x)$ parallel to the $x-$ axis by scale factor $\frac{1}{a}$.

EXAMPLE



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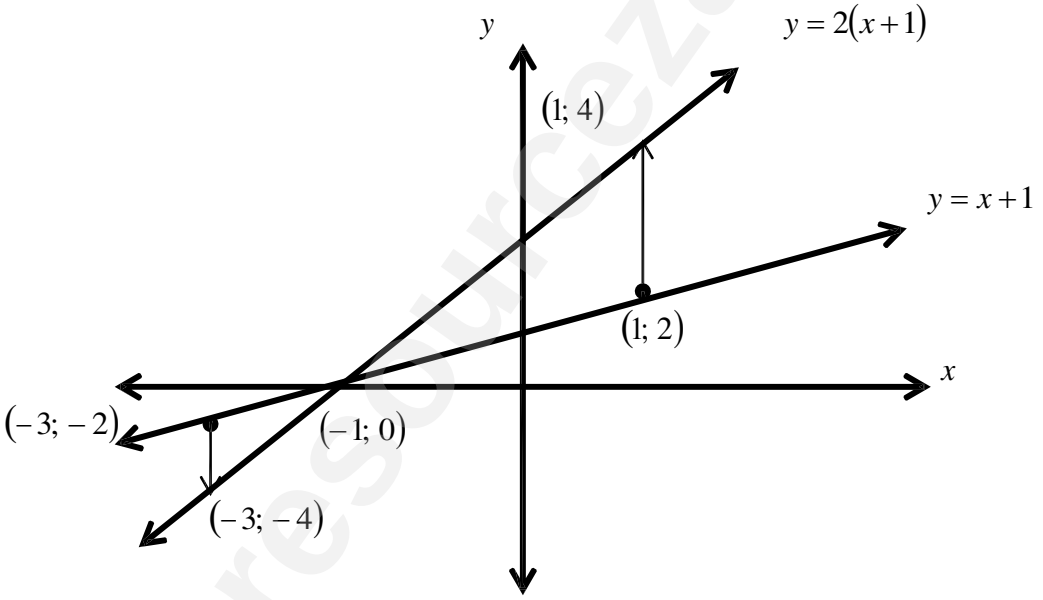
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Thus, $(x_1; y_1) \rightarrow \left(\frac{x_1}{a}; y_1\right)$ if $y_1 = f(x_1) \rightarrow y = f(ax_1)$ with $(x_1; y_1) \rightarrow \left(\frac{x_1}{\frac{1}{2}}; y_1\right) \equiv (2x_1; y_1)$ in the case $y = f\left(\frac{1}{2}x_1\right)$, yields $(1; 0) \rightarrow (2; 0)$, $(3; 4) \rightarrow (6; 4)$, $(-1; 4) \rightarrow (-2; 4)$.

The transformation: $y = af(x)$

For each point $(x_1; y_1)$ on the graph of $y = f(x)$ there exists a point $(x_1; ay_1)$ on the graph of $y = af(x)$.
Thus the graph of $y = af(x)$ can be obtained by stretching $y = f(x)$ parallel to the $y-$axis by a scale factor a.

EXAMPLE



Note: If a is negative, say -3, we can still talk of a stretch parallel to the $y-$axis by scale factor -3. This means that the graph is reflected in the $x-$axis and then stretched by a scale factor 3 because, for negative a, $y = af(x)$ is a combination of $y = -f(x)$ and $y = |a|f(x)$. Thus a reflection in the $x-$axis could be viewed as a stretch parallel to the $y-$axis with scale factor -1.

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THE RECTANGULAR HYPERBOLA

The general equation of a rectangular hyperbola is

$$y = \frac{a}{x-p} + q,\ a \neq 0,\ x \neq p$$

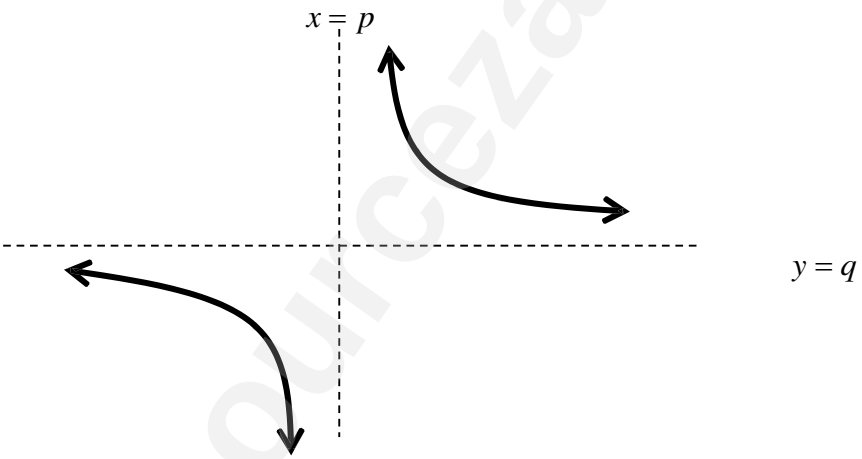
Where: (a) $x = p$ is the equation of the **vertical asymptote**

(b) $y = q$ is the equation of the **horizontal asymptote**

(c) The " a " determines the positions of the parts of the rectangular hyperbola in the respective quadrants.

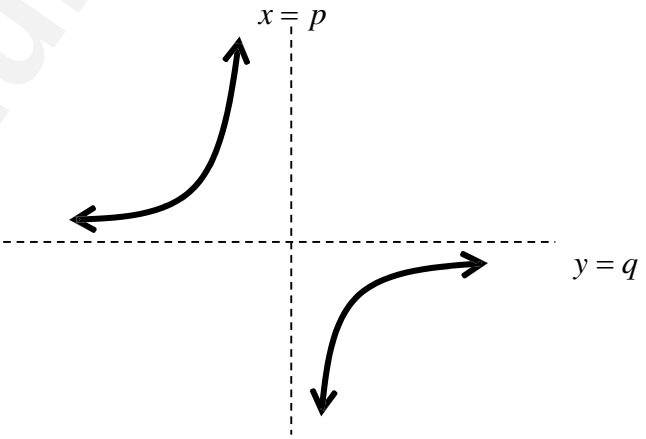
THE TWO POSSIBLE CASES

Case 1: If $a > 0$, the following is a possible graph, where only the **vertical and horizontal asymptotes** have been shown:



Note: The pieces/ parts of the rectangular hyperbola occupy the first and second quadrants.

Case 2: If $a < 0$, the following is a possible graph where only the asymptotes have been drawn:



Note: The pieces of the rectangular hyperbola, appear in the second and fourth quadrants.

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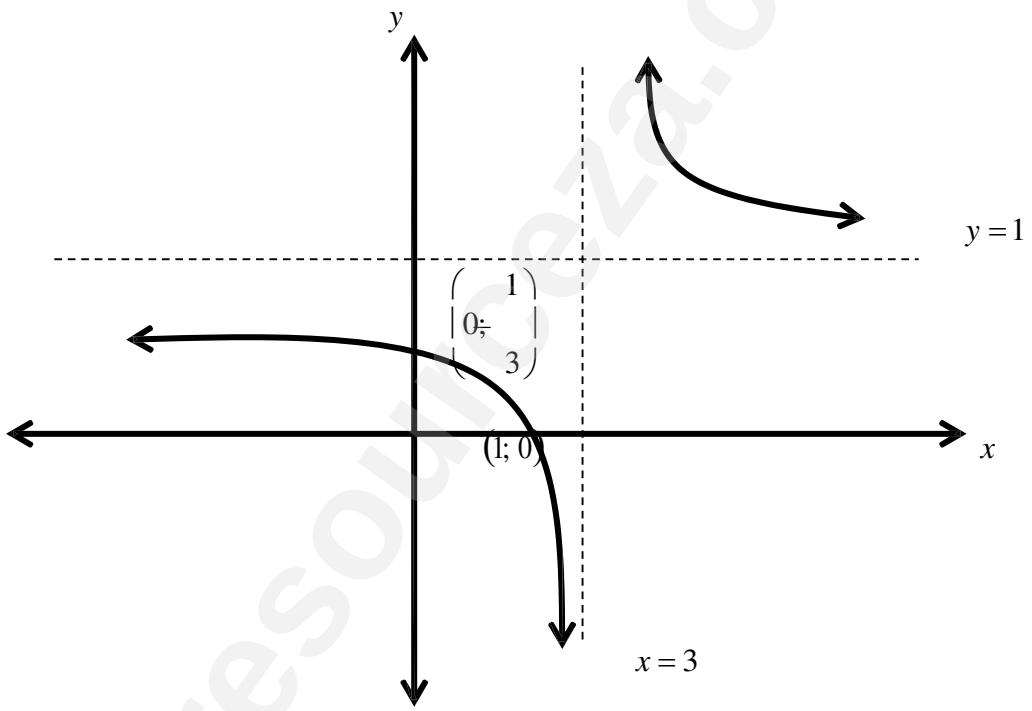
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EXAMPLE

Consider:

$$y = \frac{2}{x-3} + 1$$

1.Write down the equations of the vertical and horizontal asymptotes.
2. Determine the x – intercept of the hyperbola.
3. Determine the y – intercept of the hyperbola.
4. Hence, or otherwise, draw the graph of the rectangular hyperbola above.



EQUATIONS OF LINES OF SYMMETRY

1. Given a rectangular hyperbola whose defining equation is:

$$y = \frac{a}{x-p} + q$$

It follows that the following are lines of symmetry of the rectangular hyperbola:
(a) $y = x - p + q$

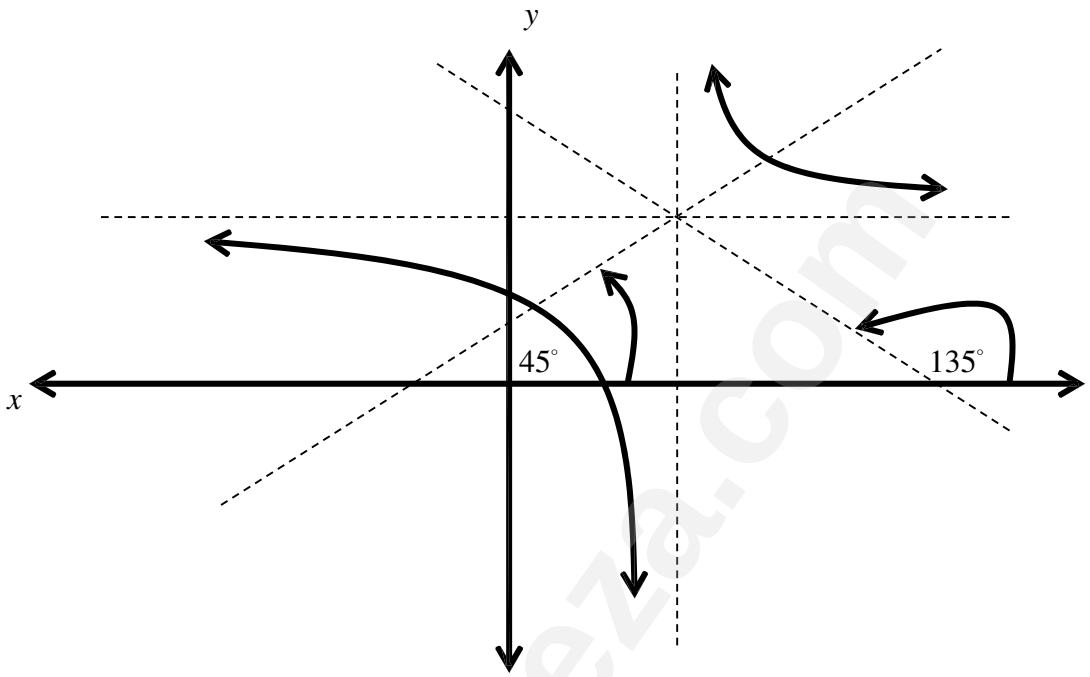
(b) $y = -x + p + q$

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Proof:

Consider the special case with $x = p > 0$, $y = q > 0$ with $a > 0$.



LINES OF SYMMETRY WITH POSITIVE GRADIENT

The formula for determine the equations of a straight line:

$y = mx + c$

The line of symmetry with positive gradient makes angle of 45° With the positive $x-$ axis so that the corresponding gradient is:

$m = \tan 45^\circ = 1$

Thus, $y = mx + c$

$\Rightarrow y = x + c$

Through $(p; q)$: $q = p + c$

$\therefore c = q - p = -p + q$

$\therefore y = x - p + q$

LINE OF SYMMETRY WITH NEGATIVE GRADIENT

The line of symmetry with negative slope of a rectangular hyperbola makes an angle of inclination of 135° with the positive $x-$axis, so that the gradient is:

$m = \tan 135^\circ = -1,$

Thus, $y = mx + c$

$\Rightarrow y = -x + c$

Through, $(p; q)$: $q = -p + c$

$\therefore c = p + q$

$\therefore y = -x + p + q$

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SUMMARY

For any rectangular hyperbola:

1. **Line of symmetry with positive gradient is given by:**

$$y = x - p + q$$

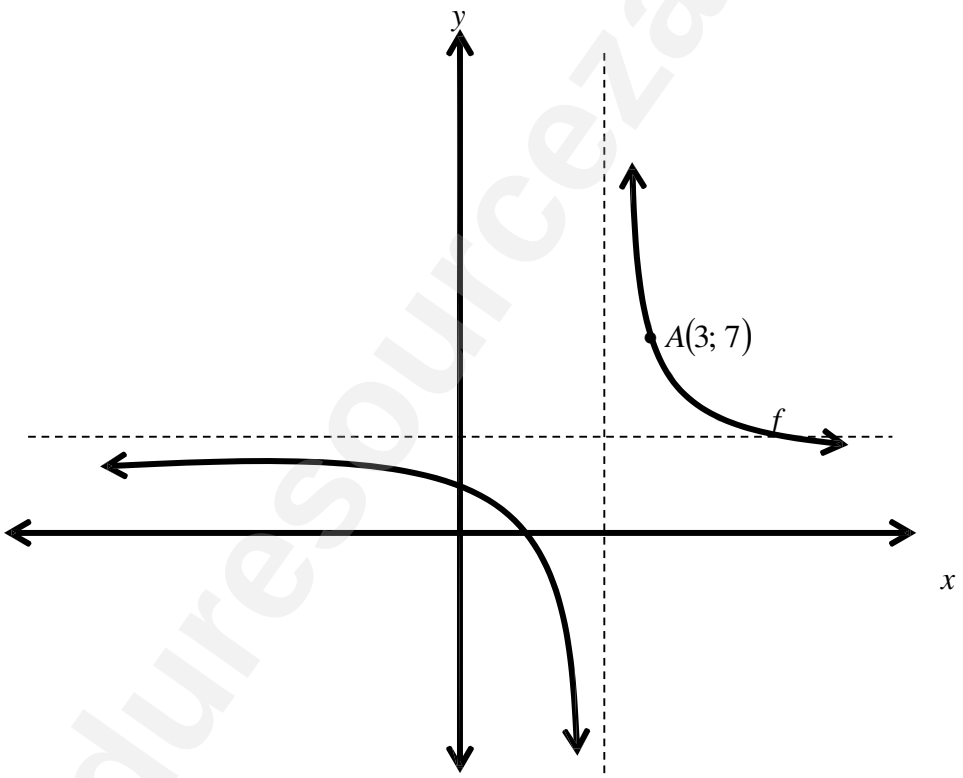
2. **Line of symmetry with negative gradient is given by**

$$y = -x + p + q$$

EXAMPLE

The diagram below shows the graph of $f(x) = \frac{p}{x-2} + 4$

The point $A(3;7)$ is given on the graph of f.



1. Write down the equations of the asymptotes of $f(x)$.

2. Find the equations of the axes of symmetry of f.

3. Determine the value of p.

4. Determine the intercept(s) of $f(x)$ with the axes.

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SOLUTION:

1. Given: $y = \frac{p}{x-2} + 4$, it follows by comparison with $y = \frac{a}{x-p} + q$ that

$x = p = 2$ **(Vertical asymptote)**

$y = q = 4$ **(Horizontal asymptote)**

2. **The equation of the axis of symmetry of the rectangular hyperbola with positive gradient:**

$y = x - p + q$

The axis of symmetry passes through: $(2; 4)$ so that

$y = x - 2 + 4$

$y = x + 2$

The line of symmetry with negative gradient of the rectangular hyperbola is given by:

$y = -x + p + q$

The axis of symmetry passes through $(2; 4)$ so that

$y = -x + 2 + 4$

$\therefore y = -x + 6$

3. Given $y = \frac{p}{x-2} + 4$ which passes through $(2; 4)$, we obtain:

$$7 = \frac{p}{3-2} + 4$$

$$\Rightarrow 7 = p + 4$$

$$\therefore p = 3$$

4. $x-$intercepts: let $y = 0$: $y = \frac{3}{x-2} + 4$

$$\Rightarrow -4 = \frac{3}{x-2}$$

$$\Rightarrow -4x + 8 = 3$$

$$\therefore x = \frac{5}{3}$$

Thus, $A\left(\frac{5}{3}; 0\right)$ is the $x-$intercept of the rectangular hyperbola.

$y-$intercept ; let $x = 0$

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$$y = \frac{3}{x-2} + 4$$

$$\Rightarrow y = \frac{3}{0-2} + 4$$

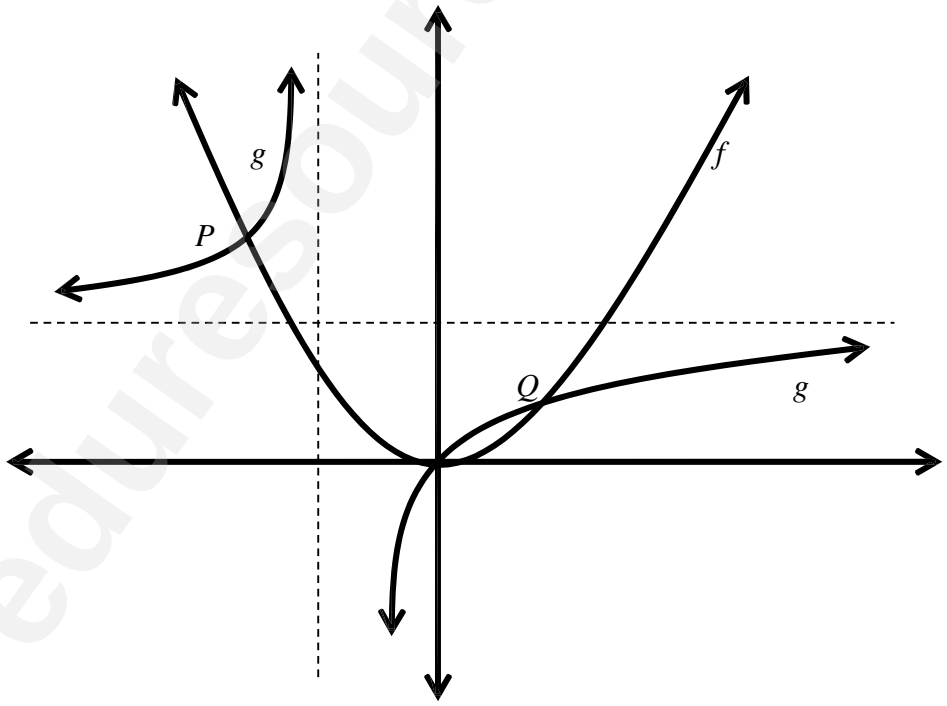
$$\therefore y = \frac{5}{2}$$

Thus, $\left(0; \frac{5}{2}\right)$ is the $y-$intercept of the rectangular hyperbola.

EXAMPLE

Sketched below are graphs $f(x) = \frac{1}{2}x^2$ and $g(x) = -\frac{1}{x+1}$

P and Q are points of intersection of f and g.



1.1 Show that the co-ordinates of P and Q are $P(-2; 2)$ and $Q\left(1; \frac{1}{2}\right)$.

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1.2 An axis of symmetry of the graph of g is a straight line defined as $y = mx + c$,

where $m > 0$. Write down the equation of this straight line in the for $y = h(x) = \ldots.$

1.3 Determine the equation of h^{-1} in the form $y = \ldots.$

1.4 Show algebraically that $g(x) + g\left|\frac{1}{x}\right| = g(-x).g(x-1)$, $(x \neq 0 \quad x \neq 1)$

,

SOLUTION:

1.1 We note that P and Q are points of intersection of f and g, so that

$$f(x) = g(x)$$

$$\Rightarrow \frac{1}{2}x^2 = -\frac{1}{x+1} + 1$$

$$\Rightarrow x^2(x+1) = -2 + 2(x+1), \qquad \times 2(x+1)$$

$$\Rightarrow x^3 + x^2 = -2 + 2x + 2$$

$$\Rightarrow x^3 + x^2 - 2x = 0$$

$$\Rightarrow x(x^2 + x - 2) = 0$$

$$\Rightarrow x(x+2)(x-1)$$

$\therefore x = 0$ or $x = -2$ or $x = 1$

$f(0) = \frac{1}{2}(0)^2$ $\quad$ $f(-2) = \frac{1}{2}(-2)^2$ $\quad$ $f(1) = \frac{1}{2}(1)^2$

$f(0) = 0$ $\quad$ $f(-2) = 2$ $\quad$ $f(1) = \frac{1}{2}$

Therefore $(0; 0)$, $(-2; 2)$ and $\left(1; \frac{1}{2}\right)$.

$\therefore P(-2; 2)$ and $Q\left(1; \frac{1}{2}\right)$

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1.2 The equation of the line of symmetry, $y = mx + c$, of $g(x) = -\dfrac{1}{x+1} + 1$ with $m > 0$ is given by $y = x - p + q$

The line of symmetry passes through $(-1; 1) \equiv (p; q)$, so that $p = -1$ and $q = 1$.

Thus, $y = x - (-1) + 1$

$\therefore y = x + 2$

$\therefore y = h(x) = x + 2$

1.3 From 1.2 above we obtained that $y = h(x) = x + 2$, so that for the inverse function, we swop x and y and obtain, $x = y + 2$

$$y = x - 2 \text{ where } h^{-1}(x) = x - 2$$

1.4 We are required to prove that $g(x) + g\left|\left(\dfrac{1}{x}\right)\right| = g(-x).g(x-1); (x \neq 0,\ x \neq 1)$.

To this end,

Let, $LHS = g(x) + g\left|\left(\dfrac{1}{x}\right)\right|$

$$= -\frac{1}{x+1} + 1 - \frac{1}{\frac{1}{x} + 1}$$

$$= \frac{-1}{x+1} + 1 - \frac{x}{1+x} + 1$$

$$= \frac{-(1+x)}{1+x} + 2$$

$$= -1 + 2$$

$$= 1$$

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$$RHS = g(-x)\cdot g(x-1)$$

$$= \left(-\frac{1}{-x+1}+1\right)\left(-\frac{1}{x-1+1}+1\right)$$

$$= \left(\frac{-1-x+1}{-x+1}\right)\left(\frac{-1+x}{x}\right)$$

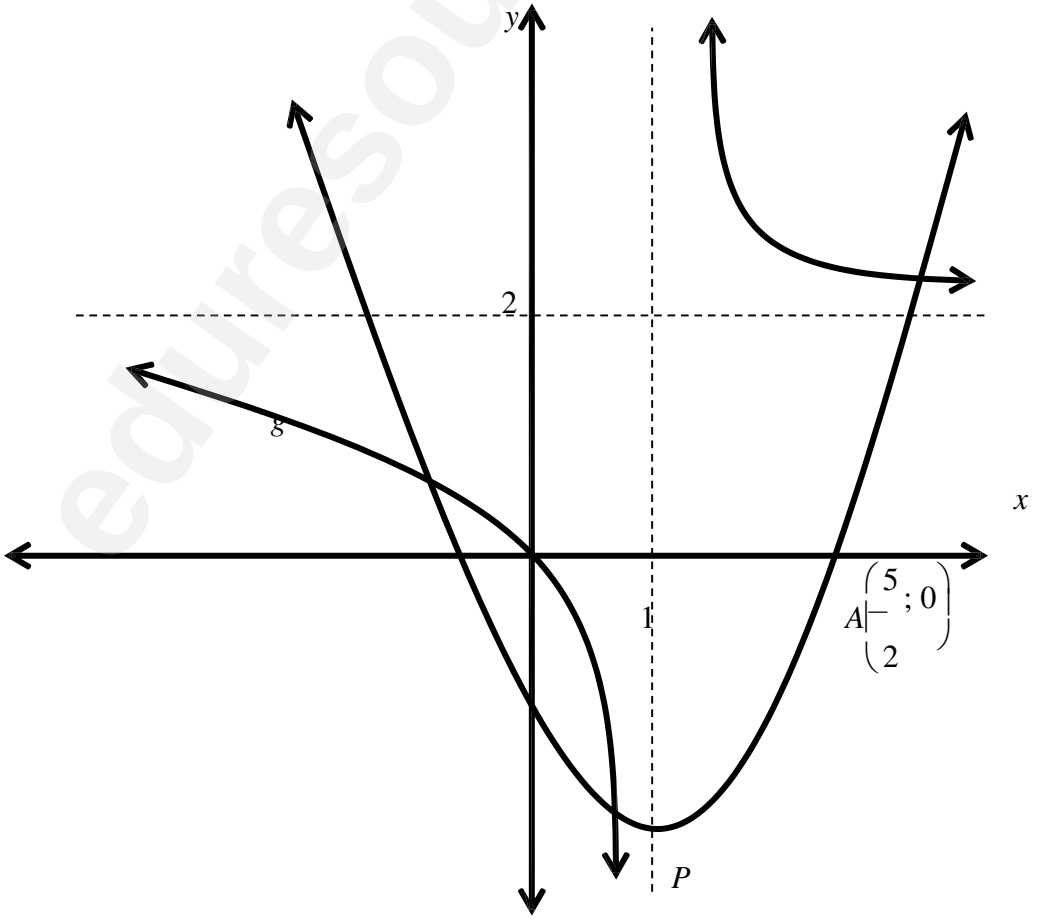
$$= \left(\frac{-x}{-x+1}\right)\left(\frac{-(1-x)}{x}\right)$$

$$= 1$$

EXAMPLE

Sketch below are graphs of $f(x)=(x-p)^2+q$ and $g(x)=\frac{a}{x-b}+c$.

$A(2\frac{1}{2};0)$ is a point on the graphs of f. P is the turning point of f. The asymptotes of f and g are represented by the dotted line. The graph of g passes through the origin.



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13.1 Determine the equation of the equation of g .

13.2 Determine the co-ordinates of P , the turning point of f .

13.3 Write down the equations of the asymptotes of $g(x-1)$.

13.4 Write down the equations h , if h is the image of f reflected on the $x-$axis.

SOLUTION:

13.1 We first note that $b=1$ (Vertical asymptote)
$c=2$ (Horizontal asymptote)

Thus, $g(x)=\dfrac{a}{x-b}+c$

$$\Rightarrow g(x)=\frac{a}{x-1}+2$$

But g passes through $(0;0)$ so that $0=\dfrac{a}{0-1}+2$

$$\Rightarrow a=2$$

$$\therefore g(x)=\frac{2}{x-1}+2$$

13.2 $f(x)=(x-p)^2+q \Rightarrow f(x)=(x-1)^2+q$

$$\Rightarrow 0=\left(\frac{5}{2}-1\right)^2+q$$

$$\therefore q=-\frac{9}{4}$$

Therefore, $f(x)=(x-1)^2-\dfrac{9}{4}$ So that

$\left(1;-\dfrac{9}{4}\right)$ is the turning point of f .

13.3 $g(x)=\dfrac{2}{x-1}+2$

$$\Rightarrow g(x-1)=\frac{2}{x-1-1}+2$$

$$\therefore g(x-1)=\frac{2}{x-2}+2$$

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Hence: $x=2$ is the equation of the vertical asymptote of $g(x-1)$

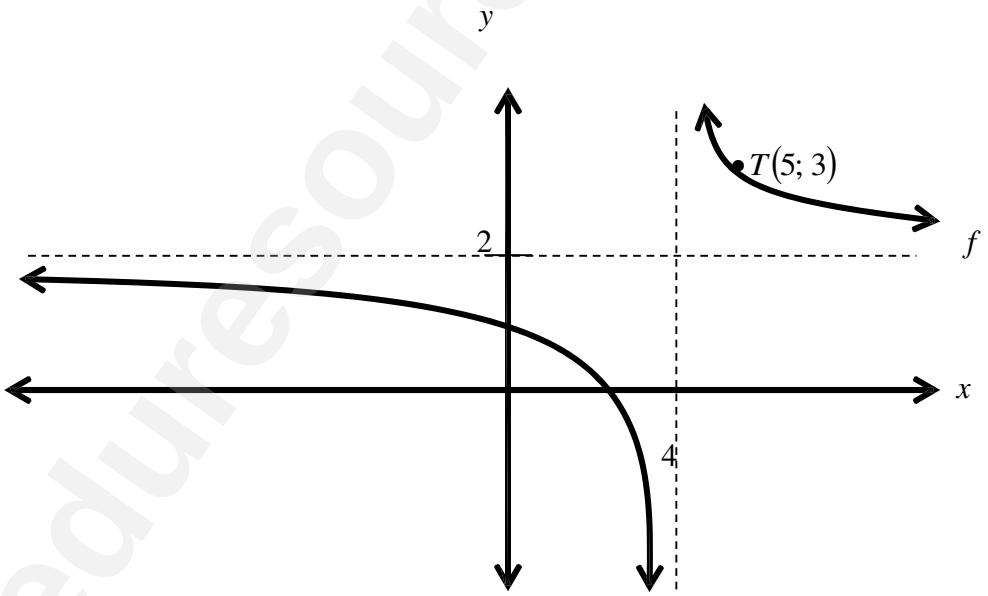
$y=2$ is the equation of the horizontal asymptote of $g(x-1)$

13.4 If $f(x)=(x-1)^2-\frac{9}{4}$ is reflected in the $x-$axis then $(x;\ y)\rightarrow(x;-y)$ is the transformation rule, so that:

$$h(x)=-f(x)$$
$$=-\left\{(x-1)^2-\frac{9}{4}\right\}$$
$$=-(x-1)^2+\frac{9}{4}, \text{ as required.}$$

EXAMPLE

The diagram below represents the graph of $g(x)=\frac{a}{x-p}+q$. $T(5;3)$ is a point on f.



1.1 Determine the values of a, p and q.

1.2 If the graph of f is reflected across the line having equation $y=-x+c$, the new graph coincides with the graph of $y=f(x)$. Determine the value of c.

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SOLUTION:

1.1 $p = 4$

$q = 2$

$3 = \frac{a}{5-4} + 2$

$1 = \frac{a}{1}$

$\therefore a = 1$

1.2 $y = -x + p + q$ is the line of symmetry through $(4; 2) \equiv (p; q)$, thus

$y = -x + 4 + 2$

$\therefore y = -x + 6$, so that $c = 6$

OR/ Alternatively

$y = -x + c$

Substitute $(4; 2)$

$2 = -4 + c$

$\therefore c = 6$

OR / Alternatively

Translation of the line $y = -x$ 2 units up and 4 units right $y = -(x-4) + 2$

$\Rightarrow y = -x + 6$

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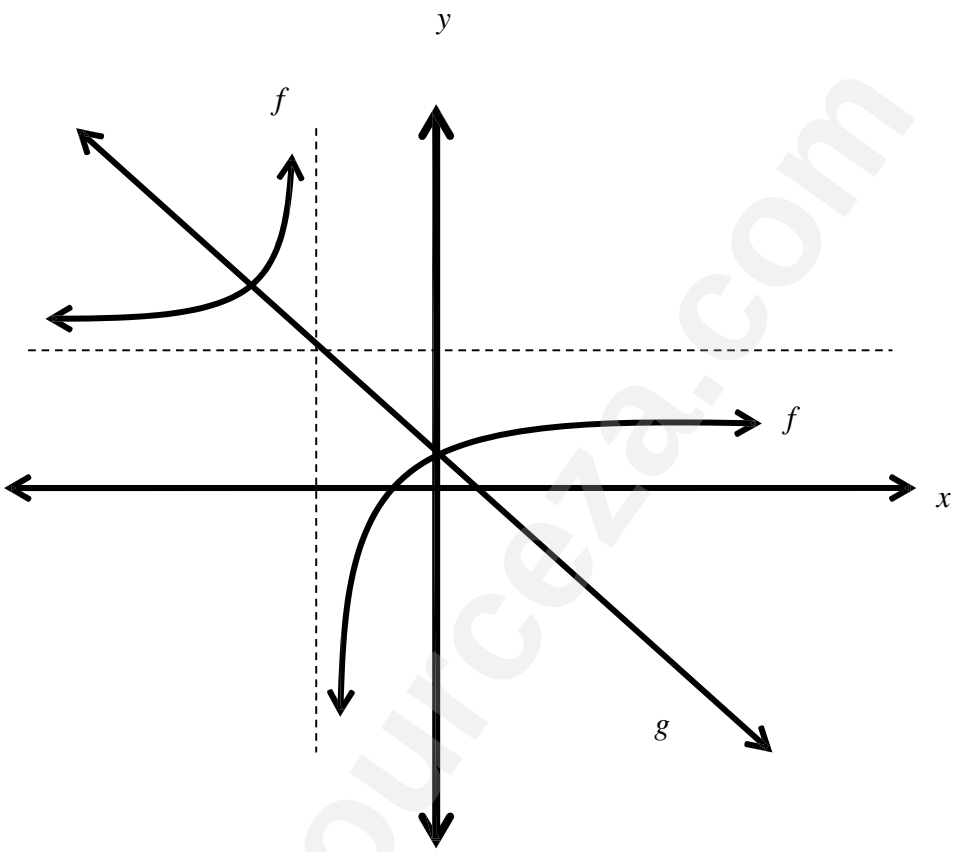
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EXAMPLE

The graphs of $f(x)=\frac{-3}{x+1}+5$ and $g(x)=-3x+2$ are sketched below.



1.1 Write down the range of $f(x)$.

1.2 Determine the co-ordinates of the points of intersection of f and g.

1.3 Describe the transformation of f to h if $h(x)=\frac{3}{x+1}+5$

Solution:

1.1 $y \in$

OR

$y \in (-\infty, 5) \cup (5, \infty)$

1.2 $\frac{-3}{x+1}+5=-3x+2$

$\Rightarrow -3+5(x+1)=-3x(x+1)+2(x+1)$

$\Rightarrow -3+5x+5=-3x^2-3x+2x+2$

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$\Rightarrow 3x^2 + 6x = 0$

$\Rightarrow 3x(x+2) = 0$

$\therefore x = 0 \quad \text{or} \quad x = -2$

$\therefore y = 2 \quad \text{or} \quad y = 8$

Thus, $(0;2)$ and $(-2;8)$ are points of intersection.

1.3 Reflection about the asymptote $x = -1$

OR/ Alternatively
Reflection about the asymptote $y = 5$

OR
Reflection about the $x-$axis and translated up by 10 units.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

THE EXPONENTIAL FUNCTION

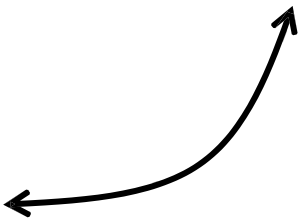
Definition:
The general equation of an exponential function is given by

$$y = ab^{x-p} + q \ , \ b > 0 .$$

where $y = q$ is the equation of the **horizontal asymptote** of the **exponential function**.
It must be pointed out that the exponential function has no vertical asymptote.

Remark:

- If $b > 1$ then the exponential function is said to be **increasing** so that the following is the **possible shape.**



- If $0 < b < 1$ then the exponential function is said to be **decreasing** so that the following is the possible shape.



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- If $p > 0$, the exponential function $y = ab^x + q$ will be **translated** p units, **horizontally**, to the **right**, otherwise to the **left**.
- If $q > 0$, the exponential function $y = ab^{x-p}$ will be **translated** q units, **vertically**, **upwards,** otherwise **downwards.**
- To find $x-$intercepts we let $y = 0$ and then solve the equation $ab^{x-p} + q = 0$ for x.
- To find a $y-$intercept we let $x = 0$ and then find the numerical value of y by using the equation $y = ab^{x-p} + q$.
- Any **exponential function** of the form $y = ab^{x-p}$ does not have an $x-$intercept.

EXAMPLE

Sketch the graph of each of the following graphs on separate sets of axes.

(a) $y = 3^x$

(b) $y = \left(\frac{1}{2}\right)^x$

(c) $y = 3.2^{x-1} + 5$

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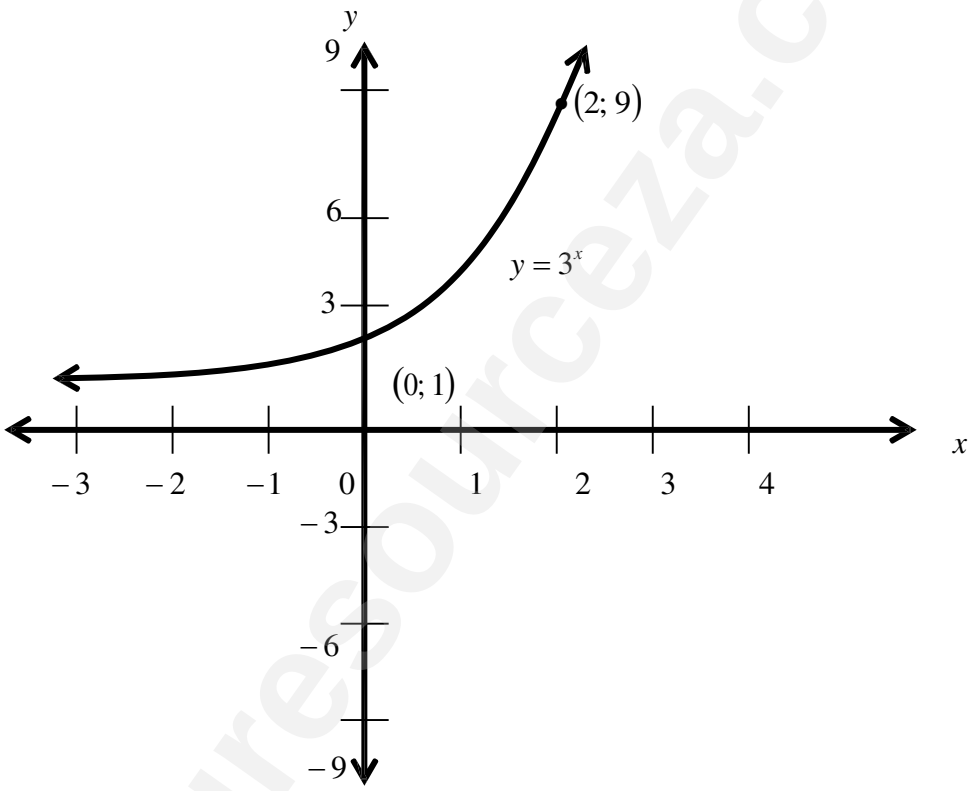
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SOLUTION:

(a) By using the point-by-point plotting method we obtain the following table of $x-$ and $y-$ values:

X	-2	-1	0	1	2
Y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Thus the following sketch can be drawn:



Only two points have been shown on the graph, as the rest of the points can be similarly plotted.

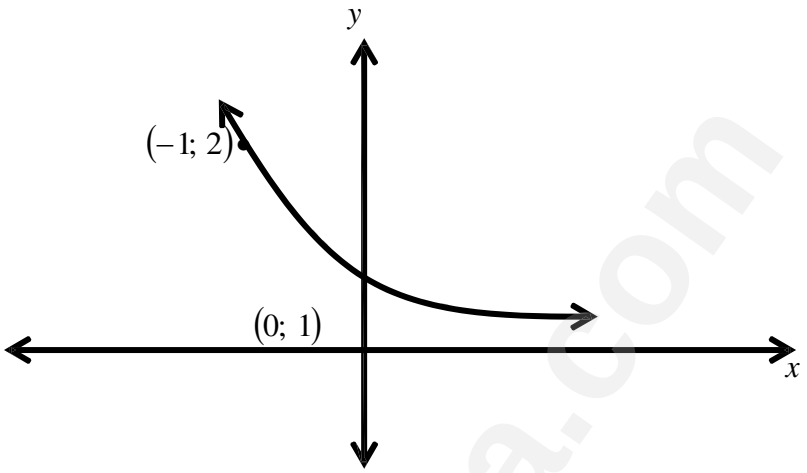
It must be noted that the **exponential function** above is said to be **increasing** as the $y-$ values increase with increasing $x-$ values.

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(b) Similar reasoning shows that, the following sketch is the graph

$$y = \left(\frac{1}{2}\right)^x .$$



(c) Given $y = 3 \cdot 2^{x-1} + 5$, we first compare with $y = ab^{x-p} + q$ so that:

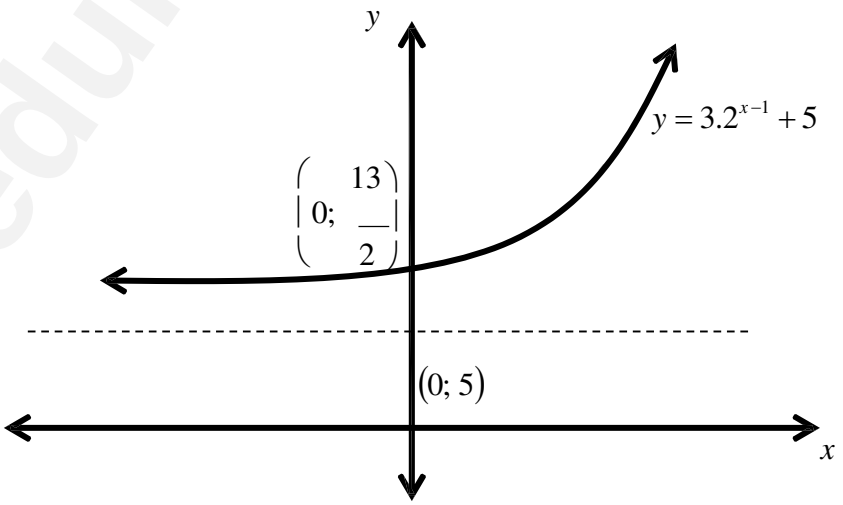
(i) $q = 5$ is the equation of the horizontal asymptote.

(ii) $p = 1 > 0$ indicates a **horizontal translation of** 1 unit to the right of the function $y = 3 \cdot 2^x + 5$

(iii) $y-$ intercept, let $x = 0$; $y = 3.2^{x-1} + 5 \Rightarrow y = 3 \cdot 2^{0-1} + 5 = \frac{3}{2} + 5 = \frac{13}{2} = 6{,}5$

$x-$ intercept , let $y = 0$; $3 \cdot 2^{x-1} + 5 = 0 \Rightarrow 3 \cdot 2^{x-1} = -5$, which is meaningless since

$3 \cdot 2^{x-1} > 0$. Thus the exponential function $y = 3 \cdot 2^{x-1} + 5$, has no $x-$ intercept.



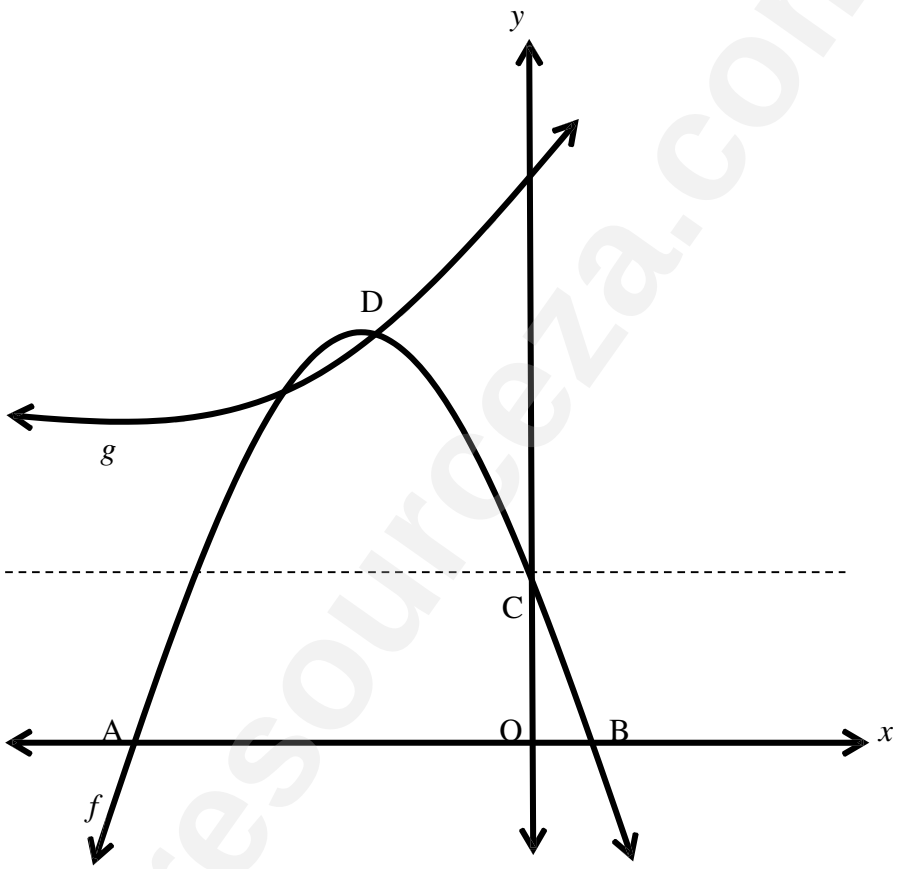
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EXAMPLE [ADDITIONAL EXEMPLE]

The graphs of $f(x)=-(x+1)^2+4$ and $g(x)=a\cdot 3^x+q$ are sketched below.

A and B are the $x-$intercepts of f. C is the $y-$intercept of f and lies on the asymptote of g. The two graphs intersect in D, the turning point of f.



1.1 Determine the coordinates of A and B.

1.2 Determine the equation of g.

1.3 Calculate the coordinates of the point on f for which the tangent to f will have a gradient of 1.

1.4 Write down the values of k for which $f(x)-k$ will always be a negative value.

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SOLUTIONS

1.1 $f(x) = -(x+1)^2 + 4$, $x-$ intercept; let $f(x) = 0$

$0 = -(x+1)^2 + 4$

$(x+1)^2 = 4$

$x+1 = 2$ or $x+1 = -2$

$\therefore x = 1$ or $x = -3$

OR/ Alternatively

$f(x) = -(x+1)^2 + 4$

$0 = -x^2 - 2x - 1 + 4$

$x^2 - 2x - 3 = 0$

$(x-1)(x+3) = 0$

$\therefore x = 1$ or $x = -3$

$A(-3; 0)$ and $B(1; 0)$

1.2 $C(0; 3)$ with

$y = a.3^x + 3$, yield

$4 = a.3^{-1} + 3$

$\frac{a}{3} = 1$

$\therefore a = 3$

$\therefore g(x) = 3.3^x + 3$

1.3 $f(x) = -x^2 - 2x + 3$

$f'(x) = -2x - 2$

$1 = -2x - 2$

$3 = -2x$

$\therefore x = -\frac{3}{2}$

Point $\left(-\frac{3}{2}; \frac{15}{4}\right)$

1.4 $k > 4$

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THE LOGARITHMIC FUNCTION

Elementary theory of logarithms

Definition:

$a^x = b \Leftrightarrow x = \log_a b$, $a, b \in \mathbb{R}^+$ (set of positive real numbers)

Thus the logarithm of x to the base a is the power to which a must be raised to give x.

If $p = \log_a x$ and $q = \log_a y$ then $a^p = x$ and $a^q = y$.

Thus $xy = a^{(x+y)}$ i.e. $\log_a(xy) = p + q \Rightarrow \boxed{\log_a(xy) = \log_a x + \log_a y}$

$\frac{x}{y} = a^{(p-q)}$ i.e. $\log_a\left(\frac{x}{y}\right) = p - q \Rightarrow \boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$

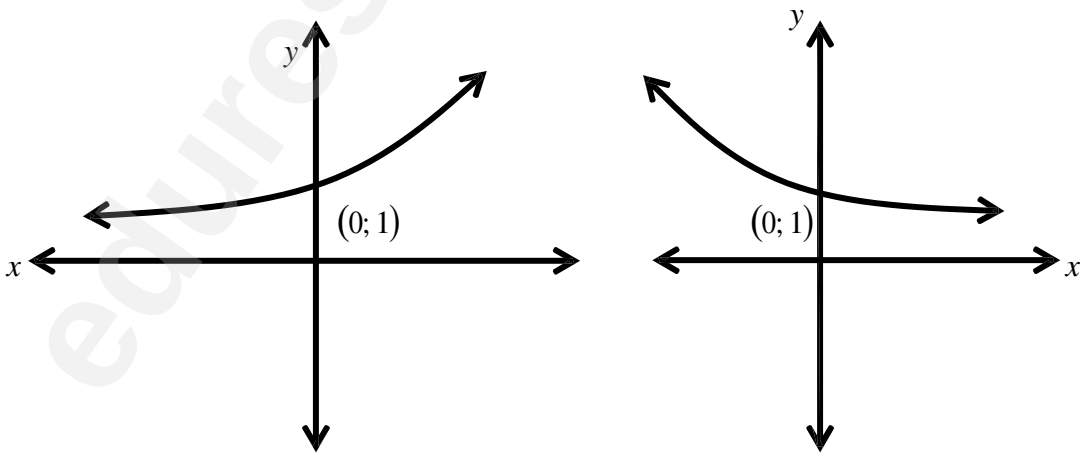
$x^n = a^{pn}$ i.e. $\log_a x^n = pn \Rightarrow \boxed{\log_a x^n = n \log_a n}$

If $r = \log_a a$, then $a^r = a$, i.e. $r = 1 \Rightarrow \boxed{\log_a a = 1}$

Graphs

Exponential functions revisited

1. $f : x \mapsto a^x$, with $a > 0$; $a \neq 1$; $x \in$



EXAMPLE

Draw $f(x) = 3^x$ and $g(x) = 3^{-x}$ on the same system of axes.

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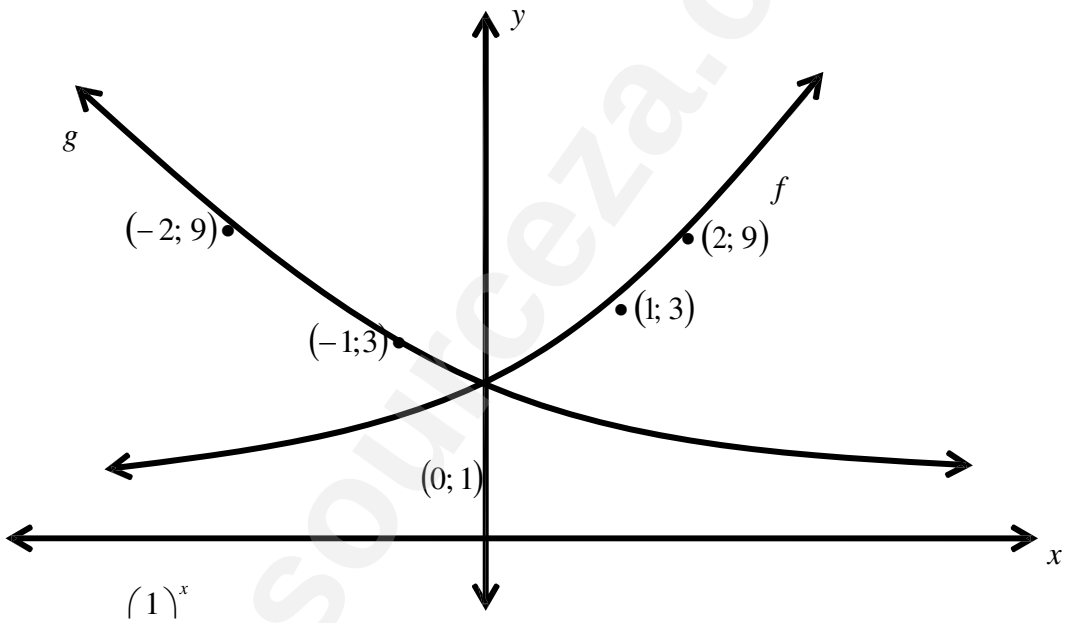
Solution:

By the point by point plotting method we have the following tables:

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	-2	-1	0	1	2
$g(x)$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

Thus plotting yields



Note: 1. $y = 3^{-x} = \left(\frac{1}{3}\right)^x : 0 < a < 1$

2. $y = 3^x$ and $y = 3^{-x}$ are mirror images with respect to the $y-$axis.
Since the transformation rule $(x;\ y) \to (-x;\ y)$ applies in this case.

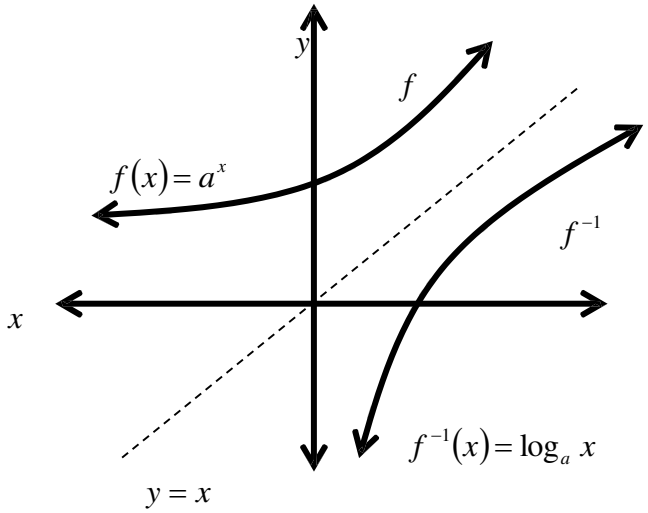
Logarithmic function

We first recall that the reflection of any function say f about the line $y = x$ yields a the corresponding inverse function f^{-1}.
We must also not that: $(x;\ y) \to (y;\ x)$ for ant reflection in the line $y = x$.
Thus the inverse of $f : x \to a^x$ is $f^{-1} : x \to \log_a x$.

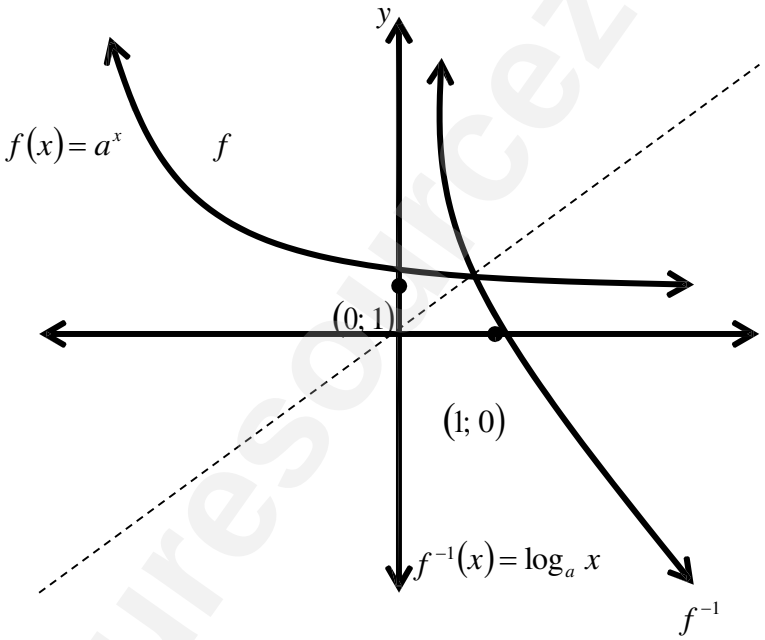
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Graphically we have that:



$a>0$



$0<a<1$

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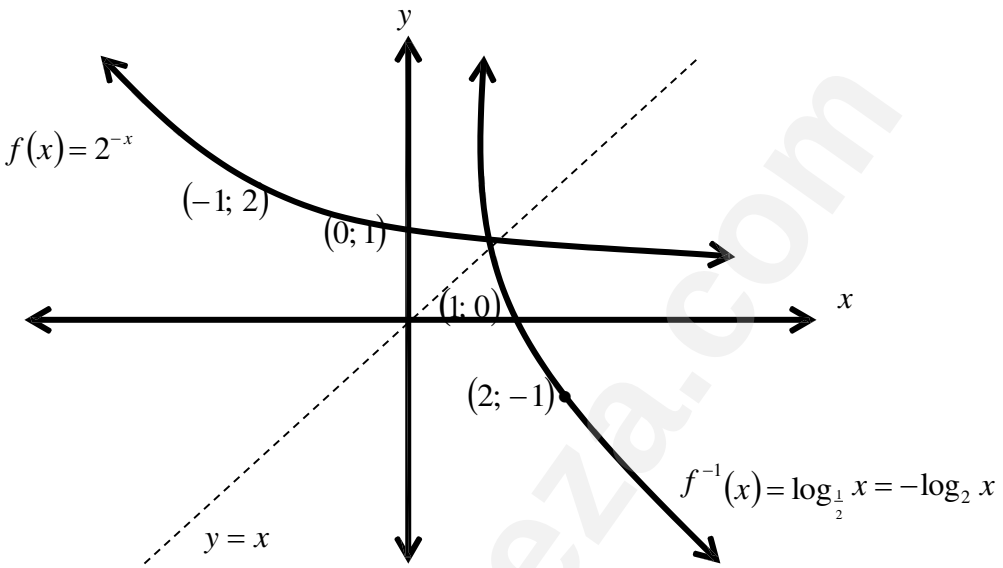
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EXAMPLE

Draw $f: x \to 2^{-x}$ and $f^{-1}(x)$ on the same system of axes.

Solution:



Note: f and f^{-1} are symmetrical about the line with equation $y = x$.

Mirror images

(a) The equation of the mirror image of a function about the $x-$axis:
Replace y by $(-y)$
OR $(x; y) \to (x; -y)$ i.e. use the transformation rule for reflections in the $x-$ axis.

(b) The equation of the mirror image of a function about the $y-$ axis:
Replace x by $(-x)$.
OR $(x; y) \to (-x; y)$ i.e. use the transformation rule for reflections in the $y-$axis

(c) The equation of a function symmetrical to a given function (f), about the line $y = x$:
determine $f^{-1}(x)$ by replacing every x by y and every y by x.
(Therefore swop x and y around)
OR $(x; y) \to (y; x)$ for reflections in the line $y = x$.

EXAMPLE

(i) Draw the graph of $f(x) = 3^x$.
(ii) Draw and determine the equation of the mirror image of f about the $y-$axis. (g)
(iii) Draw and determine the equation of the mirror image of f about the $x-$axis. (h)
(iv) Draw and determine the equation of f^{-1}.

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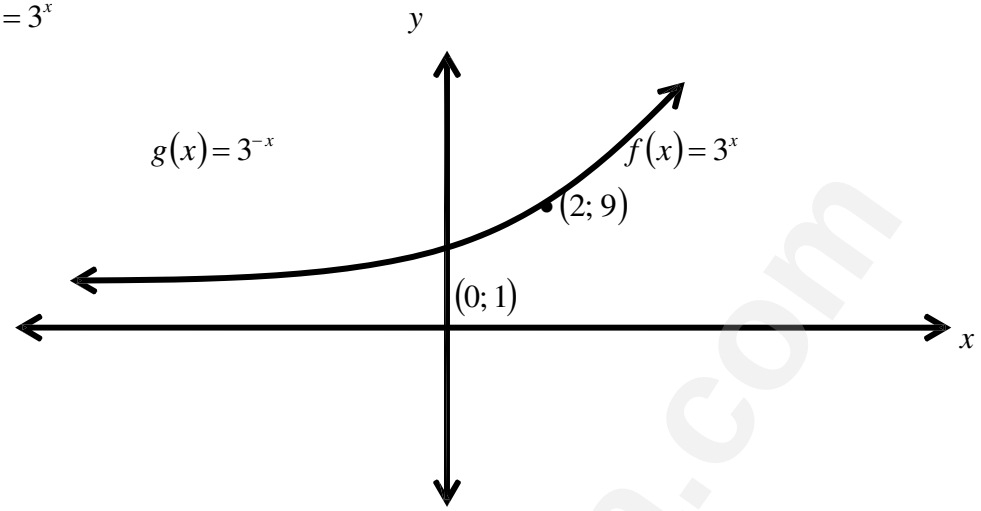
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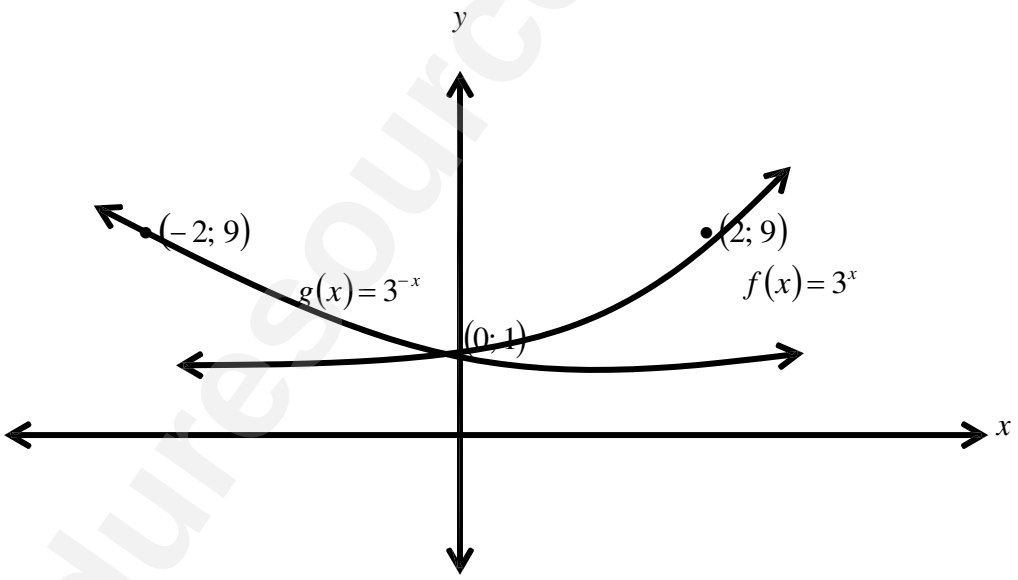
Solution:

(i) $f(x)=3^x$



(ii) f is reflected in the $y-$axis so that we replace x with $-x$,

$\therefore f(x)=3^x \rightarrow g(x)=3^{-x}$



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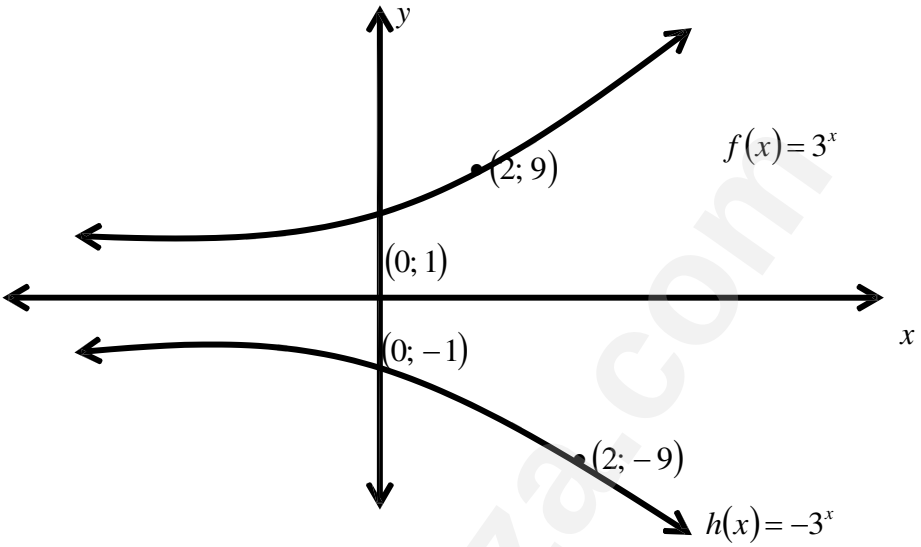
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(iii) f is reflected in the $x-$axis so that we replace y with $(-y)$;

$f(x)=3^x \rightarrow h(x)=-f(x)$

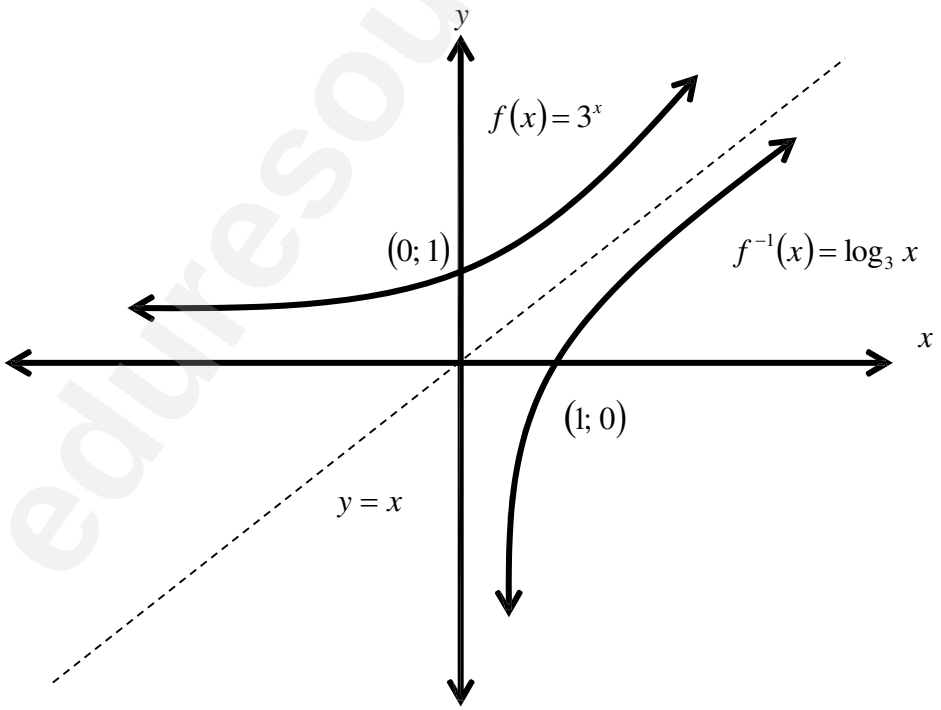
$=-3^x$



(iv) Given $f(x)=3^x=y$, swop x and y around to obtain f^{-1}.

Thus, $f(x)=3^x=y \Rightarrow x=3^y$

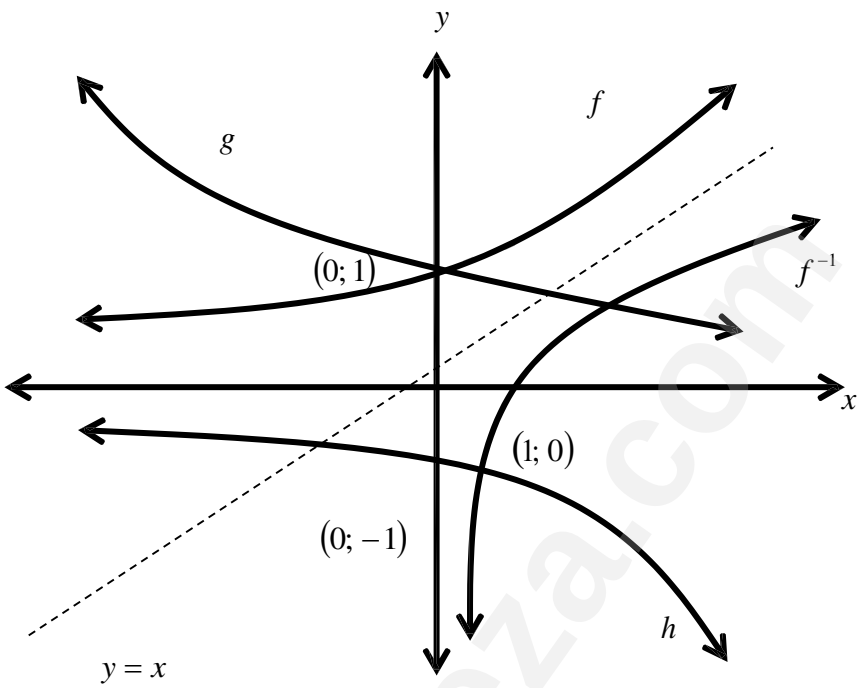
$\Rightarrow y=\log_3 x=f^{-1}(x)$



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Thus the combined reflections discussed above can be shown below:



EXAMPLE

(a) Sketch the graph of $f(x) = 2^{-x}$ for the domain: $-3 \le x \le 2,\ x \in$

(b) Use A and B to show where the solution to each of the following can be found:

(i) $2^{-a} = 7$

(ii) $\log_2 b = -\frac{1}{2}$

(c) Determine the equation of the mirror image of f about the $y-$axis.

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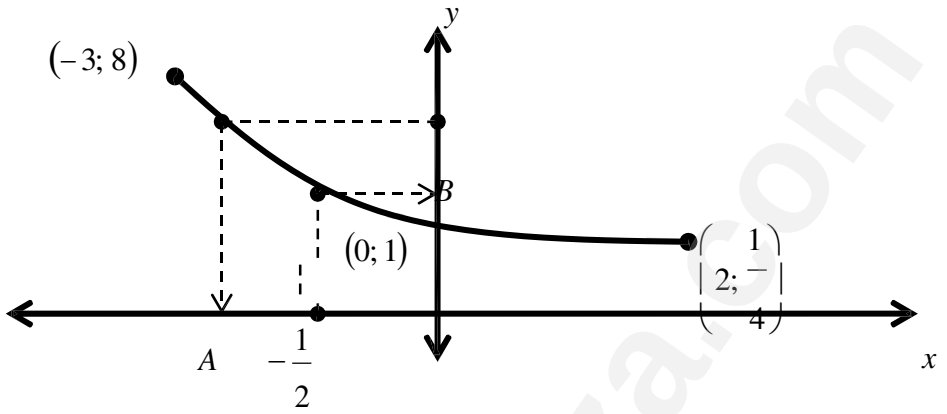
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Solution:

(a) We first note that $f(x)=2^{-x}=\left(\frac{1}{2}\right)^x$ can be compared to $y=a^x$ with $0<a<1$.



(i) $2^{-a}=7$ .....(A)

(ii) $\log_2 b=-\frac{1}{2}$

$\therefore 2^{-\frac{1}{2}}=b$ .....(B)

(c) $y=2^x$

Logarithmic inequalities

Given: $\log_a x<b$; $x>0$; $a>0$; $a\neq 1$

Then: $x<a^b$ if $a>1$

or $x>a^b$ if $0<a<1$

NB: $x>0$

e.g If $\log_3 x>2$

$\therefore x>3^2$

i.e. $x>9$

But if $\log_{\frac{1}{3}} x>2$

$\therefore x<\left(\frac{1}{3}\right)^2$

i.e. $0<x<\frac{1}{9}$

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EXAMPLE

Solve for x: $\log_{\frac{1}{2}} x < 3$

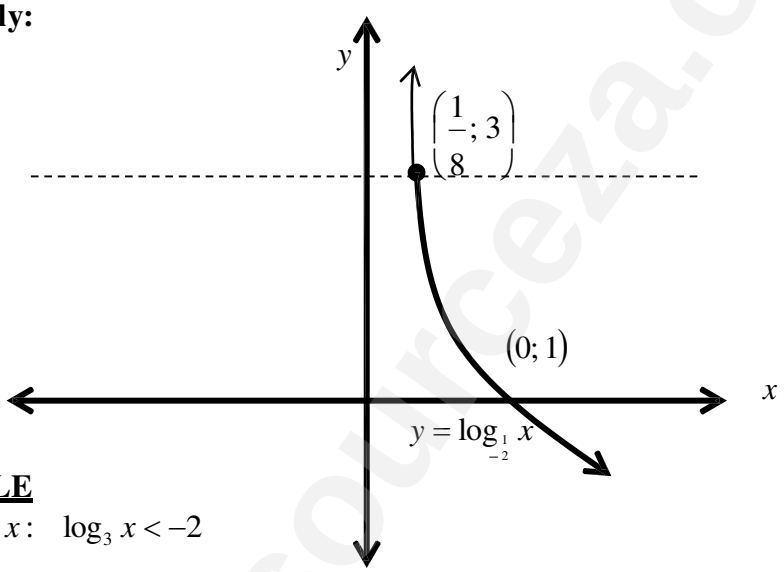
Solution:

$$\log_{\frac{1}{2}} x < 3; \quad x > 0$$

$$\Rightarrow x > \left(\frac{1}{2}\right)^3, \quad 0 < a < 1$$

$$\therefore x > \frac{1}{8}$$

Graphically:



EXAMPLE

Solve for x: $\log_3 x < -2$

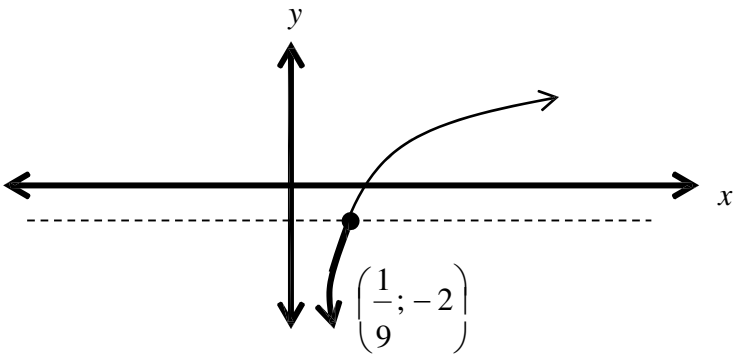
Solution:

$$\log_3 x < -2; \quad x > 0$$

$$\Rightarrow x < \frac{1}{9}; \quad a > 1$$

$$\therefore 0 < x < \frac{1}{9}$$

Graphically:



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EXAMPLE

Given: $h(x)=4^x$ and $f(x)=2(x-1)^2-8$

1.1 Sketch the graphs of h and f on the same system of axes.

Indicate ALL intercepts with the axes and any turning points.

1.2 Without any further calculations, sketch the graph of $y=\log_4 x=g(x)$

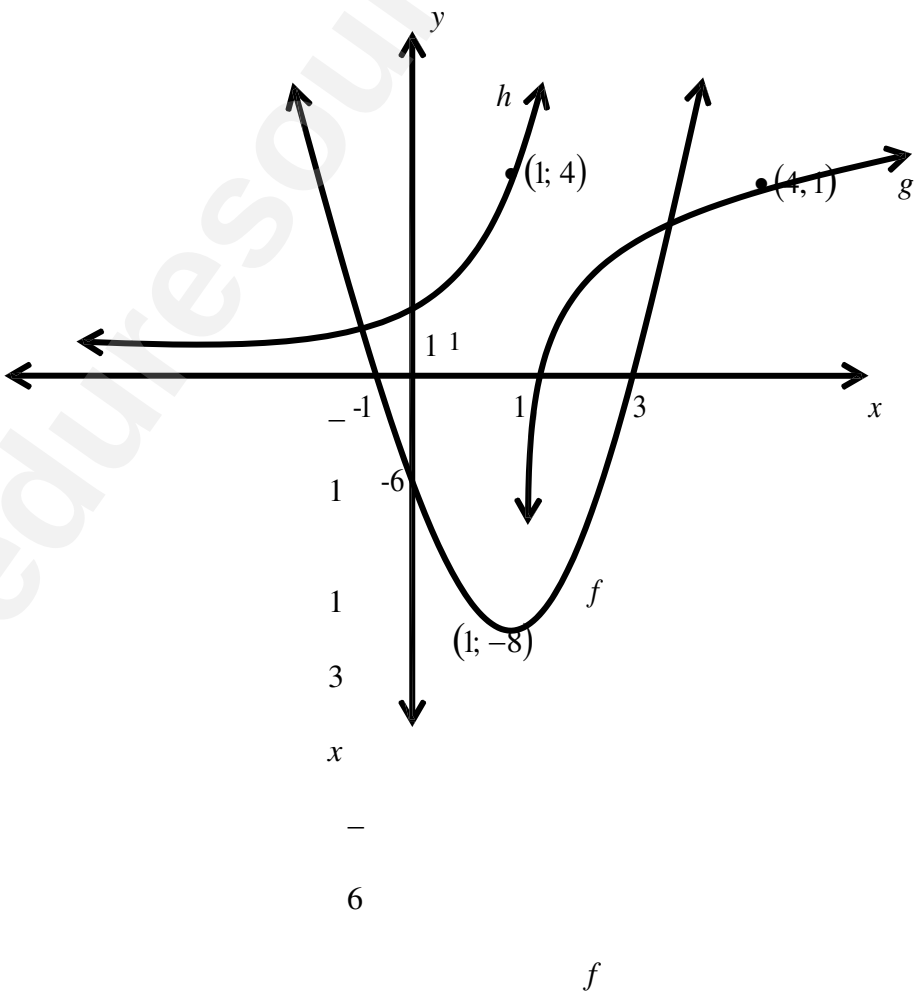
on the same system of axes.

1.3 The graph of f is shifted 2 units to the LEFT. Write down the equation of the new graph.

1.4 Show, algebraically, that $h\left|x+\ ^{1}\ \right|=2h(x)$.

SOLUTION:

1.1 and 1.2



1

3

x

–

6

f

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Calculation of $x-$intercepts of parabola:

$0=2(x-1)^2-8$

$\Rightarrow 8=2(x-1)^2$

$\Rightarrow 4=(x-1)^2$

$\Rightarrow 2=x-1 \text{ or } -2=x-1$

$\therefore x=3 \text{ or } x=-1$

OR/ Alternatively

$0=2(x-1)^2-8$

$\Rightarrow 0=2(x^2-2x+1)-8$

$\Rightarrow 0=2x^2-4x-6$

$\Rightarrow x^2-2x-3$

$\Rightarrow (x-3)(x+1)$

$\therefore x=3 \text{ or } x=-1$

1.3 $y=2(x+1)^2-8$

OR

$y=2x^2+4x-6$

1.4 $h\left(x+\frac{1}{2}\right)=4^{x+\frac{1}{2}}$

$=4^x\cdot 4^{\frac{1}{2}}$

$=2(4^x)$

$=2h(x)$

OR $h\left(x+\frac{1}{2}\right)=4^{x+\frac{1}{2}}$

$=\left(2^2\right)^{x+\frac{1}{2}}$

$=2^{2x+1}$

$=2^{2x}.2$

$=2(4^x)$

$=2h(x)$

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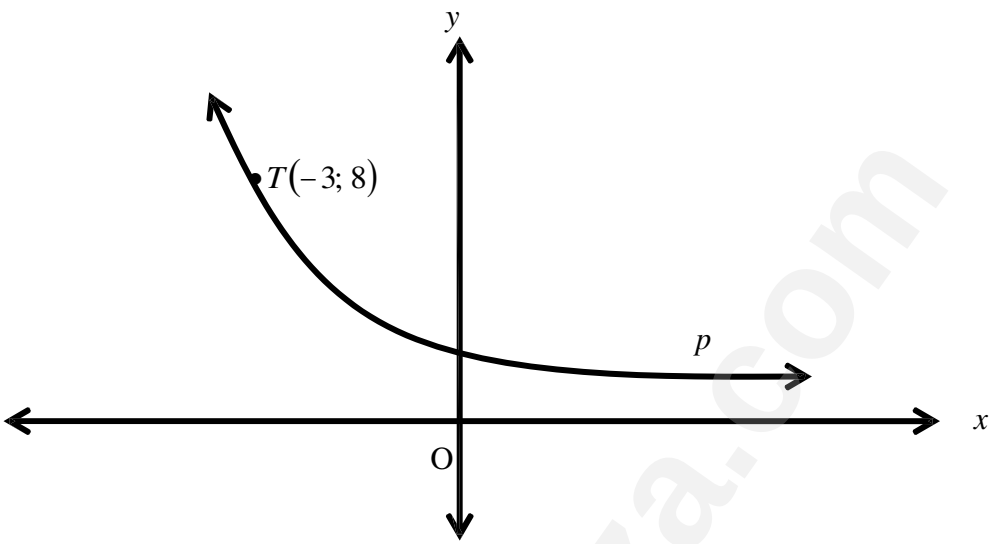
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EXAMPLE

The graph of $p(x)=a^x$ is sketched below. The point $T(-3;8)$ lies on the graph of p.



1.1 Calculate the value of a.

1.2 Write down the equation of $p^{-1}(x)$ in the form $y=...$

1.3 For which values of x will $p^{-1}(x)>-3$?

1.4 Write down the equation of q if q is the result of p shifted 3 units to the right.

SOLUTION:

1.1 $p(x)=a^x$

$8=a^{-3}$

$8=\frac{1}{a^3}$

$a^3=\frac{1}{8}$

$a=\frac{1}{2}$

1.2 $y=\log_{\frac{1}{2}} x$, or

$\Rightarrow y=-\log_2 x$, or

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$$\therefore y = \log_2\left|\frac{1}{x}\right|$$

1.3 $0 < x < 8$

1.4 $\therefore q(x) = 2^{-x+3}$

THE GENERAL LOGARITHMIC FUNCTION

The general logarithmic function is given by $y = n\log(ax - p)$,

so that: $ax - p = 0 \Rightarrow x = \frac{p}{a}$ (Equation of vertical asymptote)

$ax - p = 1 \Rightarrow x = \frac{1+p}{a}$ (x-intercept)

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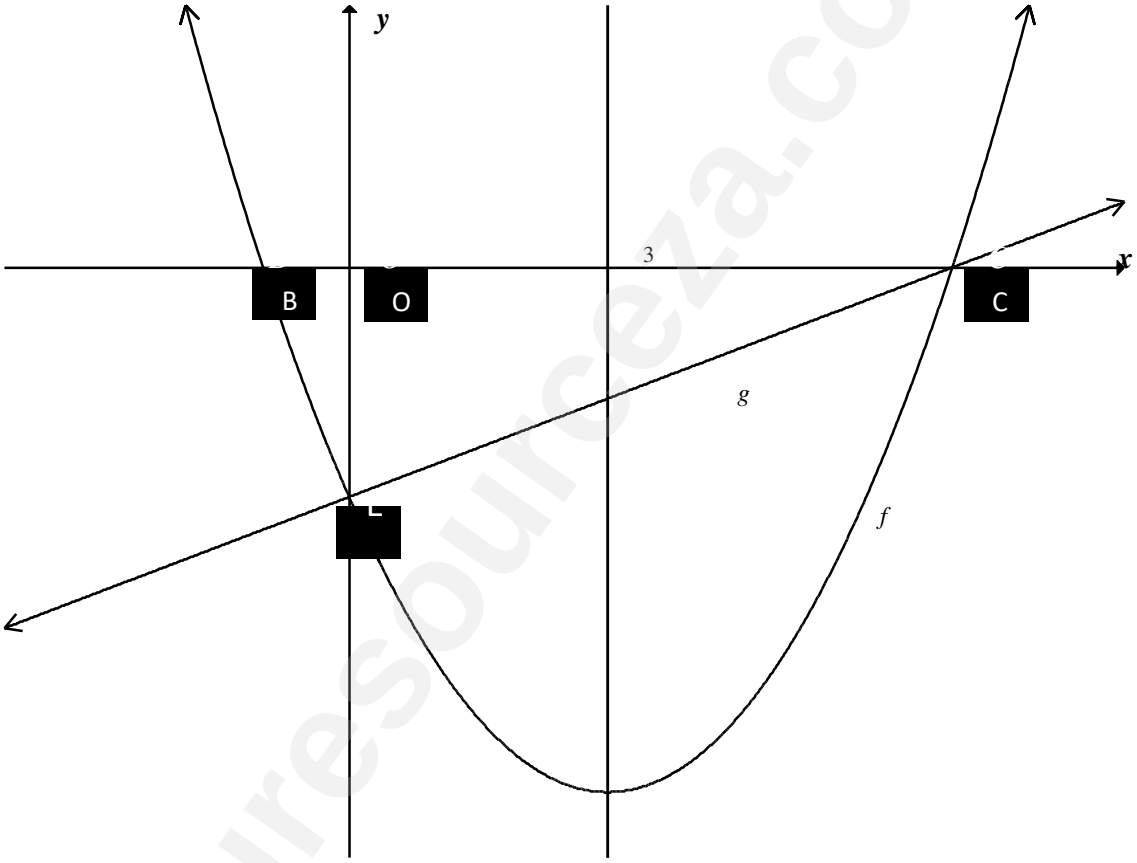
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Exercise: 1 Date:_____

Marks: 215

Question 1:

A parabola f intersects the x-axis at B and C and the y-axis at E. The axis of symmetry of the parabola has equation $x = 3$. The line through E and C has equation $g(x) = \frac{x}{2} - \frac{7}{2}$.



1.1 Show that the coordinates of C are (7 ; 0).

(1)

1.2 Calculate the x-coordinate of B. (1)

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1.3 Determine the equation of f in the form $y = a(x - p)^2 + q$. (6)

1.4 Write down the equation of the graph of h, the reflection of f in the x-axis. (1)

1.5 Write down the maximum value of $t(x)$ if $t(x) = 1 - f(x)$. (2)

1.6 Solve for x if $f(x^2 - 2) = 0$. (2)

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Question 2:

$$f(x) = \frac{3}{x-1} - 2.$$

Consider the function

2.1 Write down the equations of the asymptotes of f. (2)

2.2 Calculate the intercepts of the graph of f with the axes. (3)

2.3 Sketch the graph of f. (3)

2.4 Write down the range of $y = -f(x)$. (1)

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2.5 Describe, in words, the transformation of f to g if $g(x) = \frac{-3}{x+1} - 2$.

(2)

Question 3:

Consider the function $f(x) = \left(\frac{1}{3}\right)^x$.

3.1 Is f an increasing or decreasing function? Give a reason for your answer. (2)

3.2 Determine $f^{-1}(x)$ in the form $y = \ldots$ (2)

3.3 Write down the equation of the asymptote of $f(x) - 5$.

(1)

3.4 Describe the transformation from f to g if $g(x) = \log_3 x$. (2)

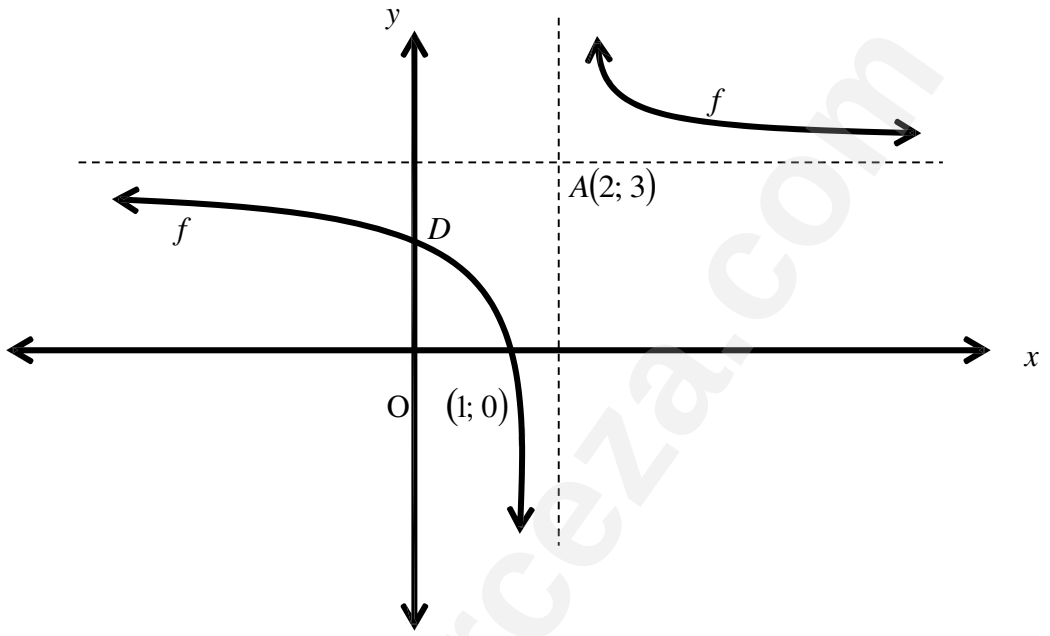
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Question 4:

Given $g(x)=\frac{a}{x-p}+q$. The point $A(2;3)$ is the point of intersection of the asymptotes of f. The graph of f intersects the $x-$axis at $(1;0)$. D is the $y-$intercept of f.



4.1 Write down the equations of the asymptotes of f. (2)

4.2 Determine the equation of f. (3)

4.3 Write down the co-ordinates of D. (2)

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4.4 Write down an equation of g if g is the straight line joining A and D. (3)

4.5 Write down the co-ordinates of the other point of intersection of f and g. (4)

Question 5:

Consider the function $f(x) = 4^{-x} - 2$.

5.1 Calculate the co-ordinates of the intercepts of the intercepts of f with the axes. (4)

5.2 Write down the equation of the asymptote of f. (1)

5.3 Sketch the graph of f. (3)

5.4 Write down the equation of g if g is the graph of f shifted 2 units upwards. (1)

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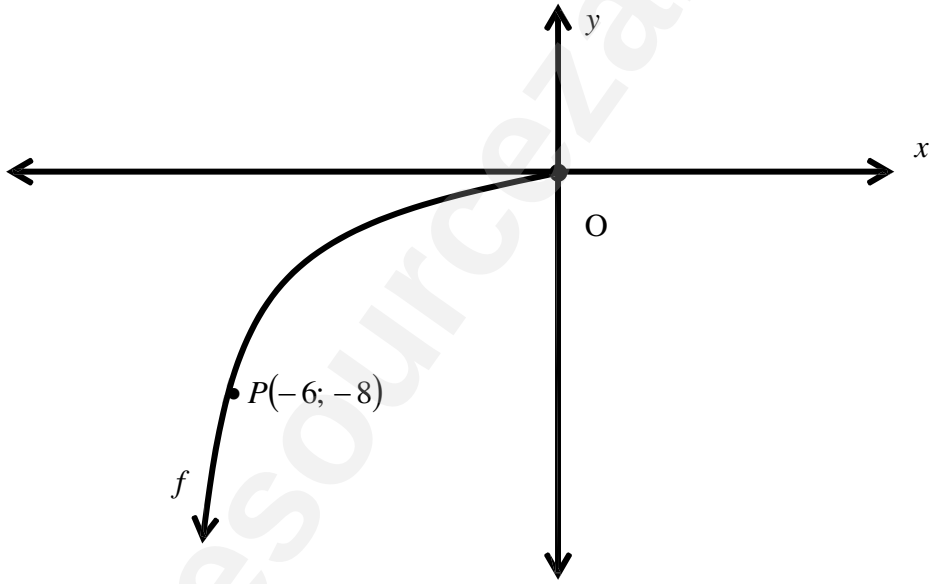
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5.5 Solve for x if $f(x)=3$.(You need not simplify your answer). (3)

Question 6:

The graph of $f(x)=ax^2$, $x\le 0$ is sketched below.
The point $P(-6;-8)$ lies on the graph of f.



6.1 Calculate the value of a. (2)

6.2 Determine the equation of f^{-1}, in the form $y=...$ (3)

6.3 Write down the range of f^{-1}. (1)

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f^{-1}. Indicate the co-ordinates of a point on the graph different

6.4 Draw the graph of

from $(0;0)$. (2)

6.5 The graph of f is reflected across the line $y=x$ and thereafter it is reflected across the $x-$axis. Determine the equation of the new function in the form $y=\ldots$ (3)

Question 7:

Given: $f(x)=\dfrac{2}{x-3}+1$

7.1 Write down the equation of the asymptotes of f. (2)

7.2 Calculate the co-ordinates of the $x-$and $y-$intercepts of f. (2)

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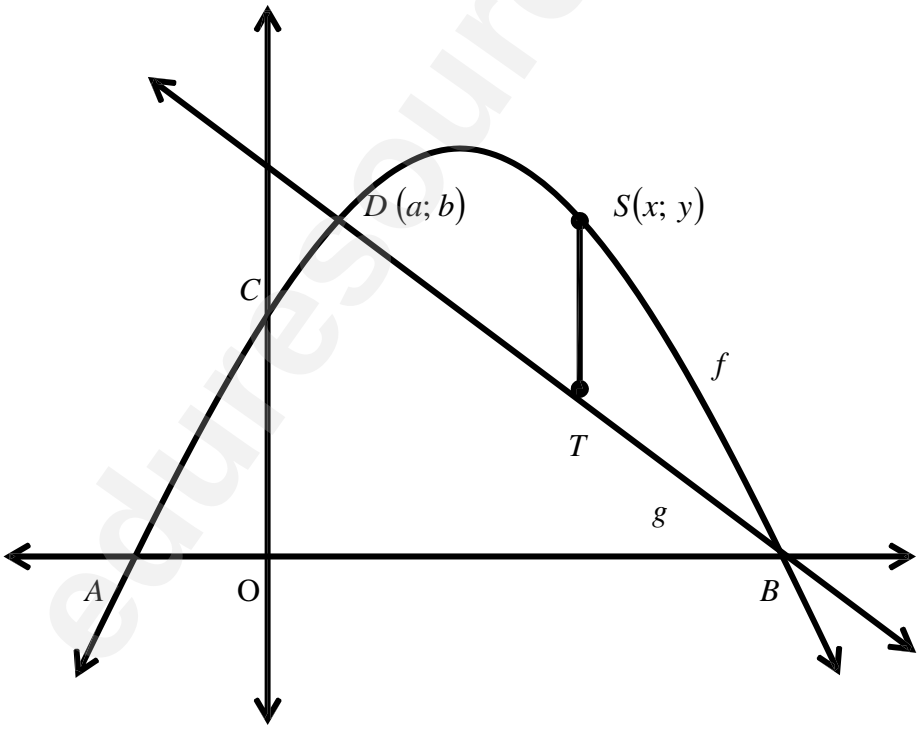
7.3 Sketch the graph of f.

Show all intercepts with the axes and the asymptotes. (3)

Question 8:

The graphs of $f(x) = -x^2 + 7x + 8$ and $g(x) = -3x + 24$ are sketched below.

f and g intersect in D and B. A and B are the $x-$intercepts of f.



8.1 Determine the co-ordinates of A and B. (4)

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x – co-ordinate of D. (4)

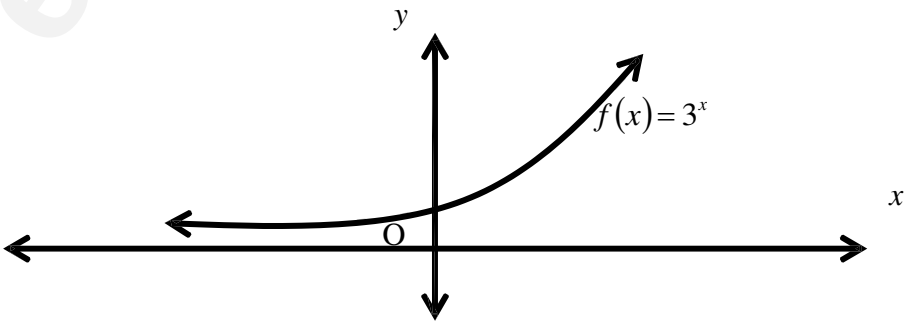
8.2 Calculate a, the

8.3 $S(x; y)$ is a point on the graph of f, where $a \leq x \leq 8$. ST is drawn parallel to the y – axis with T on the graph of g. Determine ST in terms of x. (2)

8.4 Calculate the maximum length of ST. (2)

Question 9:

The graphs of $f(x) = 3^x$ is drawn below.



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9.1 Write f^{-1} in the form $y=\ldots$ (1)

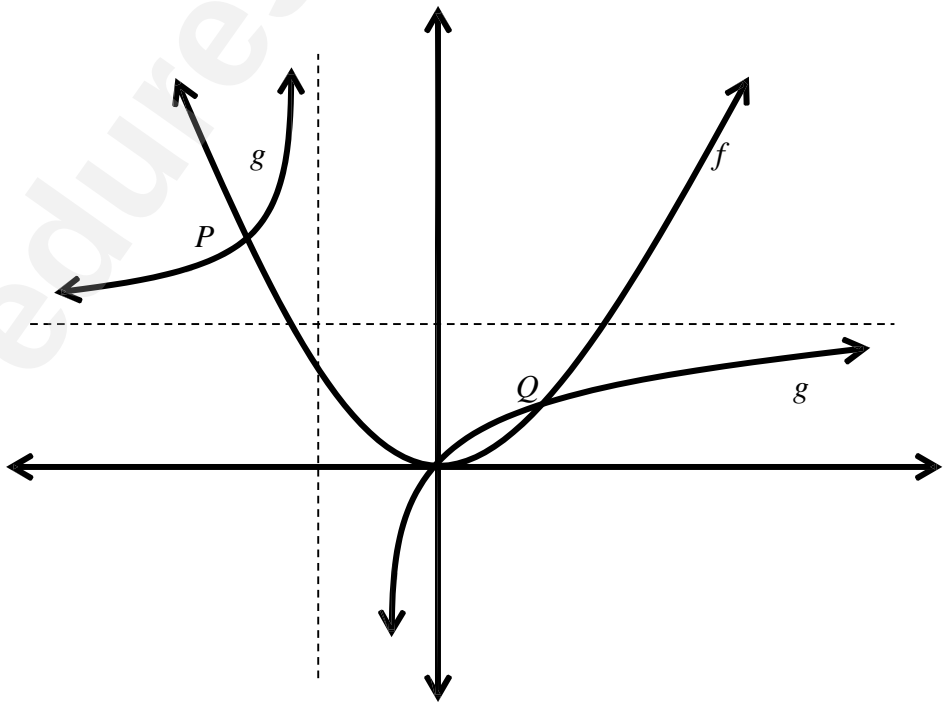
9.2 Sketch the graph of $y=f^{-1}(x)$ and $y=f^{-1}(x-2)$ on the same of axes. (1)

9.3 Use your graphs to solve for x if $\log_3(x-2)<1$. (2)

Question 10:

Sketched below are graphs $f(x)=\frac{1}{2}x^2$ and $g(x)=-\frac{1}{x+1}+1$.

P and Q are points of intersection of f and g.



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10.1 Show that the co-ordinates of P and Q are $P(-2;2)$ and $Q\left(1;-\frac{1}{2}\right)$. (6)

10.2 An axis of symmetry of the graph of g is a straight line defined as $y = mx + c$,

where $m > 0$. Write down the equation of this straight line in the form

$y = h(x) = \ldots.$ (2)

10.3 Determine the equation of h^{-1} in the form $y = \ldots.$ (2)

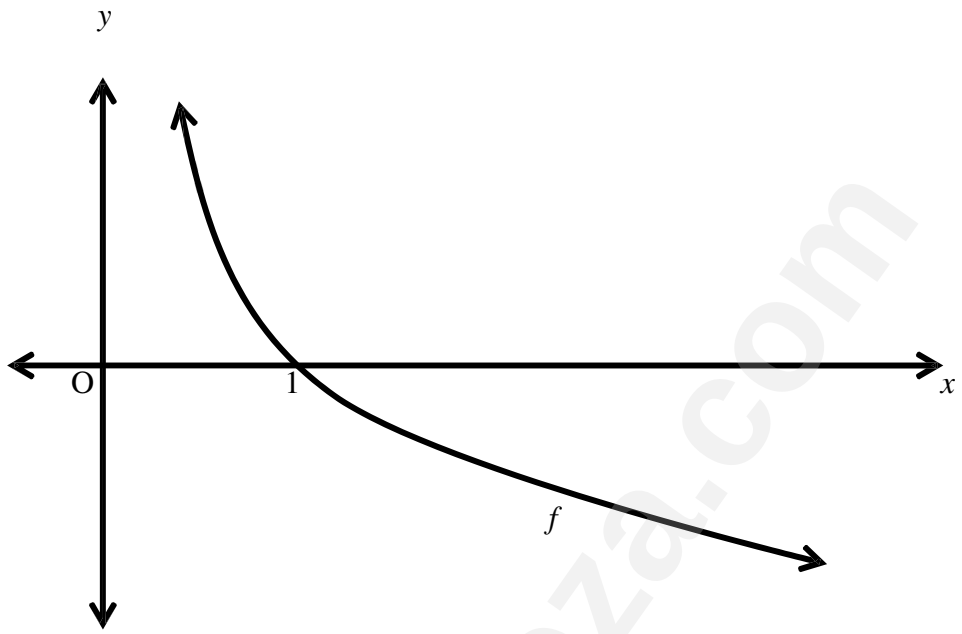
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Question 11:

Sketched below is the graph of $f(x) = -\log_2 x$



11.1 Write down the domain of f . (1)

11.2 Write down the equation of f^{-1} in the form $y = \ldots$. (1)

11.3 Write down the equation of the asymptote of f^{-1} . (1)

11.4 Explain how, using the graph of f , you should sketch the graphs of:

11.4.1 $g(x) = \log_2 x$ (1)

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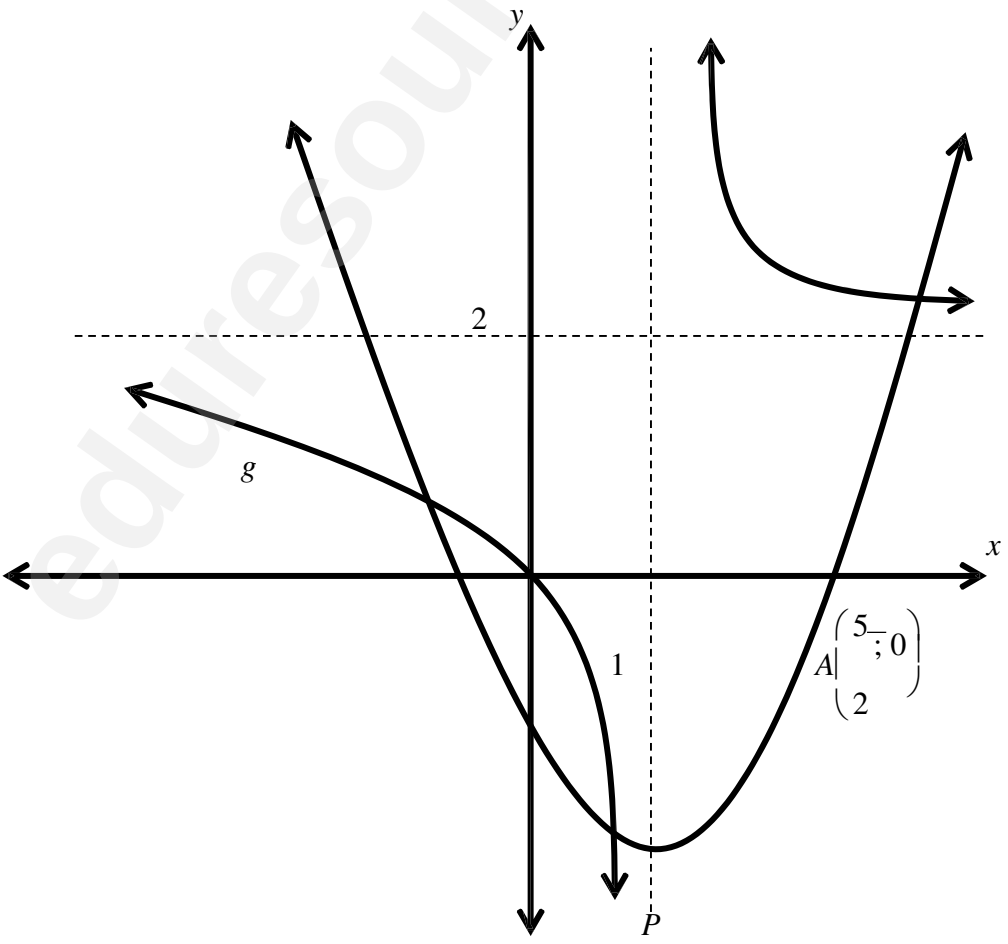
11.4.2 $h(x)=2^{-x}-5$ (3)

11.5 Use the graph of f to solve for x where $\log_2 x<3$. (3)

Question 12:

Sketch below are graphs of $f(x)=(x-p)^2+q$ and $g(x)=\frac{a}{x-b}+c$.

$A(2\frac{1}{2};0)$ is a point on the graphs of f. P is the turning point of f. The asymptotes of f and g are represented by the dotted line. The graph of g passes through the origin.



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12.1 Determine the equation of the equation of g . (4)

12.2 Determine the co-ordinates of P , the turning point of f . (4)

12.3 Write down the equations of the asymptotes of $g(x-1)$. (2)

12.4 Write down the equations h , if h is the image of f reflected on the $x-$axis. (1)

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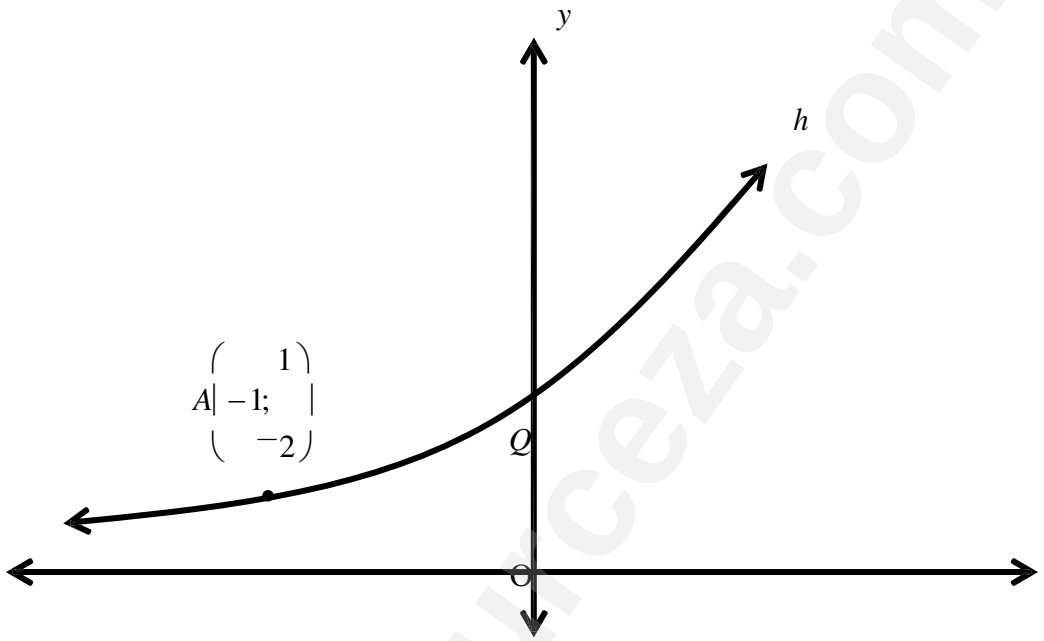
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Question 13:

The graph of $h(x) = a^x$ is sketched below.

$A\left(-1; \frac{1}{2}\right)$ is a point on the graph of h.



13.1 Explain why the co-ordinates of Q are $(0; 1)$. (2)

13.2 Calculate the value of a. (2)

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13.3 Write down the equation for the inverse function, h^{-1}, in the form

$y = \ldots .$ (2)

13.4 Draw a sketch graph, of h^{-1}.

Indicate on this graph the co-ordinates of two points that lie on this graph. (3)

13.5 Read off from the your graph the values of x for which $\log_2 x > -1$. (2)

13.6 If $g(x) = (100).3^x$, determine the values of x for which $h(x) = g(x)$. (3)

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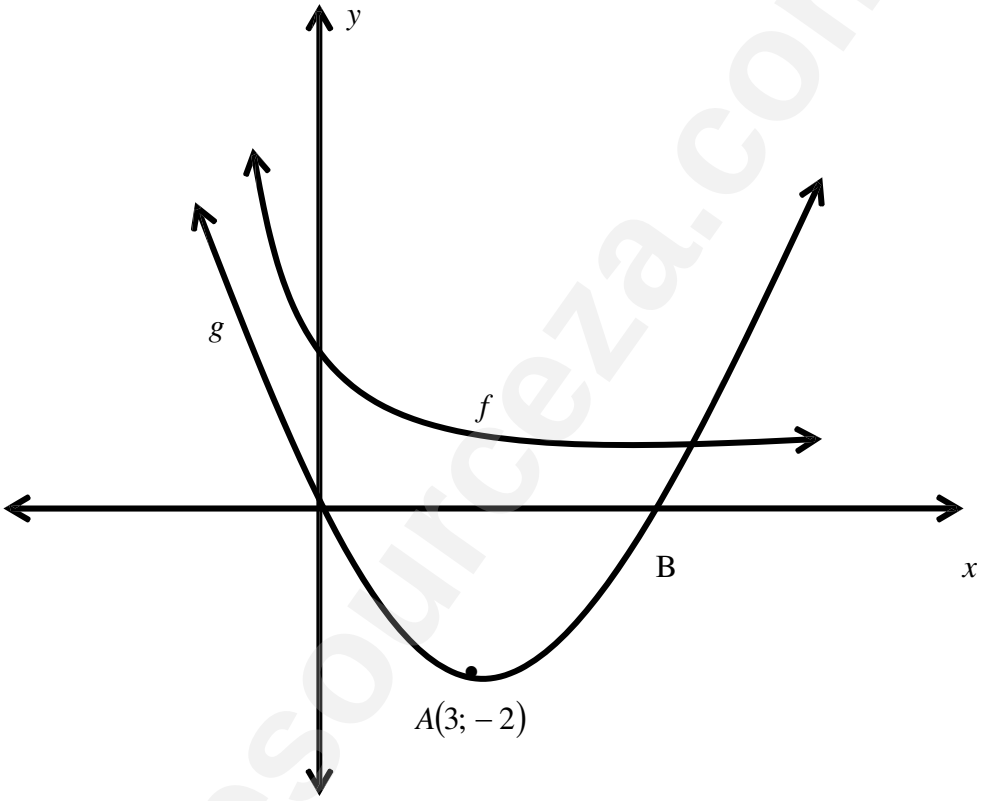
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Question 14:

14.1 The diagram below shows the curves of $f(x)=\left(\frac{1}{2}\right)^x$ and $g(x)$.

$g(x)$ is quadratic function with a turning point at $A(3;-2)$ and passes through the origin.



14.1.1 Show that the equation of $g(x)=\frac{2}{9}(x-3)^2-2$. (2)

14.1.2 Write down the co-ordinates of point B. (1)

14.1.3 Is $g(x)$ a one-to-one function or a one-to-many function? Substantiate your answer. (1)

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14.1.4 Draw a sketch graph, of g^{-1}. On this graph, indicate the co-ordinates of the turning point as well as all intercepts with axes. (3)

14.1.5 Determine the range of g^{-1} so that g^{-1} will be a function. (2)

14.1.6 Determine the equation of g in the form $y = \ldots$ if the graph of g is shifted horizontally to the left by 2 units. (2)

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14.1.7 Using the graph of g, or otherwise, determine the values of k for which the Equation $\frac{2}{9}(x-3)^2 = k$ has two positive distinct roots. (2)

14.2 Given: $f(x) = \left(\frac{1}{2}\right)^x$

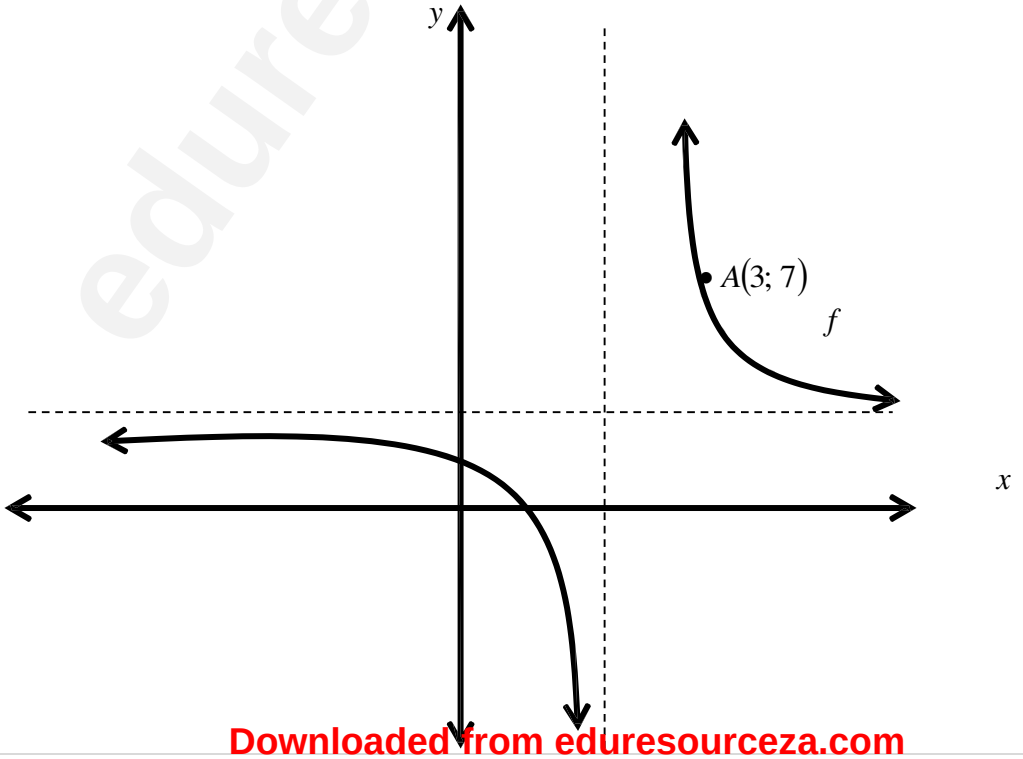
14.2.1 Write down the equation of f^{-1} in the form $y = \ldots$ (2)

14.2.2 Determine for which value(s) of x, $g(x) > f(x)$. (1)

Question 15:

The diagram below shows the graph of $f(x) = \frac{p}{x-2} + 4$

The point $A(3; 7)$ is given on the graph of f.



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15.1 Write down the equations of the asymptotes of $f(x)$. (2)

15.2 Determine the value of p. (2)

15.3 Determine the intercept(s) of $f(x)$ with the axes. (3)

Question 16:

16. Sketch $f(x)=\left(\frac{1}{2}\right)^x$ and its inverse f^{-1} on the same system of axes.

16.1 Determine $f^{-1}(x)=..$ (3)

16.2 h is the mirror image of f about the $x-$axis. Determine $h(x)$. (2)

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16.3 k is the mirror image of f about the $y-$axis. Determine $k(x)$. (3)

Question 17:

Sketch the graph of : $g(x)=\log_2 x$ (3)

17.1 Indicate where the value of $\log_2 6$ can be read off……..(A) (2)

17.2 Show where the value of $2^{1,5}$ can be read off ……(B) (2)

17.3 Give the equation of the inverse of g. (3)

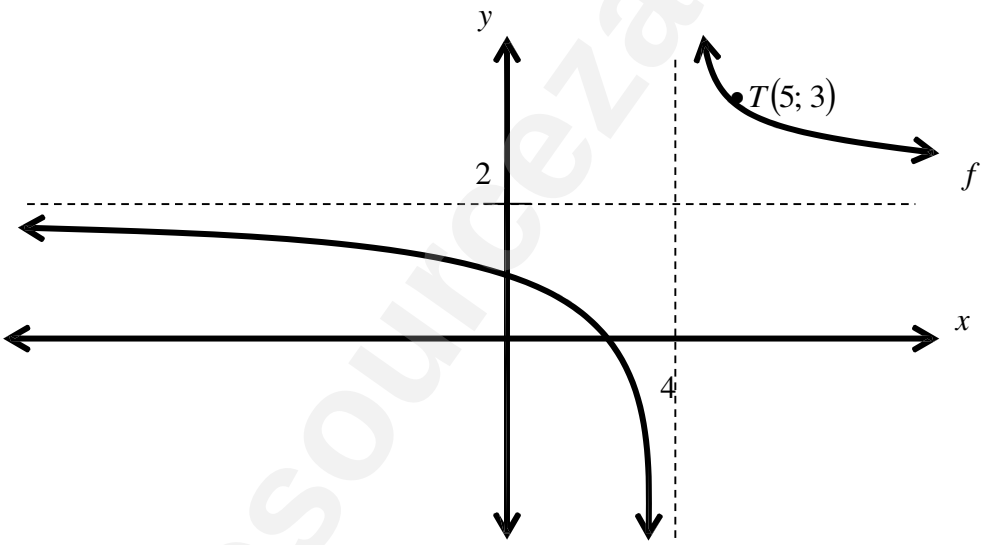
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17.4 Give the equation of the mirror image of g about the $x-$axis. (3)

Question 18:

The diagram below represents the graph of $g(x)=\frac{a}{x-p}+q$. $T(5;3)$ is a point on f.



18.1 Determine the values of a, p and q. (4)

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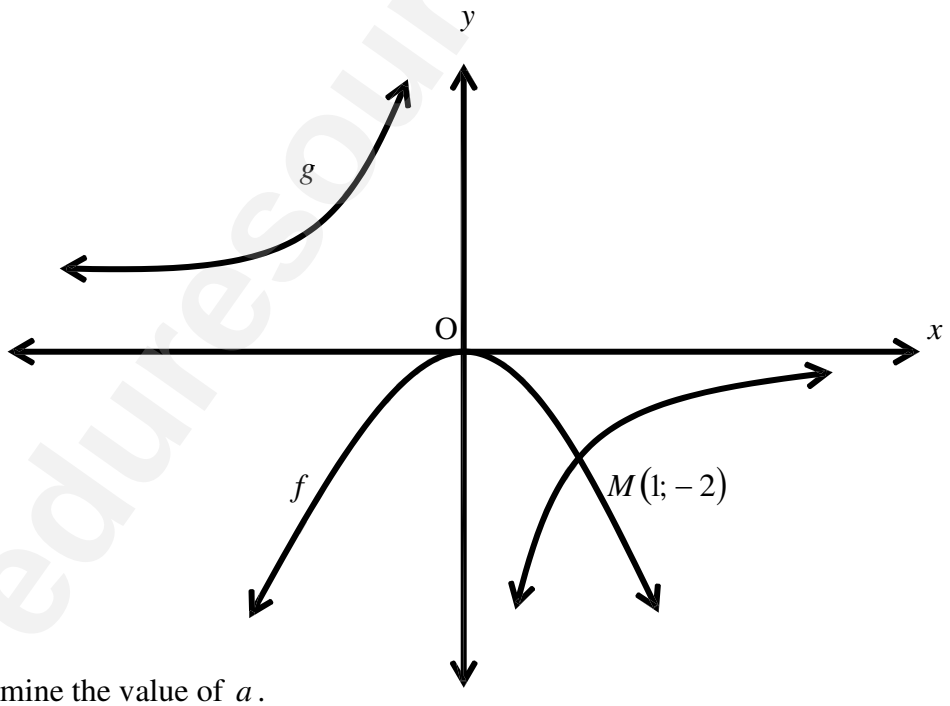
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18.2 If the graph of f is reflected across the line having equation $y = -x + c$, the new graph coincides with the graph of $y = f(x)$. Determine the value of c. (3)

Question 19:

The diagram below shows the graphs of $f(x) = ax^2$ and $g(x) = -\frac{2}{x}$.

The point $M(1; -2)$ is the point of intersection of f and g.



19.1 Determine the value of a. (1)

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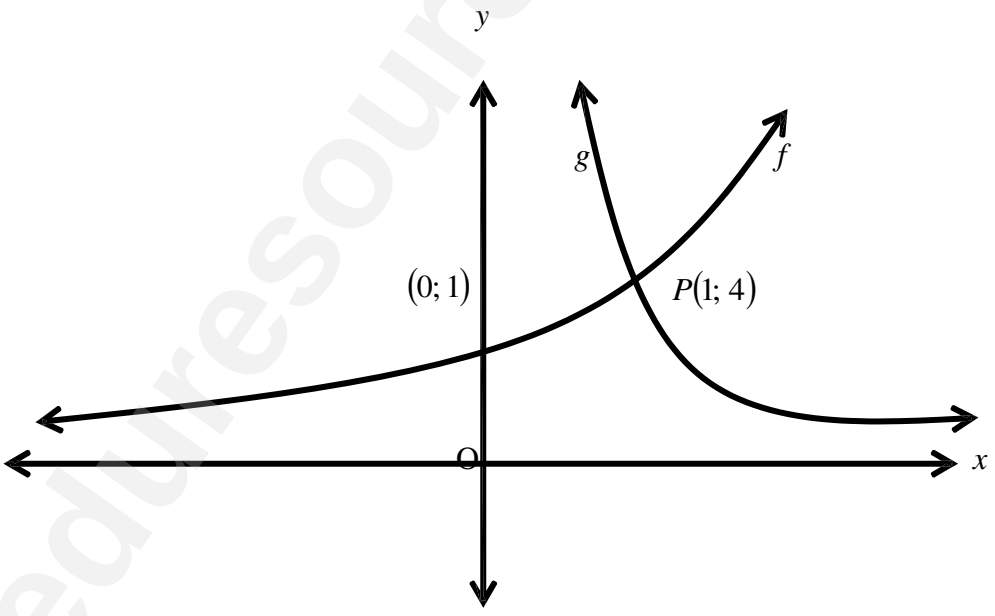
19.2 If $g(x)$ translated to give $h(x)=\dfrac{-2}{x-2}+1$, write down the asymptotes of $h(x)$. (2)

19.3 Sketch the graph of $h(x)=\dfrac{-2}{x-2}+1$. (3)

Question 20:

20.1 The diagram below shows the graphs of $f(x)=4^x$ and $g(x)=\dfrac{a}{x}$.

The point $P(1;4)$ is the point of intersection of f and g.



20.1.1 Write down the equation of f^{-1} in the form $y=...$ (2)

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20.1.2 Is f^{-1} a function?

Substantiate your answer. (2)

20.1.3 Determine the equation of $h(x)$, the resultant function when $f(x)$ is reflected about the $y-$axis. (2)

20.2 Determine the value of a in $g(x)$. (2)

20.3 Determine the equation of $m(x)$, the resultant function when $g(x)$ is shifted horizontally 2 units to the right and vertically 1 unit down. (2)

20.4 Calculate the intercepts of $m(x)$ with the axes. (3)

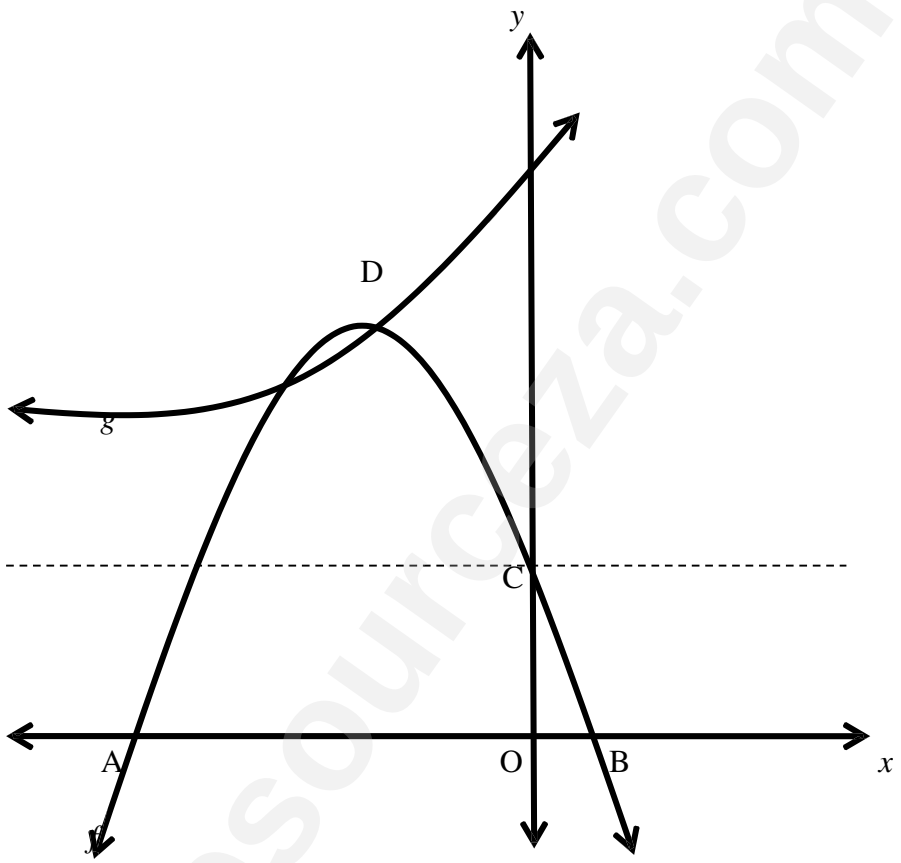
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Question 21:

The graphs of $f(x) = -(x+1)^2 + 4$ and $g(x) = a \cdot 3^x + q$ are sketched below.

A and B are the $x-$intercepts of f. C is the $y-$intercept of f and lies on the asymptote of g. The two graphs intersect in D, the turning point of f.



21.1 Determine the coordinates of A and B. (4)

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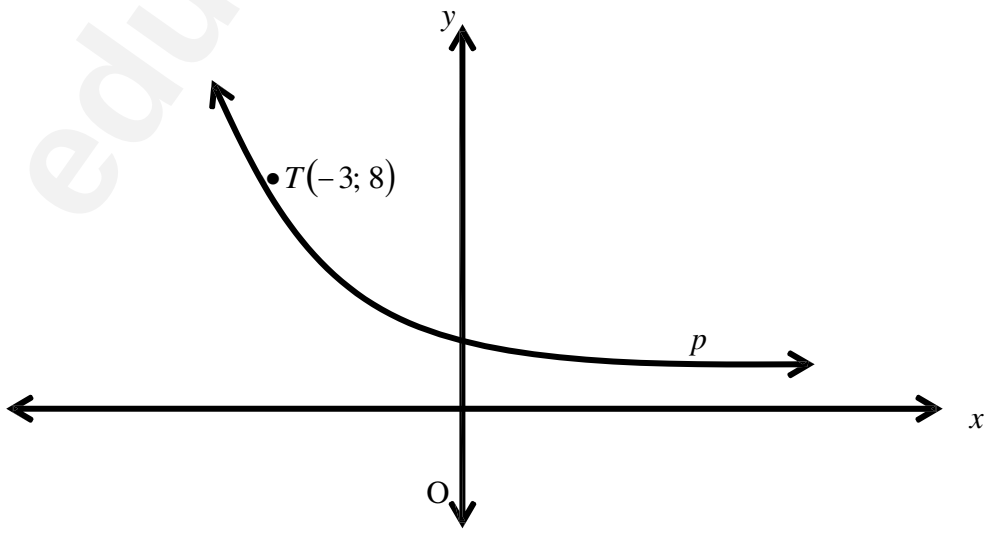
21.2 Determine the equation of g. (3)

21.3 Calculate the coordinates of the point on f for which the tangent to f will have a gradient of 1. (4)

21.4 Write down the values of k for which $f(x)-k$ will always be a negative value. (2)

Question 22:

The graph of $p(x)=a^x$ is sketched below. The point $T(-3;8)$ lies on the graph of p.



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22.1 Calculate the value of a. (3)

22.2 Write down the equation of $p^{-1}(x)$ in the form $y = ...$ (2)

22.3 For which values of x will $p^{-1}(x) > -3$? (2)

22.4 Write down the equation of q if q is the result of p shifted 3 units to the right. (2)

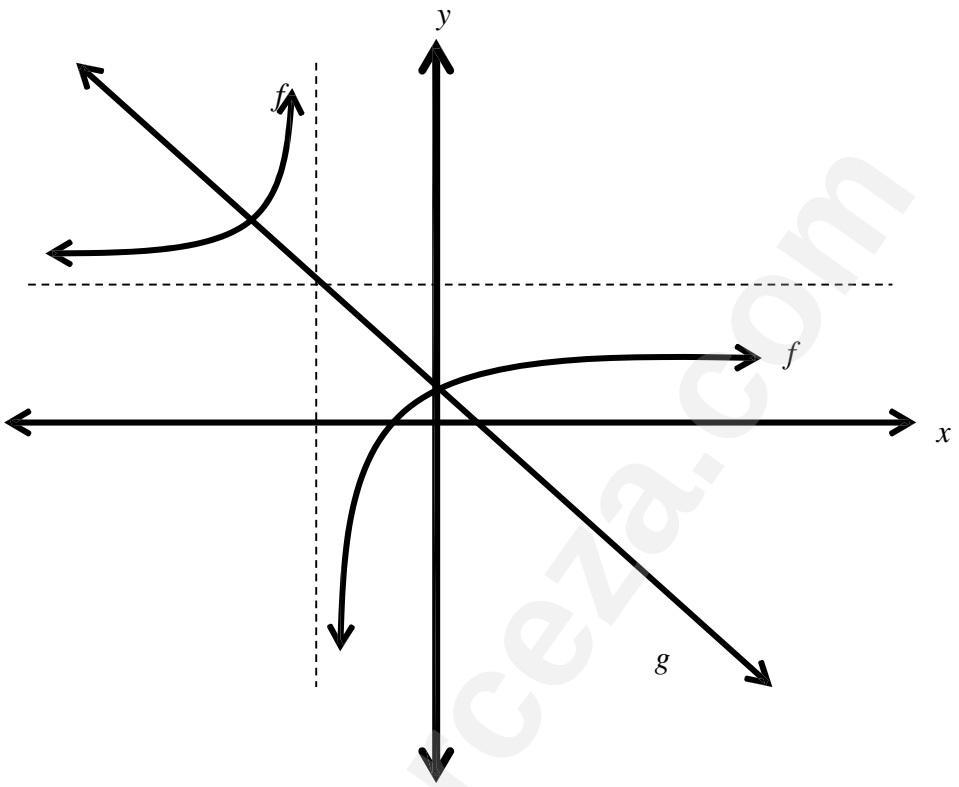
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Question 23:

The graphs of $f(x)=\frac{-3}{x+1}+5$ and $g(x)=-3x+2$ are sketched below.



23.1 Write down the range of $f(x)$. (2)

23.2 Determine the co-ordinates of the points of intersection of f and g. (4)

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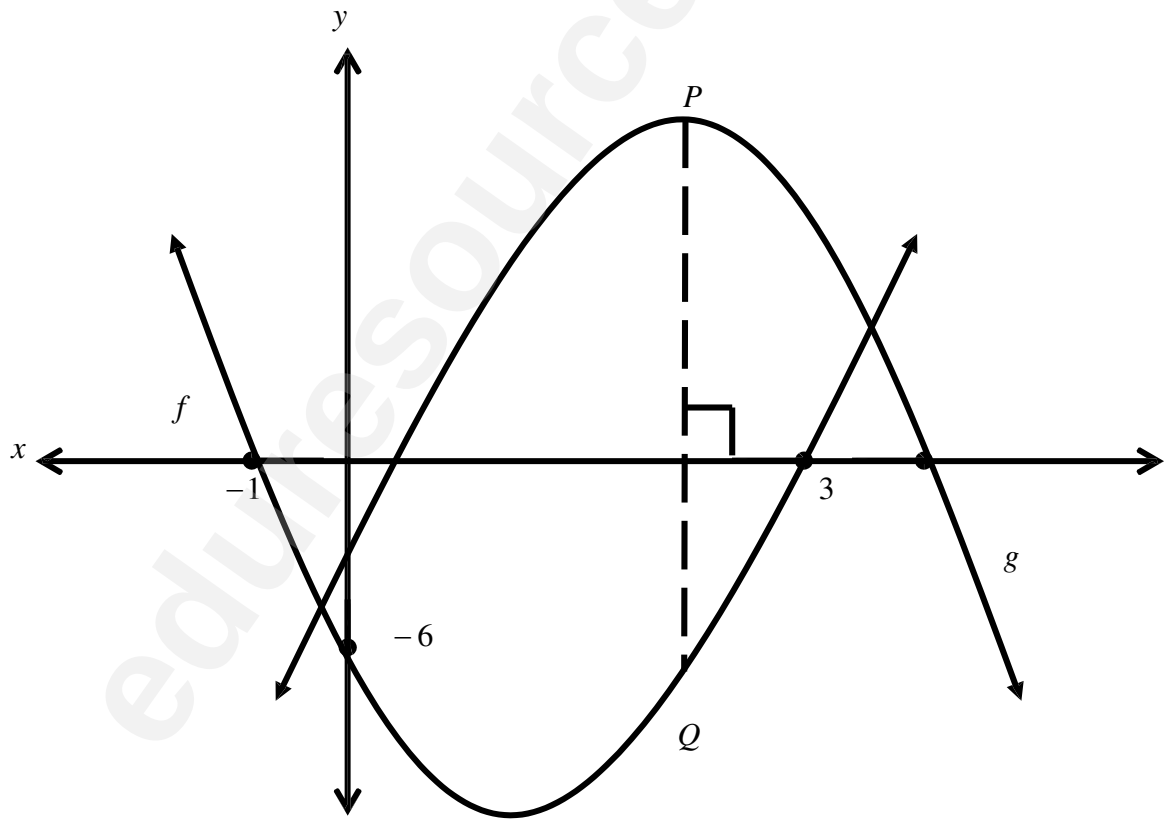
23.3 Describe the transformation o f f to h if $h(x)=\dfrac{3}{x+1}+5$. (2)

Question 24:

The sketch shows the graphs of:

$f=\{(x;\ y)|y=ax^2+bx+c\}$ and

$g=\{(x;\ y)|y=-x^2+5x-1\}$.



24.1 Determine the values of a; b and c. (3)

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24.2 P and Q are two points on g and f respectively, so that $g(x) > f(x)$ and PQ is perpendicular to the $x-$axis.

Calculate the maximum length of PQ, as well as the value of x at the point where PQ cuts the $x-$axis.

(5)

24.3 Determine the equation of the tangent to g, if the gradient of the tangent is -2.

(2)

24.4 Determine the equation of g if the axes are moved in such a way that (3; 2) is the new origin.

(3)

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Module: 4

FINANCIAL MATHEMATICS

Simple Interest Calculations
Definitions:

(1) Interest which is paid on the initial amount only is called **simple interest**.

(2) If the interest earned is added to the initial amount at specified intervals, and interest is paid on the total amount (i.e. the initial amount plus the interest earned) then the interest is called **compound interest**

EXAMPLE 1

R 100 is invested at 10% p.a. simple interest. Calculate what value the investment will be worth at the end of 3 years.

Solution:

Data:

$P = R100$

$n = 3$ years

$i = 10\% = \dfrac{10}{100} = 0{,}1$

$A = ?$

Thus, using the **simple interest formula** yields:

$A = P(1 + ni)$

$A = R100(1 + 3(0{,}1))$ where $P = R100$, $n = 3$ years and $i = 10\% = \dfrac{10}{100} = 0{,}1$

$A = R\,130$

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EXAMPLE 2

A car costs R80 000.
Determine its value after 5 years if the depreciation is calculated at 18% p.a. on a straight line basis.

Solution:

Data:

$P = R\ 80\ 000$

$n = 5$ years

$i = 18\% = \dfrac{18}{100} = 0{,}18$

$A = ?$

Thus, applying the **straight-line depreciation** formula gives:

$A = P(1 - ni)$

$A = R\ 80\ 000(1 - 0{,}18(5))$

$A = R8\ 000$

CALCULATING THE DURATION OF A LOAN OR INVESTMENT INVOLVING COMPOUND INTEREST

Definition of compound interest revisited

If the interest earned is added to the initial amount at specified intervals, and interest is paid on the total amount (i.e. the initial amount plus the interest earned) then the interest is called **compound interest**

The section on **Common logarithms** and **sequences** is important.

ELEMENTARY THEORY OF COMMON LOGARITHMS

We briefly review **common logarithms,** below**:**

Given any two **positive numbers** a and b, there exists a third number c such that

$a^c = b$.

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The number c is said to be the **logarithm** of b to the base a and we write

$\log_a b = c$.

$\therefore a^c = b \Leftrightarrow \log_a b = c$.

The logarithm of x to the base a is the power to which a must be raised to give x.

Laws of logarithms

1. $\log_c (ab) = \log_c a + \log_c b;\ c \neq 1$

2. $\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b;\ c \neq 1$

3. $\log_c a^b = b\log_c a;\ c \neq 1$

4. $\log_a a = 1;\ a \neq 1$

5. $\log_a b = \frac{\log_c b}{\log_c a};\ a \neq 1\ \&\ b \neq 1$

6. $\log_c 1 = 0;\ c \neq 1$

Examples

Solve for x :

1. $\log x = 1{,}6$

2. $3^{x+1} = 15$

3. $x = \log_9 27$

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Module: 5

COMPOUND INTEREST CALCULATIONS

Examples

(a) Principal Montsoku invests R3000 in an account paying 8% per annum compounded **annually**.
How long will it take for the investment to double?

Solution:

$A = P(1+i)^n$

$\Rightarrow 6000 = 3000(1+0{,}08)^n$

$\Rightarrow 2 = (1{,}08)^n$

$\Rightarrow \log 2 = n\log(1{,}08)$ {apply the common logarithm on both sides of the equation}

$\Rightarrow n = \dfrac{\log 2}{\log(1{,}08)}$

$\therefore n \approx 9{,}006468342$

Therefore the investment will take approximately 9 years to double at an interest rate of 8% per annum compounded **annually**.

(b) Suppose that the interest rate for Matthew's investment was 8% per annum compounded quarterly. How long will it take for the investment to double in this case?

Solution:

Interest will now be calculated four times per year. This means that the quoted annual rate must be divided by 4.

The number of quarters in 1 year is 4.

The number of quarters in 2 years is 2×4 quarters $= 8$ quarters

The number of quarters in 3 years is 3×4 quarters $= 12$ quarters

Therefore, the number of quarters in n years is $n\times 4$ quarters $= 4n$

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$A = P\left(1+\frac{i}{4}\right)^{4n}$ i.e. divide i by 4 and multiply n by 4.

$\Rightarrow 6000 = 3000\left(1+\frac{0,08}{4}\right)^{4n}$

$\Rightarrow 6000 = 3000(1,02)^{4n}$

$\Rightarrow 2 = (1,02)^{4n}$

$\Rightarrow 4n\log(1,02) = \log 2$

$\Rightarrow n = \frac{\log 2}{4\log(1,02)}$

$\therefore n \approx 8,750697195$

The investment will take approximately 8 years and 9 months to double at an interest rate of 8% per annum compounded **quarterly**.

Example 2

A motor vehicle costing R 15 000 depreciated at a rate of 9% per annum on the reducing balance method.
Calculate how long it took for the car to depreciate to a value of R60 000 under these conditions.

Solution:

$A = P(1-i)^n$

$\Rightarrow 60\ 000 = 150\ 000(1-0,09)^n$

$\Rightarrow \frac{60\ 000}{150\ 000} = (0,91)^n$

$\Rightarrow 0,4 = (0,91)^n$

$\Rightarrow n\log(0,91) = \log(0,4)$

$\Rightarrow n = \frac{\log(0,4)}{\log(0,91)}$

$\therefore n \approx 9,7156625$

It will take approximately 9 years and 8 months to depreciate to R60 000.

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Exercise: 1 Date:_____

Marks: 33

Question 1:

In this exercise, you will be required to make use of the following formulae:

$A = P(1 + in)$ **(Simple interest)**

$A = P(1 + i)^n$ **(Compound interest)**

$A = P(1 - in)$ **(Linear depreciation)**

$A = P(1 - i)^n$ **(Reducing balance depreciation)**

1. Mrs. Bella deposits R9000 in a savings account. **Calculate** how long it will take her to double her money if the interest rate is:

1.1 6% per annum compounded annually. (3)

1.2 6% per annum compounded quarterly. (3)

1.3 6% per annum compounded half-yearly. (3)

1.4 6% per annum compounded monthly. (3)

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1.5 6% per annum simple interest. (3)

Question 2:

The computers in a school's computer centre originally cost R 70 000. The computers were sold for R30 000 a few years later. **Calculate** how long it took the computers to depreciate to R30 000 if the rate of depreciation was:

2.1 12% per annum reducing balance depreciation. (3)

2.2 12% per annum linear depreciation (3)

Question 3:

3.1 Find the time taken for a certain sum of money to double if the interest rate is 11,2% per annum compounded semi-annually.

(3)

3.2 Mr. Samuel opens an account at Dynamic Fashions International and spends R5000. The interest rate charged is 24% per annum compounded monthly. How long will it take him to owe the company R8 000, if he makes no prior repayments.

(3)

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3.3. Determine how many years it will take an investment of R2 000 to earn R1 920 in interest, if the investment was made at an interest rate of 13% per annum compounded monthly. (3)

3.4. R3 000 depreciates at 9% per annum on the reducing balance scale to R1 872,10 over a period of time. What will the value of R3 000 be over the same time period, if depreciation took place on a straight-line basis at 9% per annum? (3)

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Module: 6

THE FUTURE AND PRESENT VALUE FORMULAE

Definition: An **annuity** is a series of equal payments at regular intervals of time subject to a rate of interest.

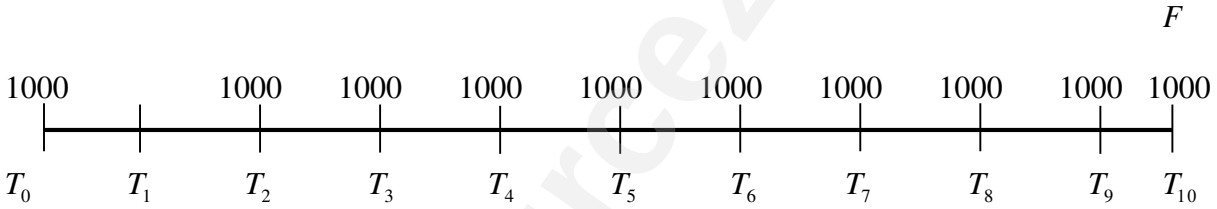
THE FUTURE VALUE ANNUITY FORMULA

In a **future value annuity**, money is invested at regular intervals in order to save money for the future. The magic of compound interest makes the investment grow in value, especially if the interest rate is above the current inflation rate.

Example 1

Suppose that R1 000 is invested every month, starting in one month's time, for a period of 10 months. Interest is 18% per annum compounded monthly. The future value of the investment after 10 months would be the sum of all payments together with the interest earned.

Solution:



$$i = \frac{0{,}18}{12} = 0{,}015$$

$$F = 1000 + 1000(1{,}015)^1 + 1000(1{,}015)^2 + 1000(1{,}015)^3 + \ldots + 1000(1{,}015)^9$$

Payment at T_{10} (no interest) Payment at T_9 (1 month interest) Payment at T_8 (2 month interest) Payment at T_7 (3 month interest) Payment at T_1 (9 month interest)

A useful formula which helps us to add up these amounts quickly is called the **Future Value Annuity formula.**

It would take forever to add up hundreds of payments which often occur in an annuity

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NOTE:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

Where:

$F =$ **Future value annuity**

$x =$ **payment per period**

$n =$ **number of payments**

$i =$ **interest rate**

$i =$ **interest**

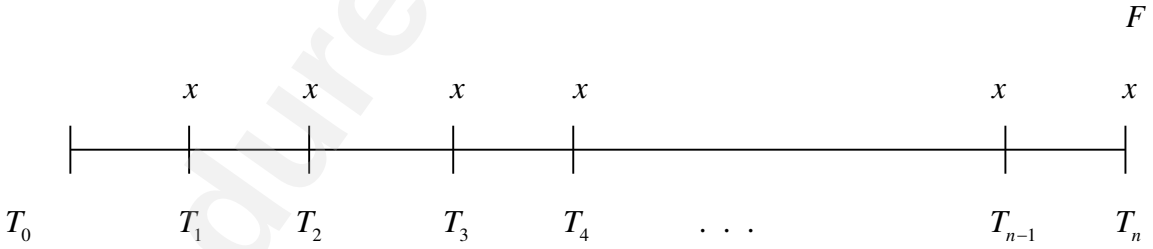
In the previous example we now can easily calculate the future value of the annuity.

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$F = \frac{1000\left[(1{,}015)^{10} - 1\right]}{0{,}015}, \text{ where } x = \text{R}1\,000,\ i = 0{,}015 \text{ per month and } n = 10 \text{ months}$$

$$\therefore F = R10\,702{,}72$$

DERIVATION OF THE FUTURE VALUE ANNUITY FORMULA (FOR INTEREST)



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$F = x + x(1+i)^1 + x(1+i)^2 + x(1+i)^3 + ... + x(1+i)^{n-1}$ **(Geometric series)**

$F = \dfrac{a(r^n - 1)}{r-1}$, $r \neq 1$ **formula from:** $S_n = \dfrac{a(r^n - 1)}{r-1}; r \neq 1$

$F = \dfrac{x\left[(1+i)^n - 1\right]}{(1+i)-1}$ where $a = x$ and $r = 1+i$

$\therefore F = \dfrac{x\left[(1+i)^n - 1\right]}{i}$

NOTE:

The formula for F can only be used if there is a final payment at the end, which does not earn interest. F coincides with the x at T_n.

Example 2

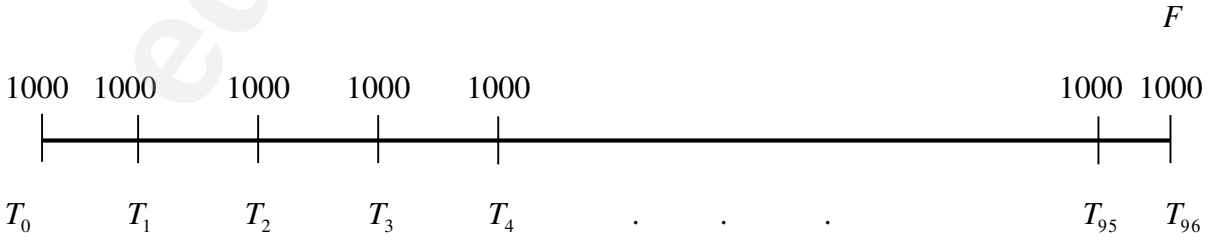
Mr. Ndlovu decides to open a savings account for his son's future education. On opening the account, he immediately deposits R1 000 into the account and continues to make monthly payments at the end of each month thereafter for a period of 8 years. The interest rate remains fixed at 24% per annum compounded monthly.

(a) How much will he have accumulated at the end of the 8th year?

(b) At the end of the 8-year period, he leaves the money in the account for a further year. How much money will he then have accumulated?

Solutions:

(a) First consider the **time line** below:



$= \dfrac{0{,}24}{12} = 0{,}02$

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, where $x = R1000$, $n = 97$, since there are 97 payments

$$F = \frac{1000\left[(1+0{,}02)^{97}-1\right]}{0{,}02}$$

$$F = R291\ 339{,}59$$

(b) $F = \dfrac{x\left[(1+i)^{n}-1\right]}{i}$

$$F = \frac{1000\left[(1+0{,}02)^{97}-1\right]}{0{,}02}\times(1+0{,}02)^{12}$$

$$\therefore F = R369\ 489{,}05$$

Example 3

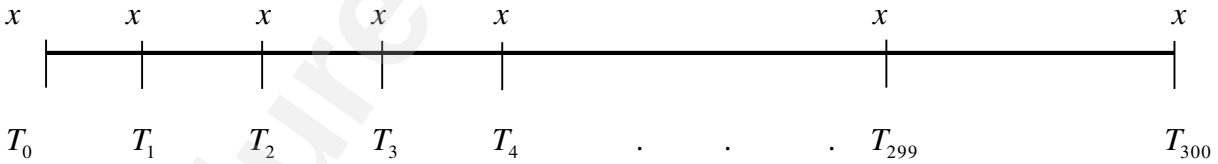
Emmah Mondi aged 25 wishes to accumulate R10 000 000 by her 50th birthday. She will pay equal monthly payments into an account paying 15% per annum **compounded monthly**. Payments start on her 25th birthday and end on her 50th birthday.

Find the **monthly payments**.

Solution:

Consider the **time-line** below:

$F = 10\ 000\ 000$



$$= \frac{0{,}15}{12} = 0{,}0125$$

There are 25 years $\times 12 = 300$ months plus the first one at T_0.

$$F = \frac{x\left[(1+i)^{n}-1\right]}{i}$$

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$$\Rightarrow 10\ 000\ 000 = \frac{x\left[(1+0{,}0125)^{301}-1\right]}{0{,}0125} \quad \text{(there are 301 payments of } x\text{)}$$

$$\Rightarrow 10\ 000\ 000 \times 0{,}0125 = x\left[(1{,}0125)^{301}-1\right]$$

$$\Rightarrow \frac{10\ 000\ 000}{\left[(1{,}0125)^{301}-1\right]} = x$$

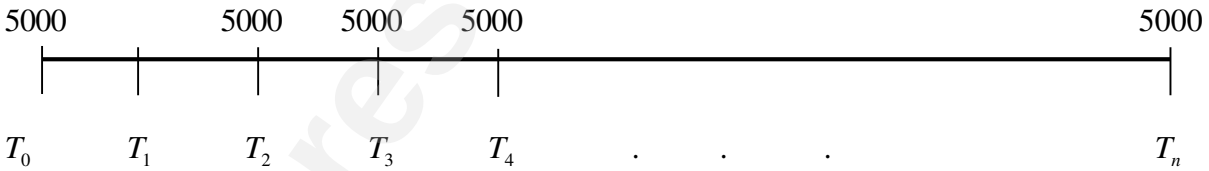
$$\therefore x \approx R3\ 044{,}07$$

Example 4

R5 000 is invested each month, starting in one month's time, into an account paying 18% p.a. compounded monthly. How long will it take to accumulate R300 000?

Solution:

Consider the **time line** below:



$$\frac{0{,}18}{12} = 0{,}015$$

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$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$300000 = \frac{5000\left[(1+0{,}015)^n - 1\right]}{0{,}015}$$

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$$\frac{300\ 000 \times 0{,}015}{5000} = (1{,}015)^n - 1$$

$$(1{,}015)^n = 1{,}9$$

$$n\log(1{,}015) = \log(1{,}9)$$

$$n = \frac{\log(1{,}9)}{\log(1{,}015)}$$

$$n \approx 43{,}11038966$$

It takes **about** 43 months (3 years and 7 months) to accumulate R 300 000.

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Exercise: 1 Date:_____

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Question 1:

1.1 Mr. Bongers decides to save money for ten years in a unit trust fund. He immediately deposits R800 into a savings account. Thereafter, at the end of each month, he deposits R800 into the fund and continues to do this for the ten year period. Interest is 15% p.a. compounded monthly. Calculate the final value of this investment.

(6)

1.2 Mrs. Rakgomu decides to start saving for a car. On her 16th birthday, she deposits R5000 into a bank account with an interest rate of 18% p.a. compounded quarterly.

She continues to make quarterly payments until the last payment on her 24th birthday.

How much money will she then have at her disposal to finance the purchase of a new car?

(6)

1.3 Mr. Martin decides to invest money into the share market in order to become a millionaire in ten years' time. He believes that he can average a return of 25% p.a. compounded monthly. In one month's time, he wishes to start making monthly payments into an account. How much must he invest per month in order to obtain his R1 000 000?

(6)

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1.4 Gugu wants to save up to R250 000 in 5 years' time in order to purchase a car.

She starts making monthly payments into an account paying 13% per annum compounded monthly, starting immediately. How much will she pay each month?

(6)

1. 5 R500 is invested each month, starting in one month's time, into an account paying 16% p.a. compounded monthly. How long will it take to accumulate R10 000?

(4)

1.6 R2 000 is invested each month, starting in six months' time, into an account paying 16%

p.a. compounded semi-annually. How long will it take to accumulate R 100 000?

(4)

7. Lebogang deposits R5000 into an account paying 14% per annum compounded half- yearly. Six months later, she deposits R400 into the account. Six months after this, she deposits a further R400 into the account. She then continues to make half-yearly deposits of R400 into the account for a further nine years.

Calculate the value of her savings at the end of the savings period?

(4)

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Module: 7

THE PRESENT VALUE ANNUITY FORMULA

In a **present value annuity**, a sum of money is normally **borrowed** from a financial institution and paid back with **interest** by means of **regular payments** at **equal intervals** over a **period of time**. The **loan** is said to be **amortised** (paid off) when it is paid off together with **interest charges**. In a present value annuity, compound interest becomes your enemy. People borrow money to finance cars, houses, furniture, overseas trips and other luxuries. They can spend many years paying off these loans. They will often pay more than double for these things. Every year, South Africans are borrowing money up to 300 billion rand. South Africa is heading for a huge debt trap in the near future and the crisis will seriously affect everyone (especially in the form of increased taxes). Being debt free and investment conscious will enable people to survive the future in the New South Africa.

Example

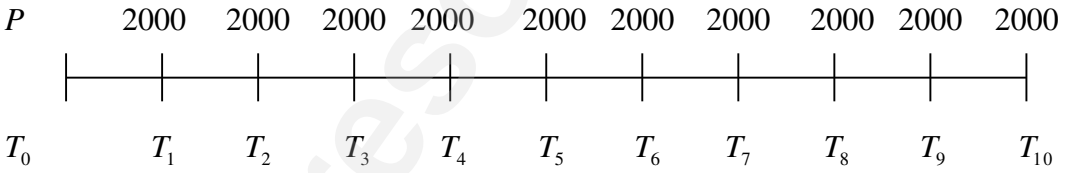
A loan is required for the purchase of equipment. The borrower can afford to pay R2 000 per month starting one month after the granting of the loan. The payments will continue for 10 months. The interest rate is 24% per annum compounded monthly.

How much can he borrow?

Solution:

In order to determine what the loan will be, we need to remove the interest charge on each payment of R2 000(finding their present values at T_0 and then add up the interest free amounts. In other words we must determine the **present value** of each payment at T_0.

Consider the **time line** below:



$$i = \frac{0{,}24}{12} = 0{,}02$$

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$$P = 2000(1+0{,}02)^{-1} + 2000(1+0{,}02)^{-2} + 2000(1+0{,}02)^{-3} + \ldots + 2000(1+0{,}02)^{-10}$$

Remove the interest from the payment at T_1	Remove the interest from the payment at T_2	Remove interest from the payment at T_3	Remove interest from the payment at T_{10}

A useful formula to calculate the sum of the present value of these payments at T_0 is called the **Present Value Annuity formula**:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$P =$ Present value annuity

$x =$ Payment per period

$n =$ Number of payments

$i =$ Interest

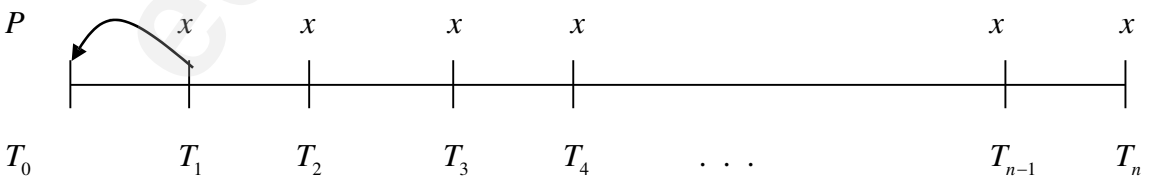
In the previous example the sum of all the **present values** of each payment is:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$P = \frac{2000\left[1-(1+0{,}02)^{-10}\right]}{0{,}02}, \text{ where } x = 2000,\ i = 0{,}02 \text{ and } n = 10 \text{ months}$$

$$\therefore P \approx R17965{,}17$$

DERIVATION OF THE PRESENT VALUE ANNUITY FORMULA(FOR INTEREST)



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$P = x(1+i)^{-1} + x(1+i)^{-2} + x(1+i)^{-3} + ... + x(1+i)^{-n}$ (Geometric series)

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ where } r \neq 1$$

$$P = \frac{x(1+i)^{-1}\left[(1+i)^{-n} - 1\right]}{(1+i)^{-1} - 1} \text{ where } a = x(1+i)^{-1},\ r = (1+i)^{-1}$$

$$\Leftrightarrow P = \frac{x\left[(1+i)^{-n} - 1\right]}{(1+i)^{1}\left\{(1+i)^{-1} - 1\right\}}$$

$$\Leftrightarrow P = \frac{x\left[(1+i)^{-n} - 1\right]}{(1+i)^{0} - (1+i)^{1}}$$

$$\Leftrightarrow P = \frac{x\left[(1+i)^{-n} - 1\right]}{1 - (1+i)}$$

$$\Leftrightarrow P = \frac{x\left[(1+i)^{-n} - 1\right]}{-i}$$

$$\therefore P = \frac{x\left[1 - (1+i)^{-n}\right]}{i}$$

NOTE

The formula for P can only be used if there is a **gap** between the loan and the first payment.

Example 2

How much can be borrowed now if a person agrees to repay a loan by means of quarterly payments of R 7 000, staring in three months from now?

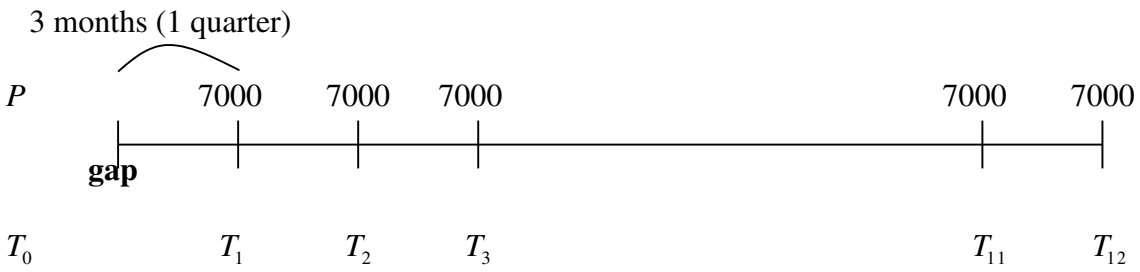
The payments will continue for 3 years and the interest rate is 16% per annum compounded quarterly.

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Solution:



$$i = \frac{0,16}{4} = 0,04$$

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

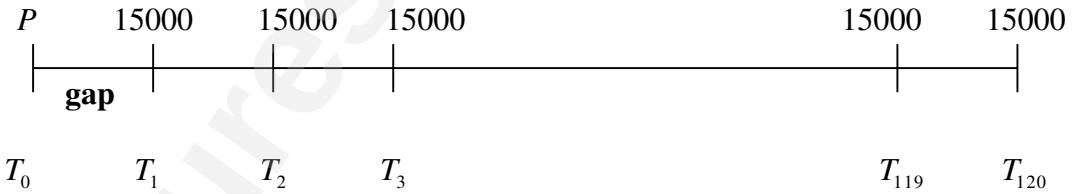
$$P = \frac{7000\left[1-(1+0,04)^{-12}\right]}{0,04}$$

$\therefore P \approx R\ 65\ 695,52$

Example 3

How much money will you need to win in the Lottery so as to receive equal monthly payments of R 15 000 per month for a period of 10 years starting one month after winning the money? Assume that you are able to receive an interest rate of 15% per annum compounded monthly.

Solution:



$$\frac{0,15}{12} = 0,0125$$

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\Leftrightarrow P = \frac{15000\left[1-(1+0,0125)^{-120}\right]}{0,0125}$$

$\therefore P \approx R929\ 742,71$

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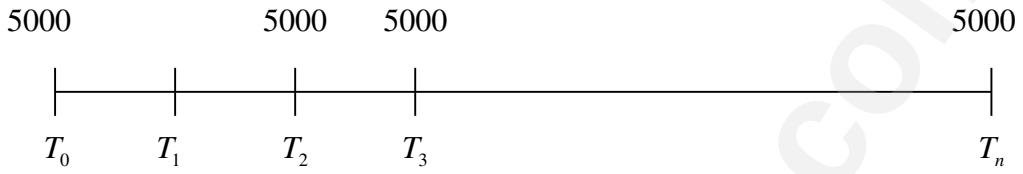
Example

A loan of R300 000 is to be repaid by means of monthly payments of R5000, starting one month after the granting of the loan. Interest is fixed at 18% per annum compounded monthly.

How long will it take to repay the loan?

Solution:

First consider the **time line** below:



$$i = \frac{0{,}18}{12} = 0{,}015$$

$$P = x\,1 - \frac{\left[(1+i)^{-n}\right]}{i}$$

$$\Rightarrow 300\,000 = \frac{5000\left[1-(1+0{,}015)^{-n}\right]}{0{,}015}$$

$$\Leftrightarrow \frac{300\,000 \times 0{,}015}{5000} = 1-(1{,}015)^{-n}$$

$$\Leftrightarrow (1{,}015)^{-n} = 0{,}1$$

$$\Leftrightarrow -n\log(1{,}015) = \log(0{,}1)$$

$$\Leftrightarrow n = \frac{\log(0{,}1)}{-\log(1{,}015)}$$

$$\therefore n \approx 154{,}6541086$$

It will take approximately 155 months to repay the loan.

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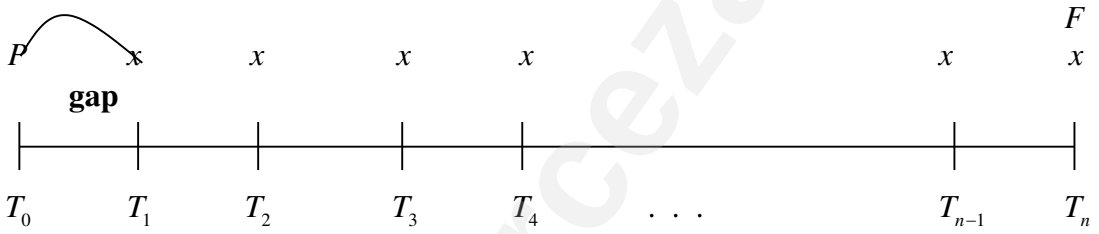
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Summary of the Annuity Formulae

Future Value Annuity formula	Present Value Annuity formula
$F = \frac{x\left[(1+i)^n - 1\right]}{i}$	$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$
Where:	Where:
F = Future value annuity	P = Present value annuity
x = Payment per period	x = Payment per period
n = Number of payments	n = Number of payments
i = Interest	i = Interest



NOTE

The formula for P can only be used if there is a gap between the loan and the first payment.

The formula for F can only be used if there is a final payment at the end, which does not earn interest. F coincides with x at T_n.

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Exercise: 1 Date:_____

Marks: 38

Question 1:

1.1 How much can be borrowed from a bank if the borrower repays the loan by means of 30 equal monthly payments of R 1250, starting in one month's time, if interest is 14% p.a. compounded monthly?

(6)

1.2 How much can be borrowed from a bank if the borrower repays the loan by means of equal quarterly payments of R2000, starting in three months' time?
The interest rate is 18% p.a. compounded quarterly and the duration of the loan is ten years.

(6)

1.3 Twenty-five semi-annual payments are made, starting six months from now, in order to repay a loan of R100 000. What is the value of each payment if interest is 18,6% p.a. compounded semi-annually?

(6)

1.4 What amount must be invested now in order for the investor to receive equal payments of R2000 per month from the bank for 3 years, starting in one month's time? Interest is 18% p.a.

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compounded monthly.

(6)

1.5 Lencelot inherits R 1 000 000 from his late father. He invests the money at an interest rate of

14% per annum compounded monthly. He wishes to earn a monthly salary from the investment for a period of 20 years starting one month from now. How much will he receive each month?

(6)

1.6 Mark takes out a retirement annuity that will supplement his pension when he retires, thirty years from now. He estimates that he will need R2,5 million in this retirement fund at that stage. The interest rate he earns is 9% per annum compounded monthly.

6.1.1 Calculate his monthly payment into this fund if he starts paying immediately and makes his final payment in 30 years' time.

(4)

6.1.2 The retirement fund does not pay out the R2,5 million when Mark retires. Instead he will be paid monthly amounts, for a period of 20 years, starting one month after his retirement. If the interest rate that he earns over this period is calculated at 7% per annum compounded monthly, determine the monthly payments he will receive.

(4)

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Module: 8

SINKING FUNDS

Companies often purchase equipment and use it for a special period of time.
The old equipment is then sold at a **scrap value** and new upgraded equipment is bought.
In order to finance the purchasing of the new equipment, the company will have, well in advance, set up an annuity called a **sinking fund**.
A **sinking fund** is nothing more than a normal **savings plan** or **future value annuity**.
The proceeds from the sinking fund and the sale of the old equipment is the used to purchase the new equipment.

Example

A school purchases a photocopying machine for R 150 000. The machine depreciates in value at 22% per annum on a reducing balance.
The school wants to buy a new machine in 5 years' time.
A new machine will cost much more in the future and its cost will escalate at 19% per annum effective.
The old machine will be sold at **scrap value** after 5 years.
A sinking fund is set up immediately in order to save up for the new machine.
The proceeds from the sale of
the old will be used together with the sinking fund to by the new machine.
The school will
pay equal monthly amounts into the **sinking fund** and the interest earned is 14,4% per annum compounded monthly.
The first payment will be made immediately and the last payment will be made at the end of the 5-year period.

(a) Find the scrap value of the old machine.

(b) Find the cost of the new machine 5 years from now.

(c) Find the amount required in the sinking fund 5 years from now.

(d) Find the equal monthly payments made into the sinking fund.

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Solution:

(a) $A = P(1-i)^n$

$\Rightarrow A = R150\ 000(1-0{,}22)^5$

$\therefore A \approx R\ 43\ 307{,}62$

(b) Cost of new machine:

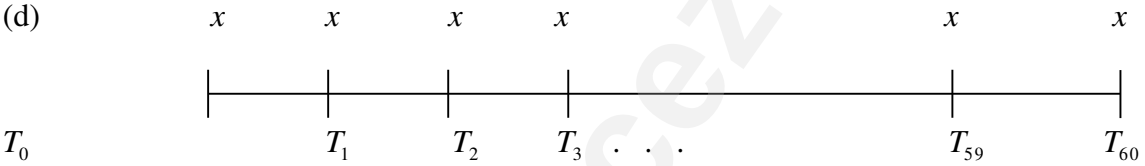
$A = P(1+i)^n$

$\Rightarrow A = R150\ 000(1+0{,}19)^5$

$\therefore A \approx R\ 357\ 953{,}05$

(c) Sinking fund + scrap value = cost of new machine
Sinking fund = Cost of new machine – scrap value
Sinking fund = R 357 953,05 – 43 307,62
Sinking fund = R 314 645,43

(d)



$$i = \frac{0{,}144}{12} = 0{,}012$$

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\Rightarrow 314\ 645{,}43 = \frac{x\left[(1+0{,}012)^{61} - 1\right]}{0{,}012}$$

$\therefore x \approx R\ 3528{,}09$

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Exercise: 1 Date:_____

Marks: 21

Question 1:

1. A printing press is bought for R 140 000. The cost of a new press is expected to rise by

18% p.a. while the rate of depreciation is 20% p.a. on the reducing-balance. The life span of the press is 6 years.

1.1 Find the scrap value of the old press.

(3)

1.2 Find the cost of a new press in 6 years' time.

(3)

1.3 Find the value of the sinking fund that will be required to purchase the press in six years' time, if the proceeds from the sale of the old press (at scrap value) will be utilized.

(3)

1.4 The company sets up a sinking fund to pay for the new press. Payments are to be made into an account paying 13,2 % p.a. compounded monthly .
Find the monthly payments, if they are to commence one month after the purchase of the old press and cease at the end of six year period.

(3)

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Question 2:

2. A company bought a large generator for R 227 851. It depreciates at 23% p.a on a reducing balance. A new generator is expected to appreciate in value at a rate of 17% p.a.. A new machine will be purchased in five years' time.

2.1 Find the scrap value of the old machine in five years from now.

(3)

2.2 Find the cost of a new machine in five years from now.

(3)

2.3 The company will use the money received from the sale of the old machine (at scrap value) as part payment for the new one. The rest of the money will come from a sinking fund that was set up when the old machine was bought. Monthly payments, which started one month after the purchase of the old machine, have been paid into a sinking fund account paying 11,4% p.a. compounded monthly. The payments will finish three months before the purchase of the new machine. Calculate the monthly payments into the sinking fund that will provide the required money for the purchasing of the new machine.

(3)

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Module: 9
DEFERRED ANNUITIES

In many situations, the repayment of a loan begins one month after the loan has been granted. However, circumstances arise where the repayments are deferred for an agreed period of time.

For example, if you borrow money to start a small business, it could be a few months before the business is set up and begins to acquire income.

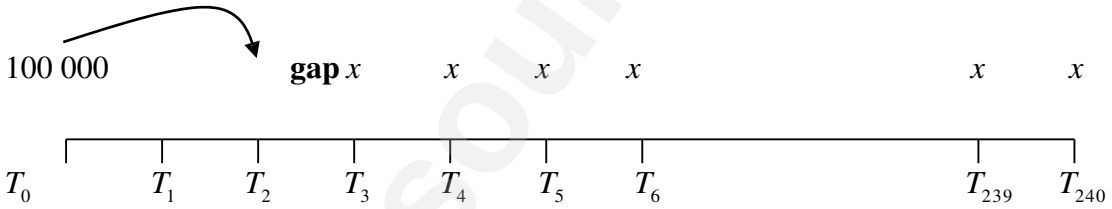
In such a situation the repayment of the loan could be structured so that the first payment begins after an agreed period of time.

Example 1

A twenty year loan of R 100 000 is repaid by means of equal monthly payments starting three months after the granting of the loan.

The interest rate is 18% per annum compounded monthly. Calculate the monthly payments.

Solution:



$$i = \frac{0{,}18}{12} = 0{,}015$$

The present value formula only works if there is a gap between the loan and the first payment.

Therefore, it is necessary to first grow the loan to T_2, which is a gap before the first payment.
The number of payments in this deferred annuity will therefore only be 238,
because two are missing (at T_1 and T_2).

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Method 1

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\Rightarrow 100\ 000(1{,}015)^2 = \frac{x\left[1-(1{,}015)^{-238}\right]}{0{,}015}$$

$$\Leftrightarrow x = \frac{100\ 000(1{,}015)^2(0{,}015)}{\left[1-(1{,}015)^{-238}\right]}$$

$\therefore x \approx R\ 1591{,}35$

Example 2

An investor pays R 3000 at the end of each month starting 3 months from now into an account paying 18% per annum compounded monthly.

He pays his final R3000 six months before the time he wishes to withdraw the money.

If the investment period, starting from now, is 8 years, calculate the future value of the investment at the end of the 8th year.

Solution:

There are 88 payments of R3000 rather than 90, because the investor only started saving at T_3, which means that two payments were not made. The accumulated amount at T_{96}.

$$F = \frac{x\left[(1+i)^n-1\right]}{i}$$

$$\Leftrightarrow F = \frac{3000\left[(1{,}015)^{88}-1\right]}{0{,}015} \times (1{,}015)^6$$

$\therefore F \approx R\ 591\ 969{,}90$

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Exercise: 1 Date:_____

Marks: 24

Question 1:

1.Thirty-two semi-annual payments of R 6 000 are made in order to repay a loan.

The payments start in two years from now. Interest is 18,6% p.a. compounded semi-annually. Find the size of the loan.

(4)

1.2 A loan of R120 454 is repaid by means of fourteen equal monthly payments.

Starting four years after the granting of the loan. Interest is 15% p.a. compounded monthly. Find the value of the payments.

(4)

1.3 A loan is repaid, starting in five years' time, by means of 12 quarterly payments of R7000.

What is the amount of the loan, if the interest is 24% p.a. compounded quarterly?

(4)

Question 2:

2.1 An investor requires R 4 500 000 in eight years from now. He decides to make quarterly payments, starting in fifteen months' time from now. The payments continue to be made for

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the remaining years. Calculate the value of each payment if interest is 24%p.a. compounded quarterly.

(4)

2.2 R2600 is invested into an account paying 16% p.a. compounded monthly.

The first payment is made nine months from now and continues up to the end of the ninth year. The investor requires a lump sum in ten years from now. What is the value of the lump sum at the end of the tenth year?

(4)

2.3 An investor pays R4500 per month into an account paying 17% p.a. compounded monthly.

He wishes to receive a large sum of money at the end of five years from now. His first payment is made in 7 months from now. Payments will continue and the last payment will be made three months before the investment is withdrawn. Determine the amount of money available to the investor at the end of the fifth year.

(4)

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Module: 10
PROBLEMS INVOLVING INTEREST RATE CHANGE(S)

Example

Mr. MAFADA decides to start saving money.

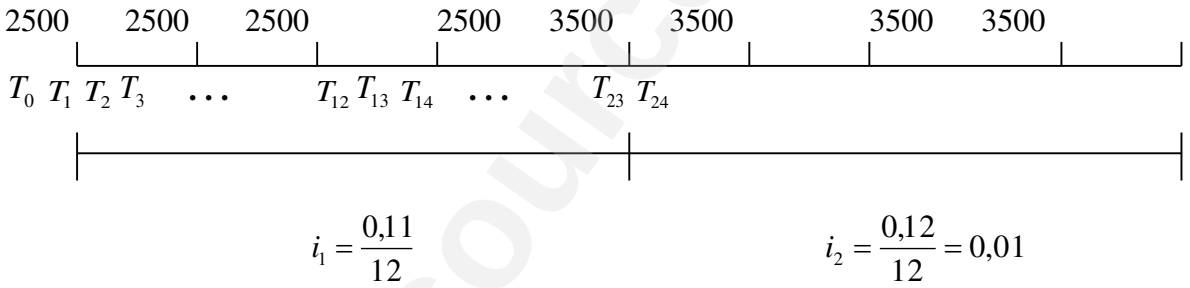
One month after opening the account, he invests R2500 per month into an account paying 11% per annum compounded monthly.

At the beginning of the second year, the interest rate changes to 12% per annum compounded monthly.

He takes advantage of this opportunity to make his money grow and increases his monthly payment to R3500.

The revised payments start one month after the interest rate changed.
How much money will he have saved by the end of the second year?

Solution:



$$i_1 = \frac{0{,}11}{12} \qquad i_2 = \frac{0{,}12}{12} = 0{,}01$$

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\Leftrightarrow F = \frac{2500\left[\left(1+\frac{0{,}11}{12}\right)^{12} - 1\right]}{\left(\frac{0{,}11}{12}\right)} \times (1{,}01)^{12} + \frac{3500\left[(1{,}01)^{12} - 1\right]}{0{,}01}$$

$$\therefore F \approx R79\,951$$

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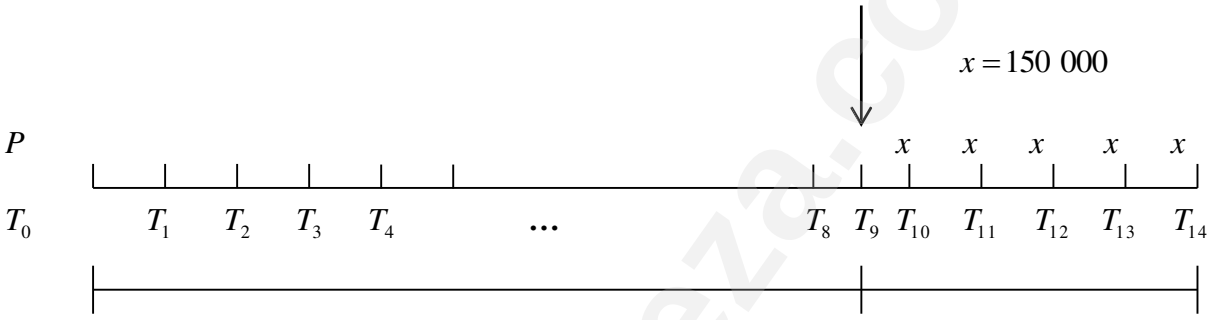
Example 2

What amount must be invested now so as to realize equal withdrawals of R 150 00 at the end of each year for 5 years, if the first withdrawal is made in 10 years' time?

The interest rate during the first 9 years is 16% per annum compounded semi-annually.

Thereafter it is 18% per annum effective.

Solution:



$$i_1 = \frac{0{,}16}{2} \qquad i_2 = 0{,}18$$

$$P = \frac{x\,1-\left[(1+i)^{-n}\right]}{i}$$

$$\Leftrightarrow P = \frac{150\ 000\left[1-(1+0{,}18)^{-5}\right]}{0{,}18} \times (1+\ 0{,}08)^{-18}$$

$$\therefore P \approx R\,117\ 385{,}73$$

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Exercise: 1 Date:_____

Marks: 9

Question 1:

1.1 Mrs. Ndlovu opens a savings account and immediately deposits R 1 000 into the account.

He continues to make monthly payments of R 1 000 into the account for a period of three years. The interest rate for the first year is 18% per annum compounded monthly. Thereafter, the interest rate changes to 19% per annum compounded monthly for the next two years. Calculate the value of his investment at the end of the savings period.

(3)

1.2 What amount must be invested now if an investor wishes to receive 12 monthly payments of R 6000 in the future? The payments will be received starting in four years' time. Interest is 15% p.a. compounded monthly for the first 47 months(from now). Thereafter, the interest is 18% p.a. compounded monthly.

(3)

1.3 Samantha has plans to buy a house, but can only afford a bond with monthly payments of

R 2500 for the first five years at the current rate of 16,5% per annum compounded monthly. Thereafter, the interest rate is adjusted to 17,8% per annum compounded monthly and her monthly payment increases to R 3200 per month. Determine the cash value of the house she can afford to buy under these circumstances, if the first payment is due one month from now, and her last payment is made at the end of twenty years.

(3)

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Self Test: 1 Date:_____

Marks: 95

Time: 2 hrs and 20 min

Question 1:

1.1 Mr. Samuel is an entrepreneur and a Mathematician at Jeppe College in the republic

of South Africa. He took a loan for his small photocopying business in Durban. The loan of R 18 000 is repaid over a period of 1 year in the following manner: R 3000 after 2 months, R 7500 at the end of 5 months and R 4200 three months later. If the interest rate is 12% p.a. compounded monthly, what would the final payment be?

(3)

1.2 As a mathematician and company manager, Mr. William decided to invest funds for future utilization. He invested R 8000 for 10 years. For first two years the interest rate was 12 % p.a, the rate the changes by to 13% p.a. compounded monthly for the next three years and the to 14% p.a. compounded quarterly for the final 5 years. Calculate the final amount paid out at the end of 10 years.

(3)

1.3 Mr. Malefane owns a massive house in Mafikeng.
As a Mathematician he determined with precision how much he must borrow from a financial institution in order to purchase furniture for his house.

Mr. Malefane borrows R 12 800 to buy furniture for his huge house. He can afford to pay

R 700 per month. The interest rate is 18% *p.a. compounded monthly*

1.3.1 Calculate the number of months that Mr. Malefane will be required to pay R 700.

(4)

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1.3.2 Hence or otherwise prove that the final payment, which is less than R 700, and is made one month after the last payment of R 700 is $R\ 374{,}51$.

(3)

Question 2:

2.1 Ms. Musawenkosi invested R 1570 at 12% p.a. , compound interest. After how many years will the investment be worth 75% more than the initial investment.

(3)

2.2 Mr. Brighton is a businessman in Pretoria; South Africa.
In view of load shedding by Eskom, He decided to buy a large generator for R 235 652. It depreciates at 12% p.a. on a reducing balance.
He wants to buy a new generator in 5 years' time. The old generator will be sold at a scrap value after five years. He sets up a sinking fund in order to save for the new generator. The proceeds from the sale of the old generator will be used together with the sinking fund to buy the new generator.

2.2.1 Determine the scrap value of the old machine in 5 years' time.

(3)

2.2.2 If the price of new machine increases by 17% per annum, determine how much a new machine will cost in 5 years' time.

(3)

2.2.3 Determine the amount required in the sinking fund after 5 years.

(2)

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2.2.4 Determine the equal monthly payments made into the sinking fund if the

Interest earned is 13,2% p.a., compounded monthly and the first payment is made at the end of the month.

(4)

Question 3:

3.1 Leone is given a loan of R300 000. The interest rate paid is 12% per annum compounded monthly. The repayment is to be made in equal monthly payments over a period of 8 years, starting one month after the loan is granted. Calculate the monthly repayment.

(4)

3.2 Mary invests R2000 annually, starting on her 21st birthday. The investment earns12% interest compounded annually. Calculate the value of Mary's investment directly after her deposit on her 26th birthday.

(3)

3.3 Calculate how many years it will take an investment to triple if it is invested at 12% per annum compounded half- yearly.

(4)

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Question 4:

4.1 Zobi bought a new Ford Figo for R12500000.It depreciates at arate of 6, 5% per annum on the reducing-balance method. After how many years will its book value be R100000?

(6)

4.2 Clive applies for a student loan of R120000 to cover the costs of his first two years at University. The loan is approved at an interest rate of 14, 25%p.a, compounded monthly. He prefers to repay the loan in 24 equal monthly payments. These payments start 2 months after receiving the loan. Calculate:

4.2.1 his monthly repayments.

(4)

4.2.2 the outstanding balance immediately after his 16th payment has been made.

(4)

Question 5:

5.1 An amount of R16000 is invested at an interest rate of 7, 5%p. a.
Determine the nominal interest, semi-yearly compounded, which is earned on the investment.

(4)

5.2 Peter decides to save a constant amount at the end of each month.
He opens a saving account at the end of the month he receives his first salary.
The bank offers an interest rate of 4, 7% p.a. compounded monthly.

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5.3.1 Determine the amount he has to save monthly to have R30000 in his savings account at the end of 4years.

(4)

5.2.2 What interest will he receive on his savings during the 4 years?

(4)

Question 6:

6.1 A motor vehicle, costing R250000, depreciates at a rate of **a**%per annum on the reducing balance method. Calculate **a** if it takes 7 years for the value of the vehicle to decrease to R90000 under these circumstances.

(4)

6.2 A loan of R240 000 is repaid over 60 months by equal monthly payments, Starting one month after the loan was granted.

6.2.1 Calculate the monthly repayments if the interest on the loan is 9%p.a Compounded monthly on a reducing balance basis.

(4)

6.2.2 The client experiences financial difficulties and makes only 17 payments. Calculate the balance of the loan at the 17th month.

(4)

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Question 7:

7.1 A couple of years ago, Tlali's uncle had bought a house for R300 500. In his will, he left the house to Tlali. The uncle passed away and Tlali decided to sell the house. He got R780 000 for the house.

Inflation is calculated at 16% p.a. calculate annually (yearly).
Calculate how many years ago did Tlali's uncle buy the house?

(4)

7.2 Different investment opportunities are given. Determine which is the most favourable effective interest rate over a period of one or more years.

7.2.1 9, 8% p.a. compounded monthly.

(3)

7.2.2 9, 5% p.a. compounded semi-annually (half yearly).

(3)

7.3 During December 2007 Charl borrowed R28 OOO. He arranged to Amortise (pay off) the loan of R28 000 over a period of 3 years. The monthly payments are R1098, 52 and the interest rate is 24% p.a. compounded monthly.

YOU ARE ADVISED TO MAKE USE OF A TIM,E LINE THAT REPRESENTS THE 3 YEARS

7.3.1 After 2years Charl won R2034 in a competition.
He immediately paid this amount off on his loan.

Calculate what amount he owed the bank after this once off payment.

(4)

7.3.2 Suppose he owed the bank R9 583, 22. Calculate what his new monthly payments would be from the 25th month onwards, if the bank then offered him an interest rate of 18% compounded monthly.

(4)

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Self Test: 2 **Date:**_____

Marks: 62

Time: 1 hrs and 20 min

Question 1:

How long any amount of money must be invested for, in order for it to double at interest rate of 8.5% p.a compounded monthly. Give your answer to the nearest month (4)

Question 2:

Solomon invests R800 into a savings account everymonth.If the annual interest rateis12.5% compounded monthly, how much will he have after 6years? (3)

Question 3:

Mpho deposits R2 000 into a savings account at the end of every quarter for 16 years. The account offers an interest rate of 8.5% p.a compounded quarterly. Due to financial difficulty, he is unable to make his last two payments. How much will he have in his account after 16years? (5)

Question 4:

Sipho wishes to purchase a car for R80 000.If he borrows the money from Mr .Samuel at an interest rate of 8.5% per annum compounded monthly, calculate:

4.1 the monthly instalments if the loan is to be paid back over 5 years, (5)

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4.2 the outstanding balance (settlement amount) if he decides to settle the loan after 3 years, (4)

4.3 the amount saved by paying the loan off after 3 years. (2)

Question 5:

Tshepo wishes to buy a fridge for R6 000.The fridge can either be paid off over a period of 6 months at an interest rate of 12% p.a. compounded monthly; or he can pay cash and receive a 12.5% discount. Calculate:

5.1 the cash price (2)

5.2 the monthly instalments and savings made if cash is not paid (6)

Question 6:

Samantha takes out a loan of R100 000 to buy a car. The loan agreement stipulates that it has to repaid by means of 238 equal monthly payments at an interest rate of 18% per annum

Compounded monthly starting 3 months after the loan is granted. Ermine her monthly payments. (4)

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Question 7:

A loan of R50 000 is repaid over a period of 5 years by equal monthly payments, at an interest rate of 5.8% per annum compounds monthly. If payments start 6 months after the loan is granted, what monthly payments are required to repay the loan? (6)

Question 8:

Mr .Rogers a mathematician at Jeppe College plans to purchase a brand new car. He borrows R100 000 at an interest rate of 10%p.a. compounded monthly. He repays the loan by means of equal monthly payments of R5 500.

8.1 how many months will it take for the loan to be repaid? (5)

8.2 What will the outstanding balance on the loan be after the 19th payment? (4)

8.3 by calculation show that the final payment is approximately R4 394.89 (3)

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Question 9:

Thapelo borrows R10 000 on the **1st of March 2013** at an interest rate of9.5% p.a. Compounded monthly. He pays the loan back via monthly instalments of R450.The loan agreement allows. Thapelo to make his first payment on the 1st **ofseptember2013**.

9.1 Calculate how much Thapelo is owing on 1st September (2)

9.2 Determine how many months it will take Thapelo to pay back the loan once payments Start (5)

9.3 what is the balance of loan immediately after the 25th payment (3)

9.4 what is the value of final payment (3)

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Module: 11
DIFFERENTIAL CALCULUS

THE POWER RULE OF DIFFERENTIATION

Differentiation of cx^n

SIMPLE DEDUCTION FROM THE POWER RULE

If $y = cx^n$, then $\frac{dy}{dx} = cnx^{n-1}$

(1) **Sum rule of differentiation**

If $y = f(x) + g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) + g(x)\}$

$$= \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}$$

$$= f'(x) + g'(x)$$

(2) **Difference rule of differentiation**

If $y = f(x) - g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) - g(x)\}$

$$= \frac{d}{dx}\{f(x)\} - \frac{d}{dx}\{g(x)\}$$

$$= f'(x) - g'(x)$$

(3) **Derivative of a constant**

If $y = c$ where c is a constant, we can write this as

$$y = cx^0$$

So that $\frac{dy}{dx} = (0)cx^{0-1} = 0$

Thus the **differential coefficient of a constant is zero.**

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Remark: Rules for differentiating $y = f(x) \times g(x)$ and $y = \dfrac{f(x)}{g(x)}$ i.e. products and quotients respectively will be seen at university or possibly at college(tertiary)

(4) **The constant multiple rule of differentiation**

If $y = cf(x)$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}\{cf(x)\}$, where c is a constant.

$$= c\frac{d}{dx}\{f(x)\}$$

$$= cf'(x)$$

EXAMPLES 5

Differentiate each of the following with respect to x.

(a) $y = x^{12}$ (b) $y = 7$ (c) $f(x) = -5x^4$

(d) $y = 3x^2 - 13x^{-4} + \dfrac{5}{x^4}$ (e) $f(x) = 18\sqrt{x}$ (f) $y = (3x-5)(2+4x)$

(g) $h(x) = \dfrac{5x^7 - 3x^4 + \pi^2 x^3}{x^3}$ (h) $g(x) = \dfrac{4x^2 - 36}{5x + 15}$

SOLUTIONS:

(a) $y = x^{12}$

$\Rightarrow \dfrac{dy}{dx} = 12x^{12-1}$ (Power rule of differentiation)

$\therefore \dfrac{dy}{dx} = 12x^{11}$

(b) $y = 6$

$\therefore \dfrac{dy}{dx} = 0$. (Derivative of a constant relation/ function is zero)

(c) $f(x) = -5x^4$

$\Rightarrow f'(x) = 4 \times (-5)x^{4-1}$

$\therefore f'(x) = -20x^3$

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(d) $y = 3x^2 - 13x^{-4} + \dfrac{5}{x^4}$

$\Rightarrow y = 3x^2 - 13x^{-4} + 5x^{-4}$

$\Rightarrow y = 3x^2 - 8x^{-4}$

$\Rightarrow \dfrac{dy}{dx} = 6x + 32x^{-5}$

$\therefore \dfrac{dy}{dx} = 6x + \dfrac{32}{x^5}$ 'It is **recommended** that your **final answers** are written with **positive indices'**

(e) $y = 18\sqrt{x}$

$\Rightarrow y = 18x^{\frac{1}{2}}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2} \times 18x^{\frac{1}{2}-1}$

$\Rightarrow \dfrac{dy}{dx} = 9x^{-\frac{1}{2}}$

$\therefore \dfrac{dy}{dx} = \dfrac{9}{\sqrt{x}}$

(f) $y = (3x-5)(2+4x)$

$\Rightarrow y = 6x + 12x^2 - 10 - 20x$ 'products must be simplified to powers of x for the power rule to apply'

$\Rightarrow y = 12x^2 - 14x - 10$

$\therefore \dfrac{dy}{dx} = 24x - 14$

(g) $h(x) = \dfrac{5x^7 - 3x^4 + \pi^2 x^3}{x^3}$

$\Rightarrow h(x) = 5x^4 - 3x + \pi^2$ 'fractions must be simplified to powers of x'

$\therefore h'(x) = 20x^3 - 3$ 'Power rule then gives the answer'

(h) $g(x) = \dfrac{4x^2 - 36}{5x + 15}$

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$$\Rightarrow g(x)=\frac{4(x^2-9)}{5(x+3)}$$ ' Simplify quotients for the power rule to apply'

$$\Rightarrow g(x)=\frac{4(x-3)(x+3)}{x+3}$$

$$\Rightarrow g(x)=\frac{4}{5}(x-3)$$

$$\Rightarrow g(x)=\frac{4}{5}x-\frac{12}{5}$$

$$\therefore g'(x)=\frac{4}{5}$$

EXAMPLE 6

Find the gradient of the curve $y=-3x^2+5x-20$ at the point $(2;16)$.

SOLUTION:

$$y=-3x^2+5x-20$$

$$\Rightarrow \frac{dy}{dx}=-6x+5$$

Thus at $(2;16)$: $\left.\frac{dy}{dx}\right|_{x=2}=-6(2)+5=-7$

EXAMPLE 7

Find the points on the curve $y=x^3+3x^2-6x-10$ where the gradient is 3.

SOLUTION:

$$y=x^3+3x^2-6x-10$$

$$\Rightarrow \frac{dy}{dx}=3x^2+6x-6$$

If the gradient is 3, then $\frac{dy}{dx}=3$.

i.e. $3x^2+6x-6=3$

$\Leftrightarrow x^2+2x-3=0$

$\Leftrightarrow (x+3)(x-1)=0$

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$\therefore x = -3 \quad \text{or} \quad x = 1$

If $x = -3$, then $y = x^3 + 3x^2 - 6x - 10$

$$= (-3)^3 + 3(-3)^2 - 6(-3) - 10$$

$$= -27 + 27 + 18 - 10$$

$$= 8$$

And if $x = 1$, then $y = x^3 + 3x^2 - 6x - 10$

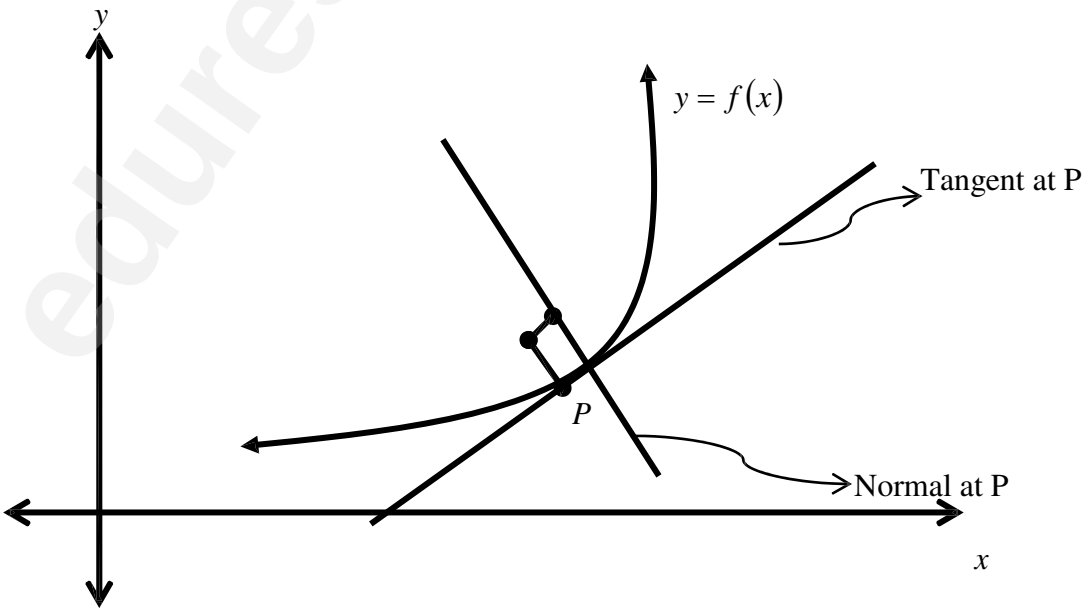
$$\Rightarrow y = (1)^2 + 3(1)^2 - 6(1) - 10$$

$$\therefore y = -12$$

Thus the curve $y = x^3 + 3x^2 - 6x - 10$ has a gradient of 3 at $(-3; 8)$ and at $(1; -12)$.

TANGENTS AND NORMALS

A ***tangent*** to a curve is a straight line which touches the curve at most at one point. Suppose that some point P lies on a curve $y = f(x)$. The line passing through P, perpendicular to the tangent and to the curve at P, is said to be the normal to the curve at P. To find the equation of the tangent or normal to a curve at some point $P(x_1; y_1)$ on the curve we use $y - y_1 = m(x - x_1)$ with the gradient m defined by differentiation, or we may use $y = mx + c$.



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EXAMPLE 8

Find the equation of the tangent and normal to the curve $y = x^2 - 4x + 1$ at the point $(-2; 13)$.

SOLUTION:

$y = x^2 - 4x + 1$

$\frac{dy}{dx} = 2x - 4$

At $(-2; 13)$: $\left.\frac{dy}{dx}\right|_{x=-2} = 2(-2) - 4 = -8$

Thus the tangent at $(-2; 13)$ has gradient -8.

But $m_{\tan gent} \times m_{normal} = -1$ (Tangent $\perp$ Normal)

$\Rightarrow -8 \times m_{normal} = -1$

$\therefore m_{normal} = \frac{1}{8}$.

Thus the equation of the tangent is:

$y - 13 = -8(x - (-2))$

$\therefore y = -8x - 3$

Equation of normal is:

$y - 13 = \frac{1}{8}(x - (-2))$

$\therefore 8y = x + 106$

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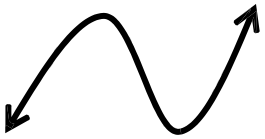
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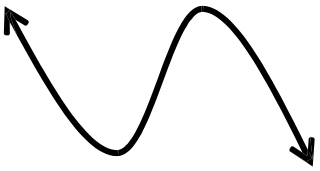
CUBIC FUNCTIONS

Graphs of $f(x) = ax^3 + bx^2 + cx + d$, *are called* *cubic curves* *or graphs of* *third degree polynomials.*

1. If $a > 0$ (**Positive**) : Shape of graph:

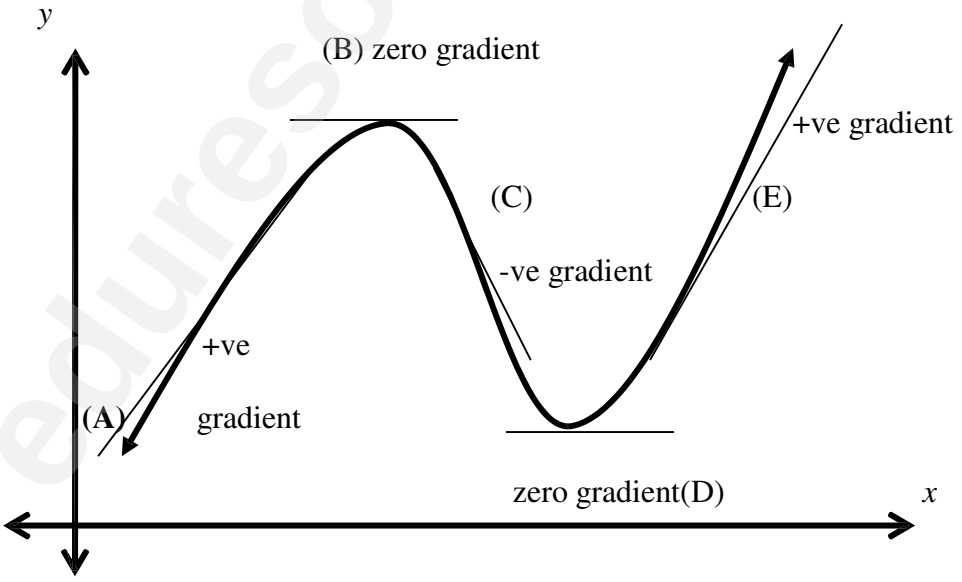


If $a < 0$ (**Negative**): shape of graph:



2. $f'(x)$ is the gradient of the curve $y = f(x)$.

Consider the following hypothetical curve:



At points B and D the tangent to the curve is parallel to the $x-$axis and therefore the gradient of the curve at B and D is zero.

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i.e. $\frac{dy}{dx} = 0$ at B and D.

The point is said to be a maximum point on the curve since the value of y at B is greater than the values of y at points close to B.

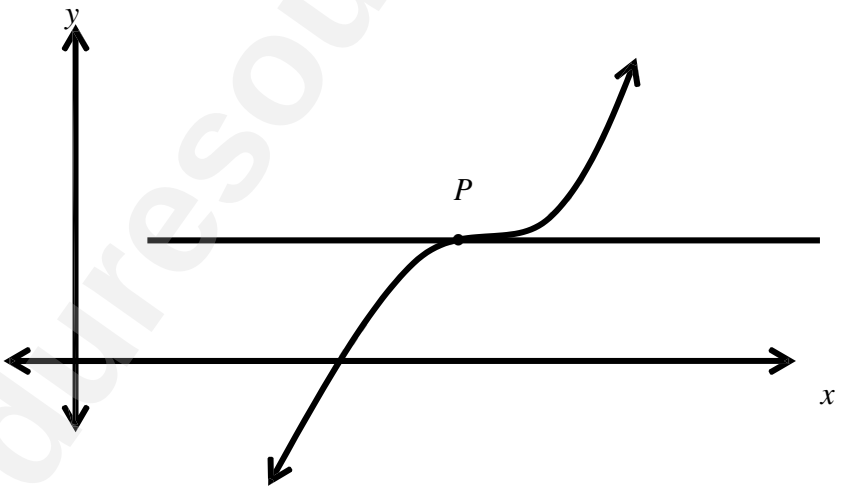
It is important to realise that at a maximum point such as B, the value of y may not be the greatest value of y on the entire curve. The important fact is that the value of y at B is greater than at points close to B. Thus B is really a ***local maximum*** but we usually refer to such points simply as maximum points.

In a similar way D is a ***(local) minimum*** point on the curve. The value of y at D is less than the values of y at points on the curve close to D and so D is said to be a (local) minimum point on the curve.

Such maximum and minimum points are said to turning points on the curve.

To locate such points without having to draw the graph we have only to find points at which $\frac{dy}{dx} = 0$. However there may be some point on a curve at which $\frac{dy}{dx} = 0$, but the point is neither a maximum nor a minimum.

In the figure below, P, is such a point because the tangent is parallel to the $x-$axis but P is neither a maximum point nor a minimum point. P is called a point of ***inflection***.



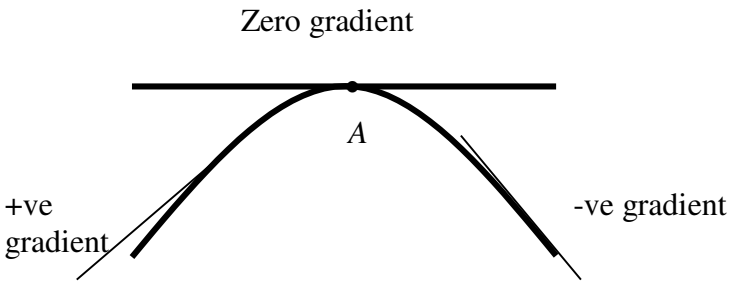
A point on a curve at which $\frac{dy}{dx} = 0$, i.e. maximum point like B , minimum point like D, or point of inflection such as P are called ***stationary*** points.

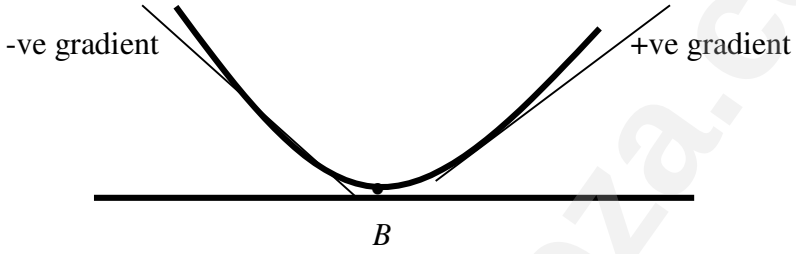
Having located these stationary points, we can distinguish between them i.e. determine whether they are maximum points or minimum points or points of inflection, by considering the sign of the gradient at points close to, and on either side of these points.

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For increasing function values of x, as we approach and pass through a maximum point such as A, the gradient of the curve changes from positive ('uphill' or for increasing x) to zero, at A, to negative ('down hill' for increasing x).

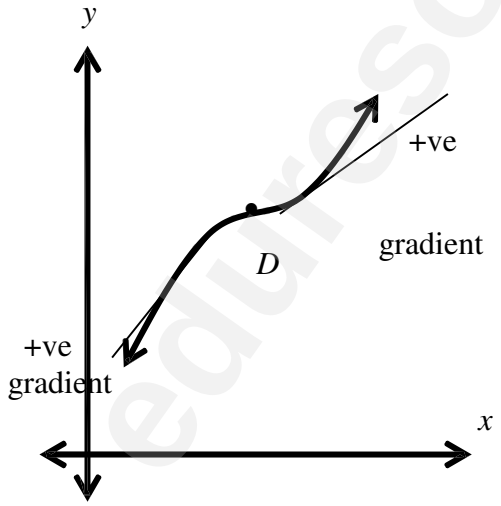


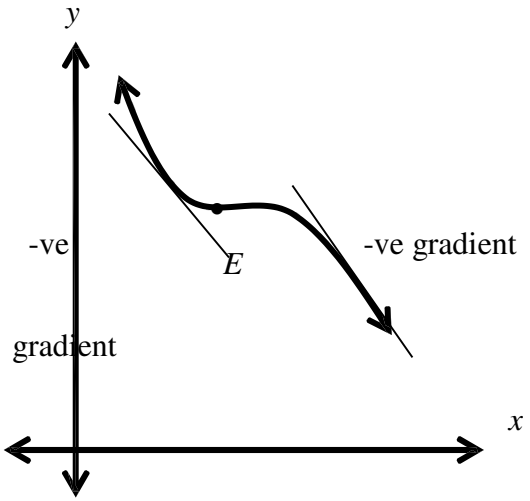


zero gradient

On the other hand, as we approach and pass through a minimum point such as B, the gradient changes from positive to negative through zero to positive.

For each points of inflection, such as Q and R, the gradient of the curve does not change sign as we pass through the point.





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Thus, to find the stationary points on a curve $y = f(x)$ and to distinguish between them, we proceed as follows:

1. find the gradient function $\frac{dy}{dx}$ of the curve .

2. Equate to zero the expression for $\frac{dy}{dx}$,i.e. let $\frac{dy}{dx} = 0$.

3. Find the values of x (i.e. x_1, x_2, x_3, . . .) which satisfy this equation.

4. Consider the sign of $\frac{dy}{dx}$ on either side of these points.

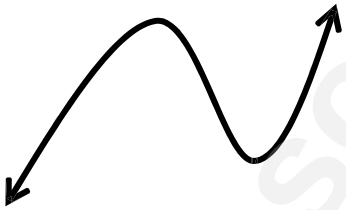
5. Find the values y_1, y_2, y_3, . . . which correspond to x_1, x_2, x_3, . . .

EXAMPLE 9

Draw a neat sketch graph of $f(x) = x^3 + 3x^2$.

SOLUTION:

1. a positive (> 0):



(Shape of graph)

2. Determine the $y-$intercept by substituting $x = 0$.(Value of the constant)

$y = x^3 + 3x^2$

$\Rightarrow y = (0)^3 + 3(0)^2$

$\therefore y = 0$

Thus , $(0; 0)$ is the $y-$intercept.

3. Determine the $x-$intercepts by substituting $y = 0$ and factorising the expression.
First try to find a common factor, or to group terms together. If this fails, use the factor theorem.

$x-$intercepts: $x^3 + 3x^2 = 0$

$\Rightarrow x^2(x+3) = 0$

$\therefore x = 0$ or $x = 0$ or $x = -3$

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Thus, $(0;\,0)$ and $(-3;\,0)$ are $x-$intercepts.

4. Turning points at $\dfrac{dy}{dx}=f'(x)=0$: $\quad f(x)=x^3+3x^2$

$$\Rightarrow f'(x)=3x^2+6x$$

Thus, $\quad 3x^2+6x=0$

$$\Rightarrow 3x(x+2)=0$$

$$\therefore x=0 \text{ or } x=-2$$

5. Determine the $y-$co-ordinates of the turning points.

$f(x)=x^3+3x^2$ with $x=0$ and $f(x)=x^3+3x^2$ with $x=-2$

$\Rightarrow f(0)=(0)^3+3(0)^2$ $\Rightarrow f(-2)=(-2)^3+3(-2)^2$

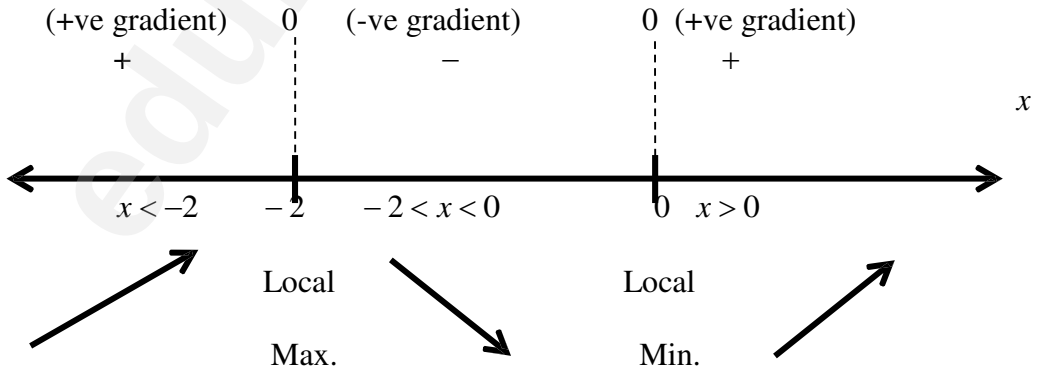
$\therefore f(0)=0$ $\therefore f(-2)=4$

Therefore $(0;\,0)$ and $(-2;\,4)$ are turning points of the function $f(x)=x^3+3x^2$.

Now, to classify the turning points as local minimum and local maximum, it helps to draw the number line below: **(a)** take any $x<-2$, e.g. $x=-5$, so that $f'(-5)=3(-5)^2+6(-5)=+45$

(b) take any $-2<x<0$, e.g. $x=-1$, so that $f'(-1)=3(-1)^2+6(-1)=-3$

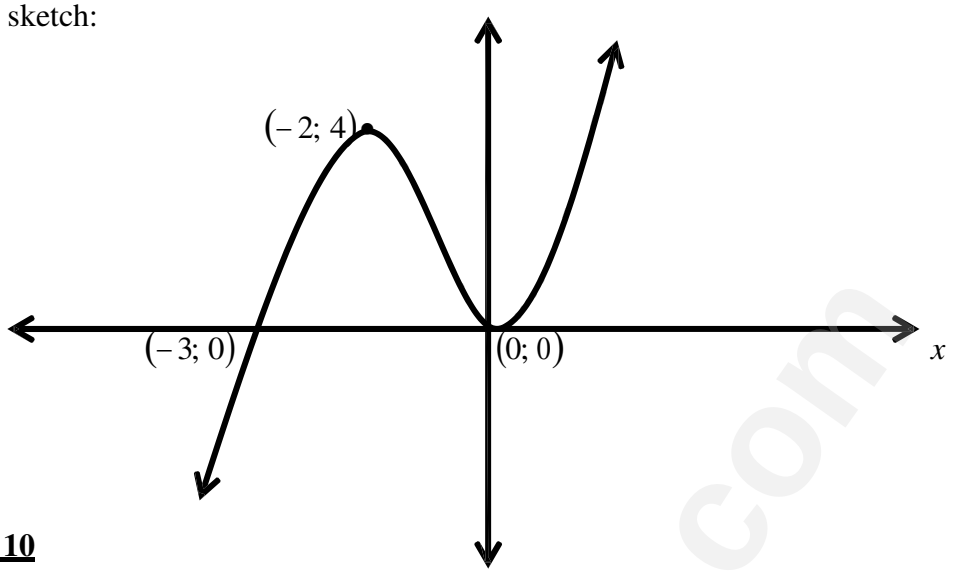
(c) take any $x>0$, e.g. $x=2$, so that $f'(2)=3(2)^2+6(2)=+24$



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6. Draw neat sketch:



EXAMPLE 10

If $f(x) = x^3 - 6x^2 + 9x - 4$, determine the coordinates of

10.1 the intercepts with the axes.

10.2 the local maximum and minimum points of the graph of f.

10.3 Draw a sketch graph of f.

10.4 Determine the equation of the tangent to the curve at the point $(2; -2)$.

10.5 Use the graph to determine the values of k for which the equation

$x^3 - 6x^2 + 9x = k$ will have three real and unequal roots.

10.6 Determine x if $f(x) < 0$.

10.7 Determine x if $f'(x) < 0$.

SOLUTION:

10.1 y – intercept: Let $x = 0$; $\quad f(x) = x^3 - 6x^2 + 9x - 4$

$$\Rightarrow f(0) = (0)^3 - 6(0)^2 + 9(0) - 4$$

$$\therefore f(0) = -4$$

Thus, $(0; -4)$ is the y – intercept of f.

x – intercept; let $y = 0$: $x^3 - 6x^2 + 9x - 4 = 0$ with $f(1) = (1)^3 - 6(1)^2 + 9(1) - 4 = 0$

Thus, $(x - 1)$ is a factor of $f(x)$.

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The **division algorithm** yields:

$$\begin{array}{r|l} & x^2-5x+4 \\ \hline x-1 & x^3-6x^2+9x-4 \\ & -(x^3-x^2) \\ \hline & -5x^2+9x-4 \\ & -(-5x^2+5x) \\ \hline & 4x-4 \\ & -(4x-4) \\ \hline & 0 \end{array}$$

Hence $x^3-6x^2+9x-4 \equiv (x-1)(x^2-5x+4)=0$

$\Rightarrow (x-1)(x-4)(x-1)=0$

$\therefore x=1$ or $x=4$ or $x=1$

Therefore $(1;0)$ and $(4;0)$ are $x-$intercepts of f.

Alternatively, the ***Synthetic division method*** can be used to determine the factors of a cubic function.

(divisor)	1	1	-6	9	-4
		1	-5	4	
	1	-5	4	0 (Remainder)	

Thus $x^3-6x^2+9x-4 \equiv (x-1)(x^2-5x+4)=0$

$\Rightarrow (x-1)(x-4)(x-1)=0$

$x=1$ or $x=4$ or $x=1$

Hence, $(1;0)$ and $(4;0)$ are $x-$intercepts of f.

10.2 $f(x)=x^3-6x^2+9x-4$

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$\Rightarrow f'(x) = 3x^2 - 12x + 9$

Local maximum and local minimum occur where $f'(x) = 0$**:**

$\therefore 3x^2 - 12x + 9 = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-3)(x-1) = 0$

$\therefore x = 3$ **or** $x = 1$

Thus, $f(x) = x^3 - 6x^2 + 9x - 4$ then gives;

$f(3) = (3)^3 - 6(3)^2 + 9(3) - 4$ and $f(1) = (1)^3 - 6(1)^2 + 9(1) - 4$

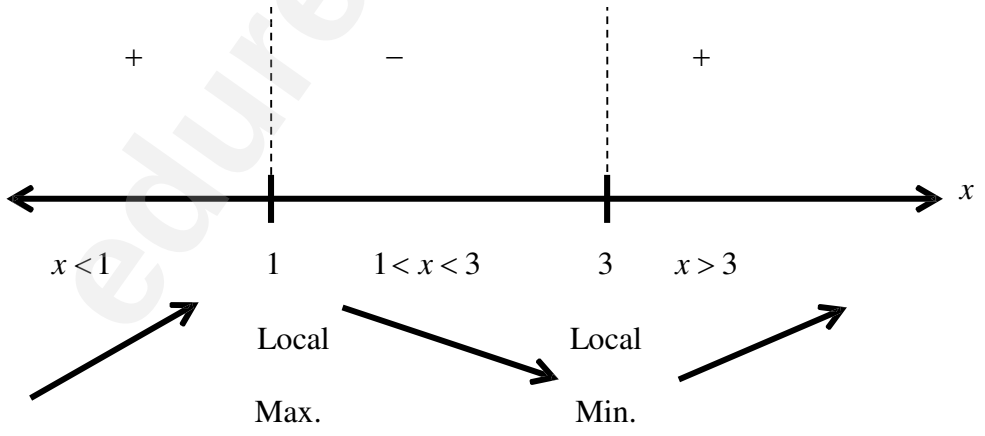
$= 27 - 54 + 27 - 4$ and $f(1) = 0$

$= -4$

Hence $(3; -4)$ and $(1; 0)$ are turning points. To classify them as local maximum and local minimum we proceed as follows: (a) take any $x < 1$, e.g. $x = 0$,so that $f'(0) = +9$

(b) take any $1 < x < 3$, e.g. $x = 2$, so that $f'(2) = -3$

(c) take any $x > 3$, e.g. $x = 4$, so that $f'(4) = +9$

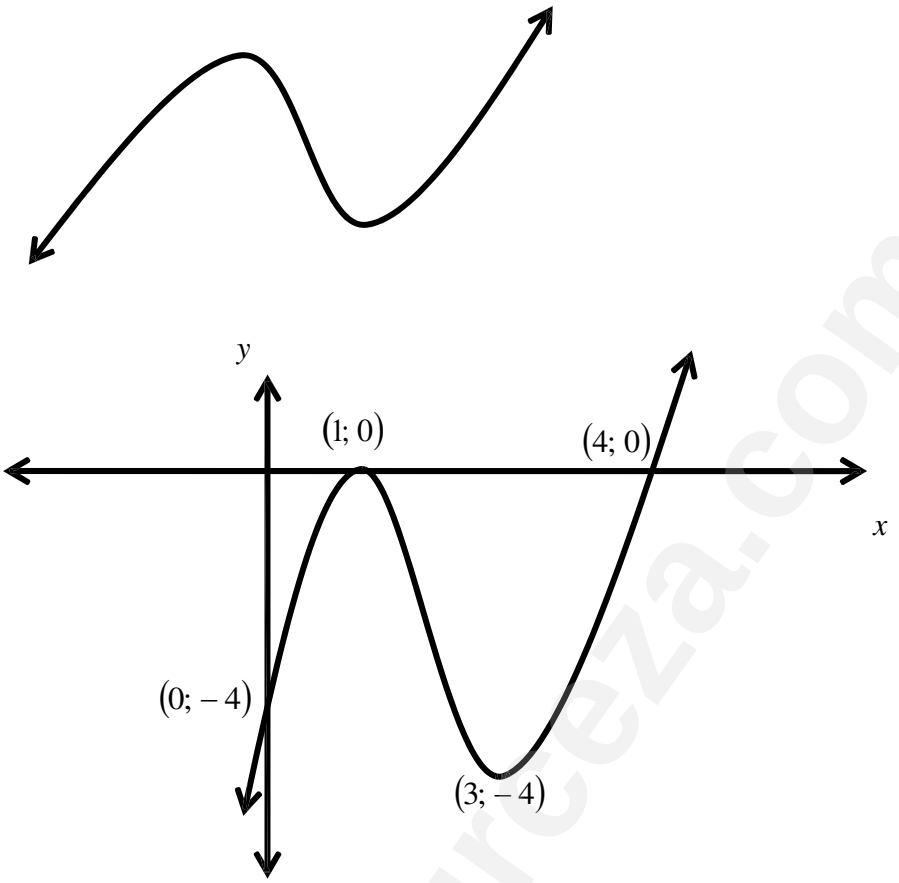


Thus $(3; -4)$ is the local minimum and $(1; 0)$ is the local maximum

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10.3 $a > 0$ (Positive) implies the shape:



10.4 Gradient of tangent at any point x is given by: $f'(x) = 3x^2 - 12x + 9$

Therefore gradient of the tangent at $(2; -2)$: $f'(2) = 3(2)^2 - 12(2) + 9$

$= 12 - 24 + 9$

$= -3$

Equation: $y = -3x + c$ through $(2; -2)$

$\Rightarrow -2 = -3(2) + c$

$\therefore c = 4$

10.5 $x^3 - 6x^2 + 9x = k$

$\Rightarrow x^3 - 6x^2 + 9x - 4 = k - 4$

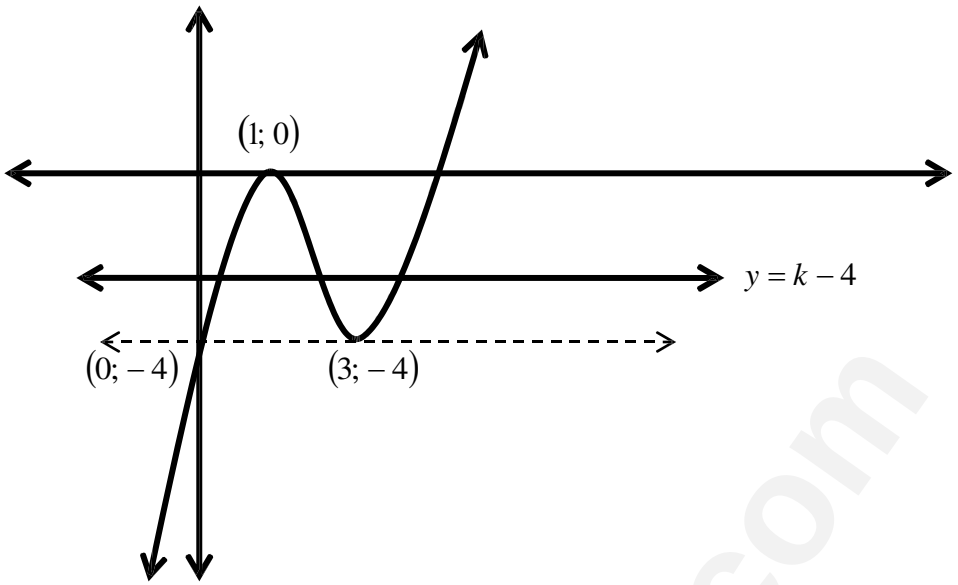
The line $y = k - 4$ intersects the graph of $y = x^3 - 6x^2 + 9x - 4$ at three different places if equation (1) has three real, unequal roots.

$\therefore -4 < k - 4 < 0$

$\therefore 0 < k < 4$; $k \in \mathbb{R}$

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10.6 $f(x) < 0$

$\therefore x < 4\,;\ x \neq 1\,;\ x \in$

10.7 $f'(x) < 0$:

$\therefore 1 < x < 3$

FINDING THE EQUATION OF A CUBIC FUNCTION

To find the equation of a drawn cubic function whose roots are known we use:

$$y = a(x - x_1)(x - x_2)(x - x_3);$$

where x_1, x_2 and x_3 are the roots of the cubic function.

To find the equation we proceed as follows:

1. Substitute the numerical values of x_1, x_2 and x_3 in the formula above.

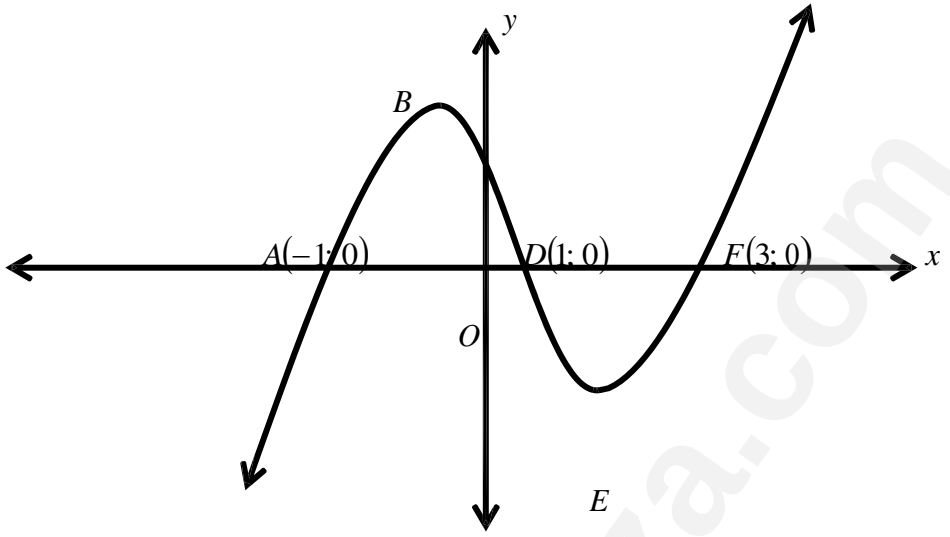
2. Use any extra/ additional point to find the numerical value of a.

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EXAMPLE 11

The sketch shows the graph of $y = x^3 + ax^2 + bx + c$



11.1 find the values of a, b and c.

11.2 Determine OC.

11.3 Find the co-ordinates of the turning points B and E (Leave your answer in surd form)

11.4 Determine the equation of the tangent to the curve at $(2; -3)$.

SOLUTION:

11.1 $f(x) = (x+1)(x-1)(x-3)$

$= (x^2 - 1)(x-3)$

$= x^3 - 3x^2 - x + 3$

$\therefore a = -3,\ b = -1$ and $c = -1$

11.2 $OC = 3$ **units**

11.3 $f'(x) = 3x^2 - 6x - 1$

$3x^2 - 6x - 1 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{6 \pm \sqrt{48}}{6}$$

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$$= \frac{6 \pm 4\sqrt{3}}{6}$$

$$= \frac{3 \pm 2\sqrt{3}}{3}$$

11.4 $f'(x) = 3x^2 - 6x - 1$

$$f'(2) = 3(2)^2 - 6(2) - 1$$

$$= -1$$

Thus, $y = mx + c$

$\Rightarrow y = -x + c$ through $(2; -3)$

$\Rightarrow -3 = -2 + c$

$\therefore c = -1$

$\therefore y = -x - 1$

INFLECTION POINTS

Definition:

A point of inflection on a curve is a point where y'' (**second derivative**) is positive on one side and negative on the other.

Thus, on a graph of a twice-differentiable function, $y'' = \frac{d^2y}{dx^2} = 0$ at a point of inflection.

EXAMPLE

Find the coordinates of any points of inflection on the curve $y = x^3(x-1)$ and determine whether they are horizontal points of inflection.

SOLUTION:

$y = x^3(x-1)$

$= x^4 - x^3$

$\frac{dy}{dx} = y' = 4x^3 - 3x^2 = x^2(4x-3)$

$\frac{d^2y}{dx^2} = y'' = 12x^2 - 6x = 6x(2x-1)$

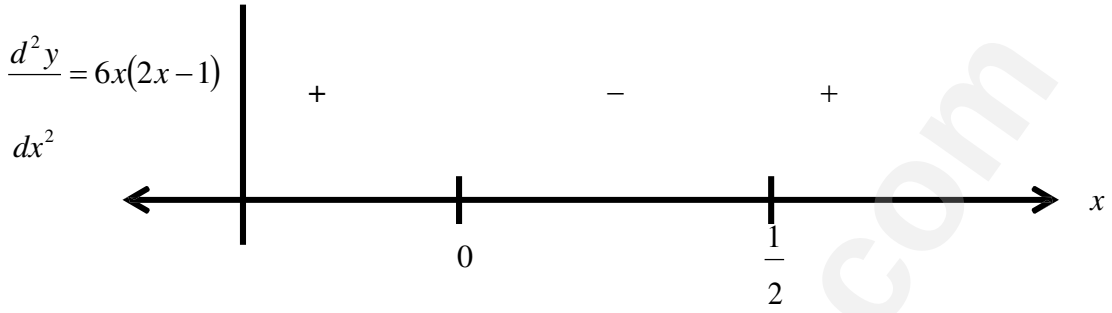
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At points of inflection $y' = 0$ i.e. $6x(2x-1) = 0$

$$\text{giving } x = 0 \text{ or } x = \frac{1}{2}$$

We the draw the following sign table:



Thus since the sign of $\frac{d_2y}{dx} = 6x(2x-1)$, changes between on either side of the critical values it follows that we have inflection points at $x = 0$ and $x = \frac{1}{2}$.

However at: $x = 0$, $y' = 0$ and $y = 0$

$$x = \frac{1}{2},\ y' \neq 0 \text{ and } y = -\frac{1}{16}$$

Thus at: $\left(-\frac{1}{2}; -\frac{1}{16}\right)$, $y' = 0$ but $y' \neq 0 \Rightarrow$ Point of inflection but not a horizontal one.

$(0, 0)$, $y' = 0$ and $y' = 0 \Rightarrow$ Stationary point, i.e. maximum, minimum or pint of inflection.

Therefore points of inflection occur at $(0; 0)$ and $\left(\frac{1}{2}; -\frac{1}{16}\right)$. The point of inflection at $(0; 0)$ is horizontal.

More about points of inflection

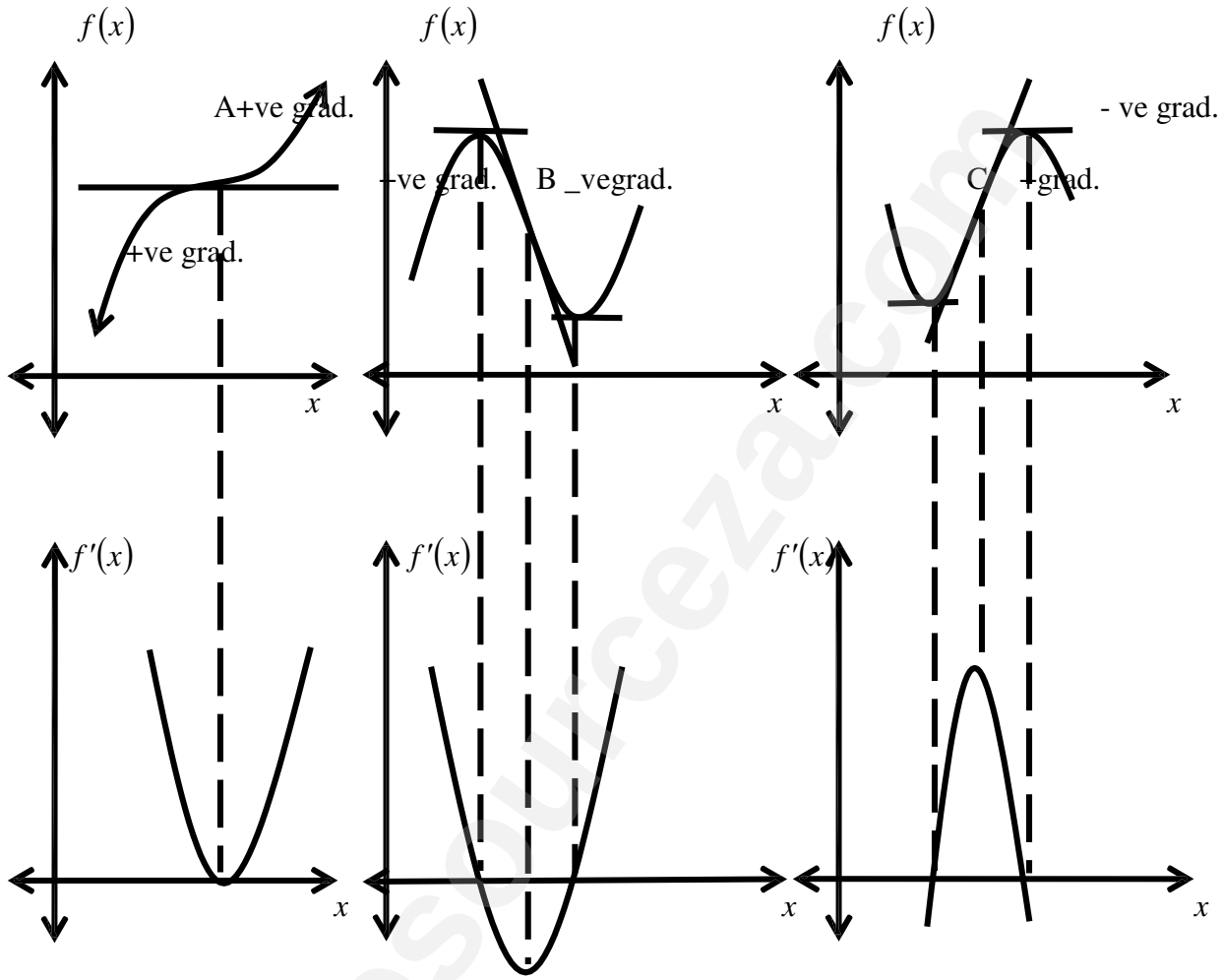
When we examine points on the curve $y = f(x)$ for which $f'(x) = 0$, if $f'(x) < 0$ we have a maximum point, if $f'(x) > 0$ we have a minimum point and if $f'(x) = 0$ we could have a point of inflection.

Indeed at all points of inflection $f'(x) = 0$ but it is possible to have a point of inflection $(a; b)$ i.e. $f'(a) = 0$, for which $f'(a) \neq 0$. A point of inflection is said to occur at any point P on a curve at which the tangent to the curve at P crosses the curve at P. Thus in the diagrams shown below A, B and C are all points of inflection, but only A is a stationary point on the curve as the tangent at A is

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parallel to the $x-$ axis, (i.e. $f'(x)=0$ at A). The second set of diagrams show that the gradient of $f'(x)$, i.e. $f'(x)$, is zero at A, B and C.



Thus if $y=f(x)$, stationary points occur where $f'(x)=0$. These stationary points may be turning points (maximum or minimum) or points of inflection. Points of inflection that are not turning points may also occur if $f'(x)\neq 0$ but $f'(x)=0$.

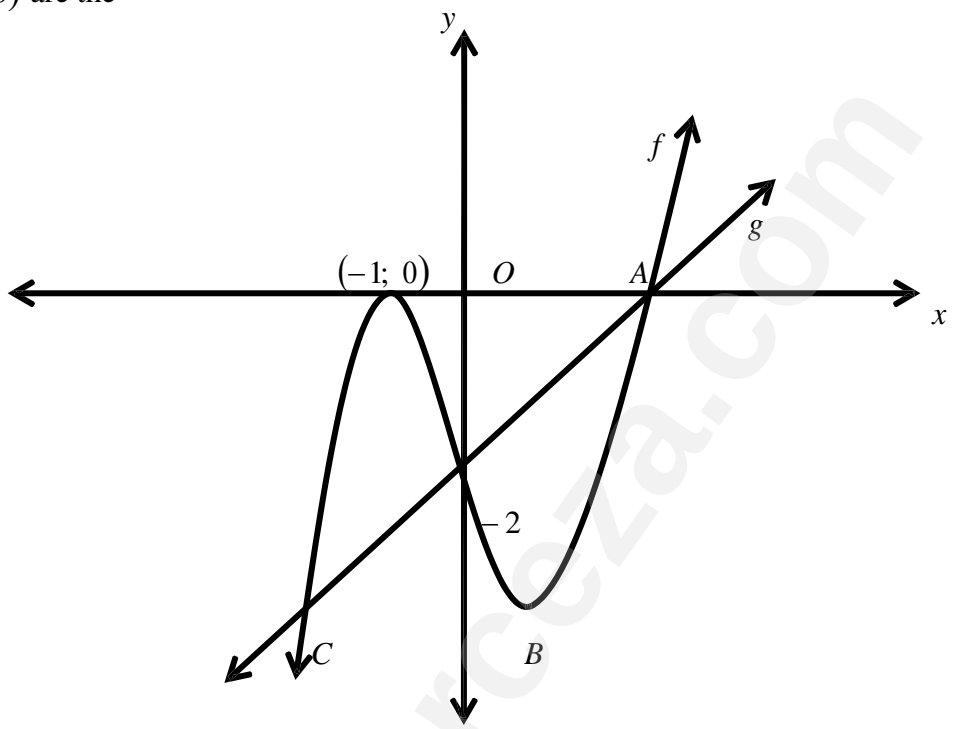
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EXAMPLE [ ]

The graph below represents the functions f and g with $f(x)=ax^3-cx-2$ and $g(x)=x-2$.

A and $(-1;\ 0)$ are the $x-$intercepts of f. The graphs of f and g intersect at A and C.



8.1 Determine the coordinates of A.

8.2 Show by calculation that $a=1$ and $c=-3$.

8.3 Determine the coordinates of B, a turning point of f.

8.4 Show that the line BC is parallel to the $x-$axis.

8.5 Find the $x-$coordinate of the point of inflection of f.

8.6 Write down the values of k for which $f(x)=k$ will have only ONE root.

8.7 Write down the values of x for which $f'(x)<0$.

SOLUTION:

8.1 $0=x-2$

$x=2$

$A(2;\ 0)$

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8.2 $f(-1)=0$**:** $-a+c=2$

$f(2)=0$**:** $8a+-2c=2$

$\therefore a=1$ **or** $c=3$

OR

$a(x+1)(x+1)(x-2)=0$

$a(0+1)(0+1)(0-2)=0$

$-2a=2$

$a=1$

$f(x)=(x^2+2x+1)(x-2)$

$=x^3-3x-2$

$c=3$

8.3 $f'(x)=0$

$3x^2-3=0$

$x^2-1=0$

$(x-1)(x+1)=0$

$\therefore B(1;-4)$

8.4 $x-2=x^3-3x-2$

$0=x^3-4x$

$0=x(x^2-4)$

$0=x(x-2)(x+2)$

$x_C=-2, \quad y_C=(-2)^2-3(-2)-2=-4$

$C(-2;-4)$

$m_{BC}=\frac{y_C-y_B}{x_B-x_C}$

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$$= \frac{-4-(-4)}{1-(-2)}$$

$$= 0$$

BC // to the $x-$axis

OR

Following from $C(-2;-4)$, B and C have the same $y-$ coordinates viz -4, so BC is parallel to the $x-$axis.

OR

$$(x-2) = (x-2)(x+1)^2$$

$$(x+1)^2 = 1 \text{ for } x \neq 2$$

$$x+1 = \pm 1$$

$$\therefore x = 0 \text{ or } x = 2$$

$$\therefore y = -4$$

PRACTICAL APPLICATIONS

1. In these problems something has to be maximised or minimised. Determine an expression in one variable for this unknown entity. We use the following familiar formulae:

   Rectangle: Area = length × breadth

   Perimeter = 2(length + length)

   Cylinder: Volume $= \pi r^2 h$

   Total surface area $= 2\pi rh + 2\pi r^2$

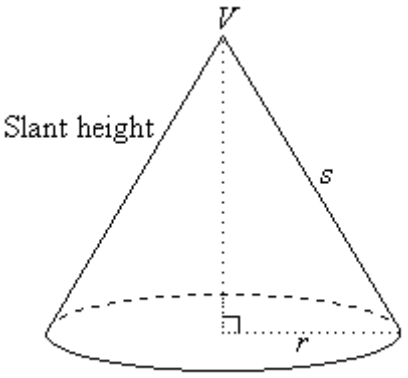
   Sphere: $Volume = \frac{4}{3}\pi r^3$ and Total surface area $= 4\pi r^2$

   Cone: $Volume = \frac{1}{3}\pi r^2 h$ and Total surface area $= \pi rs + \pi r^2$

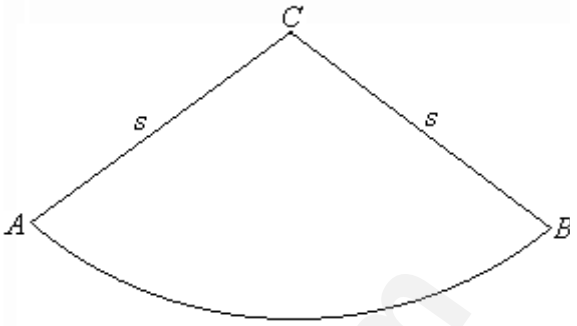
   If a **cone** of base radius r and slant height s is cut along the slant height and opened out flat, then the radius of the sector formed is s and the arc length AB is $2\pi r$.

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Circumference of the base $= 2\pi r$



Length of arc $AB = 2\pi r$

$$\text{Now, } \frac{\text{Area of sector } ABC}{\text{Area of circle with centre at } C} = \frac{\text{Arc length } AB \text{ of sector } ABC}{\text{Circumference of circle with centre at } C}$$

$$\therefore \frac{\text{Area of sector } ABC}{\pi s^2} = \frac{2\pi r}{2\pi s}$$

$$= \frac{r}{s}$$

$$\therefore \text{Area of sector } ABC = \frac{r}{s} \times \pi s^2$$

$$= \pi r s$$

If the curved surface area of a cone is equal to the area of sector ABC, then:
The area of the curved surface of a cone $= \pi r s$

$\therefore$ Total surface area of the cone = Area of the base + Area of curved surface

$$\therefore TSA = \pi r^2 + \pi r s$$

$$TSA = \pi r(r + s)$$

Note:

A cone does not have uniform (or congruent) cross-sections.

EXAMPLE

A conical tent has a radius of 3 m and a perpendicular height of 4 m. Find:
a. the slant height of the tent
b. the curved surface area of the cone using $\pi = 3.142$.

SOLUTION:

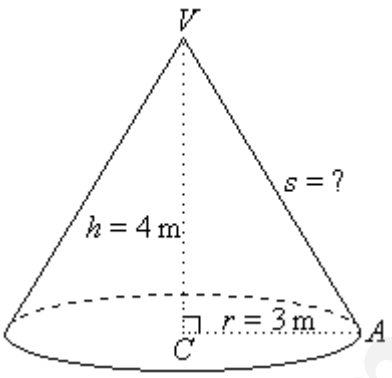
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a. By Pythagoras' Theorem from triangle VCA,

$$s^2 = h^2 + r^2$$
$$= 4^2 + 3^2$$
$$= 16 + 9$$
$$= 25$$
$$\therefore s = \sqrt{25}$$
$$= 5$$

So, the slant height of the tent is 5 m.



b. $CSA = \pi rs$
$= 3.142 \times 3 \times 5$
$= 47.13$

So, the curved surface area is $47.13\ m^2$.

2. Determine the derivative with respect to the specific variable.

3. Maximum or Minimum values occur when the **derivative = 0.**

4. Calculate the max. or min. values by means of substitution.

EXAMPLE 12

The formula shows the profit P, in cents, if a taxi covers a certain distance at an average speed of

v km/h: $P = 10v - \frac{v^2}{10}$.

Determine:

12.1 the speed leading to a maximum profit.

12.2 the maximum profit.

12.3 the value(s) of v which will lead to a loss.

SOLUTION:

12.1 $$P = 10v - \frac{v^2}{10}$$

$$\frac{dP}{dv} = 10 - \frac{2v}{10}$$

$$= 10 - \frac{v}{5}$$

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Maximum P if $\frac{dP}{dv} = 0$

$$\Rightarrow 10 - \frac{v}{5} = 0$$

$$\therefore v = 50 \text{ km/h}$$

12.2 Maximum $P = 10(50) - \frac{(50)}{10}$

$$= 250 \text{ cents}$$
$$= \text{R}2{,}50$$

12.3 $10v - \frac{v^2}{10} < 0$

$$\Rightarrow 100v - v^2 < 0$$
$$\Rightarrow v(100 - v) < 0$$
$$\Rightarrow v < 0 \text{ or } v > 100$$

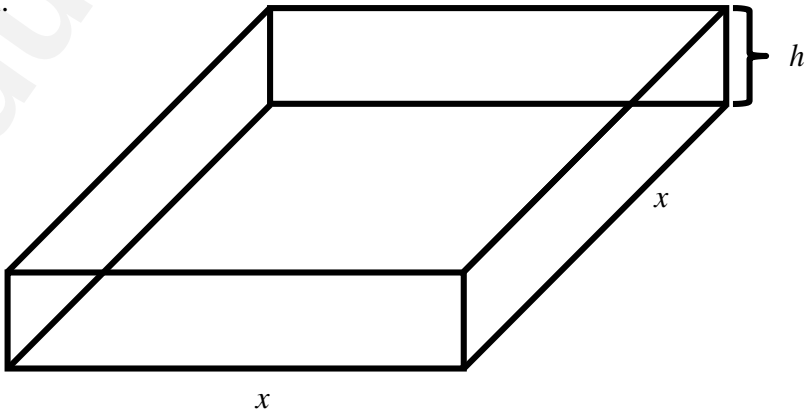
$$\therefore v > 100 \text{ km/h}$$

$v < 0$ means he reverses, away from his destinations.

EXAMPLE 13

A firm wants to manufacture an open stainless steel container with a capacity of $4l$ $\left(4000\ cm^3\right)$.

Determine the dimensions of the container with a square base if they want to use the minimum amount of sheet metal.



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SOLUTION:

We have to minimise the **Area.**
Therefore express area in terms of one variable.

Let one side of the square base $= x$ cm and let the height $= h$ cm.

A = Area of metal $= x^2 + 4xh$ ……(1)

[The container has no lid]
Since we want one variable, we will have to express h in terms of x by using the additional data, namely $V = 4\ 000\ cm^3$.

Volume $= x^2h$

$\Rightarrow 4000 = x^2h$

$\therefore h = \dfrac{4000}{x^2}$ ……(2)

Thus, $A = x^2 + \dfrac{4x(4000)}{x^2}$ ((1) is substitute into (1))

$$\frac{dA}{dx} = 2x - \frac{16000}{x^2}$$

$$2x - \frac{16000}{x^2} = 0$$

$$\Rightarrow 2x^3 - 16000 = 0$$

$$\Rightarrow x^3 = 8000$$

$$\Rightarrow x = 20 \text{ cm}$$

$$\therefore h = \frac{4000}{400}$$

$$= 10 \text{ cm}$$

$\therefore$ Side length of square base = 10 cm.

EXAMPLE 14

An object is projected vertically upwards from the ground and its movement is described by $S = 112t - 16t^2$, where S = distance from the starting point in metres and $t =$ time in seconds.

14.1 Determine its initial velocity.

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14.2 Determine its velocity after 3 seconds.
14.3 Find the maximum height the object reaches.
14.4 When will it be more than 96 m above the ground?

SOLUTION:

14.1 $S = 112t - 16t^2$

$$\text{Speed} = \frac{dS}{dt} = 112t - 32t$$

$\therefore$ Initial speed = 112 m/s At beginning $t = 0$

14.2 Speed after 3 sec. $= 112 - 32(3)$
$= 16\ m/s$

14.3 Maximum S if $\frac{dS}{dt} = 0$

$$\Rightarrow 112 - 32t = 0$$

$$\therefore t = \frac{112}{32} = 3{,}5\ s$$

Thus, $S = 112(3{,}5) - 16(3{,}5)^2$ Let $t = 3{,}5$ in $S = 112t - 16t^2$

$= 196\ m$

14.4 $112t - 16t^2 > 96$

$\Rightarrow -16t^2 + 112t - 96 > 0$

$\Rightarrow t^2 - 7t + 6 < 0$

$\Rightarrow (t - 6)(t - 1) < 0$

$\therefore 1 < t < 6$

$\therefore$ height > 96 m between the first and sixth seconds.

EXAMPLE 15

Determine the area, A, of the largest rectangle which can be included in the region between the $x-$ axis and the graph of $y = -x^2 + 9$.

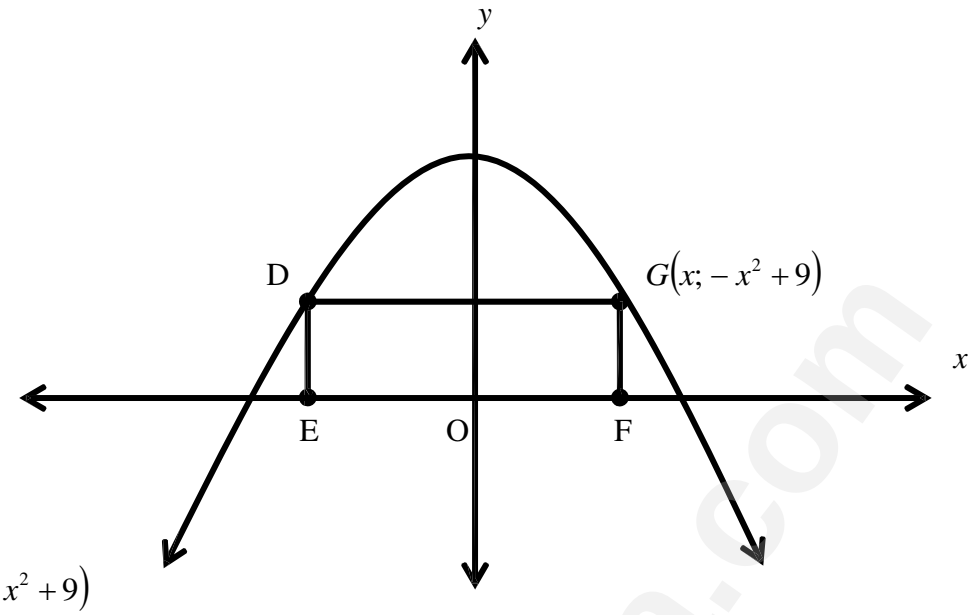
SOLUTION:

Let $G(x; -x^2 + 9)$

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$\therefore DG = 2x$ and $GF = -x^2 + 9$



$\therefore A = 2x(-x^2 + 9)$

$= -2x^3 + 18x$

$\frac{dA}{dx} = -6x^2 + 18$

Maximum if $\frac{dA}{dx} = 0$

$\therefore -6x^2 + 18 = 0$

$\Rightarrow x^2 = 3$

$\therefore x = +\sqrt{3} \quad [\ x > 0\ ]$

Maximum area: $A = -2\left(\sqrt{3}\right)^3 + 18\left(\sqrt{3}\right)$

$= -6\sqrt{3} + 18\sqrt{3}$

$= 12\sqrt{3}$ Square units.

$\approx 20{,}78$ square units

EXAMPLE 16

A rectangular container, made of sheet metal, and which has no lid, has a length of $2x$ cm, breadth x cm and height h cm.

16.1 Form an expression in terms of x and h, of the total area of sheet metal needed to make the container.

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16.2 If the total area is 600 cm^2, show that: $V = \dfrac{600 - 2x^2}{6x}$.

16.3 Also prove that the volume, V, of the container can be expressed by the formula:

$$V = 200x - \frac{2x^3}{3}.$$

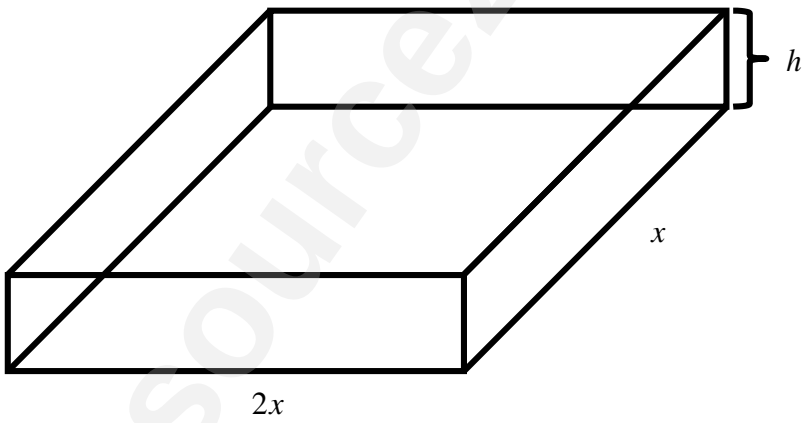
16.4 Determine the value of x for which V will be a maximum.

16.5 Determine the volume of the largest possible container which can be made with exactly 600 cm^2 sheet metal.

SOLUTION:

16.1 Total area $= 2(2xh) + 2(xh) + 2x^2$

$= 6xh + 2x^2$



16.2 $6xh + 2x^2 = 600$

$\Rightarrow 6xh = 600 - 2x^2$

$\therefore h = \dfrac{600 - 2x^2}{6x}$

16.3 $V = 2x^2h$

$= 2x^2 \cdot \dfrac{600 - 2x^2}{6x}$

$= 200x - \dfrac{2x^3}{3}$

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16.4 $\dfrac{dV}{dx} = 200 - \dfrac{6x_2}{3}$

$= 200 - 2x^2$

Maximum volume if $\dfrac{dV}{dx} = 0$

$\Rightarrow 200 - 2x^2 = 0$
$\Rightarrow x^2 = 100$
$\Rightarrow x = \pm 10$

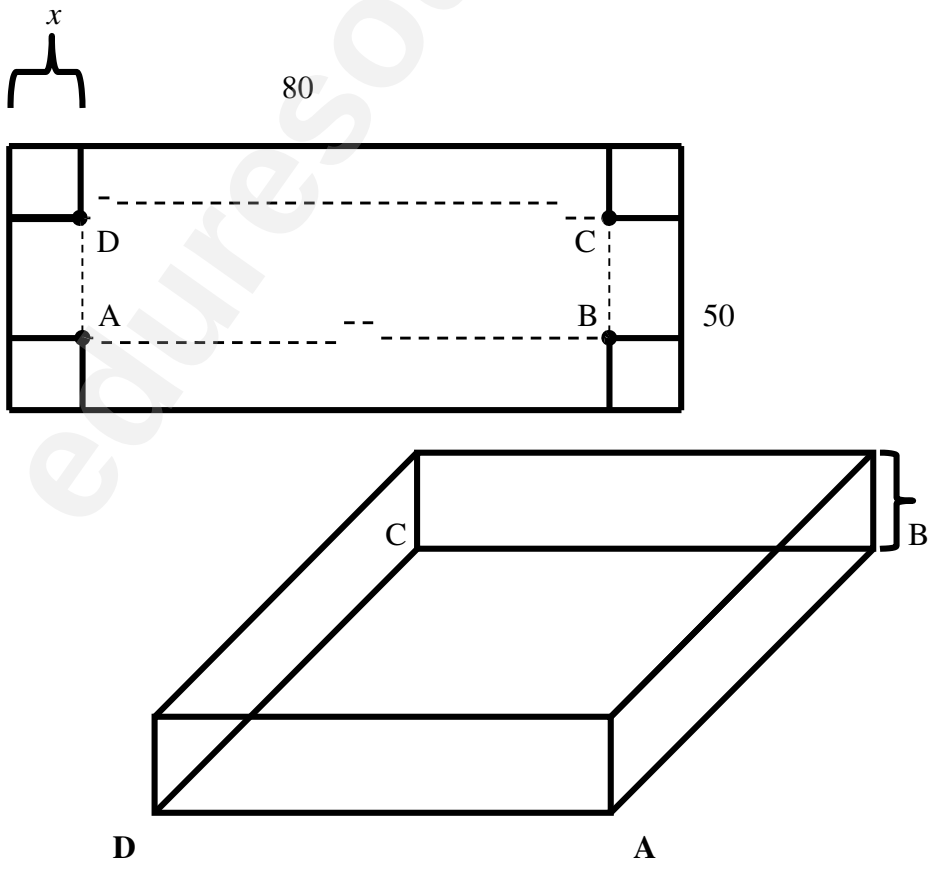
$\therefore x = 10$ ($x > 0$, breadth of container)

16.5 Maximum Volume: $V = 200(10) - \dfrac{2(10)^3}{3}$

$\approx 1333{,}33 \text{ cm}^3$

EXAMPLE 17

A manufacturer obtained sheets of rectangular cardboard measuring 80 cm by 50 cm. He wants to cut out equal squares at each of the corners so that he can fold the remaining piece to form an open box as in the accompanying sketch.



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The squares that are cut out each have sides of x units.

17.1 Write down the length, breadth and depth of the box in terms of x.

17.2 What is the volume of the box.

17.3 For which value of x will the volume of the box be a maximum?

SOLUTION:

17.1 Length $= (80 - 2x)$ cm

Breadth $= (50 - 2x)$ cm

Depth $= x$ cm

17.2 Volume $= (80 - 2x)(50 - 2x)(x)$

$$= 4000x - 160x^2 - 100x^2 + 4x^3$$

$$= 4000x - 260x^2 + 4x^3$$

17.3 $V = 4000x - 260x^2 + 4x^3$ as obtained in 17.2 above.

$$\frac{dV}{dx} = 4000x - 520x + 12x^2$$

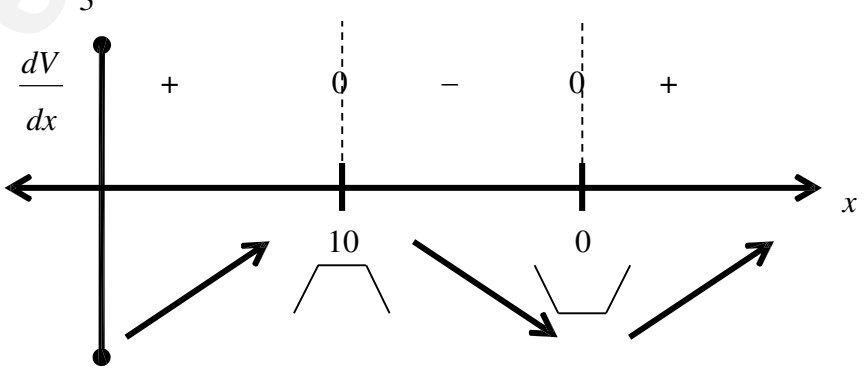
For Maximum volume: $\frac{dV}{dx} = 0$;

$$\Rightarrow 4000 - 520x + 12x^2 = 0$$

$$\Rightarrow 3x^2 - 130x + 1000 = 0$$

$$\Rightarrow (3x - 100)(x - 10) = 0$$

$$\therefore x = \frac{100}{3} = 33\frac{1}{3} \text{ or } x = 10$$



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If $x = 33\frac{1}{3}$ then the breadth is negative, which is impossible. As x increases through $x = 10$ the function changes from increasing to decreasing,

$\therefore x = 10$ cm, gives a maximum volume.

EXAMPLE 18

The sum of two positive numbers is 12. Find the numbers when their product is a maximum.

SOLUTION:

Let the numbers be x and y.

Then $x + y = 12 \qquad x, y \in$

$$\therefore y = 12 - x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

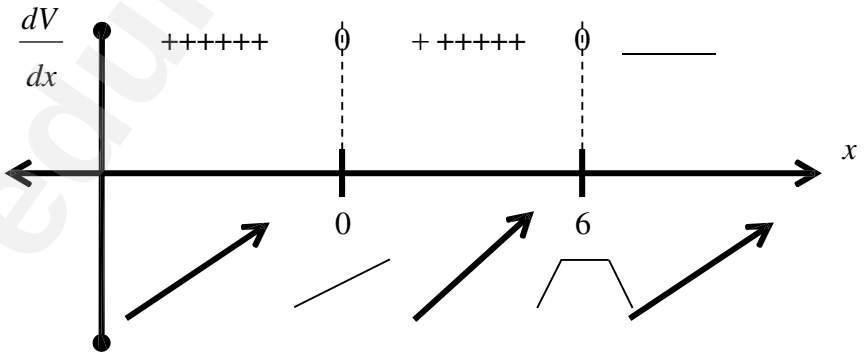
Put $z = xy$

From(1): $z = x(12 - x)$

$= 12x - x^2$ (In order to differentiate there must be only one variable on the RHS)

$$\frac{dz}{dx} = 12 - 12x$$

$$\therefore \frac{dz}{dx} = 0 \text{ for } x = 6$$



From the table of signs, z, is increasing on the interval $(0; 6)$ and decreasing on $(6; \infty)$.

Thus, maximum $xy = 12(6) - (6)^2 = 36$ when x and y are both equal to 6.

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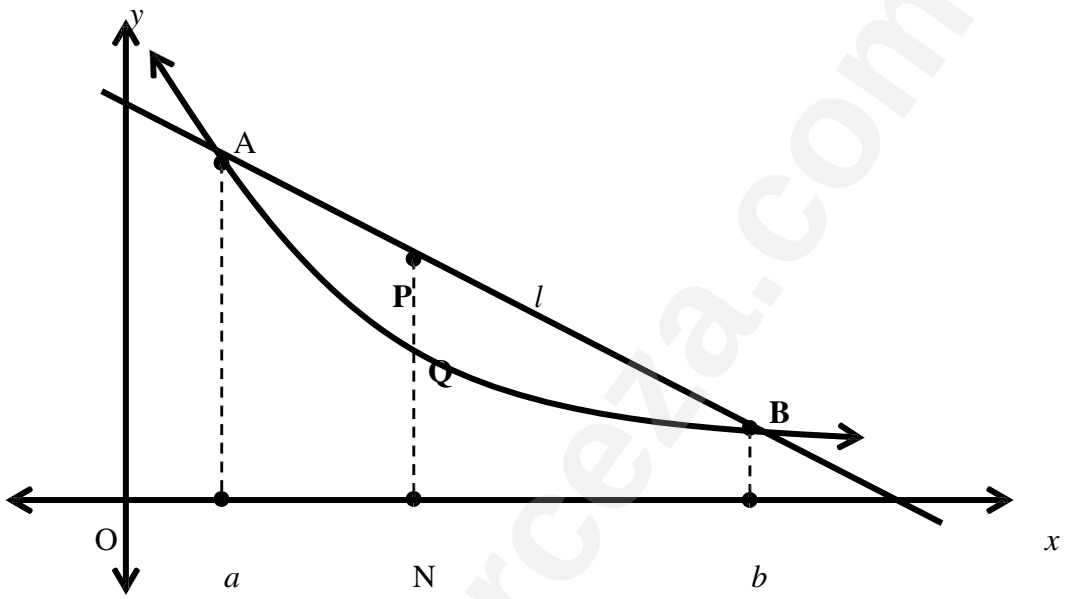
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EXAMPLE 19

The equation of l is $2y + x - 12 = 0$.

The equation of h is $xy = 2 \ (x > 0)$.

PN is perpendicular to the $x-$axis.
Find the maximum length of PQ, for P between A and B.



SOLUTION:

$PQ = PN - QN$

Let $PQ = z$

So $z = \left|-\frac{1}{2}x + 6\right| - \left|\frac{2}{x}\right|, \ [x \in (a; b)]$

$\therefore \frac{dz}{dx} = -\frac{1}{2} + \frac{2}{x^2} = -\frac{1}{2} + 2x^{-2}$

The maximum should occur when:

$-\frac{1}{2} + \frac{2}{x^2} = 0$ or $x^2 = 4$ or $x = 2$ ($x = -2$ is not applicable.)

$\frac{d^2z}{dx} = -4x^{-3} = -\frac{4}{x^3}$

At $x = 2$: $\frac{d_2z}{dx^2} = -\frac{1}{2} < 0$

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$\therefore$ A maximum is confirmed at $x = 2$.

$\therefore$ Maximum $PQ = \left[-\frac{1}{2}(2)+6\right] - \left[\frac{2}{2}\right] = 5 - 1 = 4$ units.

RATES OF CHANGE

Differentiation can be applied as a study of change.

For $y = f(x)$

We have: $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

We can write this as: $\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$

$\frac{\Delta y}{\Delta x}$ = average rate of change of y with respect to x.

With Δx getting closer and closer to zero $\left(\lim_{\Delta x\to 0}\right)$ we get:

$\frac{dy}{dx}$ = the instantaneous rate of change of y with respect to x.

EXAMPLE 20

A water tank has both an inlet and an outlet pipe which regulates the depth of water it contains. The depth is given by the function:

$D(t) = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$; where D is in metres and t in hours measured form 8h00.

20.1 Determine the rate at which the depth is changing at 11:00 and state whether it is increasing or decreasing.

20.2 At what rate will the inflow of water be the same as the outflow?

SOLUTION:

20.1 $D(t) = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$

$\Rightarrow D'(t) = t - \frac{3}{4}t^2$

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$\Rightarrow D'(3) = 3 - \frac{3}{4}(3)^2$, $t = 3$ hours between 08h00 and 11h00

$= -\frac{15}{4}$

Decreasing, since the derivative is **negative** at $t = 3$ hours.

20.2 $D(t) = 3 + \frac{1}{2}t^2 - \frac{1}{4}t^3$ **, Inflow = Outflow means that the depth of the water is constant,**

so that:

$D'(t) = t - \frac{3}{4}t^2 = 0$

$\Rightarrow 4t - 3t^2 = 0$

$\Rightarrow t(4 - 3t) = 0$

$\therefore t = 0$ hours or $t = \frac{4}{3}$ hours

Inflow = Outflow at 9:20 (8:00 + $\frac{4}{3} \approx 9{,}33$ hours)

(0.33×60 minutes = 19,8 minutes)

At 9:20 a.m.

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Self Test: 1 **Date:_____**

Marks: 130

Time: 3 hrs

Question 1:

1.1 Use the definition to differentiate $f(x) = 1 - 3x^2$. (Use first principles.)

(4)

1.2 Determine $\frac{dy}{dx}$ if $y = \left(1 + \sqrt{x}\right)^2$.

(4)

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Question 2:

Given: $g(x) = (x-6)(x-3)(x+2)$

2.1 Calculate the y-intercept of g. (2)

2.2 Write down the x-intercepts of g. (2)

2.3 Determine the turning points of g. (5)

2.4 Sketch the graph of g. (4)

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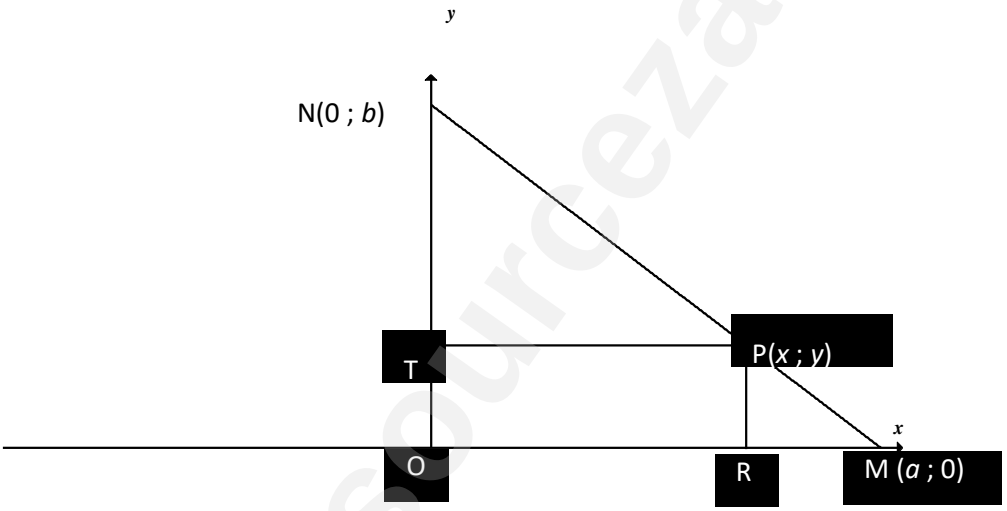
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2.5 For which values of x is $g(x).g'(x) < 0$? (3)

Question 3:

A farmer has a piece of land in the shape of a right-angled triangle OMN, as shown in the figure below. He allocates a rectangular piece of land PTOR to his daughter, giving her the freedom to choose P anywhere along the boundary MN. Let OM = a, ON = b and P(x ; y) be any point on MN.



3.1 Determine an equation of MN in terms of a and b. (2)

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3.2 Prove that the daughter's land will have a maximum area if she chooses P at the midpoint of MN. (6)

Question 4:

4.1 Differentiate $g(x) = x^2 - 5$ from first principles. (4)

4.2 Evaluate $\frac{dy}{dx}$ if $y = \frac{x^6}{2} + 4\sqrt{x}$. (3)

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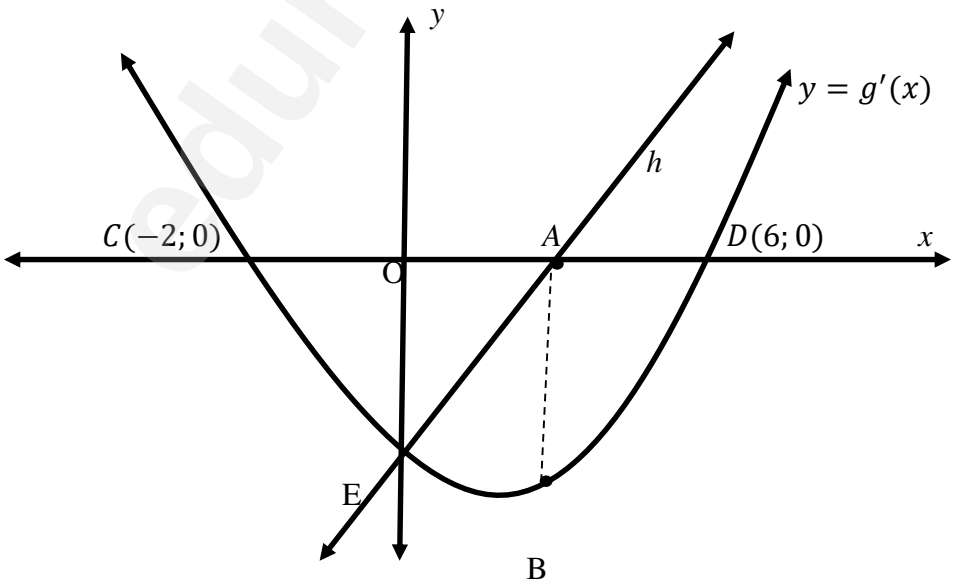
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4.3 A function $g(x) = ax^2 + \frac{b}{x}$ has a minimum value at $x = 4$. The function value at $x = 4$ is 96. Calculate the values of a and b. (6)

Question 5:

The graphs of $y = g'(x) = ax^2 + bx + c$ and $h(x) = 2x - 4$ are sketched below. The graph of $y = g'(x) = ax^2 + bx + c$ is the derivative graph of a cubic function g.

The graphs of h and g' have a common y – intercept at E.
$C(-2; 0)$ and $D(6; 0)$ are the intercepts of the graph of g'.
A is the x – intercept of h and B is the turning point of g'.
AB $\parallel y$-axis.



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5.1 Write down the coordinates of E. (1)

5.2 Determine the equation of the graph of g' in the form $y = ax^2 + bx + c$. (3)

5.3 Write down the $x-$coordinates of the turning point of g. (2)

5.4 Write down the $x-$coordinate of the point of inflection of the graph of g. (2)

5.5 Explain why g has a local maximum at $x = -2$. (3)

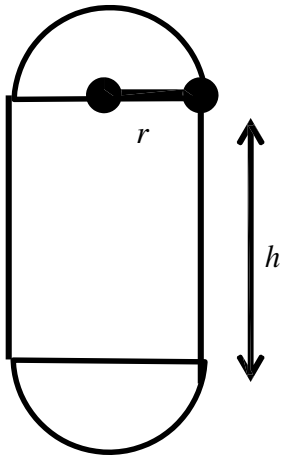
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Question 6:

A satellite is to be constructed in the shape of a cylinder with a hemisphere at each end. The radius of the cylinder is r metres and its height is h metres (see diagram below). The outer surface area of the satellite is to be coated with heat-resistant material which is very expensive.

The volume of the satellite has to be $\frac{\pi}{6}$ cubic metres.



Outer surface area of a sphere = $4\pi r^2$

Curved surface area of a cylinder = $2\pi rh$

Volume of a sphere = $4/3\pi r^3$

Volume of a cylinder = $\pi r^2 h$

6.1 Show that $h = \frac{1}{6r^2} - \frac{4r}{3}$. (3)

6.2 Hence, show that the outer surface area of the satellite can be given as

$$S = \frac{4\pi r^2}{3} + \frac{\pi}{3r}.$$ (3)

6.3 Calculate the minimum outer surface area of the satellite. (6)

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Question 7:

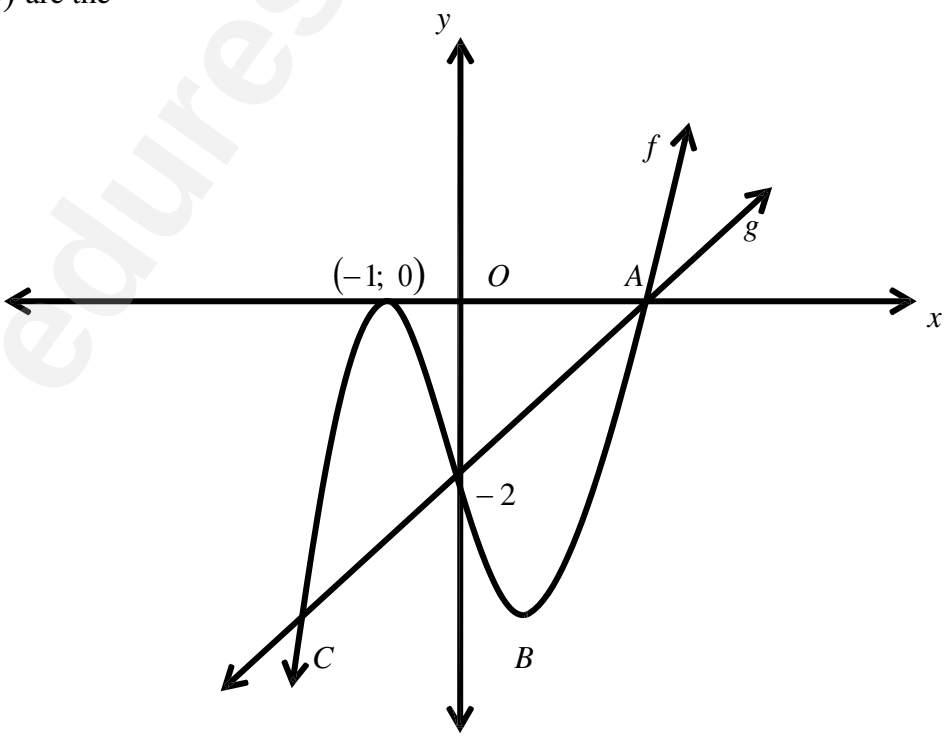
7.1 Differentiate f from first principles: $f(x)=\frac{1}{x}$. (3)

7.2 Use the rules of differentiation to determine $\frac{dy}{dx}$ if $y=(2-5x)^2$. (3)

Question 8:

The graph below represents the functions f and g with $f(x)=ax^3-cx-2$ and $g(x)=x-2$.

A and $(-1;\ 0)$ are the $x-$intercepts of f. The graphs of f and g intersect at A and C.



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8.2 Show by calculation that $a = 1$ and $c = -3$. (4)

8.3 Determine the coordinates of B, a turning point of f. (3)

8.4 Show that the line BC is parallel to the $x-$axis. (6)

8.5 Find the $x-$coordinate of the point of inflection of f. (2)

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8.6 Write down the values of k for which $f(x)=k$ will have only ONE root. (3)

8.7 Write down the values of x for which $f'(x)<0$. (2)

Question 9:

A wire, 4 metres long, is cut into two pieces. One is bent into the shape of a square and the other into the shape of a circle.

9.1 If the length of wire used to make the circle is x metres, write in terms of x the length of the sides of the square in metres. (1)

9.2 Show that the sum of the areas of the circle and the square is given by

$$f(x)=\left(\frac{1}{16}+\frac{1}{4\pi}\right)x^2-\frac{x}{2}+1 \text{ square metres.}$$ (3)

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9.3 How should the wire be cut so that the sum of the areas of the circle and the square is a minimum? (3)

Question 10:

10.1 Given: $f(x)=1-x^2$. Determine $f'(x)$ from first principles. (4)

10.2 Determine $\frac{dy}{dx}$ if:

10.2.1 $y=\sqrt{x}-\frac{1}{2}x^2$ (3)

10.2.2 $y=\left(x+\frac{1}{x}\right)\left(x-\frac{3}{x}\right)$ (3)

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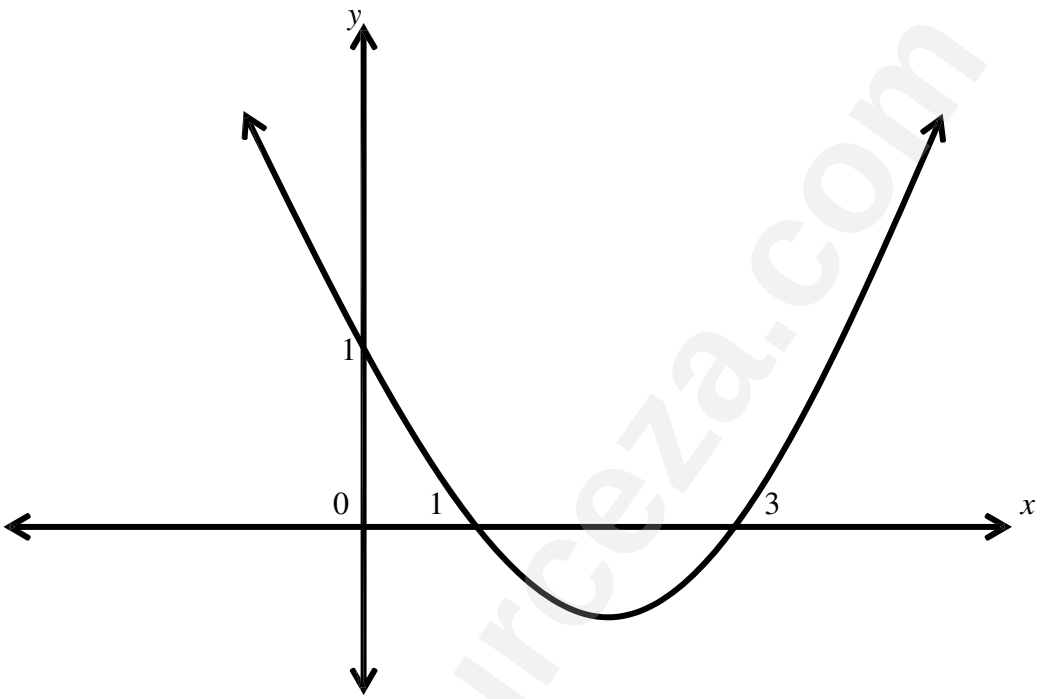
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Question 11:

The parabola in the accompanying sketch represents the curve of $y = f'(x)$.

The parabola is the derivative of a cubic function $f(x) = ax^3 + bx^2 + cx + d$.



11.1 Write down the gradient of the tangent to f at the point where $x = 0$. (2)

11.2 Write down the $x-$intercepts of the curve of f . (2)

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11.3 Show that $x = -\frac{b}{3a}$ is the $x-$coordinate of the point of inflection of f. (3)

11.4 For which values of x is f decreasing? (2)

11.5 Sketch the graph of f if $f(x) = 0$ when $x = 0$, $x = 2$ and $x = 5$.

(It is not necessary to determine the $y-$coordinates of the turning points.) (3)

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Question 12:

12.1 Differentiate $f(x)$ from first principles if $f(x) = -2x^2 + 3$. (4)

12.2 Evaluate $\frac{dy}{dx}$ if $y = x^2 - \frac{1}{2x^3}$. (2)

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Self Test: 2 **Date:**_____

Marks: 100

Time: 2 hrs 30 min

Question 1:

Given: $f(x) = -x^3 + x^2 + 8x - 12$

1.1 Calculate the $x-$intercepts of the graph of f. (5)

1.2 Calculate the coordinates of the turning point of the graph of f. (5)

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1.3 Sketch the graph of f, showing clearly all the intercepts with the axes and turning points. (3)

1.4 Write down the $x-$ coordinate of the point of inflection of f. (2)

1.5 Write down the coordinates of the turning points of $h(x)=f(x)-3$. (2)

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Question 2:

A tourist travels in a car over a mountainous pass during his trip. The height above sea level of the car, after t minutes, is given as $s(t) = 5t^3 - 65t^2 + 200t + 100$ metres.

The journey lasts 8 minutes.

2.1 How high is the cat above sea level when it starts its journey on the mountainous pass? (2)

2.2 Calculate the car's rate of change of height above sea level with respect to time,

4 minutes after starting the journey on the mountainous pass. (3)

2.3 Interpret your answer to QUESTION 2.2. (2)

2.4 How many minutes after the journey has started will the rate of change of height with respect to time be a minimum? (2)

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Question 3:

3.1 Differentiate f by first principles where $f(x) = x^2 - 2x$. (5)

3.2 Evaluate:

3.2.1 $D_x\left[(x^3 - 3)^2\right]$ (3)

3.2.2 $\frac{dy}{dx}$ if $y = \frac{4}{\sqrt{x}} - \frac{x_3}{9}$ (3)

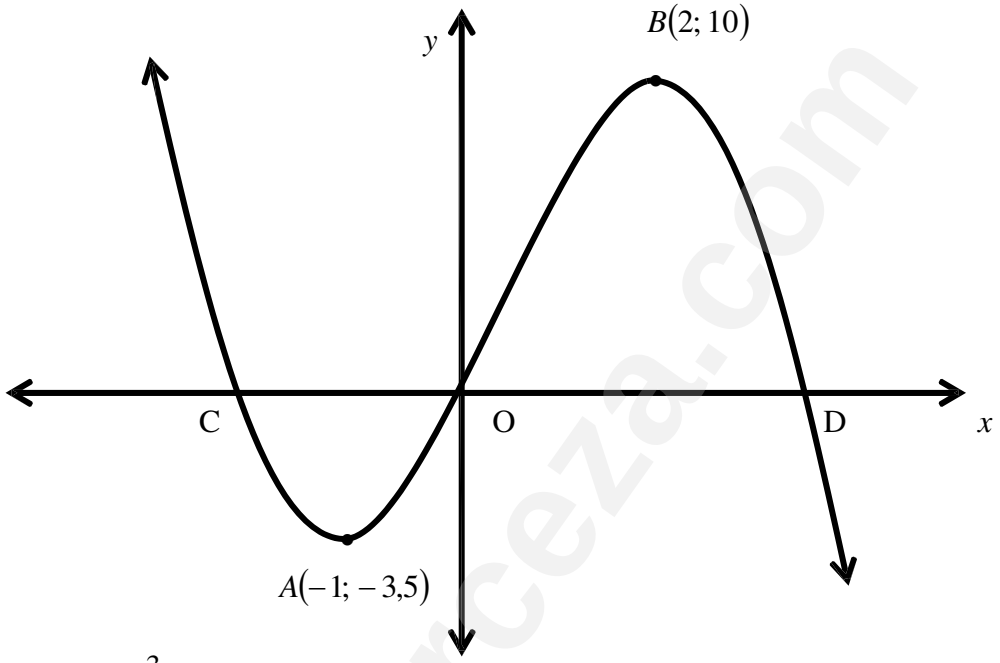
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Question 4:

The graph of $h(x) = -x^3 + ax^2 + bx$ is shown below. $A(-1; -3,5)$ and $B(2; 10)$ are the turning points of h. The graph passes through the origin and further cuts the $x-$axis at C and D.



4.1 Show that $a = \frac{3}{2}$ and $b = 6$. (5)

4.2 Calculate the average gradient between A and B. (2)

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4.3 Determine the equation of the tangent to h at $x = -2$. (5)

4.4 Determine the $x-$value of the point of inflection of h. (3)

4.5 Use the graph to determine the values of p for which the equation

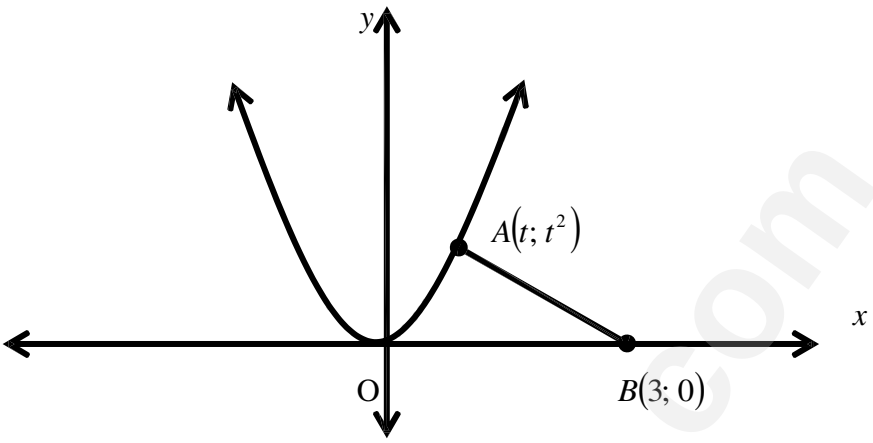
$-x^3 + \frac{3}{2}x^2 + 6x + p = 0$ will have ONE real root. (2)

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Question 5:

Sketched is the graph of $y = x^2$ $A(t; t^2)$ and $B(3; 0)$ are shown.



5.1 $A(t; t^2)$ is a point on the curve $y = x^2$ and the point $B(3; 0)$ lies on the $x-$axis.

Show that $AB^2 = t^4 + t^2 - 6t + 9$. (2)

5.2 Hence, determine the value of t which minimises the distance AB. (5)

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Question 6:

6.1 Determine $f'(x)$ from first principles if $f(x)=x^3$. (5)

6.2 Determine $\frac{dy}{dx}$ by means of differentiation principles, if:

6.2.1 (3)

$y = 2\sqrt{x} - x^3$

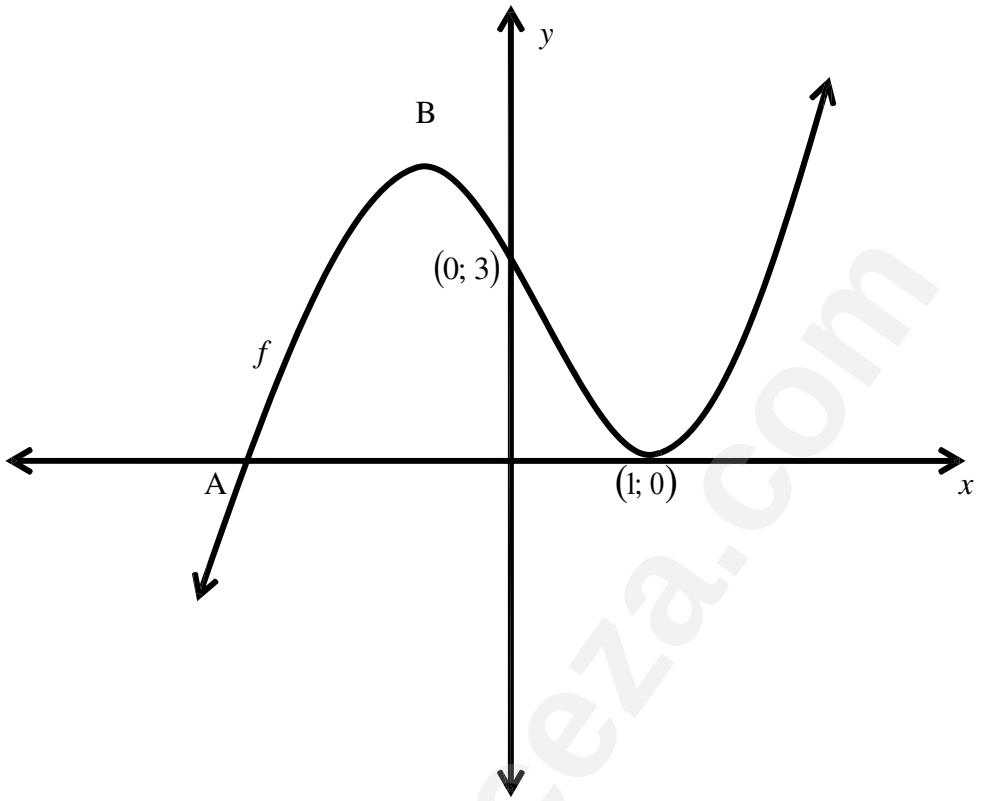
6.2.2 $3 = x^2 y$ (3)

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Question 7:



The diagram above represents the curve of $f(x) = x^3 + x^2 - 5x + 3$.
The curve has a y – intercept at $(0; 3)$ and turning points at $(1; 0)$ and B.
The point A is an x – intercept of f.

7.1 Calculate the coordinates of A. (5)

7.2 Calculate the x – coordinate of B. (4)

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7.3 Write down values of x for which $f'(x) > 0$. (2)

7.4 If $k < 0$, how many real roots will the equation $x^3 + x^2 - 5x + 3 = k$ have? (2)

7.5 Determine the equation of the tangent to the graph of f at $x = -1$. (4)

7.6 Determine the point(s) of inflection of f. (4)

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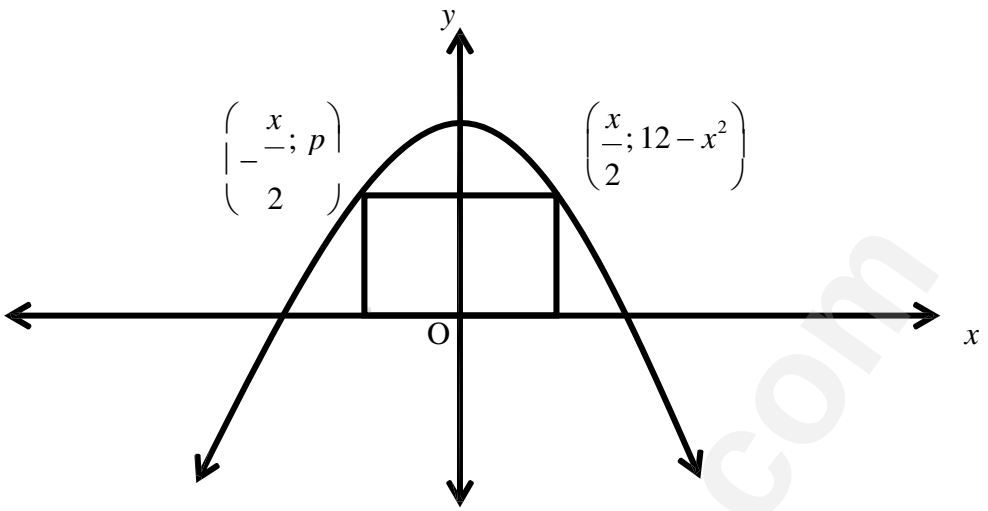
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Question 8:



The figure shows a shaded rectangle that has two, $\left(-\frac{x}{2}; p\right)$ and $\left(\frac{x}{2}; 12-x^2\right)$, vertices on the curve $y=16-x^2;\ x>0$.

20.1 Express p in terms of x. (1)

20.2 Show that the area of the shaded rectangle is $A=12x-x^3$. (2)

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20.3 The line $g(x) = 5x+1$ is a tangent to the curve of a function f at the point where $x = 2$. Calculate the numerical value of $f(2) + f^{l}(2)$ (4)

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Self Test: 3 **Date:_____**

Marks: 100

Time: 2 hrs 30 min

Question 1:

1.1 Determine $f'(x)$ form first principles if $f(x) = -3x^2$. (6)

1.2 Determine, using rules of differentiation $\frac{dy}{dx}$ if

$$y = \frac{\sqrt{x}}{2} - \frac{1}{6x^3}.$$

Show ALL calculations. (4)

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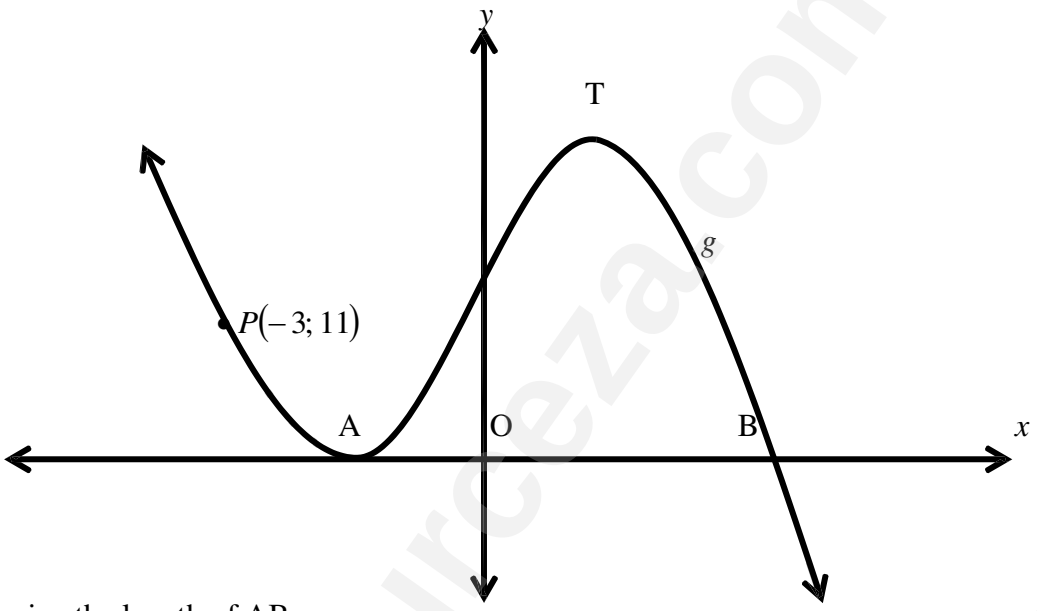
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Question 2:

Sketched below is the graph of $g(x) = -2x^3 - 3x^2 + 12x + 20$

$$= -(2x - 5)(x + 2)^2$$

A and T are the turning points of g. A and B are the $x-$intercepts of g.
$P(-3; 11)$ is a point on the graph.



2.1 Determine the length of AB. (3)

2.2 Determine the $x-$ coordinate of T (4)

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2.3 Determine the equation of the tangent to g at $P(-3;11)$ in the form

$y = \ldots$ (5)

2.4 Determine the value(s) of k for which $-2x^3 - 3x^2 + 12x + 20 = k$ has

three distinct roots. (4)

2.5 Determine the x – coordinate of the point of inflection. (4)

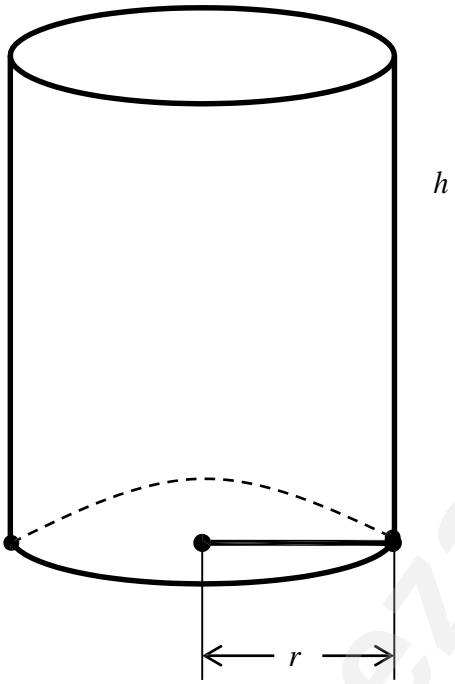
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Question 3:

A drinking glass, in the shape of a cylinder must hold 200 ml of liquid when full.



3.1 Show that the height of the glass, h, can be expressed as $h = \frac{200}{\pi r^2}$. (2)

3.2 Show that the total surface area of the glass can be expressed as

$$S(r) = \pi r^2 + \frac{400}{r}.$$ (2)

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3.3 Hence determine the value of r for which the total surface area of the glass is a minimum. (5)

Question 4:

4.1 Given: $f(x) = -x^2$

4.1.1 Determine $f'(x)$ from first principles. (5)

4.1.2 Determine the average gradient of $f(x)$ between $x = 1$ and $x = 3$. (4)

4.2 Determine $\frac{dy}{dx}$ if:

4.2.1 $y = (x^3 + 1)(x^2 - 2)$ (4)

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4.2.2 $y = \dfrac{\sqrt{x}-4}{\sqrt{x}}$ (4)

Question 5:

Given: $f(x) = x^3 + x^2 - 5x + 3$

5.1 Determine all intercepts of $f(x)$. (4)

5.2 Determine the coordinates of the turning points of $f(x)$. (5)

5.3 Draw a sketch of $f(x)$ (4)

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5.4 What is the equation of the tangent to $f(x)$ when $x = 2$? (3)

5.5 What are the coordinates of the turning points of $f(x-1)$? (2)

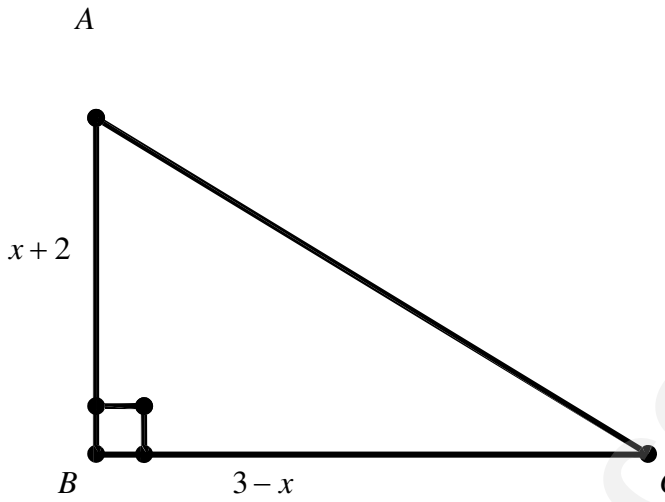
5.6 Determine the x – value of the point of inflection. (4)

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Question 6:



6.1 Show that the area of the triangle in the figure above is given by

$$A(x) = -\frac{x^2}{2} + \frac{x}{2} + 3.$$ (3)

6.2 Calculate the value of x for which the area will be maximum. (3)

6.3 Hence, calculate the maximum area of the given triangle. (3)

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Question 7:

7.1 Determine $f'(x)$ by using first principles if $f(x)=x^3$. (5)

7.2 Use differentiation rules to differentiate the following:

27.2.1 $y=\frac{2}{5\sqrt{x}}-\sqrt[3]{x}$ (4)

7.2.2 $y=\frac{x^4-3x^2+7}{x}$ (4)

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Self Test: 4 **Date:_____**

Marks: 90

Time: 2 hrs 20 min

Question 1:

Given: $f(x) = x^3 + x^2 - 5x + 3$

1.1 Calculate the $x-$ and $y-$ intercepts of f. (5)

1.2 Determine the turning points of f. (5)

1.3 Determine the $x-$ value of the point of inflection. (4)

1.4 Hence, sketch the graph of f.
Show clearly ALL intercepts with the axes and any turning points. (5)

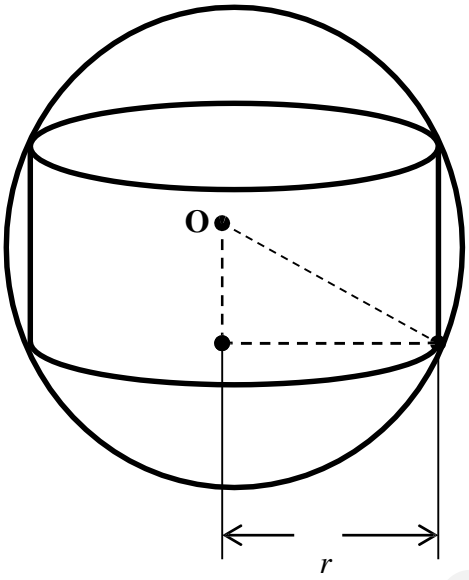
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Question 2:

A cylinder with height $2x$ units is placed inside a sphere with radius $5\ 3$ units.
O is the centre of the sphere.



2.1 Show that the volume of the cylinder can be expressed as

$V = 150\pi x - 2\pi x^3$. (4)

2.2 Calculate the height of the cylinder of the cylinder if it is of maximum volume. (4)

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Question 3:

3.1 Prove from first principles of differentiation that, given the function

$$f(x)=\frac{15-19\sqrt{x}}{3\sqrt{x}-5}+\frac{6x}{3\sqrt{x}-5} \quad , \quad f'(x)=\frac{1}{\sqrt{x}}. \qquad (5)$$

3.2 Hence or otherwise find the equation of the tangent to f at the point $(x;\ 7)$. (3)

3.3 Find $D_x\left[\left(x^3-2x^2\right)(2-x)\right]$ (3)

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3.4 If $z = \frac{1}{y} + \sqrt[3]{y}$ and $y = \frac{8}{x^2}$, find $\frac{dz}{dx}$. (4)

3.5 Find: $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x+2} \right) \div \frac{1}{2}$ (4)

3.6 Determine the equation of the tangent to the curve of f at the point $x = 3$.

$$f : x \to \frac{2}{3}x^3 - x^2 - 3x$$

(5)

3.7 For what value of the constant m will the curve of $y = mx - 2x^3$ have a local minimum at $x = -\frac{1}{2}$. (5)

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Question 4:

4.1 $f(x) = ax^3 + bx^2 + cx$

The following information is given about f :

(i) $f(-3) = f(8) = 0$

(ii) $f'(-1) = f'(5) = 0$

(iii) $f(x)$ is increasing for $-1 < x < 5$

4.1.1 Use the above information to sketch f . (5)

4.1.2 Use the graph to find the values of x for which $f'(x) \cdot f(x) \leq 0$. (4)

4.2 A particle starts from rest at A and moves in a straight line, coming to rest again at another point B. The distance in a straight line from A is given by the formula

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$s = 3t^2 - \frac{1}{2}t^4$ where t is measured in seconds and s in metres.

If the velocity (v) and acceleration (a) are given by $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$ respectively, determine:

4.2.1 an expression for the velocity of the particle in terms of t. (2)

4.2.2 the time taken to reach B. (3)

4.2.3 the time taken to reach the maximum velocity (3)

4.2.4 the maximum acceleration of the particle (3)

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Question 5:

5.1 Given: $8x^3 - 2xy + y - 1 = 0 \quad \left(x \neq -\frac{1}{2}\right)$

Evaluate $\frac{dy}{dx}$ when $x = 8$ (5)

5.2 $f(x) = ax^3 + bx^2 + cx - 5$. The gradient at any point $(x;\ f(x))$ is given by $(6x^2 - 24)$.

Find the values of a, b and c. (5)

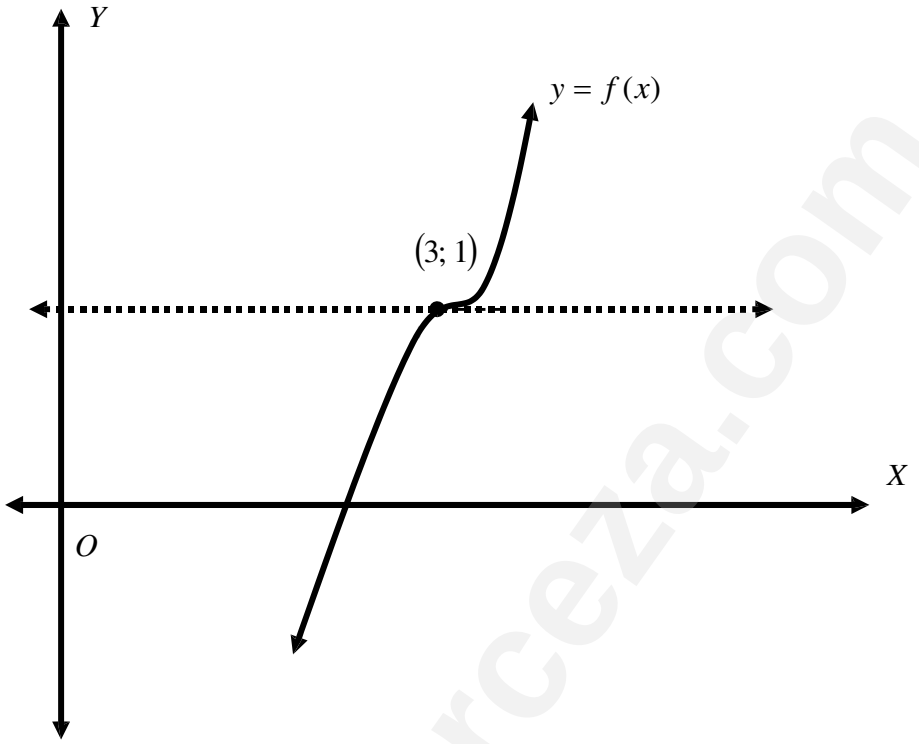
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5.3 Refer to the figure.

f is a cubic function. The tangent at $(3;1)$ has a gradient of zero. Draw a sketch of $y = f'(x)$, indicating the co-ordinates of any point that may be read accurately from the sketch.



(3)

5.4 The depth, d, of water in a kettle t minutes after it starts to boil, is given by

$d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

5.4.1 How many millimetres of water are there in the kettle just before it starts to boil? (3)

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5.4.2 As the water boils, the level in the kettle drops. Find the rate at which the water

Level is decreasing when $t = 2$ minutes. (4)

5.4.3 How many minutes after the kettle starts boiling will the water level be dropping

at a rate of $12\frac{1}{8}$ mm/minute? (4)

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PROBABILITY
TERMINOLOGY

1. **Outcome:** a single observation of a random process (called an experiment).
   For example, when you accidentally drop a book, it might fall on its cover, on its back or on its side. Each of these options is a **possible outcome.**

2. **Sample Space of an experiment:** This is a set of all possible outcomes of experiment.
   Eg: when you roll a die; $S = \{1; 2; 3; 4; 5; 6\}$.The sample space is denoted is denoted by letter S.

3. **Event**: a set of outcomes of an experiment. Eg: if you have a standard deck of 52 playing cards, an event may be picking a spade card or a king card, etc.

4. **Probability of an event:** $P(E)$ lies between 0 and 1 inclusive, i.e $\mathbf{0 \leq P(E) \leq 1}$
   Therefore; to calculate probability, you divide the number of favourable outcomes by the total number of possible outcomes.

$$P(E) = \frac{number\ of\ favourable\ outcomes}{total\ number\ of\ possible\ outcomes.}$$

$$P(E) = \frac{n(E)}{n(S)}$$

5. **Union of events:** this is a set of all outcomes that occur in at least one of the events. For 2 events A and B; we write the union as $(A\ or\ B)$.

6. **Intersection of events**: this is a set of all outcomes that occur in all of the events. For 2 events called A and B, we write the intersection as $(A\ and B)$.

7. **Mutually Exclusive events:** these are events with no outcome (element) in common. That is A and B is an empty set. Mutually exclusive events cannot occur simultaneously.

8. **Complementary events:** two mutually exclusive events that together contain the outcomes in the sample space. For an event A, we write the complement of A as **" not A".** Another way of writing complement is as A'.

9. **Independent events:** two events, A and B, are independent if the outcome of the first event does not influence (affect) the outcome of the second event.

IDENTITIES

1. The addition rule (also called the **sum rule**) for any two events A and B is:

$$\boldsymbol{P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B).}$$

This rule relates probabilities of two events with the probabilities of their union and intersection.

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2. The addition rule for two mutually exclusive events is:
$\boldsymbol{P(A\ or\ B) = P(A) + P(B).}$

This rule is a special case of the previous rule. Because the events are mutually exclusive; $\boldsymbol{P(A\ and\ B) = 0}$

3. The complementary rule is
$\boldsymbol{P(not\ A\ = 1 - P\ (A).}$

This rule is a special case of the previous rule since A and $(not\ A)$ are complementary;
$\boldsymbol{P\ (A\ or\ (not\ A))\ =\ 1.}$

4. The Product rule (Multiplicative rule) for independent events A and B is
$\boldsymbol{P\ (A\ and\ B)\ =\ P\ (A) \times P\ (B).}$

Note: If two events A and B are dependent, then $\boldsymbol{P\ (A\ and\ B)\ \neq\ P\ (A)\ \times\ P\ (B).}$

Very Important:

Just because two events are mutually exclusive does not necessarily mean they are independent.
To test whether events are mutually exclusive; always check that $\boldsymbol{P\ (A\ and\ B)\ = 0.}$

To test whether events are independent, always check that $\boldsymbol{P\ (A\ and\ B)\ = P\ (A)\ \times\ P\ (B).}$

Check your understanding of the concepts:

Dependent and independent events:

Question 1:

Write down which of the following events are dependent or which are independent.

a) The student council chooses a head prefect and then an assistant head prefect.

b) A bag contains blue beads and red beads. You take a red bead out of the bag and then throw it back in again before you take another bead out of the bag.

ANSWER: 1

Ask yourself a question: Did the available choices change for the second event because of the first event?

If yes; implies dependent, if no; implies Independent.

a) Yes; because after selecting the head prefect, there are fewer council members available to choose for the assistant head prefect position. Therefore; two events are dependent.

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b) No, because when you throw the first bead back into the bag, there are the same number and colour composition of choices for the second bead. Therefore the two events are independent.

Question 2:

A bag contains 3 yellow and 4 black beads.
We remove a bead at random from the bag, record its colour and put it back into the bag.
A second bead is removed from the bag and its colour its colour is recorded.

What is the probability that:

a) The first bead is yellow?

b) The second bead is black?

c) The first bead is yellow and the second bead is black?

d) Of the first bead being yellow and the second bead being black independent events?

ANSWER: 2

a) (Probability of a yellow bead first)

Since there is a total of 7 beads, of which 3 are yellow, $P\ (first\ bead\ yellow) = \frac{3}{7}$ $\therefore$ $P\ (Y_1) = \frac{3}{7}$

b) Probability of a black bead second.

The problem states that the first bead is placed back into the bag before we take the second bead. This means that when we draw the second bead, there are again a total of 7 beads in the bag, of which 4 are black. Therefore the probability of drawing a black bead is

$$P\ (second\ bead\ black) = \frac{4.}{7}$$

c) Probability of yellow first and black second:

When drawing two beads from the bag, there are 4 possibilities. We can get

- *A yellow bead and then another yellow bead;*
- *A yellow bead and then a black bead;*
- *A black bead and then a yellow bead;*
- *A black bead and then another black bead;*

Now we want to know the probability of the second outcome, where we have to get a yellow bead first. Since there are 3 yellow beads, and 7 beads in total, there are $\frac{3}{7}$ ways to get a yellow bead first. Now we put the first bead back, so there are again 3 yellow beads and 4 black beads in the bag.

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Therefore there are $\frac{4}{7}$ ways to get a black bead second; if the first bead was yellow. This means that there are $\frac{3}{7} \times \frac{4}{7} = \frac{12}{49}$ ways to get a yellow bead first and a black bead second. So the probability of getting a yellow bead first and a black bed second is $\frac{12}{49}$.

d) Dependent or independent?

According to the definition, events are independent if and only if:

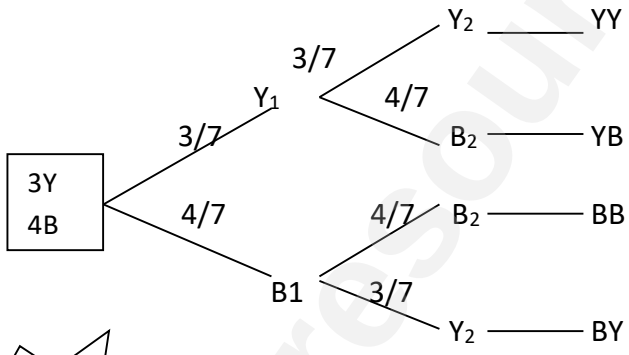
$$P\ (A\ and\ B)\ =\ P\ (A)\ \times\ P\ (B).$$

In this problem: $P\ (Y_1)\ = \frac{3}{7}$, $P\ (B_2)\ = \frac{4}{7}$

$$P\ (Y_1\ and\ B_2)\ =\ P\ (Y_1 B_2)\ =\ \frac{12}{49}$$

Since $\frac{12}{49} = \frac{3}{7} \times \frac{4}{7}$ the events are independent.

Very Important: (The above question can also be answered using probability tree (Tree diagram)



Take Note

In the above example, we picked a bead at random and put it back into the bag before continuing.

This is called Sampling with replacement.

But if we picked the first bead and it was not put back into the bag; this is called Sampling without replacement.

Question: 3

A bag contains 3 red 5 green beads, a bead is selected at random and its colour is recorded. Then without putting back the first bead, a second bead is selected from the bag and its colour is recorded.

What is the probability that

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a) The first bead is red?
b) The second bead is green?
c) The first bead is red and the second bead is green?
d) The first bead being red and the second bead being green independent events?

ANSWER: 3

The possible outcomes when removing two beads. $(RR), (RG), (GR), (GG)$

a) $P\ (first\ bead\ red)\ =\ P(RR)\ or\ P(RG).$

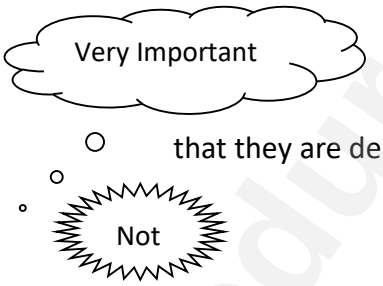
$$= (\frac{3}{8} \times \frac{2}{7}) + (\frac{3}{8} \times \frac{5}{7})$$
$$= \frac{6}{56} + \frac{15}{56} = \frac{3}{8}$$

b) $P\ (second\ bead\ green)\ =\ P\ (RG)\ or\ P\ (GG)$

$$(\frac{3}{8} \times \frac{5}{7}) + (\frac{5}{8} \times \frac{4}{7})$$
$$= \frac{5}{8}$$

c) $P(R_1\ and\ G_2) = \frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$

d) Dependent or independent?
Since $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64} \neq \frac{15}{56}$; the events are dependent.



It does not necessarily mean that if events are not independent; that they are dependent.

Question: 4

Above can also be answered with the help of a probability tree diagram. Try and see

<u>The addition rule for 2 mutually exclusive events</u>

Example:

A sample space, S consists of all natural numbers less than 16.A is the event of drawing an even number at random. B is the event of randomly drawing a prime number.

Are A and B mutually exclusive events?

Prove this using the addition rule.

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ANSWER: 4

Write down the sample space (Possible outcomes).
Sample space contains all natural numbers less than16.

$$S = \{1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ 11;\ 12;\ 13;\ 14;\ 15\}.$$

Write down the events:

The even natural numbers less than 16 are

$$A = \{2;\ 4;\ 6;\ 8;\ 10;\ 12;\ 14\}$$

The prime numbers less than 16 are

$$B = \{2;\ 3;\ 5;\ 7;\ 11;\ 13\}$$

Compute the probabilities

$$n(A) = 7;\ n(s) = 15;\ \therefore P(A) = \frac{n(A)}{n(s)} = \frac{7}{15}$$

$$n(B) = 6;\ P(B) = \frac{n(B)}{n(s)} = \frac{6}{15} = \frac{2}{5}$$

Obtain: $(A\ or\ B)$

$$(A\ or\ B) = \{2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ 11;\ 12;\ 13\}$$
$$n\ (A\ or\ B) = 12$$
$$\therefore p\ (A\ or\ B) = \frac{n\ (A\ or\ B)}{n(s)} = \frac{12}{15}$$

<u>Are the two events mutually exclusive?</u>

To test whether the two events are mutually exclusive; we can use the addition rule.
For two mutually exclusive events

$$P\ (A\ and\ B) = 0.$$

$$P(A\ or\ B) = P(A) + P(B)$$

Since $P(A\ or\ B) = \frac{12}{15}\ and P(A) + P(B) = \frac{7}{15} + \frac{6}{15} = \frac{13}{15}$
$\therefore P(A\ or\ B) \neq P(A) + P(B).$

This means that the intersection of events A and B is non zero; Therefore A and B are not mutually exclusive.

Alternatively:

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Check if $P(A\ and\ B) = 0$
First check if intersection exists: $(A\ and\ B) = \{2\}$

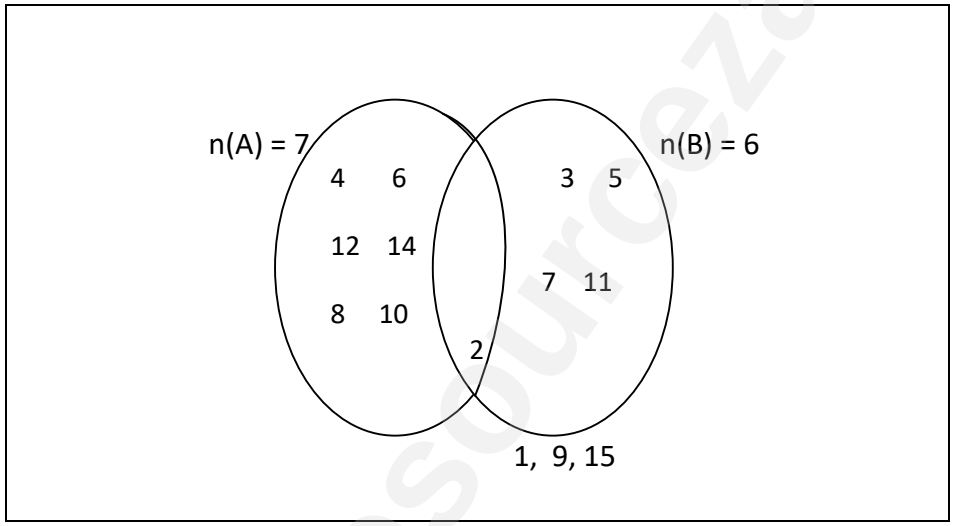
$$n(A\ and\ B) = 1.$$
$$\therefore P(A\ and\ B) = \frac{1}{15}$$

Therefore since $P(A\ and\ B) \neq 0$; Events A and B are not mutually exclusive.

Note

The above example can also be represented on a Venn diagram.

n(s) = 15



<u>The general sum (addition) rule for events</u>

Question: 5

The probability that a person drinks tea is $0{,}5$. The probability that a person drinks coffee is $0{,}4$. The probability that a person drinks tea; coffee or both is $0{,}8$. Determine the probability that a person drinks tea and coffee.

ANSWER: 5

Let the probability that a person drinks tea $= P(T)$ and the probability that a person drinks coffee$= P(C)$

$$P(T) = 0{,}5\ ;\ P(C) = 0{,}4\ ; P(T\ or\ C) = 0{,}8$$
$$P(T\ and\ C) = ?$$

Using the addition rule;

$$P(T\ or\ C) = P(T) + P(C) - P(T\ and\ C)$$
$$0{,}8 = 0{,}4 + 0{,}5 - P(T and C)$$
$$0{,}8 = 0{,}9 - P(T\ and\ C)$$

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$$\therefore P(T \text{ and } C) = 0{,}9 - 0{,}8 = 0{,}1.$$

Question: 6

The complementary rule:

Mr. Samuel wants to open a tuck-shop at Jeppe Education Centre but is not sure which cool drinks to stock. Before opening, he interviewed a sample of learners to determine what types of cool drinks they like. From his research, he determined that the probability that a learners drinks cola is $0{,}3$; the probability that a learner drinks a lemonade is $0{,}6$ and the probability that a learner drink neither of the two drinks is $0{,}2$.Determine the probability that a learner drinks

a) Cola and lemonade
b) Only cola or only lemonade.

ANSWER: 6

$$P(C) = 0{,}3\ ;\ P(L) = 0{,}6\ ; P(not\ (C \text{ or } L)) = 0{,}2.$$

Using the complementary rule:

$$P(not\ (C \text{ or } L)) = 1 - P(C \text{ or } L)$$
$$P(C \text{ or } L) = 1 - P(not(C \text{ or } L))$$
$$= 1 - 0{,}2$$
$$\therefore P(C \text{ or } L) = 0{,}8$$

a) $P(C \text{ and } L) = ?$

Using ;$P(C \text{ or } L) = P(C) + P(L) - P(C \text{ and } L)$

$0{,}8 = 0{,}3 + 0{,}6 - P(C \text{ and } L)$

$$P(C \text{ and } L) = 0{,}3 + 0{,}6 - 0{,}8$$
$$= 0{,}1$$

The probability that a learner drinks both cola and Lemonade is $0{,}1$.

b) $P(only\ C \text{ or } only\ L) = P(C \text{ or } L) - P(C \text{ and } L).$

This question requires us to calculate the probability that a Learner likes lemonade or Cola but not both of them. Since a learner can like either Cola or Lemonade but not both.

$$P(only\ C \text{ or } only\ L) = 0{,}8 - 0{,}1 = 0{,}7.$$

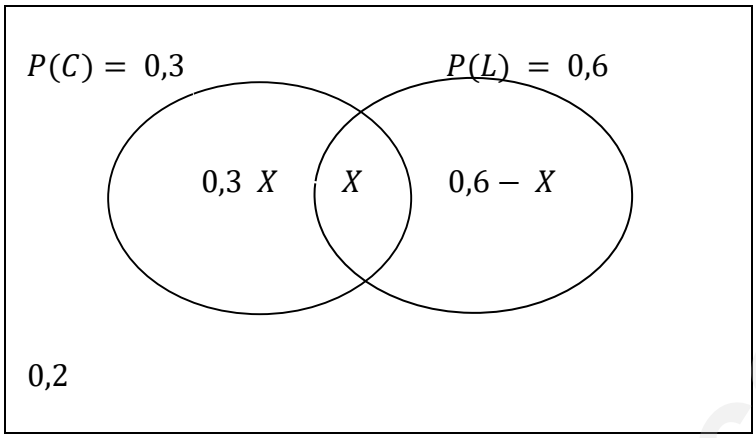
Alternatively:

Using a Venn diagram:

Let $P(C \text{ and } L) = X$

a)

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$$P(S) = 1$$
$$0{,}3 - x + 0{,}6 - X + X + 0{,}2 = 1$$
$$1{,}1 - X = 1$$
$$1{,}1 - 1 = X$$
$$0{,}1 = X.$$
$$Therefore\ P(C\ and\ L) = 0{,}1.$$

b) $P(only\ Cola\ or\ only\ Lemonade) = (0{,}3 - x) + (0{,}6 - x)$
$= (0{,}3 - 0{,}1) + (0{,}6 - 0{,}1)$
$= 0{,}2 + 0{,}5$
$= 0{,}7$

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Student No:_____ **Cell No:**_____

Exercise: 1 Date:_____
Marks: 143

Question 1:

1.1 Determine whether the following events are dependent or independent and give a reason for your answer.

1.1.1 Lorraine has a box of yellow, green and orange sweets. She takes out a yellow sweet and eats it. Then, she chooses another sweet and eat it.

(3)

1.2 Chauke throws a die twice.

(2)

1.3 Mrs Bella chooses a card at random from a deck of 52 cards. She is unhappy with her choice, so she places the back in the deck, shuffles it and chooses a second card.

(3)

1.4 MAGOGO has a bag of beads. She randomly chooses a yellow bead, looks at it and then puts it back in the bag. Then randomly chooses another bead and sees that it is red and puts it in the bag.

(4)

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1.5 Dr. Bongers has a container with calculators. Some of them work and some are broken. He randomly chooses a calculator and sees that it doesn't work and throws it away. He then chooses another calculator, sees that it works and keeps it.

(3)

Question 2:

2.1 Given that $P(A) = 0{,}7\,; P(B) = 0{,}4$ *and* $P(A \text{ and } B) = 0{,}28$.

1.1.1 Are the events A and B mutually exclusive? Give a reason for your answer.

(3)

1.1.2 Are the events A and B independent? Give a reason for your answer.

(3)

Question 3:

3.1 In the following questions; prove whether the events A and B are independent or not.

3.1.1 $P(A) = 0{,}2\,;\ P(B) = 0{,}7;\ \ P(A \text{ and } B) = 0{,}21$

(3)

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3.1.2 $P(A) = 0{,}2\ ;\ P(B) = 0{,}7\ ; P(B\ and\ A) = 0{,}14.$

(3)

Question 4:

4.1 Given that $n(A) = 5\ ; n(B) = 4\ ;\ n(S) = 20\ and\ n(A\ or\ B) = 8.$

4.1.1 Represent the information above in a Venn diagram. Hence determine $n(A\ and\ B)$.

(5)

4.1.2 Calculate the $P(A);\ P(B);\ P(A\ or\ B);\ P(A\ and\ B)$.

(6)

4.1.3 Are the events mutually exclusive or not?

(2)

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4.1.4 Are the events A and B independent?

(2)

Question 5:

5.1 Joseph rolls a die twice. What is the probability of getting?

5.1.1 Two threes

(3)

5.1.2 A prime number then an even number

(3)

5.1.3 No threes

(3)

5.1.4 Only one three

(3)

5.1.5 At least one three

(3)

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Student No:_____ **Cell No:**_____

Question 6:

Jeppe Education Center soccer team has to win both of their next two matches in order to qualify for the finals. The probability that Jeppe will win their first soccer match against supreme high is $\frac{2}{5}$ and the

Probability of winning their second match against Summit College is $\frac{3}{7}$. Assume each match is an independent event.

6.1 What is the probability that they will?

6.1.1 Progress to the finals?

(2)

6.1.2 Not win either match?

(3)

6.1.3 Win only one of their matches?

(2)

Question 7:

7.1 A pencil bag contains 2 red pens and 4 green pens. A pen is drawn from the bag at random and then replaced before the second pen is drawn.
Calculate the probability of drawing:

7.1.1 A red pen first if a green pen is drawn second.

(3)

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7.1.2 A green pen second if the first pen drawn was red.

(3)

7.1.3 A red pen first and a green pen second.

(2)

Question 8:

8.1 A lunch box contains 4 sandwiches and 2 apples. Mr Mfuphi chooses a food item randomly and eats it. She chooses another food item at random and eats that. Determine the following; probability that

8.1.1 The first item is a sandwich.

(4)

8.1.2 The first item is a sandwich and the second item is an apple.

(4)

8.1.3 The second item is an apple.

(3)

8.1.4 Are the events in (a) and (c) dependent, confirm your answer with a calculation.

(3)

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Question 9:

Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

9.1 Given that $P(A) = 0{,}5$; $P(B) = 0{,}4$ and $P(A \text{ or } B) = 0{,}7$.
Determine by calculation, whether the events are

9.1.1 Mutually exclusive

(2)

9.1.2 Independent

(2)

Question 10:

10.1 A and B are two events in a sample space, where
$$P(A) = 0{,}3\ ; P(A \text{ and } B) = 0{,}8\ ; P(B) = K.$$
Determine the value of K if

10.1.1 A and B are mutually exclusive.

(3)

10.1.2 A and B are independent.

(3)

Question 11:

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

11.1 A and B are two events in a sample space S, where $n(S) = 36, n(A) = 9, n(B) = 4$; $n(not(A\ or\ B)) = 24$
Using a venn diagram determine:

11.1.1 P(A or B)

(2)

11.1.2 P(A and B)

(2)

11.1.3 Whether events A and B are independent. Justify your answer with a calculation.

(3)

Question 12:

12.1 The probability that a mathematics teacher is absent from school on a certain day is $0{,}2$. The probability that a science teacher will be absent on that same day is $0{,}3$.

12.1.1 Do you think these two events are independent?
(Give a reason for your answer).

(3)

12.1.2 Assuming the events are independent, what is the probability that the mathematics teacher or the science teacher is absent?

(3)

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12.1.3 What is the probability that neither the mathematics nor the science teacher is absent?

(4)

Question 13:

13.1 Jeppe College cricket club plays two cricket matches against different clubs. The probability of winning the first match is $\frac{3}{5}$ and the probability of winning the second match is $\frac{4}{9}$. Assuming the results of the matches are independent, calculate the probability that Jeppe College club will...

13.1.1 Win both matches

(3)

13.1.2 Not win the first match

(3)

13.1.3 Win one or both of the two matches.

(3)

13.1.4 Win neither match.

(2)

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13.1.5 Not win the first match and win the second match.

(2)

Question 14:

14.1 The groups in grade 12 class of Jeppe Education Centre are working on the last question in a Mathematics June exam paper. They have 10 minutes remaining to finish the question. The probability that group A will complete the question in time is 40% and the probability that group B will finish the question in time is 25%.
Calculate the probability that both groups will finish before they run out of time.

(5)

Question 15:

15.1 Mr. Edward and Mr. Martin were arguing about whether the educators during a workshop at Jeppe College prefer tea or coffee.
Mr. Edward suggested that they do survey to settle the dispute. In total; they surveyed 24 people and found out that; 8 of them preferred to drink coffee and 12 of them preferred to drink tea.
The number of people who drink tea, coffee or both is 16.

Determine the probability that a person:

15.1.1 Drinks tea, coffee or both.

(2)

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15.1.2 Drinks neither tea nor coffee.

(2)

15.1.3 Doesn't drink coffee.

(2)

15.1.4 Determine whether the event that a person drinks coffee and the event that a person drinks tea are independent.

(3)

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Module: 14
3 - VENN DIAGRAMS

A Venn diagram is used to show how events are related to one another.

Tree diagram:

This is also useful in organizing and visualizing the different possible outcomes of a sequence of events.

Each branch in the tree shows on outcome of an event, along with the probability of that outcome.

NB: the probability of a sequence of outcomes is calculated as the product of the probabilities along the branches of the sequence.

Two-way contingency tables:

These are tools for keeping a record of the counts or percentages in a probability problem.

These tables are especially helpful in figuring out whether events are dependent or independent.

Example: 1

There are 200 boys in grade 12 at a certain school.
Their participation in sports can be broken as follows:

- 107 play rugby
- 90 play soccer
- 63 play cricket
- 35 play rugby and soccer
- 23 play rugby cricket
- 15 play rugby, soccer or cricket
- 190 boys play rugby, soccer or cricket.

a) How many boys do not play any of these sports?

b) Draw a Venn diagram to illustrate the given information and use it to answer the following questions:

   i. How many boys play soccer and cricket, but no rugby?

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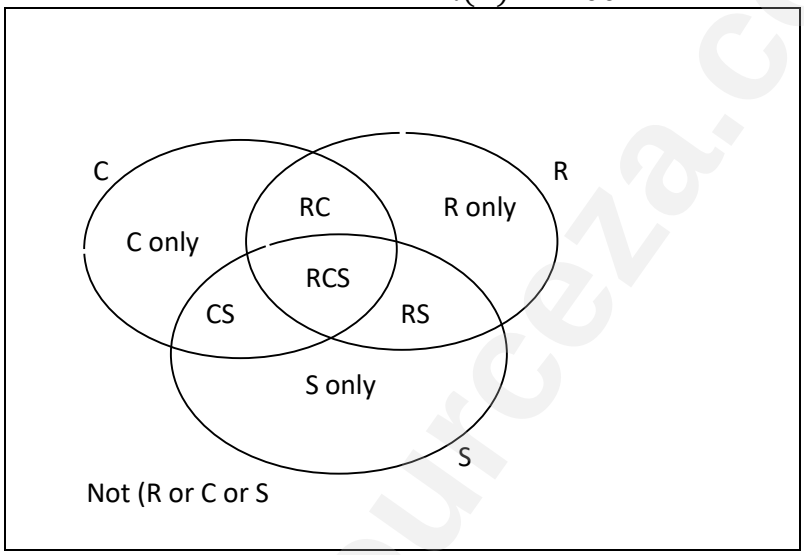
ii. What is the probability that randomly chosen grade 12 boy will take part in at least two of the sport:

ANSWER: 1

a) $n(not\ rugby;\ cricket\ nor\ soccer) = 200 - 190 = 10\ boys$

b) $Let\ R = rugby, S = soccer, C = Cricket.$

$$n(E) = 200$$



Rugby, Cricket and Soccer $= RCS = 15$

Rugby and Soccer but not Cricket = RS only:

$$RS = (R\ and\ S) - RCS$$

$$= 35 - 15 = 20.$$

Rugby and Cricket but not Soccer$= RC$ only

$$RC\ only = (R\ and\ C) - RCS$$

$$= 23 - 15 = 8$$

Cricket and Soccer but not rugby $= CS$ only

$$CS\ only = (C\ and\ S) - RCS$$

$$Let\ (S\ and\ C) = x.$$

Therefore $CS\ only = (X - 15)$

Only Soccer $= S - RS - CS - RCS$

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$= 90 - 20 - (x - 15) - 15 = 70 - x$

Only Cricket $= C - RC - CS - RCS$

$= 63 - 8 - (x - 15) - 15 = 55 - x$

$Not\ (rugby, cricket\ or\ Soccer);\ not(R\ or\ C\ or\ S) = 10$

Since 190 of boys play atleast one of the sports:

$$64 + 8 + 15 + 20 + x - 15 + 70 - x + 55 - x = 190$$
$$217 - x = 190$$
$$x = 27$$

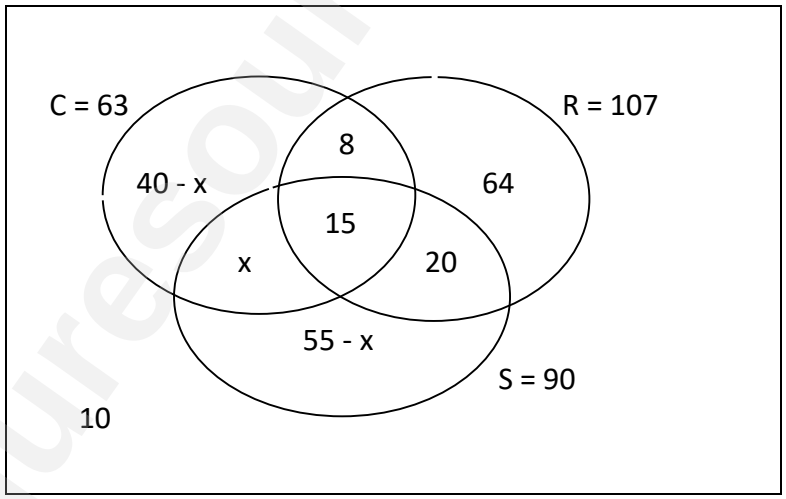
i. $(C\ and\ S\ but\ not\ rugby) = x - 15 = 27 - 15 = 12.$

ii. Number of boys who play two or more sports. (≥ 2)

$$P(atleast\ two\ sports) = \frac{8 + 20 + 12 + 15}{200} = \frac{55}{200}$$
$$= 0{,}275 = 27{,}5\%.$$

$$n(E) = 200$$

OR:



Summation $= 200.$

$$8 + 64 + 15 + 20 + x + 40 - x + 55 - x + 10 = 200$$
$$212 - x = 200$$
$$x = 12\ boys$$

Example 2: (Tree diagrams)

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The probability that the floor of a supermarket will be wet when it rains in the morning is 30% and there is a 10% probability of the floor being very wet. The probability that a person will slip and fall if the floor is dry is $12\%$and a person is three times as likely to fall if the floor is wet. If the floor is very wet, the probability that a person will fall is 0,6.
Draw a tree diagram to represent the given information, showing the probabilities of each outcome; and use it to answer the following questions.

a) What is the probability that a person will fall on any given day?
b) What is the probability that a person will not fall on any given day?
c) Are the events of the floor being dry and a person falling independent? Justify your answer with calculation.

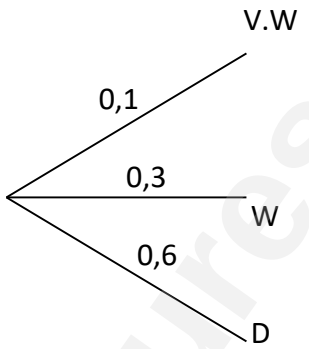
ANSWER: 2

Identify the events:

There are 3 outcomes for the floor, namely; dry, wet and very wet; and two outcomes for a person:

Namely to fall or not fall

Draw the first level of the tree diagram



This tree diagram shows the possible outcomes and probabilities of the status of the floor.

Draw the second level of the tree diagram

Note

Remember that the sum of probabilities for any set of branches is 1.
(Use this as a logical check whenever you are constructing a tree diagram).

a) $P(F) = P((V.W)F)\ or\ P(W.F)\ or\ P(D.F)$
$= 0,06 + 0,108 + 0,072 = 0,24\ or\ 24\%$

b) $P(not\ fall) = P(not\ F) = 1 - P(F)$

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$$= 1 - 0{,}24 \ = \ 0{,}76 \ or \ 76\%$$

Alternatively: from the tree diagram:

$$P(not\ fall) \ = \ P((V.W).(NF)) \ or \ P(w(NF)) \ or \ P(D(NF))$$
$$= 0{,}04 \ + \ 0{,}192 \ + \ 0{,}528$$
$$= 0{,}76$$

c) Logically, it appears that these events are dependent but the question asked us to prove this using a calculation. We can do this using the independent events:

$P(A\ and\ B) = \ P(A) \ \times \ P(B).$
$P(dry\ and\ fall) \ = \ P(DF) \ = \ 0{,}072.$
$P(dry) \ \times \ P(fall) \ = \ 0{,}6 \ \times \ 0{,}24 \ = \ 0{,}144.$
Therefore $P(D\ and\ F) \ \neq P(D) \ \times \ P(F).$
The floor being dry and a person falling are dependent events.

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Exercise: 1 **Date:**_____
Marks: 35

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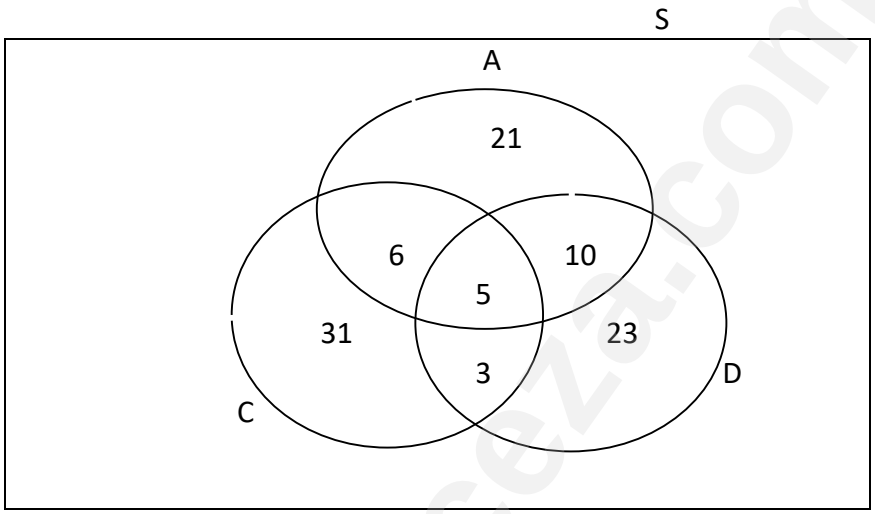
Question 1:

Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

A survey was done on a group of learners to determine which type of TV shows they enjoy: action, comedy or drama. Let A= action, C = comedy and D = drama.

The results of the survey are shown in the Venn diagram below



1.1 Study the venn diagram and determine the following:

1.1.1 The total number of learners surveyed.

(2)

1.1.2 The total number of learners who do not enjoy any of the mentioned types of TV shows.

(1)

1.1.3 P(not A)

(2)

1.1.4 P(A or D)

(2)

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1.1.5 P(A and C and D)

(2)

1.1.6 P(A or not C)

(2)

1.1.7 P(not (A or C)

(2)

1.1.8 P(not (A and D)

(2)

1.1.9 The probability of a learner enjoying at least two types of TV shows.

(3)

1.1.10 Describe, in words, the meaning of each of the questions 1.1.3 to 1.1.8 in the context of this question

(5)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

Question 2:

2.1 During July in Cape Town, the probability that it will rain on a randomly chosen day $\frac{4}{5}$. Dlamini either walks to school or gets a ride with his parents in their car. If it rains; the probability that Dlamini's parents will take him to school by car is $\frac{5}{6}$. If it doesn't rain the probability that Dlamini's parents will take him to school by car is $\frac{1}{12}$.

2.1.1 Represent the above information on a tree diagram.

(6)

2.1.2 What the probability that is a rainy day and Dlamini's walks to school?

(2)

2.1.3 What the probability that Dlamini's parents take her to school by car?

(2)

Exercise: 2 Date:_____

Marks: 35

Question 1:

1.1 Let A and B be two events in a sample space. Suppose that P(A) = 0.4; P(A or B) = 0.7 and P(B) = k

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

1.1.1 For what value of k are A and B mutually exclusive? (2)

1.1.2 For what value of k are A and B independent? (4)

1.2 In a random experiment it was found that:
P(A) = 0.25; P(B) = 0.5 and P(A or B) = 0.625

1.2.1 Calculate P(A and B) (3)

1.2.2 Determine, giving reasons, if events A and B are:

1.2.2.1 mutually exclusive or inclusive (2)

1.2.2.2 complementary (2)

Question 2:

The data below was obtained from the financial aid office at a certain university.

	RECEIVING FINANCIAL AID	NOT RECEIVING FINANCIAL AID	TOTAL
Undergraduate	4222	3898	8120
Postgraduate	1879	731	2610
TOTAL	6101	4629	10730

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

2.1 Determine the probability that a student at random is

2.1.1 receiving financial aid. (3)

2.1.2 a postgraduate student and not receiving financial aid. (3)

2.1.3 an undergraduate student and receiving financial aid. (3)

2.1.4 not receiving financial aid given that the student is postgraduate. (3)

2.2 Are the events of being an undergraduate and receiving financial aid independent? Show ALL relevant workings to support your answer. (4)

Question 3:

In a survey 1530 skydivers were asked if they had broken a limb. The results of the survey were as follows:

	Broken a limb	**Not broken a limb**	**TOTAL**
Male	463	*b*	782
Female	*a*	*c*	*d*
TOTAL	913	617	1530

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

3.1 Calculate the values of *a, b, c* and *d*. (4)

3.2 Calculate the probability of choosing at random in the survey, a female skydiver who has not broken a limb. (3)

3.3 Suppose that only male skydivers are considered. Calculate the probability of choosing at random in the survey, a male skydiver with a broken limb. (3)

3.4 Is being a female skydiver and having broken a limb independent? Use calculations, correct to TWO decimal places, to motivate your answer. (4)

Question 4:

The probability that it will rain on an given day is 63%. A Child has a 12% chance of falling in dry weather and is three times as likely to fall in wet weather.

4.1 Draw a tree diagram to represent all outcomes of the above information. (6)

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4.2 What is the probability that a child will not fall on any given day? (3)

4.3 What is the probability that a child will fall in dry weather? (2)

Question 5:

At a school for boys there are 240 learners in Grade 12. The following information was gathered about participation in school sport.

122 boys play rugby (R)

58 boys play basketball (B)

96 boys play cricket (C)

16 boys play all three sports

22 boys play rugby and basketball

26 boys play cricket and basketball

26 boys do not play any of these sports

Let the number of learners who play rugby and cricket only be x.

5.1 Draw a Venn diagram to represent the above information. (8)

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5.2 Determine the number of boys who play rugby and cricket. (3)

5.3 Determine the probability that a learner in Grade 12 selected at random:
(Leave your answers correct to THREE decimal places.)

5.3.1 only plays basketball. (2)

5.3.2 does not play cricket. (2)

5.3.3 participate in at least two of these sports. (2)

Module: 15
COUNTING PRINCIPLES

The fundamental counting rule:

a) The number of ways of making several decisions in succession is determined by multiplying the numbers of choices that can be made in each decision. That is successive choices from $\mathbf{m_1}$, then $\mathbf{m_2}$ then $\mathbf{m_3}$ --- options,
Create $\mathbf{m_1 \times m_2 \times m_3 \times}$ --- different combined options.

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

b) All arrangements of a number of objects or items is called the **permutations** of the objects or items.

c) The number of permutations of **n** different items is **n!**
Where n! = n x (n-1) x (n-2) x--- x 1.

Eg. The number of ways that 7 digits of a telephone number can be arranged =7 x 6 x 5 x 4 x 3 x 2 x 1. Ie the number of ways = 7! =5040 ways.

d) The number of ways that **n** items, taken **r** at a time, can be arranged is written as ${}_nP_r = \frac{n!}{(n-r)!}$ For $n > r$.

Eg. ${}_5P_r = \frac{5!}{(5-2)!} = \frac{5!}{3!}$=5 x 4= 20.

e) The number of permutations of **n** items ,of which **a** are alike ,another **b** are alike, another **c** are alike is $\frac{n!}{a!\ x\ b!\ x\ c!}$

Eg.If in a 5- digit number there are three fours and two sixes, the number of ways / arrangements is $\frac{5!}{3!\ 2!}$ =10

f) The number of permutations of **n** items , taken **r** at a time ,when each of the items may be repeated any number of times is n x n x n x --- to **r** factors.ie $\mathbf{n^r}$

Exercise: 1 Date:_____
Marks: 36

Question 1:

1.1 Consider the word **MATHS**.

1.1.1 How many different 5-letter arrangements can be made using all the above letters?

(2)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

1.1.2 Determine the probability that the letters **S** and **T** will always be the first two letters of the arrangements in1 (a) above.

(2)

Question 2:

2.1 How many three- character codes can be formed if the first character must be a letter and the second two characters must be digits?

(2)

2.2 What is the probability that a random arrangement of the letters **BAFANA** starts and ends with an 'A'?

(2)

2.3 Suppose you are given a test in Mathematics and you have not studied for it. Each question in the test can only be answered true or false. If there are 20 questions in the test and you answer all the questions by guessing.
Calculate how many ways of answering the test are there.

(3)

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2.4 License plates in Eastern Cape follow the pattern **LLLDDD EC** where L is a letter and D is a digit between 0 and 9.Calculate how many different possible combinations are there for Eastern Cape license plates.

(3)

2.5 How would the answer in question 2.4 be different if vowels could not be used, and no numbers or letters could be repeated?

(3)

2.6 In how many ways can the letters of the word **SOCCER** be arranged to form different arrangements?

(2)

2.7 In how many ways can the letters of the word **STATISTICS** be arranged to form different arrangements?

(3)

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2.8 In how many ways can the letters of the word **SOCCER** be arranged so that the arrangements start with **'c'** and end with**'s'**

(3)

Question 3:

3.1 Using the letters of the word **'MILLENIUM'** determine:

3.1.1 The number of nine letter 'words' that can be formed.

(2)

3.1.2 The probability that the new word will start and end with letter 'L'.

(2)

3.2 Four different glasses and five different bottles are to be arranged on a shelf. How many arrangements can be made if all the bottles are to be together and all the glasses are to be together?

(3)

3.3 In how many ways can 5 letters be arranged to form a 7 letter 'word', if letters may be used more than once?

(3)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

Exercise: 2 **Date:**_____

Marks: 47

Question 1:

During the time of registration of a student at the University of South Africa, three subject groups are presented to him for the B.Com course that he wants to follow.

SUBJECT GROUP A	SUBJECT GROUP B	SUBJECT GROUP C
Accounting	Financial Management	Applied mathematics
Audit	Financial Mathematics	Physics
taxation	Office Management	Chemistry
Statistics	Psychology	Biochemistry
Mathematics		Physiology

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

1.1 If a student decides to choose one subject from group A and one subject from group B, how many different combinations are possible? (2)

1.2 If one subject is chosen from each of the three subject groups and you know that you will definitely not choose Biochemistry and Chemistry, how many combinations will there be to choose from? (2)

Question 2:

Consider the digits 1, 2, 3, 4, 5, 6, 7 and 8 and answer the following questions:

2.1 How many 8-digit numbers can be formed if repetition is allowed? (2)

2.2 How many 8-digit numbers can be formed if repetition is NOT allowed? (2)

2.3 How many 2-digit numbers can be formed if repetition is allowed? (2)

2.4 How many 4-digit numbers can be formed if repetition is NOT allowed? (2)

2.5 How many numbers between 4000 and 5000 can be formed? (2)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

Question 3:

There are 7 different shirts and 4 different pairs of trousers in a cupboard. The clothes have to be hung on the rail.

3.1 In how many different ways can the clothes be arranged on the rail? (2)

3.2 In how many different ways can the clothes be arranged if all the shirts are to be hung next to each another and the pairs of trousers are to be hung next to each another on the rail? (3)

3.3 What is the probability that a pair of trousers will hang at the beginning of the rail and a shirt will hang at the end of the rail? (4)

Question 4:

A South African band is planning a concert tour with performances in Durban, East London, Port Elizabeth, Cape Town, Bloemfontein, Johannesburg and Polokwane.

4.1 There are no restrictions. (2)

4.2 The first performance must be in Cape Town and the Last performance must be in Polokwane. (2)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

4.3 The performances in the four coastal cities (the cities close to the sea or ocean) must be grouped together? (4)

Question 5:

Ping pong balls are numbered 1 to 12 and placed in a hat. Every time a ball is drawn it is placed on a rack, one next to the other.

5.1 How many different arrangements of the 12 balls is possible? (2)

5.2 Assume that the numbers 8 and 11 must be placed next to each other, in any order. In how many ways can the numbers then be rearranged? (3)

5.3 The numbers 2, 5, 6, and 9 are taken from the group of balls. Two-digit numbers must be formed out of these four numbers. How many numbers can be formed? (2)

Question 6:

In Gauteng number plates are designed with 3 alphabetical letters, excluding the 5 vowels, next to one another and then any 3 digits, from 0 to 9, next to one another. The GP is constant in all Gauteng number plates, for example, BBV 023 GP. Letters and digits may be repeated in a number plate.

6.1 How many unique number plates are available? (3)

6.2 What is the probability that a car's number plate will start with a Y? (3)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

6.3 What is the probability that a car's number plate will contain only one 7? (3)

6.4 How many unique number plates will be available if the letters and numbers are not repeated? (3)

Exercise: 3 Date:_____

Marks: 47

Question 1:

Consider the word ACTION

1.1 How many six-letter word arrangements can be made if the letters may be repeated? (2)

1.2 How many six-letter word arrangements can be made if the letters may not be repeated? (2)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

1.3 How many four-letter word arrangements can be made if the letters may be repeated? (2)

1.4 How many six-letter word arrangements can be made if the letters may NOT be repeated? (2)

1.5 How many word arrangements can be made with this word if the word starts with C and ends with N? Letters may NOT be repeated? (2)

1.6 What is the probability that the word starts with C and ends with N? Letters may NOT be repeated. (2)

1.7 How many word arrangements can be made with this word if the word starts with C and ends with N? Letters may be repeated? (2)

1.8 What is the probability that the word starts with C and ends with N? Letters may be repeated. (2)

Question 2:

Consider the word WOW

2.1 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Letters may NOT be used more than once. (2)

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2.2 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Letters may be used more than once. (2)

2.3 How many word arrangements can be made with this word if the repeated letters are treated as identical? Each word arrangement must not contain more than the number of given letters. (4)

2.4 How many word arrangements can be made with this word if the repeated letters are treated as identical? All letters may be used more than once. (4)

Question 3:

Consider the word CONCENTRATION

3.1 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Letters may NOT be used more than once. (2)

3.2 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Letters may be used more than once. (2)

3.3 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Each word arrangement must not contain more than the number of given letters. (4)

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

3.4 How many word arrangements can be made with this word if the repeated letters are treated as different letters? Letters may be used more than once. (4)

Learning material

MATHEMATICS P1

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

EXEMPLAR I

GRADE 12

MARKS: 150

TIME: 3 HOURS

This question paper consists of 8 pages and 1 Data sheet.

INSTRUCTIONS AND INFORMATION:

1. This question paper consists of 11 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, etc. that you have used in determining your answers.
4. AN answer ONLY will not necessarily be awarded full marks.
5. You may use an approved scientific calculator (non-programmable and non-graphic), unless stated otherwise.
6. Round off to TWO decimal places unless stated otherwise.
7. Diagrams are not necessarily drawn to scale.
8. An information sheet, with formulae, is included at the end of the question paper.
9. Number the answers correctly according to the numbering system in the question paper.
10. Write legibly and present your work neatly.

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Name:_____ **Surname:**_____

Student No:_____ **Cell No:**_____

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QUESTION 1

1.1 Solve for x

1.1.1 $2 - 3x^2 + 5x = 0$ (2)

1.1.2 $5x^2 - x - 2 = 0$ (3)

1.1.3 $x - 2 = \sqrt{2x - 5}$ (4)

1.1.4 $(2 - x)(x - 6) > 0$ (3)

1.2 Solve for x and y simultaneously.

$x - 2y = -3$

$xy = 20$ (5)

1.3 Simplify

$$\frac{7^{a-2} \cdot 2^{a-2}}{14^{a-1} \cdot 2}$$ (3)

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1.4 If $x = \frac{3 \pm \sqrt{1+12r}}{2}$, write down ONE integer value of r so that x is rational. (2)

[22]

QUESTION 2

2.1 The following is a combination of a linear and a geometric series:

$$3b - 6 + 6b - 10 + 12b - 14 + \cdots$$

2.1.1 Write down the next two terms of the series. (2)

2.1.2 Determine in terms of b, the sum of the first 30 terms of this series. (6)

2.2 Given the series $54 + 18 + 6 + \ldots$

2.2.1 Determine the nth term of this series. (2)

2.2.2 Hence determine the 12th term of the series. (2)

2.2.3 Show that the sum to n terms of this series is $81 - 81(\frac{1}{3})^n$. (3)

2.2.4 Calculate: $\sum_{n=0}^{10} 54\left(\frac{1}{3}\right)^n$ (3)

2.2.5 Determine the maximum value of: $\sum_{n=0}^{k} 54\left(\frac{1}{3}\right)^n$ (3)

[20]

QUESTION 3

Given the quadratic sequence:

$x;\ y;\ 16;\ 28;\ 42;\ 58;\ \ldots$

3.1 Determine the values of x and y, the first two terms of the sequence. (2)

3.2 Determine the 45th term of this sequence. (5)

[7]

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QUESTION 4

Given the equation of the graph $g(x) = \frac{1}{4}(x-4)(x+2)$

4.1 Determine the coordinates of:

4.1.1 the y-intercept of the graph. (1)

4.1.2 the x- intercepts of the graph. (2)

4.1.3 the turning point of the graph. (3)

4.2 Draw a sketch graph of $g(x)$, showing all critical points. (3)

4.3 If h is formed when g is shifted two units to the right and one unit up, determine the equation of h in the form $h(x) = a(x-p)^2 + q$. (4)

4.4 If $f(x) = mx - 2$ is a tangent to the graph of g, determine the value of m. (3)

[16]

QUESTION 5

In the diagram below are the sketches of $f(x) = \frac{2}{3}x + 2$ and $g(x) = \frac{a}{x+p} + q$.

The graph of g passes through $(0; 4)$ and its x –intercept is A.

5.1 Calculate the values of p and q. (3)

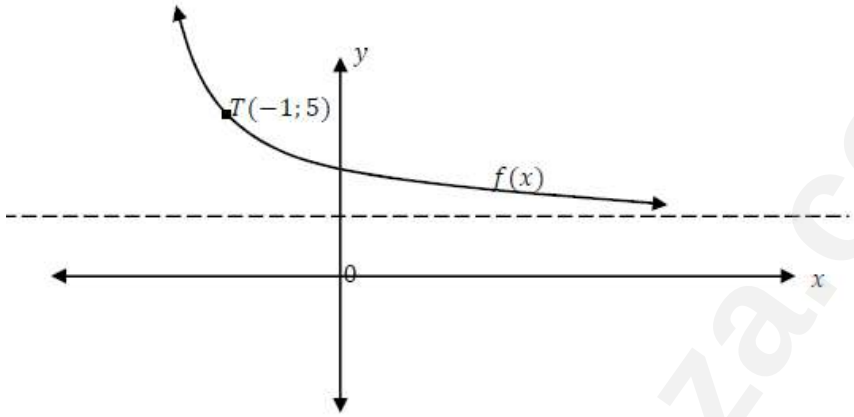
5.2 Calculate a. (2)

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QUESTION 6

In the sketch below, $f(x) = b^x + 2$, and T $(-1;5)$ is a point on f.



6.1 Calculate the value of b. (3)

6.2 Determine the equation of h, which is a reflection of f about the y- axis. (2)

6.3 Determine the domain of f^{-1} the inverse of f. (1)

6.4 If $(x;\ 8)$ is a point on the graph f calculate the value of x. (3)

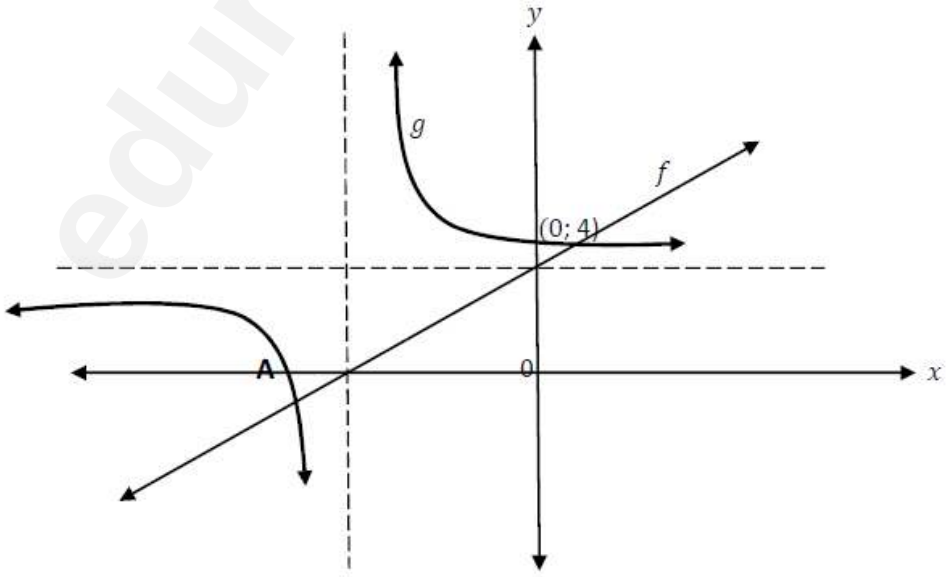
[9]

5.3 Determine the coordinates of point A. (2)

5.4 The value of p is increased by 1 unit.

What effect will that have on the graph of g? (2)

5.5 Determine the equation of f^{-1}, the inverse of f, in the form $y = \cdots$ (2)



[11]

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Student No:_____ **Cell No:**_____

QUESTION 7

7.1 A car is purchased today for R189000. What will the book value of the car be after 5 years, if the depreciation rate is 16% p.a. calculated on a straight- line method. (3)

7.2 Nelly plans to have R25 000 at the end of 10 years. If the bank offers 8% p.a. interest compounded monthly, what amount should be deposited as a lump sum now so that she will have the required amount after 10 years.
Assume that no further transactions take place on this account. (3)

7.3 Peter wants to buy a house costing R1250000. He is required to pay 10% deposit and borrow the rest from the bank as a loan. The bank offers interest at 7,5% p.a. compounded monthly and expects the loan to be repaid over 20 years.

7.3.1 What amount does he have to borrow? (1)

7.3.2 What is Peter's monthly installment? (5)

7.3.3 What is the total amount that Peter will have paid on the loan after 20 years? (1)

7.3.4 How much in interest will Peter pay in total after 20 years? (2)

[15]

QUESTION 8

8.1 Given that $f(x) = 2x - x^2$. Determine $f'(x)$ from first principles. (5)

8.2 Determine the derivatives of the following:

8.2.1 $y = \frac{2}{3x^5}$ (1)

8.2.2 $y = \frac{x-25}{\sqrt{x}-5}$ (3)

[9]

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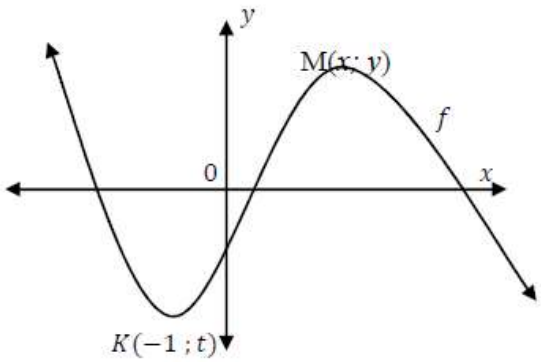
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Student No:_____ Cell No:_____

QUESTION 9

The function f below has equation $f(x) = -x^3 + 3x^2 + cx - 4$,

with K$(-1; t)$and M$(x;\ y)$ as the turning points.



9.1 Show that $c = 9$. (4)

9.2 Determine the coordinates of M$(x; y)$. (4)

9.3 Show that$f'(x) - 12 \leq 0$ for all real values of x. (5)

[13]

QUESTION 10

A pinball is shot vertically up in order to hit a target which is 90m above the ground.

The height H above the ground (in meters) that the ball reaches after t seconds is given by the formula: H$(t) = -5t^2 + 35 + 30t$.

10.1 What was the initial height from which the pinball was shot? (2)

10.2 Determine the initial speed with which the ball was shot. (3)

10.3 Is it possible for the ball to hit the target? (5)

10.4 What will the speed of the ball be when it hits the ground? (5)

[15]

QUESTION 11

11.1 5 girls and 3 boys are to sit on a bench in a straight line.

11.1.1 In how many ways can they all sit on the bench? (3)

11.1.2 In how many ways can they all sit on the bench with girls seated together? (2)

11.1.3 Determine the probability that all girls are seated next to each other. (2)

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11.2 Events A and B are independent of one another, with $P(A) = 0,4$ and $P(A \text{ and } B) = 0,12$.

Determine:

11.2.1 $P(B)$ (2)

11.2.2 $P(A \text{ or } B)$ (2)

11.2.3 $P(\text{not } A \text{ and not } B)$ (3)

[13]

TOTAL MARKS: 150

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Student No:_____ **Cell No:**_____

INFORMATION SHEET: MATHEMATICS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$A = P(1+ni) \qquad A = P(1-ni) \qquad A = P(1-i)^n \qquad A = P(1+i)^n$$

$$T_n = a + (n-1)d \qquad S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$T_n = ar^{n-1} \qquad S_n = \frac{a\left(r^n - 1\right)}{r-1} \; ; \quad r \neq 1 \qquad S_\infty = \frac{a}{1-r} \; ; -1 < r < 1$$

$$F = \frac{x\left[(1+i)^n - 1\right]}{i} \qquad P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \mathrm{M}\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$y = mx + c \qquad y - y_1 = m(x - x_1) \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \tan\theta$$

$$(x-a)^2 + (y-b)^2 = r^2$$

In ΔABC: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $a^2 = b^2 + c^2 - 2bc.\cos A$ area $\Delta ABC = \dfrac{\cdot}{2}$

$$\sin(\alpha+\beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases} \qquad \sin 2\alpha = 2\sin\alpha.\cos\alpha$$

$$\bar{x} = \frac{\sum fx}{n} \qquad \sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

$$P(A) = \frac{n(A)}{n(S)} \qquad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$\hat{y} = a + bx \qquad b = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

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Learning material

MATHEMATICS P1

EXEMPLAR II

GRADE 12

MARKS: 150

TIME: 3 HOURS

This question paper consists of 8 pages and 1 Data sheet.

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INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions.

1. This question paper consists of 12 questions.
2. Answer ALL the questions.
3. Number the answers correctly according to the numbering system used in this question paper.
4. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining your answers.
5. Answers only will not necessarily be awarded full marks.
6. You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.
7. If necessary, round off answers to TWO decimal places, unless stated otherwise.
8. Diagrams are NOT necessarily drawn to scale.
9. An information sheet with formulae is included at the end of the question paper.
10. Write neatly and legibly.

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Name:_____ **Surname:**_____

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QUESTION 1

1.1 Solve for x:

1.1.1 $3x^2 - 4x = 0$ (2)

1.1.2 $x - 6 + \frac{2}{x} = 0$; $x \neq 0$. (Leave your answer correct to TWO decimal places.) (4)

1.1.3 $x^{\frac{2}{3}} = 4$ (2)

1.1.4 $3^x(x-5) < 0$ (2)

1.2 Solve for x and y simultaneously:
$y = x^2 - x - 6$ **and** $2x - y = 2$ (6)

1.3 Simplify, without the use of a calculator:
$\sqrt{3}.\sqrt{48} - \frac{4^{x+1}}{2^{2x}}$ (3)

1.4 Given: $f(x) = 3(x-1)^2 + 5$ and $g(x) = 3$

1.4.1 Is it possible for $f(x) = g(x)$? Give a reason for your answer. (2)

1.4.2 Determine the value(s) of k for which $f(x) = g(x) + k$ has TWO unequal real roots. (2)
[23]

QUESTION 2

2.1 Given the arithmetic series: $18 + 24 + 30 + \ldots + 300$

2.1.1 Determine the number of terms in this series. (3)

2.1.2 Calculate the sum of this series. (2)

2.1.3 Calculate the sum of all the whole numbers up to and including 300 that are NOT divisible by 6. (4)

2.2 The first three terms of an infinite geometric sequence are 16, 8 and 4 respectively.

2.2.1 Determine the nth term of the sequence. (2)

2.2.2 Determine all possible values of n for which the sum of the first n terms of this sequence is greater than 31. (3)

2.2.3 Calculate the sum to infinity of this sequence. (2)
[16]

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QUESTION 3

3.1 A quadratic number pattern $T_n = an^2 + bn + c$ has a first term equal to 1.
The general term of the first differences is given by $4n + 6$.

3.1.1 Determine the value of a. (2)

3.1.2 Determine the formula for T_n. (4)

3.2 Given the series: $(1\times 2)+(5\times 6)+(9\times 10)+(13\times 14)+\ldots+(81\times 82)$

Write the series in sigma notation. (It is not necessary to calculate the value of the series.) (4)
[10]

QUESTION 4

4.1 Given: $f(x)=\frac{2}{x+1}-3$

4.1.1 Calculate the coordinates of the y-intercept of f. (2)

4.1.2 Calculate the coordinates of the x-intercept of f. (2)

4.1.3 Sketch the graph of f in your ANSWER BOOK, showing clearly the asymptotes and the intercepts with the axes. (3)

4.1.4 One of the axes of symmetry of f is a decreasing function. Write down the equation of this axis of symmetry. (2)

4.2 The graph of an increasing exponential function with equation $f(x)=a.b^x+q$ has the following properties:

- Range: $y > -3$
- The points $(0\,;-2)$ and $(1;-1)$ lie on the graph of f.

4.2.1 Determine the equation that defines f. (4)

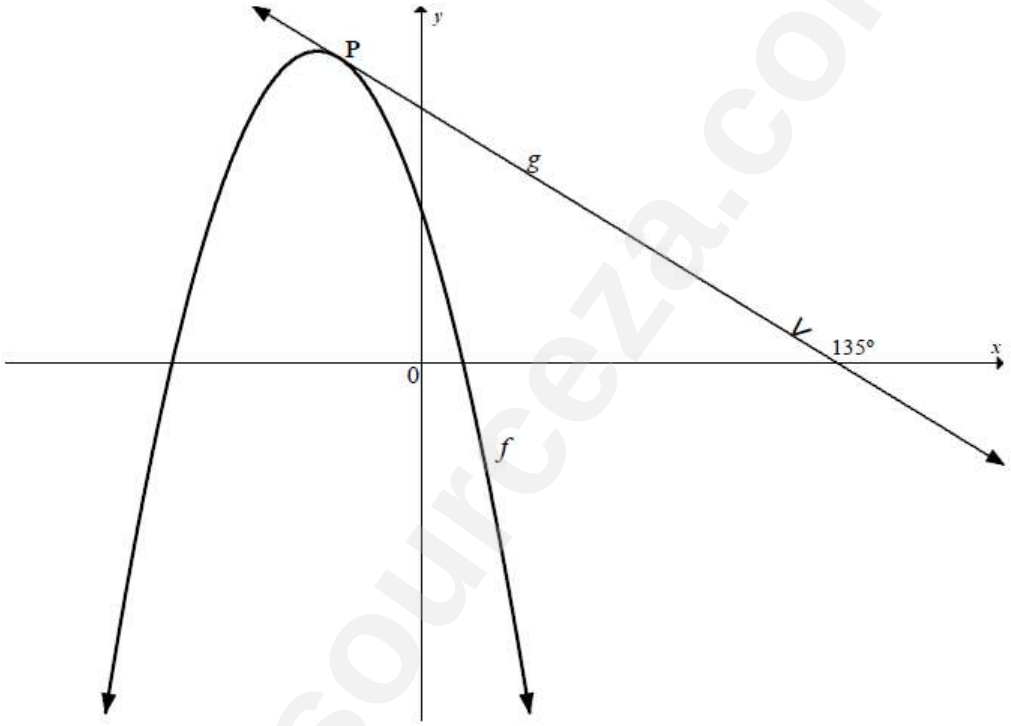
4.2.2 Describe the transformation from $f(x)$ to $h(x)=2.2^x+1$ (2)
[15]

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QUESTION 5

The sketch below shows the graphs of $f(x) = -2x^2 - 5x + 3$ and $g(x) = ax + q$. The angle of inclination of graph g is $135°$ in the direction of the positive x-axis. P is the point of intersection of f and g such that g is a tangent to the graph of f at P.



5.1 Calculate the coordinates of the turning point of the graph of f. (3)

5.2 Calculate the coordinates of P, the point of contact between f and g. (4)

5.3 Hence or otherwise, determine the equation of g. (2)

5.4 Determine the values of d for which the line $k(x) = -x + d$ will not intersect the graph of f. (1)
[10]

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QUESTION 6

The graph of g is defined by the equation $g(x) = \sqrt{ax}$. The point (8 ; 4) lies on g.

6.1 Calculate the value of a. (2)

6.2 If $g(x) > 0$, for what values of x will g be defined? (1)

6.3 Determine the range of g. (1)

6.4 Write down the equation of g^{-1}, the inverse of g, in the form $y = ...$ (2)

6.5 If $h(x) = x - 4$ is drawn, determine ALGEBRAICALLY the point(s) of intersection of h and g. (4)

6.6 Hence, or otherwise, determine the values of x for which $g(x) > h(x)$. (2)
[12]

QUESTION 7

Siphokazi bought a house. She paid a deposit of R102 000, which is equivalent to 12% of the selling price of the house. She obtained a loan from the bank to pay the balance of the selling price. The bank charges her interest of 9% per annum, compounded monthly.

7.1 Determine the selling price of the house. (1)

7.2 The period of the loan is 20 years and she starts repaying the loan one month after it was granted. Calculate her monthly instalment. (4)

7.3 How much interest will she pay over the period of 20 years? Round your answer correct to the nearest rand. (2)

7.4 Calculate the balance of her loan immediately after her 85th instalment. (3)

7.5 She experienced financial difficulties after the 85th instalment and did not pay any instalments for 4 months (that is months 86 to 89). Calculate how much Siphokazi owes on her bond at the end of the 89th month. (2)

7.6 She decides to increase her payments to R8 500 per month from the end of the 90th month. How many months will it take to repay her bond after the new payment of R8 500 per month? (4)
[16]

QUESTION 8

8.1 Determine $f'(x)$ from first principles if $f(x) = 3x^2 - 2$. (5)

8.2 Determine $\frac{dy}{dx}$ if $y = 2x^{-4} - \frac{x}{5}$. (2)
[7]

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QUESTION 9

Given: $f(x) = x^3 - 4x^2 - 11x + 30$.

9.1 Use the fact that $f(2) = 0$ to write down a factor of $f(x)$. (1)

9.2 Calculate the coordinates of the x-intercepts of f. (4)

9.3 Calculate the coordinates of the stationary points of f. (5)

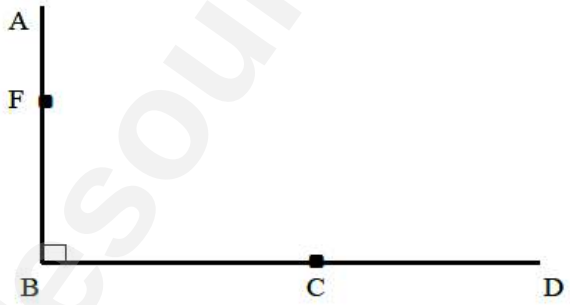
9.4 Sketch the curve of f in your ANSWER BOOK. Show all intercepts with the axes and turning points clearly. (3)

9.5 For which value(s) of x will $f'(x) < 0$? (2)

[15]

QUESTION 10

Two cyclists start to cycle at the same time. One starts at point B and is heading due north to point A, whilst the other starts at point D and is heading due west to point B. The cyclist starting from B cycles at 30 km/h while the cyclist starting from D cycles at 40 km/h. The distance between B and D is 100 km. After time t (measured in hours), they reach points F and C respectively.



10.1 Determine the distance between F and C in terms of t. (4)

10.2 After how long will the two cyclists be closest to each other? (4)

10.3 What will the distance between the cyclists be at the time determined in QUESTION 10.2? (2)

[10]

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QUESTION 11

11.1 Events A and B are mutually exclusive. It is given that:

- P(B) = 2P(A)
- P(A or B) = 0,57

Calculate P(B). (3)

11.2 Two identical bags are filled with balls. Bag A contains 3 pink and 2 yellow balls. Bag B contains 5 pink and 4 yellow balls. It is equally likely that Bag A or Bag B is chosen. Each ball has an equal chance of being chosen from the bag. A bag is chosen at random and a ball is then chosen at random from the bag.

11.2.1 Represent the information by means of a tree diagram. Clearly indicate the probability associated with each branch of the tree diagram and write down all the outcomes. (4)

11.2.2 What is the probability that a yellow ball will be chosen from Bag A? (1)

11.2.3 What is the probability that a pink ball will be chosen? (3)
[11]

QUESTION 12

Consider the word M A T H S.

12.1 How many different 5-letter arrangements can be made using all the above letters? (2)

12.2 Determine the probability that the letters S and T will always be the first two letters of the arrangements in QUESTION 12.1. (3)
[5]

TOTAL: 150

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INFORMATION SHEET

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$A = P(1+ni) \qquad A = P(1-ni) \qquad A = P(1-i)^n \qquad A = P(1+i)^n$$

$$T_n = a + (n-1)d \qquad S_n = \frac{n}{2}[2a + (n-1)d]$$

$$T_n = ar^{n-1} \qquad S_n = \frac{a(r^n - 1)}{r-1} \; ; \quad r \neq 1 \qquad S_\infty = \frac{a}{1-r} \; ; \; -1 < r < 1$$

$$F = \frac{x[(1+i)^n - 1]}{i} \qquad P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{M}\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$y = mx + c \qquad y - y_1 = m(x - x_1) \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \tan\theta$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$In\ \Delta ABC: \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad a^2 = b^2 + c^2 - 2bc.\cos A$$

$$area\ \Delta ABC = \frac{1}{2}ab.\sin C$$

$$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases} \qquad \sin 2\alpha = 2\sin\alpha.\cos\alpha$$

$$\bar{x} = \frac{\sum fx}{n} \qquad \sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

$$P(A) = \frac{n(A)}{n(S)} \qquad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$\hat{y} = a + bx \qquad b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}$$

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Learning material

MATHEMATICS P1

EXEMPLAR III

GRADE 12

MARKS: 150

TIME: 3 HOURS

This question paper consists of 9 pages and 1 Data sheet.

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INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions.

1. This question paper consists of 11 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, graphs, et cetera that you have used in determining the answers.
4. Answers only will not necessarily be awarded full marks.
5. You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.
6. If necessary, round off answers to TWO decimal places, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. An information sheet with formulae is included at the end of this question paper.
9. Number the answers correctly according to the numbering system used in this question paper.
10. Write neatly and legibly.

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QUESTION 1

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1.1 Solve for x, correct to TWO decimal places where necessary:

1.1.1 $2x^2 - 7x + 3 = 0$ (3)

1.1.2 $17x - 8 = 3x^2$ (5)

1.1.3 $(2x - 3)(4 - x) \geq 0$ (3)

1.2 Given: $4^{a+b} = 2^{b+4}$

1.2.1 Show that: $b = 4 - 2a$ (2)

1.2.2 Hence, solve for a and b simultaneously if it is further given that

$2a^2 - 3ab = -4$ (5)

1.3 Determine the value of $(0{,}04)^{0{,}5}$ without using a calculator. (2)

1.4 If $x = \sqrt{3 - 2\sqrt{2}}$ and $y = \sqrt{2} - 1$, prove that $y = x$ without using a calculator. (3)

[23]

QUESTION 2

2.1 Evaluate:

$$\sum_{n=3}^{20} (15 - 4n) \quad (4)$$

2.2 A water tank contains 216 litres of water at the end of day 1. Because of a leak, the tank loses one-sixth of the previous day's contents each day.
How many litres of water will be in the tank by the end of:

2.2.1 the 2nd day? (2)

2.2.2 the 7th day? (3)

2.3 Consider the geometric series: $2(3x - 1) + 2(3x - 1)^2 + 2(3x - 1)^3 + \ldots$

2.3.1 For which values of x is the series convergent? (3)

2.3.2 Calculate the sum to infinity of the series if $x = \frac{1}{2}$. (4)

2.4 $2 ; x ; 12 ; y ; \ldots$ are the first four terms of a quadratic sequence. If the second difference is 6, calculate the values of x and y. (5)

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[21]

QUESTION 3

Given: $h(x) = \frac{12}{x-4} + 6$ for $x > 0$

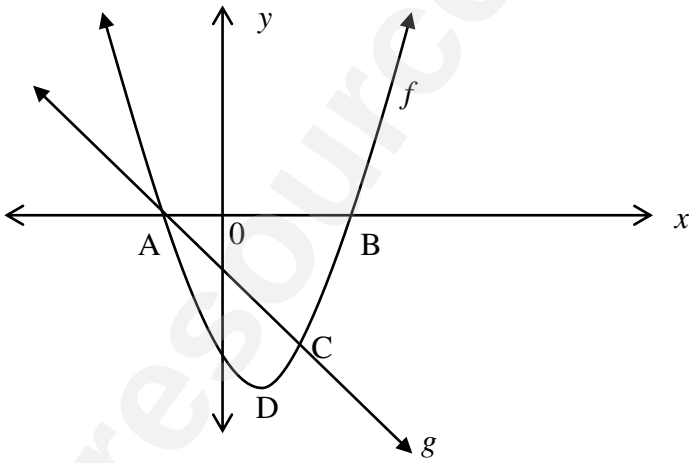
3.1 Draw a neat sketch graph of h in your ANSWER BOOK. Show all intercepts with the axes and asymptotes. (4)

3.2 Write down the equation of k if k is the reflection of h about the x-axis. (3)

[7]

QUESTION 4

4.1 Sketched below are the functions: $f(x) = 2x^2 - 6x - 20$ and $g(x) = -2x + k$.



Determine:

4.1.1 the coordinates of turning point D. (2)

4.1.2 the coordinates of A and B. (3)

4.1.3 the value of k . (2)

4.1.4 the values of p if $2x^2 - 6x + p = 0$ has no real roots. (2)

4.1.5 for which values of x is $f(x).g(x) \leq 0$. (2)

4.1.6 the value of t if $y = -2x + t$ is a tangent to f. (4)

4.2 Consider the following two functions: $p(x) = x^2 + 1$ and $r(x) = x^2 + 2x$.

4.2.1 How will you shift p to become the function r? (3)

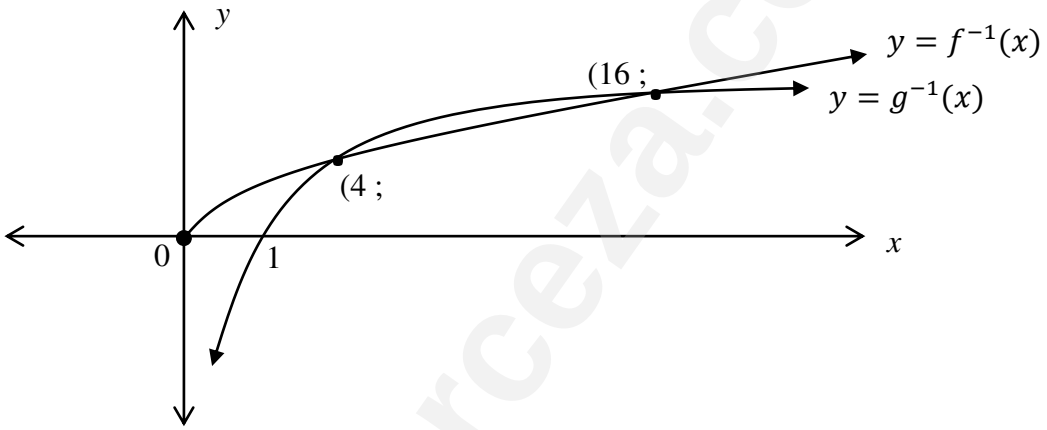
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4.2.2 Write down the range of p. (1)

[19]

QUESTION 5

The sketch below represents the inverses of $g(x) = 4^x$ and $f(x) = ax^2$; $x \geq 0$.



5.1 Write down the coordinates of ONE point through which both f and g will pass. (1)

5.2 Determine the equation of f. (3)

5.3 Calculate x if $g(x + 2) = 16$. (3)

5.4 If $h(x) = g^{-1}(x - 2)$, for which values of x will $h(x) \leq 0$? (2)

[9]

QUESTION 6

6.1 Convert an interest rate of 14% p.a. compounded monthly to an interest rate per annum compounded quarterly. (3)

6.2 Nelson deposits R3 500 into a savings account. Three years later he adds R5 700 to the account. The interest for the first two years is 7% p.a. compounded quarterly. The interest for the last three years is 8% p.a. compounded monthly. Calculate the balance in the savings account at the end of five years. (4)

[7]

QUESTION 7

Mr Daniels wants to take out a loan for a house over twenty years. He has approached two financial institutions and was offered two different options. The two options are

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shown in the table below:

VARIABLES	OPTION 1	OPTION 2
Loan amount	R950 000	R950 000
Interest rate (compounded monthly)	12%	11,8%
Repayments	Rx per month	R10 328,16 per month
Bank charges	R0	R200 per month
Commissions	R6 000	R0

7.1 Determine the total cost for Option 1. (7)

7.2 Which option is the best? Provide relevant calculations to justify your answer. (3)
[10]

QUESTION 8

8.1 Given: $f(x) = 3 - x^2$. Determine $f'(x)$ from first principles. (5)

8.2 Determine $\frac{dy}{dx}$ if:

8.2.1 $y = \frac{2}{\sqrt[3]{x}} - \pi x$ (3)

8.2.2 $xy - y = x^2 - 1$ (4)

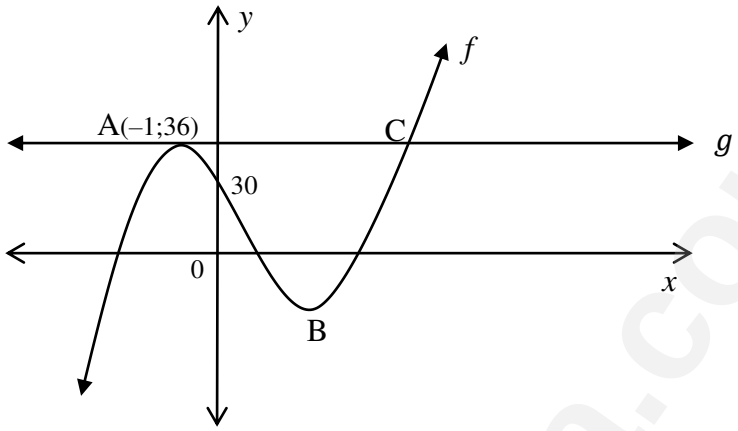
8.3 Determine the value of x where the gradient of the tangent to $f(x) = (x-5)^2$ is equal to -8. (4)
[16]

QUESTION 9

9.1 The function defined by $f(x) = x^3 + px^2 + qx + 30$ is represented by the sketch below. A $(-1; 36)$ and B are the turning points of f. g is a tangent to f at A

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which cuts f at point C.



9.1.1 Show that $p = -4$ and $q = -11$. (7)

9.1.2 Determine the coordinates of C. (3)

9.1.3 Write down the coordinates of a turning point of k, if $k(x) = f(x) - 10$. (2)

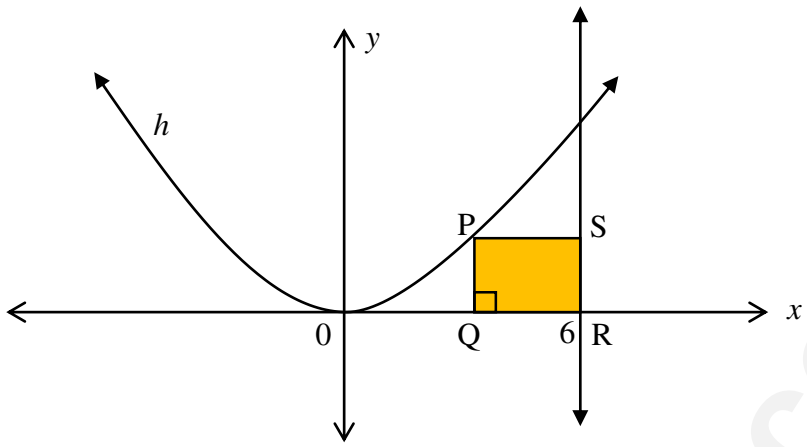
9.2 The turning points of the graph of a cubic polynomial $h(x)$ are $(2\,;\,-3)$ and $(5\,;4)$. Draw a sketch graph of the derivative function $h'(x)$, clearly showing the x-intercepts. (3)

[15]

QUESTION 10

PQRS is a rectangle with P on the curve $h(x) = x^2$ and with the x-axis and the line $x = 6$ as boundaries.

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10.1 Show that the area of rectangle PQRS can be expressed as:

$$A = 6x^2 - x^3 \ .$$ (3)

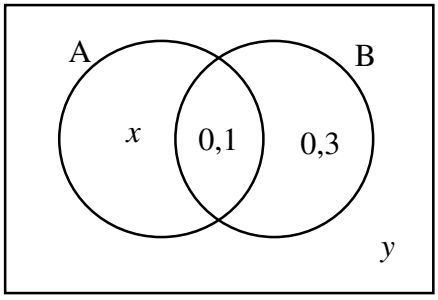
10.2 Determine the largest possible area for rectangle PQRS. Show all your calculations. (4)

[7]

QUESTION 11

11.1 The probability of getting the first answer in a quiz correct, is 60%. If the first answer is correct, the probability of getting the second answer correct, rises to 70%. However, if the first answer is wrong, the probability of getting the second answer correct, is only 40%. Determine the probability of getting the second answer correct. (4)

11.2 A and B are independent events.



Determine the values of x and y. All calculations must be shown. (5)

11.3 Ping-Pong balls are numbered 1 to 12 and placed in a hat. Every time a ball is drawn, it is placed on a rack, one next to the other.

11.3.1 How many different arrangements of the 12 balls are possible? (2)

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11.3.2 Assume that the numbers 8 and 11 must be placed next to each other, in any order. In how many different ways can the numbers then be arranged? (3)

11.3.3 The numbers 2, 5, 6 and 9 are taken from the group of balls. Two-digit numbers must be formed out of these four numbers. How many two-digit numbers can be formed? (2)

[16]

TOTAL: 150

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INFORMATION SHEET: MATHEMATICS

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$A = P(1+ni)$ $A = P(1-ni)$ $A = P(1-i)^n$ $A = P(1+i)^n$

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$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad T_n = a + (n-1)d \qquad S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$T_n = ar^{n-1} \qquad S_n = \frac{a\left(r^n - 1\right)}{r-1} \; ; \quad r \neq 1 \qquad S_\infty = \frac{a}{1-r} \; ; \; -1 < r < 1$$

$$F = \frac{x\left[(1+i)^n - 1\right]}{i} \qquad P = \frac{x[1-(1+i)^{-n}]}{i}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$y = mx + c \qquad y - y_1 = m(x - x_1) \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \qquad m = \tan\theta$$

$$(x-a)^2 + (y-b)^2 = r^2$$

In ΔABC: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $\qquad a^2 = b^2 + c^2 - 2bc.\cos A$

$$area\,\Delta ABC = \frac{1}{2}ab.\sin C$$

$$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$$

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases} \qquad \sin 2\alpha = 2\sin\alpha.\cos\alpha$$

$$\bar{x} = \frac{\sum fx}{n} \qquad \sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

$$P(A) = \frac{n(A)}{n(S)} \qquad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$\hat{y} = a + bx \qquad b = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}$$

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MATHEMATICS P1

EXEMPLAR IV

GRADE 12

MARKS: 150

TIME: 3 HOURS

This question paper consists of 11 pages.

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INSTRUCTIONS AND INFORMATION

Read the following instructions carefully before answering the questions.

1. The question paper consists of 13 questions.
2. Answer ALL the questions.
3. Show ALL calculations, diagrams, graphs, etc. that you have used in obtaining your answer.
4. Answers only will not necessarily be awarded full marks.
5. An approved scientific calculator (non-programmable, non-graphic) may be used, unless stated otherwise.
6. Where necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.
7. Diagrams are not necessarily drawn to scale.
8. A formula sheet is attached at the end of the question paper.
9. Number your answers exactly as the questions are numbered.
10. It is in your own interest to write neatly and legibly.

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QUESTION 1

1.1 Calculate the values of x:

$x^2 + 2x = 0$ (3)

1.2 Hence, determine the positive values of x for which

$x^2 \geq -2x$. (3)

1.3 Solve for x:

1.3.1 $2x^2 - 3x - 7 = 0$ (correct to TWO decimal places). (3)

1.3.2 $2^0 + 2^{x-2} + 2^{x+1} + 2^x = 53$ (5)

[14]

QUESTION 2

2.1 Given:

$k = \sqrt{(x+1)^2 - 4}$, where k is a real number.

2.1.1 Solve for x if $k = 4$. (Leave the answer in the simplest surd form). (5)

2.1.2 Write down the minimum value of k. (1)

2.2 Calculate the values of k, for which the equation

$3x^2 + 2x - k + 1 = 0$ has real roots. (4)

[10]

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QUESTION 3

Given:

$$\sum_{p=1}^{n}(101-2p)$$

3.1 Write down the first THREE terms of the series. (3)

3.2 Calculate n for which $\sum_{p=1}^{n}(101-2p)=0$ (5)

[8]

QUESTION 4

4.1 The infinite number sequence below is a combination of two sequences:

1; 1; 1; 5; 1; 9; 1; 13; ...

Determine:

4.1.1 T_{31}, the thirty-first term of the sequence. (1)

4.1.2 S_{32}, the sum of the first thirty two terms of the sequence. (4)

4.1.3 the smallest value of the even number n for which $T_n > 97$. (3)

4.2 A plant is infected with bacteria. The number pattern below represents the number of bacterial cells in the plant at the end of every hour after infection.

3; 10; 19; 30 ...

4.2.1 Determine a general formula for the nth term (T_n) of this sequence. (4)

4.2.2 Calculate the number of bacterial cells in the plant at the end of the first day (24 hours). (2)

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4.2.3 After the first day of infection, the plant undergoes treatment, which reduces the formation of new bacterial cells. The sequence below represents the number of new bacterial cells formed in the plant at the end of every day after infection.

670 ; 536 ; 428,8 ; …

The plant will perish if infected by more than 3 500 bacterial cells. If the treatment continues indefinitely, prove that the plant will survive this bacterial infection.

Show all calculations used in your answer. (3)
[17]

QUESTION 5

5.1 Convert an interest rate of 10% per annum, compounded monthly, to an annual interest rate, compounded semi-annually. (3)

5.2 Gavin purchases a house for R1 200 000. He pays a deposit of 10% of the value of the house. The bank grants him a loan for the outstanding amount, at an interest rate of 8,4% per annum compounded monthly, payable over a period of 20 years

5.2.1 Calculate the deposit Gavin pays on the house. (2)

5.2.2 Calculate Gavin's monthly repayments. (4)

5.2.3 Calculate the total amount Gavin would have paid for the house at the end of 20 years (3)

5.3 For two years, equipment worth R20 000, depreciates by 20% per annum according to the reducing-balance method. Each year thereafter, the annual depreciation is 2% less than that of the previous year. Determine the depreciated value of the equipment after 4 years. (4)
[16]

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QUESTION 6

The function $f(x)=\frac{3}{x-1}-1$ is given.

6.1 Draw a neat sketch graph of f in your answer book.
Clearly show all intercepts with the axes as well as all the asymptotes. (4)

6.2 Determine the equation(s) of the axis of symmetry of f. (2)

6.3 Determine the range of f. (1)

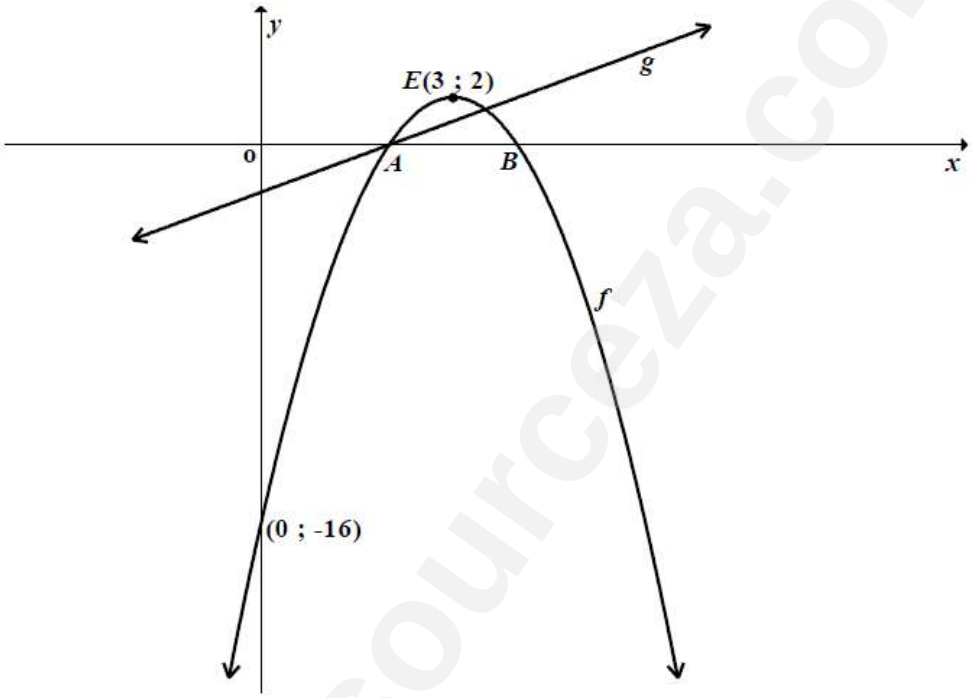
[7]

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QUESTION 7

Sketched below are the graphs of $f(x) = a(x+p)^2 + q$ and $g(x) = mx + c$, where a, p, q, m and c are real constants.
$E(3; 2)$ is the turning point and $(0; -16)$ is the y-intercept of f.
The graph of f intersects the x-axis at A and B.
C, a point on the y-axis, is symmetrical to A about the line $y = -x$.
The graph of g passes through points C and A.



7.1 Show that the equation of f can be written as $f(x) = -2x^2 + 12x - 16$ (3)

7.2 Determine the coordinates of points A, B and C. (4)

7.3 Determine the equation of g. (3)

7.4 If $h(x) = \frac{-96}{x}$, calculate the point of intersection of h and f. (4)

7.5 Write down the turning point of p, if $p(x) = -f\left(\frac{x}{2}\right)$. (2)

[16]

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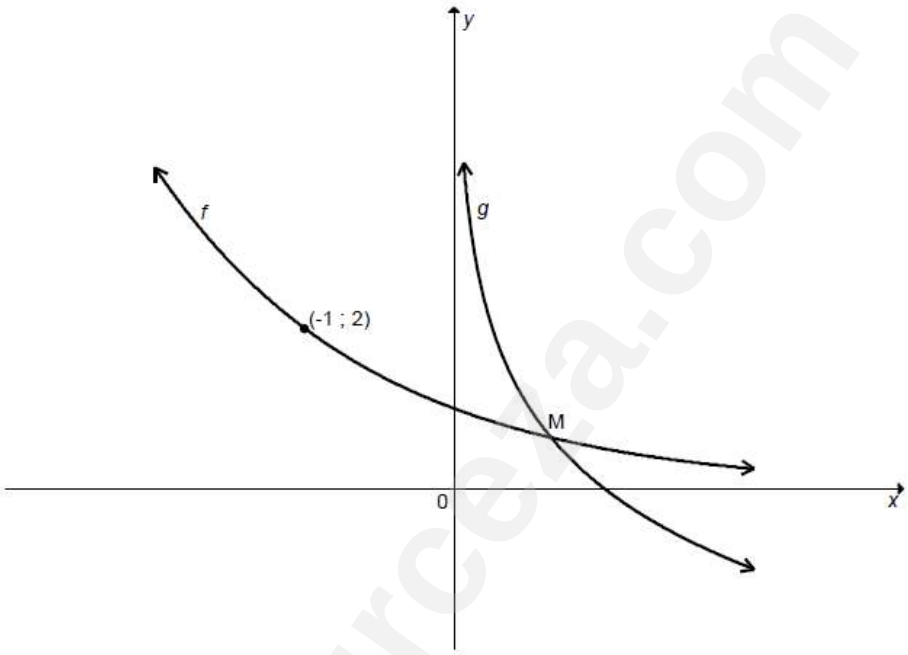
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QUESTION 8

The graphs of $f(x)=a^x$, $a>0$ and $g(x)=f^{-1}(x)$, where g is the inverse of f, are sketched below. The point $(1;-2)$ lies on f. M is the point of intersection of f and g.



8.1 Show that $a=\frac{1}{2}$. (1)

8.2 Determine the equation of g in the form $g(x) = ...$ (2)

8.3 Write down the equation of the line which divides f and g symmetrically. (1)

8.4 If M is the point of intersection of f and the line $y=x$, show that $x\log\frac{1}{2}-\log x=0$. (3)

[7]

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QUESTION 9

9.1 Given $f(x)=1-3x^2$.

9.1.1 Differentiate f from first principles (5)

9.1.2 Determine the equation of the tangent of f at $x=-2$. (3)

9.2 Evaluate:

9.2.1 $Dx\left[\left(1-x^{\frac{1}{2}}\right)\left(1-2x^{-\frac{1}{2}}\right)\right]$ (4)

9.2.2 $\frac{dy}{dp}$ if $y=\frac{p^3+1}{p+1}$ (3)

[15]

QUESTION 10

The displacement, (s) (distance from a fixed point), of a car moving towards a fixed point after t seconds, is given by

$s = t^3 - 9t^2 + 24t + 8$

10.1 Determine an expression for the velocity (v) of the car.
(rate of change of distance with respect to time) (1)

10.2 Calculate the time t for which the car was reversing. (5)

[6]

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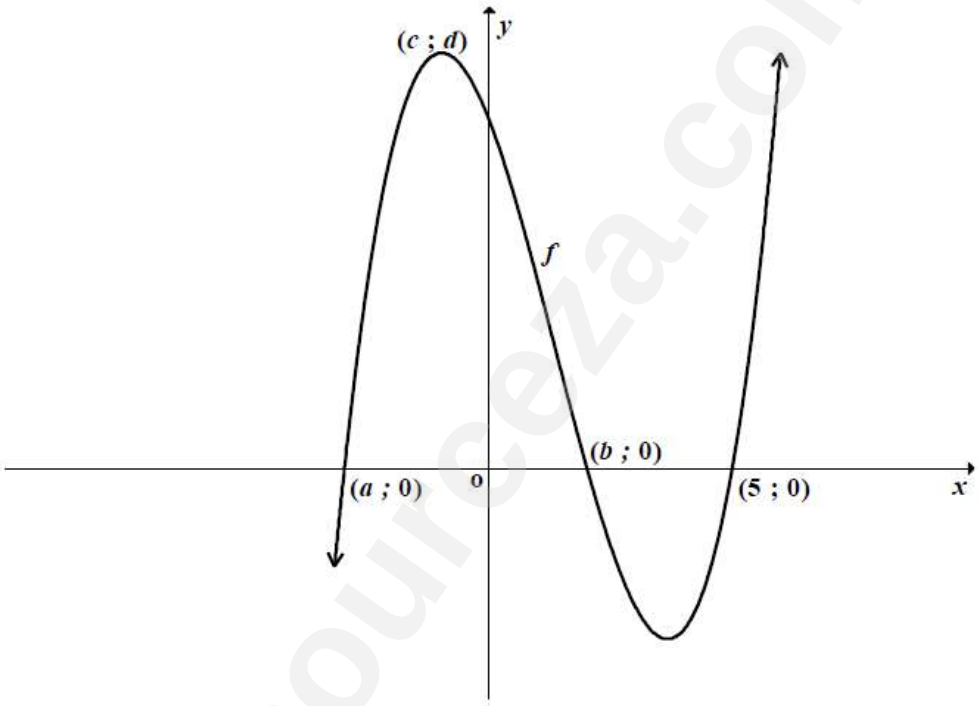
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QUESTION 11

11.1 The graph below, not drawn to scale, represents the cubic function f with equation $f(x) = x^3 - 4x^2 - 11x + 30$. The graph intersects the x-axis at $(5; 0)$.



Determine

11.1.1 the values of a and b, the x-intercepts of f. (4)

11.1.2 the values of c and d, where $(c; d)$ is a turning point of f. (4)

11.1.3 the values of x for which $f'(x) < 0$ and $f(x) \geq 0$. (2)

11.2 The turning points of a cubic graph $y = g(x)$ are $(-1; 6)$ and $(5; 4)$.
In your answer book, draw a sketch graph of y' (the derivative function of $y = g(x)$). Label only the x-intercepts clearly. (3)

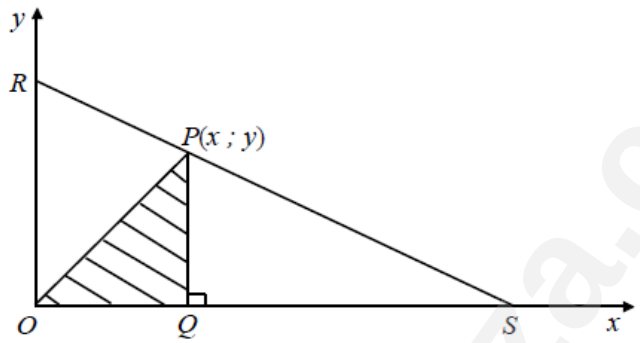
[13]

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QUESTION 12

In the diagram below, $P(x;y)$ is a point on line segment PS with equation $x+2y=18$, where $0\le x\le 18$. Q is a point on the x-axis such that ΔPOQ forms a right-angled triangle. $PQ \perp OS$



12.1 Write the y-coordinate of point P in terms of x. (1)

12.2 Calculate the coordinates of P so that the area of ΔOPQ is at its maximum. (5)
[6]

QUESTION 13

13.1 Tebo writes an Art and a Music examination. He has a 40% chance of passing the Music examination, a 60% chance of passing the Art examination and a 30% chance of passing both the Music and Art examination.

Calculate the probability that Tebo will pass the Music or Art examination. (3)

13.2 A survey was conducted asking 60 people with which hand they write and what colour hair they have. The results are summarized in the table below.

		HAND USED TO WRITE WITH		
		Right	Left	Total
HAIR COLOUR	Light	***a***	***b***	**20**
	Dark	***c***	***d***	**40**
	Total	**48**	**12**	**60**

The survey concluded that the 'hand used for writing' and 'hair colour' are independent events.
Calculate the values of a, b, and c. (5)

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13.3 The digits from 1 to 9 are used to make 5-digit codes.

13.3.1 Determine the number of 5-digit codes possible, if the digits are arranged in any order without repetition. (2)

13.3.2 Determine the number of 5-digit codes possible, if the code formed has to be an even number and the digits may not be repeated. (3)

13.3.3 Determine the number of 5-digit codes possible, if the code formed only uses even digits and repetition of digits is allowed. (2)
[15]

TOTAL: 150

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