

LEMNISCATES ACADEMY

PURE MATH 3

WORKBOOK

CAIE PAST PAPER QUESTIONS

A practice workbook covering the full syllabus.



CONCEPTS

Key pointers of what you should know.



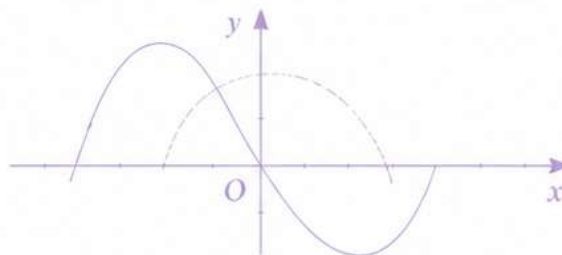
COMMON QUESTIONS

Practice questions to build confidence and understanding.



UNUSUAL QUESTIONS

Challenging questions to strengthen your skills.



Do you struggle and need a one-to-one class?

Feel free to contact:

 **+91 94261 16018**

1. Modulus

- a. Graphs
 - b. Solving inequalities (best method: Graph. Be cautious with the gradients)
-

Q. Solve the inequality $|x - a| < 2x - a$, where a is a positive constant.

[4]

LEMNISCATES ACADEMY

2. Polynomials

- Long division or Inspection method to find a quotient and remainder
- Factor and Remainder theorems to find the unknown coefficients

COMMON QUESTIONS

Type 1: W21-33 | Q1

Find the quotient and remainder when $2x^4 + 1$ is divided by $x^2 - x + 2$. [3]

Type 2: M21-32 | Q2

The polynomial $ax^3 + 5x^2 - 4x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 2)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 2.

Find the values of a and b . [5]

UNUSUAL QUESTIONS

Q1. W22-32 | Q2b

The polynomial $2x^3 - x^2 + a$, where a is a constant, is denoted by $p(x)$. It is given that $(2x + 3)$ is a factor of $p(x)$.

(a) Find the value of a . [2]

$a = 9$

(b) When a has this value, solve the inequality $p(x) < 0$. [4]

Q2. S14-32 | Q5

- (i) The polynomial $f(x)$ is of the form $(x - 2)^2 g(x)$, where $g(x)$ is another polynomial. Show that $(x - 2)$ is a factor of $f'(x)$. [2]
- (ii) The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where a and b are constants, has a factor $(x - 2)^2$. Using the factor theorem and the result of part (i), or otherwise, find the values of a and b . [5]

3. Partial Fractions

- Check if a given fraction is proper or improper fraction
 - Add an extra constant if dealing with an improper fraction
 - Must know each type (linear factors, repeated factors, quadratic factors, mixed)
-

COMMON QUESTIONS

Type 1: S21-33 | Q4

$$\text{Let } f(x) = \frac{15 - 6x}{(1 + 2x)(4 - x)}.$$

- (a) Express $f(x)$ in partial fractions. [3]

Type 2: W20-31 | Q9a

$$\text{Let } f(x) = \frac{8 + 5x + 12x^2}{(1 - x)(2 + 3x)^2}.$$

- (a) Express $f(x)$ in partial fractions. [5]

Type 3: W20-32 | Q9a

$$\text{Let } f(x) = \frac{7x + 18}{(3x + 2)(x^2 + 4)}.$$

- (a) Express $f(x)$ in partial fractions. [5]

Type 4: S18-31 | Q9i

$$\text{Let } f(x) = \frac{12x^2 + 4x - 1}{(x - 1)(3x + 2)}.$$

- (i) Express $f(x)$ in partial fractions. [5]

4. Binomial Expansion

- Convert $(a + bx)^n$ in the form of $(1 + kx)^n$ and expand (Usually paired up with Partial Fractions)
- Validity of the given binomial expansion
- Finding the coefficient of any term through the binomial expansion (similar to P1)

COMMON QUESTIONS

Type 1: S20-31 | Q2

- Expand $(2 - 3x)^{-2}$ in ascending powers of x , up to and including the term in x^2 , simplifying the coefficients. [4]
- State the set of values of x for which the expansion is valid. [1]

Type 2:

Find the coefficient of x^3 in the binomial expansion of $(3 + x)\sqrt{1 + 4x}$. [4]

UNUSUAL QUESTION

Q1. S18-31 | Q2ii

Let $f(x) = \frac{12x^2 + 4x - 1}{(x - 1)(3x + 2)}$.

- Express $f(x)$ in partial fractions. [5]

$$4 + \frac{3}{x - 1} - \frac{1}{3x + 2}$$

- Hence obtain the expansion of $f(x)$ in ascending powers of x , up to and including the term in x^2 . [5]

5. Logarithmic and Exponential Functions

- Must know all the formulae on finger tips (Multiplication/Addition, Division/Subtraction, Power rule)
- Must know when and how to introduce logs on each side and when and how to remove logs
- Common Questions (only log terms, log terms and non-log terms, only exponential terms)
- Must know how to use modulus function to solve log/exp questions
- Convert an exponential equation in the form of straight line and find unknowns (y-int and gradient)
- Inverse relationship of log and its exponential term ($\log_b x$ has its inverse as b^x)
- Graphs of logs and exponential functions (Usually paired up with numerical solutions chapter)

COMMON QUESTIONS

Type 1: M21-32 | Q1

Solve the equation $\ln(x^3 - 3) = 3 \ln x - \ln 3$. Give your answer correct to 3 significant figures. [3]

Type 2: S22-32 | Q1

Solve the equation $\ln(e^{2x} + 3) = 2x + \ln 3$, giving your answer correct to 3 decimal places. [4]

Type 3: S22-31 | Q1

Solve the equation $2(3^{2x-1}) = 4^{x+1}$, giving your answer correct to 2 decimal places. [4]

Type 4: W21-33 | Q3

Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places. [4]

Type 5: W21-31 | Q1

Solve the equation $4|5^x - 1| = 5^x$, giving your answers correct to 3 decimal places. [4]

Type 6: W20-32 | Q3

The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

- By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]
- Find the exact x -coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

UNUSUAL QUESTION

Q1. M24-32 | Q4

The positive numbers p and q are such that

$$\ln\left(\frac{p}{q}\right) = a \quad \text{and} \quad \ln(q^2 p) = b.$$

Express $\ln(p^7 q)$ in terms of a and b .

[4]

6. Trigonometry (THE MORE YOU SOLVE; THE LESS IT IS ♥ KEEP UP WITH THE PRACTICE)

- REMEMBER ALL THE FORMULAE (Pythagorean Identities, Double angle, Addition/Subtraction)
- Must know all the tricks to prove trig identities from AS level + A2 formulae and a bit of luck!
- ASTC Rule to solve for angles (P1 rules)
- Form a Harmonic Identity from something like $a\sin(x) + b\cos(x)$ (Find R and α)
- Finding minimum or maximum using Harmonic format
- Graphs of $\csc(x)$, $\sec(x)$, and $\cot(x)$ (Usually paired up with numerical solutions)
- Manipulations using double angle formulae to simplify or convert in a specific format

COMMON QUESTIONS:

Type 1: S23-32 | Q4

Solve the equation $2\cos x - \cos \frac{1}{2}x = 1$ for $0 \leq x \leq 2\pi$. [5]

Type 2: S18-33 | Q5

(i) By first expanding $(\cos^2 x + \sin^2 x)^3$, or otherwise, show that

$$\cos^6 x + \sin^6 x = 1 - \frac{3}{4} \sin^2 2x. \quad [4]$$

(ii) Hence solve the equation

$$\cos^6 x + \sin^6 x = \frac{2}{3},$$

for $0^\circ < x < 180^\circ$. [4]

Type 3: W21-31 | Q2

(a) Express $5\sin x - 3\cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the exact value of R and give α correct to 2 decimal places. [3]

(b) Hence state the greatest and least possible values of $(5\sin x - 3\cos x)^2$. [2]

Type 4: M22-32 | Q5

The angles α and β lie between 0° and 180° and are such that

$$\tan(\alpha + \beta) = 2 \quad \text{and} \quad \tan \alpha = 3 \tan \beta.$$

Find the possible values of α and β . [6]

Type 5: M22-32 | Q7a

By sketching a suitable pair of graphs, show that the equation $4 - x^2 = \sec \frac{1}{2}x$ has exactly one root in the interval $0 \leq x < \pi$. [2]

UNUSUAL QUESTIONS

Q1. W14-31 | Q8ii | Q8iii

(i) By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta. \quad [4]$$

(ii) Show that, after making the substitution $x = \frac{2 \sin \theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \frac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

7. Differentiation (Should master the trigonometry, algebraic, logarithmic and exponential manipulations)

- a. Must know differentiation of e^x , $\log(x)$, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\arctan(x)$
- b. $\sec(x)$, $\csc(x)$, $\cot(x)$ can be differentiated using the quotient rule (very important for integration)
- c. Product rule and Quotient rule should be on your finger tips
- d. Parametric and Implicit Differentiation (Equation of tangent and normal, stationary points, e.t.c..)

COMMON QUESTIONS

Type 1: W19-32 | Q2

The curve with equation $y = \frac{e^{-2x}}{1-x^2}$ has a stationary point in the interval $-1 < x < 1$. Find $\frac{dy}{dx}$ and hence find the x -coordinate of this stationary point, giving the answer correct to 3 decimal places. [5]

Type 2: M17-32 | Q5

The curve with equation $y = e^{-ax} \tan x$, where a is a positive constant, has only one point in the interval $0 < x < \frac{1}{2}\pi$ at which the tangent is parallel to the x -axis. Find the value of a and state the exact value of the x -coordinate of this point. [7]

Type 3: M16-32 | Q6

A curve has equation

$$\sin y \ln x = x - 2 \sin y,$$

for $-\frac{1}{2}\pi \leq y \leq \frac{1}{2}\pi$.

(i) Find $\frac{dy}{dx}$ in terms of x and y . [5]

(ii) Hence find the exact x -coordinate of the point on the curve at which the tangent is parallel to the x -axis. [3]

Type 4: S15-31 | Q4

The equation of a curve is

$$y = 3 \cos 2x + 7 \sin x + 2.$$

Find the x -coordinates of the stationary points in the interval $0 \leq x \leq \pi$. Give each answer correct to 3 significant figures. [7]

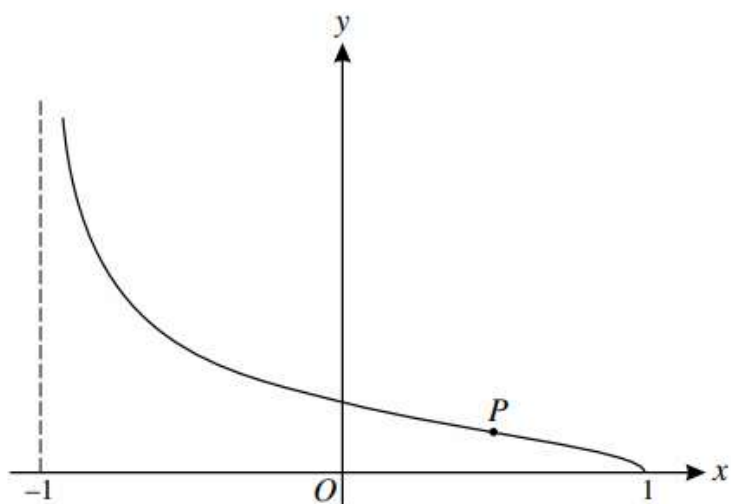
UNUSUAL QUESTIONS

Q1. M19-32 | Q5

The variables x and y satisfy the relation $\sin y = \tan x$, where $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$. Show that

$$\frac{dy}{dx} = \frac{1}{\cos x \sqrt{\cos 2x}}.$$

[5]



The diagram shows the curve $y = \sqrt{\left(\frac{1-x}{1+x}\right)}$.

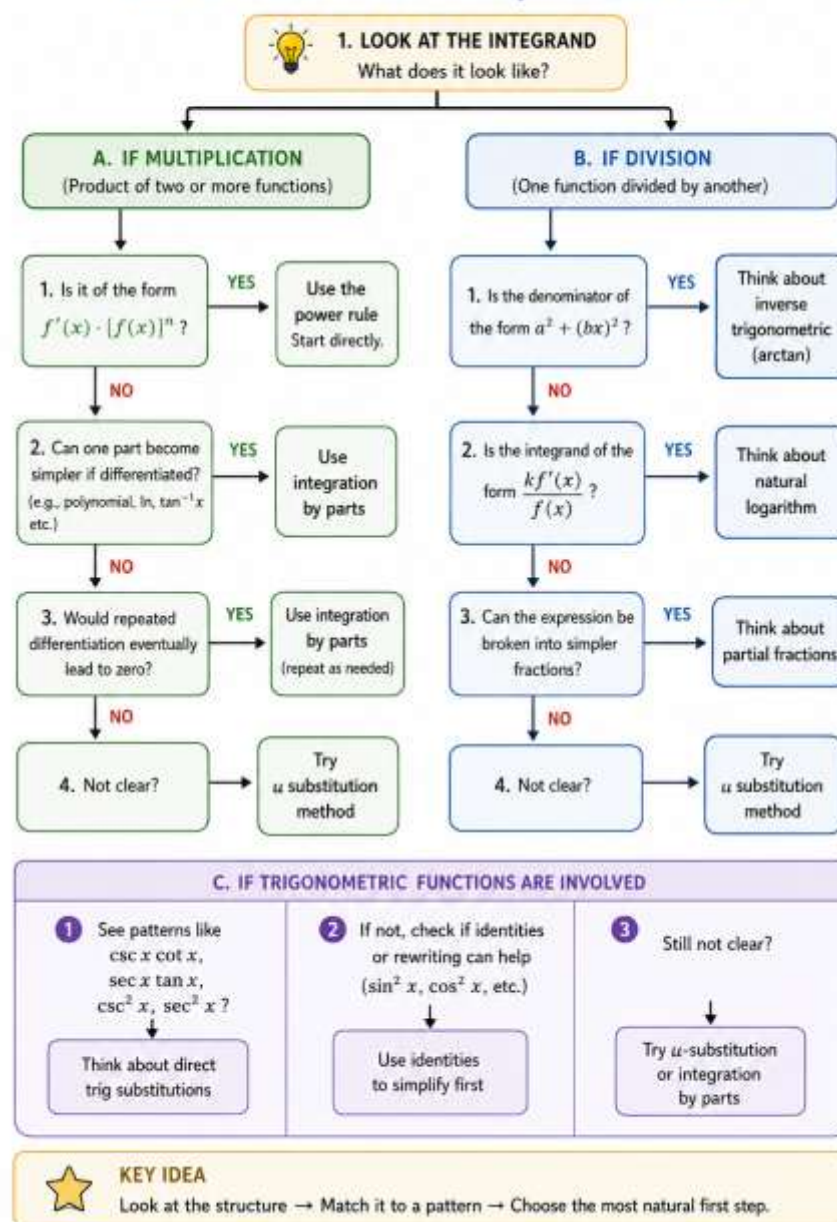
- (i) By first differentiating $\frac{1-x}{1+x}$, obtain an expression for $\frac{dy}{dx}$ in terms of x . Hence show that the gradient of the normal to the curve at the point (x, y) is $(1+x)\sqrt{(1-x^2)}$. [5]

8. Integration (Should master manipulations of algebraic, trigonometry, exponential, logarithmic functions)

- Must know how to integrate e^{ax+b} , $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$
- Integration of $\ln(ax+b)$ using integration by parts
- $\int \frac{1}{a^2+(bx)^2} dx = \frac{1}{ba} \tan^{-1}\left(\frac{bx}{a}\right) + c$
- $k \int \frac{f'(x)}{f(x)} dx = k \ln|f(x)| + c$ (Integration of $\tan(ax+b)$ and $\cot(ax+b)$ are trig examples)
- $\int f'(x) \cdot f(x)^n dx = \frac{f(x)^{n+1}}{n+1} + c$ (Example: $\int \cos(x) \sin^2(x) dx$)
- Integration of $\sin^2(ax)$ and $\cos^2(ax)$ can be done by manipulation of $\cos(2x)$
- Integration of $\sin^2(ax) \cos^2(ax)$ can be done by manipulation of $\sin(2x)$
- Integrating a polynomial fraction $\rightarrow$ Partial Fraction or Remainder-Quotient form
- I method for by parts whenever the integrand is repeated in the method (Example: $\int e^x \sin(x) dx$)
- U-substitution method is always evident in the question or it can be your last resort if nothing works
- Area and Volume Questions (P1 concept with P3 Integration)

INTEGRATION ROADMAP

How to Decide Which Technique to Start With



COMMON QUESTIONS

Type 1: S20-31 | Q5b

- (a) Find the quotient and remainder when $2x^3 - x^2 + 6x + 3$ is divided by $x^2 + 3$. [3]

Quotient: $2x - 1$

Remainder: 6

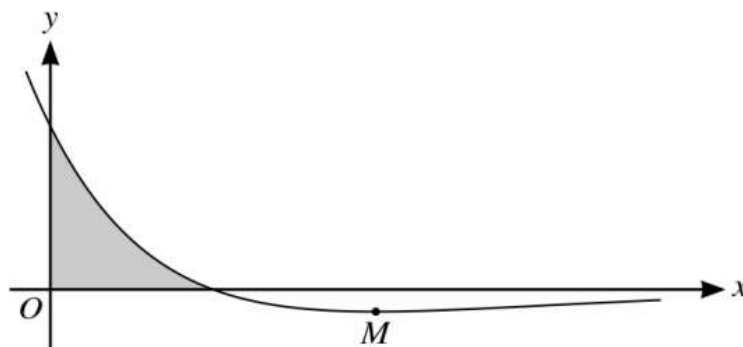
- (b) Using your answer to part (a), find the exact value of $\int_1^3 \frac{2x^3 - x^2 + 6x + 3}{x^2 + 3} dx$. [5]

Type 2: S20-31 | Q7b

Let $f(x) = \frac{\cos x}{1 + \sin x}$.

- (b) Find $\int_{\frac{1}{6}\pi}^{\frac{1}{2}\pi} f(x) dx$. Give your answer in a simplified exact form. [4]

Type 3: W20-31 | Q10b



The diagram shows the curve $y = (2 - x)e^{-\frac{1}{2}x}$, and its minimum point M .

- (b) Find the area of the shaded region bounded by the curve and the axes. Give your answer in terms of e . [5]

Type 4: W20-32 | Q9b

Let $f(x) = \frac{7x + 18}{(3x + 2)(x^2 + 4)}$.

- Hence find the exact value of $\int_0^2 f(x) dx$. [6]

Type 5: S18-31 | Q5

Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)} dx.$

(i) Using the substitution $x = \cos^2 \theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2 \cos^2 \theta d\theta.$ [4]

(ii) Hence find the exact value of $I.$ [4]

Type 6: S15-31 | Q5

(a) Find $\int (4 + \tan^2 2x) dx.$ [3]

(b) Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin(x + \frac{1}{6}\pi)}{\sin x} dx.$ [5]

Type 7: M26-32 | Q3

Find the exact value of $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} 3 \sin x \sin 2x dx.$

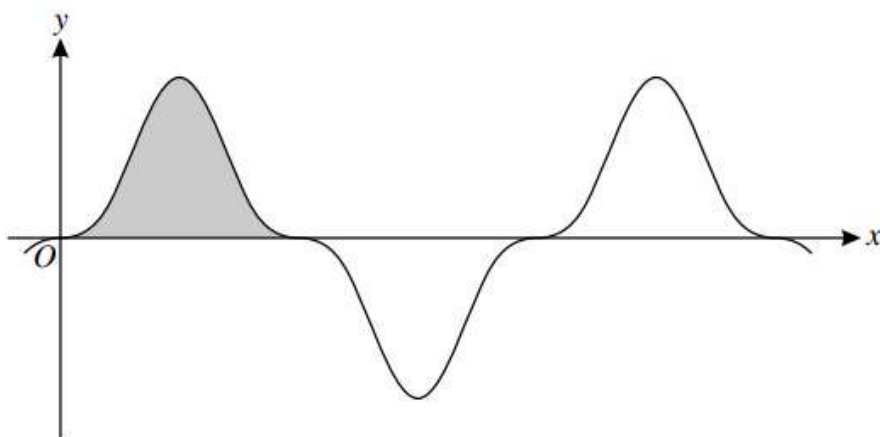
Give your answer in the form $p\sqrt{3} + q\sqrt{2}$, where p and q are rational. [4]

Type 8: M17-32 | Q10ii

Show that $\int \ln x dx = x \ln x - x + c$, where c is a constant. [1]

UNUSUAL QUESTIONS

Q1. W12-33 | Q7ii



The diagram shows part of the curve $y = \sin^3 2x \cos^3 2x$. The shaded region shown is bounded by the curve and the x -axis and its exact area is denoted by A .

- (i) Use the substitution $u = \sin 2x$ in a suitable integral to find the value of A . [6]

$$A = \frac{1}{24} \text{ for limits of } 0 \text{ to } \frac{\pi}{4}$$

- (ii) Given that $\int_0^{k\pi} |\sin^3 2x \cos^3 2x| dx = 40A$, find the value of the constant k . [2]

Q2. W11-33 | Q10ii

- (i) Use the substitution $u = \tan x$ to show that, for $n \neq -1$,

$$\int_0^{\frac{1}{4}\pi} (\tan^{n+2} x + \tan^n x) \, dx = \frac{1}{n+1}. \quad [4]$$

- (ii) Hence find the exact value of

(a) $\int_0^{\frac{1}{4}\pi} (\sec^4 x - \sec^2 x) \, dx,$ [3]

(b) $\int_0^{\frac{1}{4}\pi} (\tan^9 x + 5 \tan^7 x + 5 \tan^5 x + \tan^3 x) \, dx.$ [3]

9. Differential Equations (One chapter enough to revise most of the P3 chapters)

- You must know all the tricks of Integration to not struggle with this topic
- You should be solid at manipulating logs, exponentiation, trig and algebraic terms
- Create a differential equation using connected rates of change (Using P1 Logic)
- Use the variable separable method to solve the differential equations
- Use the particular or general solutions to conclude any modelling part of the problem

COMMON QUESTIONS

Type 1: W20-31 | Q8

The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x \frac{dy}{dx} = (1 - 2x^2)y,$$

for $x > 0$. It is given that $y = 1$ when $x = 1$.

Solve the differential equation, obtaining an expression for y in terms of x .

[6]

Type 2: W20-32 | Q7

The variables x and t satisfy the differential equation

$$e^{3t} \frac{dx}{dt} = \cos^2 2x,$$

for $t \geq 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t .

[7]

$$x = \frac{1}{2} \tan^{-1} \left(\frac{2}{3} (1 - e^{-3t}) \right)$$

(b) State what happens to the value of x when t tends to infinity.

[1]

Type 3: S19-32 | Q7

The variables x and y satisfy the differential equation $\frac{dy}{dx} = xe^{x+y}$. It is given that $y = 0$ when $x = 0$.

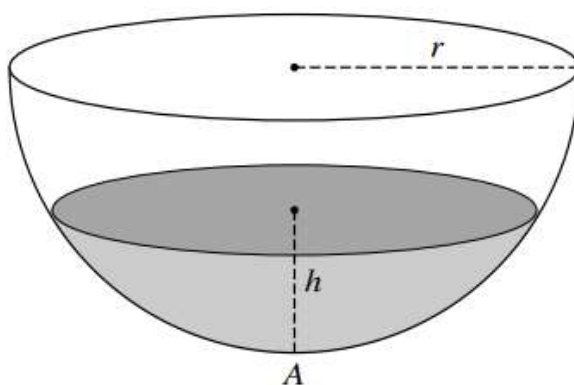
(i) Solve the differential equation, obtaining y in terms of x .

[7]

$$y = -\ln(1 - x) - x$$

(ii) Explain why x can only take values that are less than 1.

[1]



A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r , as shown in the diagram. The depth of water at time t is h . At time $t = 0$ the tank is full and the depth of the water is r . At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$.

The volume of water in the tank is V when the depth is h . It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant.

[4]

(b) Solve the differential equation and obtain an expression for t in terms of h and r .

[8]

UNUSUAL QUESTIONS

There are no as such unusual questions for this topic, you may find it difficult only if you are not able to apply any required manipulations in given time. So, it is heavily recommended to practice this topic by solving as many topicals as it doesn't just boost your confidence of this topic but also other topics like Algebra, Trigonometry, Logarithms, Exponentials, and Integration. You can post your doubts when you are stuck on any question via WhatsApp groups.

Still if you want to see one good question, give this a shot.... (Question in the next page)

Q1. W23-32 | Q11a

The variables x and y satisfy the differential equation

$$x^2 \frac{dy}{dx} + y^2 + y = 0.$$

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for y in terms of x .

[8]

10. Numerical Solutions to Equations

- Using the graphs to prove that a root exists for a given domain
- Use change of sign method to prove that root lies between two values
- Rearrange a given equation into the iterative format that question demands
- Must know the idea of iterations may fail to converge sometimes
- Use of Calculus and Circular Measures to form an iterative equation
- Find an approximate value of the solution of the given equation through an iterative process

COMMON QUESTIONS

Type 1: S20-33 | Q6

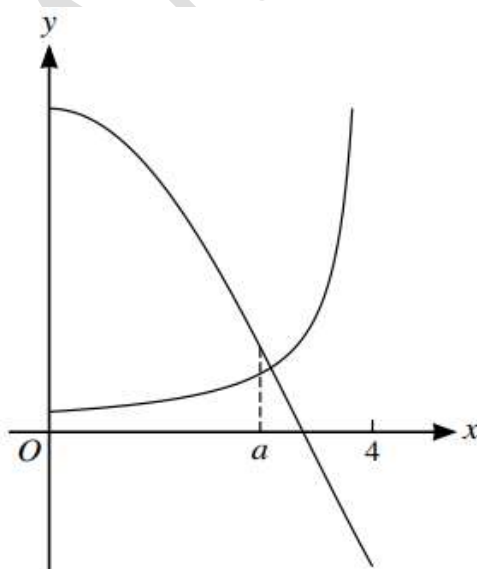
- (a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]
- (b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part (a). [2]

- (c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Type 2: S19-31 | Q7



The diagram shows the curves $y = 4 \cos \frac{1}{2}x$ and $y = \frac{1}{4-x}$, for $0 \leq x < 4$. When $x = a$, the tangents to the curves are perpendicular.

- (i) Show that $a = 4 - \sqrt{2 \sin \frac{1}{2}a}$. [4]

(ii) Verify by calculation that a lies between 2 and 3.

[2]

(iii) Use an iterative formula based on the equation in part (i) to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

[3]

UNUSUAL QUESTIONS:

Just like differential equations, the topic itself doesn't have any difficult questions. Keep practicing and use WhatsApp groups to solve your doubts if you get stuck anywhere.

11. Vectors (Overrated topic – Nothing is hard about vectors if you draw a quick sketch of what's given)

- a. Vector Notations: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $i + j + k$ (Choose one that you prefer and change it only if question asks)
- b. Addition and Subtraction of Vectors
- c. Multiplication of a vector by a scalar value
- d. Dot product (Scalar Product) of two vectors
- e. Magnitude of a vector
- f. Angle between any two vectors (direction vectors for angle between two lines)
- g. Midpoint of a vector $\overrightarrow{AB}$ is $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$
- h. Use diagrams to find the position or displacement vectors
- i. Using ratios of the vector to find the unknown position vector (B lies on AB such that $AB = 3BC$)
- j. Formation of vector equation of a line using a point on the line and direction vector of that line
- k. Formation of vector equation of a line using two points (A and B) on the line (direction vector = $\overrightarrow{AB}$)
- l. For two perpendicular vectors, the dot product of those vectors is always zero
- m. Parallel lines: Direction vector should be parallel (Should have a common scale factor)
- n. Intersecting lines: Equate two coordinates to find λ and μ and check if last coordinate is same
- o. Skew lines: If lines are neither parallel nor intersecting, they must be skew
- p. Finding the foot of the perpendicular and use it for other applications as asked in the question

COMMON QUESTIONS

Type 1: S20-31 | Q9

With respect to the origin O , the vertices of a triangle ABC have position vectors

$$\overrightarrow{OA} = 2\mathbf{i} + 5\mathbf{k}, \quad \overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

- (a) Using a scalar product, show that angle ABC is a right angle. [3]
- (b) Show that triangle ABC is isosceles. [2]
- (c) Find the exact length of the perpendicular from O to the line through B and C . [4]

Type 2:

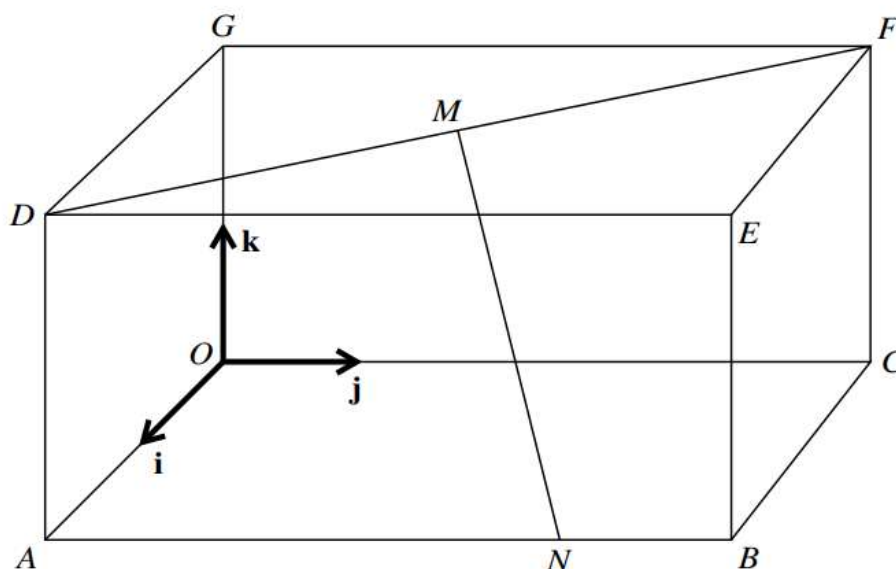
Relative to the origin O , the points A , B and D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point C is such that $ABCD$ is a parallelogram.

- (a) Find the position vector of C and verify that the parallelogram is not a rhombus. [5]
- (b) Find angle BAD , giving your answer in degrees. [3]
- (c) Find the area of the parallelogram correct to 3 significant figures. [2]

Type 3: S22-31 | Q9



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA , OC and OG respectively. The point M is the midpoint of DF . The point N on AB is such that $AN = 3NB$.

- Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]
- Find a vector equation for the line through M and N . [2]
- Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$. [4]

Type 4: M21-32 | Q7

Two lines have equations $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

- Show that the lines are skew. [5]
- Find the acute angle between the directions of the two lines. [3]

UNUSUAL QUESTIONS

Even though I mentioned that the topic is easy, it will still require you to practice and get a hang of different types of questions. They keep asking same question with a little twists and turns. Hence, I highly recommend you to practice this topic until you start understanding the patterns. Use the WhatsApp groups to solve your doubts, as usual.

Check next page to see some questions in which if you didn't draw a quick sketch, you will be clueless...

Q1. M26-32 | Q11b

The points A and B have position vectors $\overrightarrow{OA} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OB} = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ relative to the origin, O .

The point C has position vector $\overrightarrow{OC} = 3\mathbf{i} + p\mathbf{j} + q\mathbf{k}$, where p and q are constants.

Given that

- $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$ and
- angle $AOC = \text{angle } COB$,

find the values of p and q .

[7]

Q2. S23-33 | Q9b

The lines l and m have equations

$$l: \mathbf{r} = a\mathbf{i} + 3\mathbf{j} + b\mathbf{k} + \lambda(c\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}),$$

$$m: \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Relative to the origin O , the position vector of the point P is $4\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$.

- (a) Given that l is perpendicular to m and that P lies on l , find the values of the constants a , b and c . [4]

$$a = -6, b = 6, c = -5$$

- (b) The perpendicular from P meets line m at Q . The point R lies on PQ extended, with $PQ : QR = 2 : 3$.

Find the position vector of R .

[6]

12. Complex Numbers (Argand Diagrams can be a bit hard to interpret sometimes, rest is easy)

- a. Modulus and Argument of any given complex number
- b. Add, Subtract, Multiply and Divide complex numbers
- c. If a denominator has a complex number, it is not in its simplified format, you must divide
- d. Solve a cubic or quartic equation where one complex root is given
- e. Convert a cartesian form of complex number to its polar form
- f. Know that $Re^{i\theta} = R(\cos \theta + i \sin \theta)$ and $Re^{-i\theta} = R(\cos \theta - i \sin \theta)$
- g. Geometrical understanding of the complex conjugates (reflection of each other across the real axis)
- h. Know that if $z_1 = R_1e^{i\theta_1}$ and $z_2 = R_2e^{i\theta_2}$, then $z_1 \cdot z_2 = R_1R_2e^{i(\theta_1+\theta_2)}$
- i. Know that if $z_1 = R_1e^{i\theta_1}$ and $z_2 = R_2e^{i\theta_2}$, then $\frac{z_1}{z_2} = \left(\frac{R_1}{R_2}\right)e^{i(\theta_1-\theta_2)}$
- j. Geometrical understanding of points h and i
- k. Finding the square root of a complex number and its visualization on Argand diagram
- l. Argand Diagrams for: Circles, Half Lines, Perpendicular Bisectors
- m. Minimum and Maximum values of argument and modulus of a complex number in the region

COMMON QUESTIONS

Type 1: W21-31 | Q10abc

The complex number $1 + 2i$ is denoted by u . The polynomial $2x^3 + ax^2 + 4x + b$, where a and b are real constants, is denoted by $p(x)$. It is given that u is a root of the equation $p(x) = 0$.

- (a) Find the values of a and b . [4]
- (b) State a second complex root of this equation. [1]
- (c) Find the real factors of $p(x)$. [2]

Type 2: S21-31 | Q5

- (a) Solve the equation $z^2 - 2piz - q = 0$, where p and q are real constants. [2]

In an Argand diagram with origin O , the roots of this equation are represented by the distinct points A and B .

- (b) Given that A and B lie on the imaginary axis, find a relation between p and q . [2]
- (c) Given instead that triangle OAB is equilateral, express q in terms of p . [3]

Type 3: M21-32 | Q8

The complex numbers u and v are defined by $u = -4 + 2i$ and $v = 3 + i$.

(a) Find $\frac{u}{v}$ in the form $x + iy$, where x and y are real. [3]

(b) Hence express $\frac{u}{v}$ in the form $re^{i\theta}$, where r and θ are exact. [2]

In an Argand diagram, with origin O , the points A , B and C represent the complex numbers u , v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC . [2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

Type 4: S21-32 | Q5

The complex number u is given by $u = 10 - 4\sqrt{6}i$.

Find the two square roots of u , giving your answers in the form $a + ib$, where a and b are real and exact. [5]

Type 5: W25-31 | Q6

It is given that $z_1 = 3e^{\frac{1}{4}\pi i}$, $z_2 = \frac{3}{2}e^{\frac{1}{6}\pi i}$ and $\omega = 2e^{\frac{1}{2}\pi i}$.

(a) State the values of ωz_1 and ωz_2 . Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(b) On a sketch of an Argand diagram with origin O , show the points A , B , C and D representing the complex numbers z_1 , z_2 , ωz_1 and ωz_2 respectively. [2]

(c) State the geometric effects of multiplying z_1 and z_2 by ω . [2]

Type 6: W24-32 | Q5a

The complex number u is given by

$$u = \frac{(\cos \frac{1}{7}\pi + i \sin \frac{1}{7}\pi)^4}{\cos \frac{1}{7}\pi - i \sin \frac{1}{7}\pi}.$$

Find the exact value of $\arg u$. [2]

Type 7: M23-32 | Q4

Solve the equation

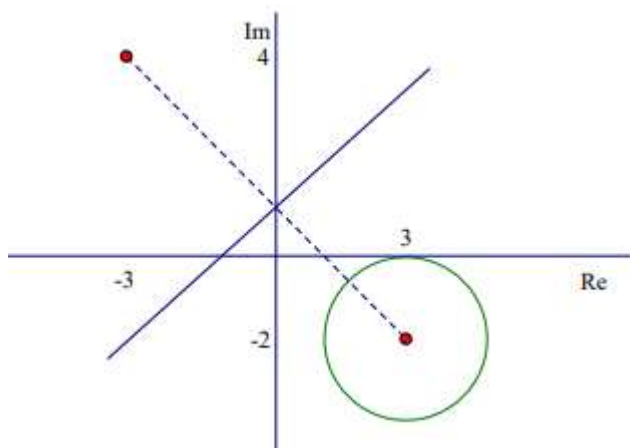
$$\frac{5z}{1+2i} - zz^* + 30 + 10i = 0,$$

giving your answers in the form $x + iy$, where x and y are real. [5]

Type 8: S24-31 | Q7

- (a) On a single Argand diagram sketch the loci given by the equations $|z - 3 + 2i| = 2$ and $|w - 3 + 2i| = |w + 3 - 4i|$ where z and w are complex numbers.

[4]



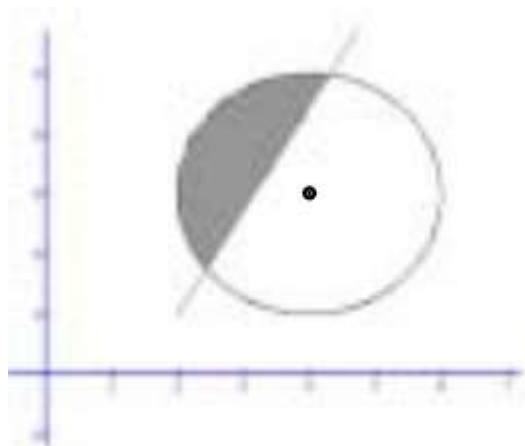
- (b) Hence find the least value of $|z - w|$ for points on these loci. Give your answer in an exact form.

[2]

Type 9: S24-33 | Q6

- (a) On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 4 - 3i| \leq 2$ and $\arg(z - 2 - i) \geq \frac{1}{3}\pi$.

[5]



- (b) Calculate the greatest value of $\arg z$ for points in this region.

[2]

Type 10: W24-33 | Q1

The complex number z satisfies $|z| = 2$ and $0 \leq \arg z \leq \frac{1}{4}\pi$.

- (a) On the Argand diagram below, sketch the locus of the points representing z .

[2]

- (b) On the **same diagram**, sketch the locus of the points representing z^2 .

[2]

UNUSUAL QUESTIONS

Q1. S15-31 | Q8ii | Q8iii

The complex number w is defined by $w = \frac{22 + 4i}{(2 - i)^2}$.

- (i) Without using a calculator, show that $w = 2 + 4i$. [3]
- (ii) It is given that p is a real number such that $\frac{1}{4}\pi \leq \arg(w + p) \leq \frac{3}{4}\pi$. Find the set of possible values of p . [3]
- (iii) The complex conjugate of w is denoted by w^* . The complex numbers w and w^* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through S , T and the origin. [3]

Q2. S17-33 | Q11b

The complex numbers u and v are given by $u = 1 + (2\sqrt{3})i$ and $v = 3 + 2i$. In an Argand diagram, u and v are represented by the points A and B . A third point C lies in the first quadrant and is such that $BC = 2AB$ and angle $ABC = 90^\circ$. Find the complex number z represented by C , giving your answer in the form $x + iy$, where x and y are real and exact. [4]

HOW TO ATTEMPT ANY QUESTION?

- Always read the question properly, spend more time understanding the question.
- Note down or underline the given data so you know what quantities are already provided.
- Now try to think what exactly is question asking me to find!
- Then use the underlined data and think about how you can utilise them to find what's asked.
- The underlined data are like the pieces of puzzle.
- The thing you are asked to find is like a completed puzzle.
- You just need to utilize your pieces of puzzle (data) and complete it!
- If you get more than one answer, always ask if any values need to be rejected.

HOW NOT TO ATTEMPT ANY QUESTION?

- Not reading the question properly because you think you are running out of time!
- Not underlining the given data and then forgetting what all things are already given to you.
- Not having confidence to solve a problem just because YOU THINK it is lengthy.
- Not asking yourself when you copy down the signs from previous step if you have copied it correctly.
- Not starting out the solution just because YOU THINK you don't know what to do!

HOW TO GET OUT IF STUCK ON A TRICKY QUESTION IN EXAMS?

- When you are stuck, you draw a quick diagram to visualize what's really going on!
- 99% of the times you will get an idea of what to actually do.
- Just scribble your ideas on rough page for difficult question can lead you to the solutions.

HOW CAN I GET THE BEST GRADE?

- It is purely based on your hard work.
- First understand the notes completely, and get your doubts solved.
- To master a topic, solve topicals until you become confident.
- Save the screenshots of the difficult questions according to you in this process.
- Repeat this for each topic, it will boost your confidence to next level.
- Regularly keep revising the older topic questions orally meanwhile you move to next topics.
- Keep helping your peers in your class or on online platforms.
- To master a concept, you explain it to someone. The more you do this, the better you become!
- Use at least 5 years of full-length past papers to learn time management skills.
- Don't spend more than two minutes on thinking how can I solve a question, skip that and come back later.
- **YOU DON'T WANT TO LOSE REMAINING MARKS JUST BECAUSE YOU WERE STUCK ON ONE QUESTION.**
- Use light handed pencil diagrams as in case if you have to erase, it doesn't look ugly later.
- **SHOW ALL YOUR WORKING, DON'T SKIP OBVIOUS STEPS.**
- Always give answer in exact format whenever possible unless asked in a specified format.

Remember, everyone finds their paper hard when they just look at the question paper for first time. But as you start understanding the questions, gather the data and then use them to find what is asked, you will realise that I can surely do this. So don't lose hope or think your question paper is hard, it is just a paper that a good student from a future batch will say, oh it was so easy!

----- **GOOD LUCK!** -----