

INTEGRAL CALCULUS

Note: <https://www.integral-calculator.com/> is a very useful site to evaluate integrals. This site can also display all the steps at arriving at a solution if you need more help.

Integration can be viewed in two ways:

- as the reverse process of differentiation called *anti-differentiation* or the *indefinite integral*
- as the mathematical process to find the exact area bound by a curve and the x-axis of the *definite integral*

Anti-differentiation

Anti-differentiation is the reverse process of differentiation.

- The general formula for differentiation if $f(x) = x^n$ is: $f'(x) = nx^{n-1}$
- Therefore the general formula for anti-differentiation if $f'(x) = x^n$ is: $f(x) = \frac{x^{n+1}}{n+1} + c$
- Formal definition: If $f(x)$ is a continuous function and $F(x)$ is the function whose derivative is $f(x)$, then $\int f(x)dx = F(x) + c$ where c is an arbitrary constant
- The formula for a Riemann sum is: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i$ and is an approximation of the integral

$$\int_a^b f(x) dx.$$

The Fundamental Theorem of Calculus

Although integration and differentiation have such apparently different origins, since integration arises from the process of finding the area under a curve, while differentiation comes from finding the slope of a curve, the two operations are in fact closely connected, as each is the inverse process of the other, i.e. integration is essentially anti-differentiation. The result by which this relationship is established is called the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus: If the function f is continuous at the point x and is integrable from a to x , then:

$$\frac{d}{dx} \int_a^x f(u) du = f(x)$$

If $f(x)$ is continuous on the interval $a \leq x \leq b$ and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

Note: Only separate terms can be integrated at this stage, not product or quotients, so simplification may be necessary in order to create separate terms.

Useful standard integrals

$\int ax^n dx$	$\frac{ax^{n+1}}{n+1} + c; \quad n \neq -1$
$\int \sin x dx$	$-\cos x + c$
$\int \cos x dx$	$\sin x + c$
$\int \sec^2 x dx$	$\tan x + c$
$\int \operatorname{cosec}^2 x dx$	$-\cot x + c$
$\int \sec x \tan x dx$	$\sec x + c$
$\int \operatorname{cosec} x \cot x dx$	$-\operatorname{cosec} x + c$
Not required by the curriculum, but useful for tertiary study.	
$\int \frac{1}{\sqrt{1-x^2}} dx$	$\arcsin x + c$
$\int \frac{-1}{\sqrt{1-x^2}} dx$	$\arccos x + c$
$\int \frac{1}{1+x^2} dx$	$\arctan x + c$

The Definite Integral

- The general notation for the area under the curve, $\int_a^b f(x) dx$, is known as the definite integral
- The definite integral is used to find the exact area bound by a curve and the x-axis.
- $\int f(x) dx$ is known as the indefinite integral with respect to x .
- The symbol $\int$ is the 'long' S and has links to the '*sum of*' concept
- Anti-differentiation is a much quicker method of determining the integral of a function.
 - when determining the indefinite integral a constant must be added to the end
 - when working with the definite integral the constant will cancel out in the subtraction step and thus does not need to be considered.

Example 2a:

Determine the indefinite integral of the function $f(x) = x^2 + 3$.

Example 2b:

Evaluate the following definite integral: $\int_{-2}^3 (x^2 + 3) dx$

Rules of the Definite Integral

- $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$
- $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
- $\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx$
- $\int \left(\frac{1}{x}\right) dx = \ln x + c$

Other general rules:

- all signs can be disregarded. A negative sign just indicates that the area found is below the x-axis.
- area broken up by intercepts with the x-axis must be calculated in parts and added up to determine the total area.

Note: $\int \left(\frac{1}{x}\right) dx = \int x^{-1} dx = \frac{x^0}{0} \dots$

which creates a major mathematical problem, since there is a *defined* area between this curve and the x-axis. This is an exception to the rule and as is defined as follows:

$$\int \left(\frac{1}{x}\right) dx = \ln x + c$$

Methods of Integration

- 1) **Integration by Inspection** – isolate each x-term in order to integrate pieces *individually*
 - basic form of integration as described above
 - no complex manipulation of terms or substitution required
- 2) **Integration of Power Functions: when a function (usually raised to a power) and its derivative appear in the integral.**

The Switch-Around Method

- not officially recognized as a separate method of integration, but highly useful
- involves substitution

- 3) **Integration of e^x and 'hyperbolic' functions leading to $\ln x$**
- 4) **Integration of TRIG Functions: Here we have 4 specific strategies**
- 5) **Integration by Parts**

Worked Examples: (Standard Integrals)

1. Find the general anti-derivatives of the following expressions by inspection:

a) $3x^2 + \sqrt{x} - x^{-2}$

b) $\sin x + \cos x$

c) $\frac{x^3 + 5x^2 - 4}{x^2}$

d) $\frac{1}{2} \operatorname{cosec}^2 x + 1$

2. Integrate the following:

a) $\int x^5 dx$

b) $\int \sqrt[3]{z} dz$

c) $\int \frac{dx}{x^2}$

d) $\int \frac{3dx}{\sqrt[3]{x^2}}$

e) $\int (\sin \theta - 2 \cos \theta) d\theta$

NB: We cannot integrate a Quotient, simplification into individual terms is needed.

NB: A Product cannot be integrated by inspection. Try to make a product into a sum rather or U-sub.

Definition of the anti-derivative:

If $f(x)$ is a continuous function and $F(x)$ is the function whose derivative is $f(x)$, that is, $F'(x) = f(x)$, then $\int f(x) dx = F(x) + c$, where c is an arbitrary constant.

EXERCISE 1:

Evaluate each of the following integrals.

a) $\int (x^2 + 3x - 2) \, dx$

b) $\int_1^4 \left(\frac{\sqrt{x} + (x-1)^2}{\sqrt{x}} \right) dx$

c) $\int \left(\frac{1}{x^2} \right) dx$

d) $\int \left(\frac{3x^5 - 2x^3 + 8x}{x^3} \right) dx$

e) $\int \left(3x^2 + 5x + 8 + \frac{7}{x^5} \right) dx$

f) $\int_0^3 (20x^3 - 2x) dx$

U-substitution

2) Integration of Power Functions: when a function (usually raised to a power) and its derivative appear in the integral.

- pertains to integrals in the form $\int f'(x).f(x)dx$ or $\int \frac{f'(x)}{f(x)}dx$
- integrated as follows: $\int f'(x).[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$
- many such integrals can be solved by using inspection, however, substitution is a useful method that can also create simpler integrals to work with.

Substitution Method

- let $u = f(x)$
- then $\frac{du}{dx} = f'(x)$
- $\therefore du = f'(x)dx$
- the integral $\int f(x).f'(x)dx$ then becomes $\int u du$

How to Integrate:

First check if standard Integral or can it be made into a standard integral but either multiplying out or by separating terms when division.

Second use U-Sub.

Example 3:

Determine the indefinite integral of the following: $\int x^2(x^3 - 3)dx$

Solution: by 'inspection'

When the derivative of the 'power function' is in the 'front', all we need to do is integrate the 'outer' part of the power function. Sometimes we may need to compensate with a constant multiple to ensure that the **exact** derivative appears at the front.

Example 3:

Determine the indefinite integral of the following: $\int x^2(x^3 - 3)dx$

EXERCISE 3:

Evaluate each of the following integrals.

a) $\int 5x.(3x^2 - 2)^{17} dx$

b) $\int (7x - 3)^9 dx$

c) $\int 3x(x^2 + 1)^2 dx$

d) $\int (5x - 3)^{12} dx$

e) $\int (3x^2 - 2)(x^3 - 2x + 1)^2 dx$

f) $\int (3x^2 + 4x)(x^3 + 2x^2)^3 dx$

g) $\int 4x(2x^2 - 1)^4 dx$

h) $\int (3 - x)^4 dx$

i) $\int (6x^3 - 8x + 2)^7 (9x^2 - 4) dx$

j) $\int \sqrt{1+2x} \, dx$

k)* $\int \cos x \cdot \sin^4 x \, dx$

A trick involving substitution:

Evaluate $\int \frac{1}{\sqrt{1-x^2}} dx$ using a trig substitution, where $x = \sin u$

The Switch-Around Method

- not officially recognized as a separate method of integration, but highly useful
- involves substitution
- best shown by example

Example 4:

Integrate the function $f(x) = \frac{x}{\sqrt{x+1}}$

3) Integration of e^x and 'hyperbolic' functions leading to $\ln x$

$f(x) = e^x$ then $f'(x) = e^x$, and in general, if k is a constant and $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$.

It follows then that the integral will follow in the same way.

An exponential function can obviously also be expressed as a composite function:

In general: $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

Example 6:

(a) $\int e^{3x-2} dx$

(b) $\int \frac{4}{e^{3x}} dx$

(c) $\int \frac{e^{3x} + 4}{e^{2x}} dx$

EXERCISE 4:

Evaluate each of the following integrals.

a) $\int e^{4x} dx$

b) $\int 3e^{4x} dx$

c) $\int -2e^{-2x} dx$

d) $\int e^{4x+3} dx$

e) $\int e^{5-2x} dx$

f) $\int 2e^{1-2x} dx$

g) $\int 3e \times e^{2+5x} dx$

h) $\int (e^x + e^{-x})^2 dx$

Integration of Hyperbolic Functions

$f(x) = \ln x$ then $f'(x) = \frac{1}{x}$ for $x > 0$, and in general, if k is a constant and $f(x) = \ln(kx)$,
 then $f'(x) = \frac{1}{kx} \times k = \frac{1}{x}$.

So, it makes sense that: $\int \frac{1}{x} dx = \ln|x| + c$

Note: The absolute value is very important.

$f(x)$	$f'(x)$
$\ln 2x+1 $	$\frac{2}{2x+1}$
$\ln x^2-x+2 $	$\frac{2x-1}{x^2-x+2}$

It is also then easy to conclude the following:

$$\int \frac{2}{2x+1} dx = \ln|2x+1| + c$$

$$\int \frac{2x-1}{x^2-x+2} dx = \ln|x^2-x+2| + c$$

NOTE: Integrals in the form $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

Example 7:

(a) $\int \frac{1}{4x} dx$

(b) $\int \frac{1}{2x-9} dx$

(c) $\int \frac{6}{3x-2} dx$

(d) $\int \frac{x}{3x^2-2} dx$

(e) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

(f) $\int \frac{\sqrt{\ln x}}{x} dx$ since $\frac{d(\ln x)}{dx} = \frac{1}{x}$

Partial fractions can be used to help integrate rational functions.

Example 8:

Evaluate $\int \frac{1}{x(x-5)} dx$.

However, $\frac{1}{x(x-5)}$ can be expressed as:

$$\frac{1}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5}$$

$$1 = A(x-5) + Bx$$

$$\text{if } x = 5: \quad B = \frac{1}{5}$$

$$\text{if } x = 0: \quad A = -\frac{1}{5}$$

$$\therefore \frac{1}{x(x-5)} = -\frac{1}{5x} + \frac{1}{5(x-5)}$$

Hence:

$$\begin{aligned}\int \frac{1}{x(x-5)} dx &= \int \left(-\frac{1}{5x} + \frac{1}{5(x-5)} \right) dx \\ &= -\frac{1}{5} \int \frac{1}{x} dx + \frac{1}{5} \int \frac{1}{x-5} dx \\ &= -\frac{1}{5} \ln|x| + \frac{1}{5} \ln|x-5| + c\end{aligned}$$

EXERCISE 5:

Evaluate each of the following integrals.

a) $\int \frac{3}{2x-5} dx$

b) $\int \frac{e}{ex-7} dx$

c) $\int \left(2 + \frac{1}{x-1} \right) dx$

d) $\int \frac{x^2}{2+x^3} dx$

e) $\int \frac{e^x}{1+e^x} dx$

f) $\int \frac{2e^{3x}}{5-e^{3x}} dx$

g) $\int \frac{x-2}{x^2-4x+1} dx$

h) $\int \frac{3}{(2x+1)(x-1)} dx$

i) $\int \frac{18-4x-x^2}{(4-3x)(1+x)^2} dx$

j) $\int \frac{15-13x+4x^2}{(1-x)^2(4-x)} dx$

4) **Integration of TRIG Functions: Here we have 4 specific strategies**

- Basic – ‘directly’ from the table of integrals (may need *minor* manipulation)
- Power functions (substitution) – refer to ‘strategy 2’ of the integration notes
- Working with square and cube functions – converting them using identities / double angles
- Special products of sin/cos – working from the formula sheet

We start with some **Useful Standard Trig Integrals**, which can be found on the **formula sheet**.

Familiarize yourself with this table alongside.

Function	Derivative
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$

So, for example, the following can be easily determined:

- $\int \cos x \cdot dx = \sin x + c$
- $\int \sin x \cdot dx = -\cos x + c$
- $\int \sec^2 x \cdot dx = \tan x + c$
- $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

Note that we can use the square identities to transform a given square identity so that it can be read from the formula sheet:

$$\tan^2 x = \sec^2 x - 1$$

$$\operatorname{cosec}^2 x = \cot^2 x + 1$$

EXERCISE 6:

Evaluate each of the following integrals.

a) $\int \frac{1}{\cos^2 4x} dx$

b) $\int \cot^2 2x dx$

c) $\int \sin \frac{x}{2} dx$

d) $\int (\cos 2x - \sin 5x) dx$

e) $\int \sec^2 3x dx$

f) $\int 3 \cos 5x dx$

g) $\int \sec \frac{x}{4} \cdot \tan \frac{x}{4} dx$

h) $\int 2 \operatorname{cosec} 2x \cdot \cot 2x dx$

$$\text{i)} \quad \int_0^{\frac{\pi}{2}} \cos 2x \cdot dx$$

$$\text{j)} \quad \int 3 \tan^2 3x \cdot dx$$

Integration of Power Functions: when a function and its derivative appear in the integral.

When the derivative of the (trig) function is in front of the power function, all we do is integrate the 'outer power function'.

Example 9:

$$\int \cos x \cdot \sin^4 x \cdot dx \quad \text{of the form} \quad \rightarrow \quad \int f'(x) \cdot [f(x)]^n \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$\therefore \int \cos x \cdot \sin^4 x \cdot dx = \frac{1}{5} \sin^5 x + c$$

Example 10:

$$\begin{aligned} \int \cos^4 2x \cdot \sin 2x \cdot dx &= \int \sin 2x \cdot \cos^4 2x \cdot dx \\ &= -\frac{1}{2} \int -2 \cdot \sin 2x \cdot \cos^4 2x \cdot dx \\ &= -\frac{1}{2} \frac{(\cos 2x)^5}{5} + c \\ &= -\frac{1}{10} \cos^5 2x + c \end{aligned}$$

Remember to ensure that the exact derivative appears before the 'power function', by compensating with a constant multiple outside the integral, so that we can just integrate the 'outer function'.

EXERCISE 7:

Evaluate each of the following integrals.

a) $\int \cos x \cdot \sin^7 x \, dx$

b) $\int \cos^3 x \cdot \sin x \, dx$

c) $\int \cos^5 \frac{x}{2} \cdot \sin \frac{x}{2} \, dx$

d) $\int \frac{\cos^4 x}{\operatorname{cosec} x} \, dx$

e) $\int \frac{\sec^5 x \cdot \sin x}{\cos^2 x} \, dx$

f) $\int \cot^2 x \cdot \operatorname{cosec}^2 x \, dx$

g) $\int x^2 \cdot \sin(x^3) \, dx$

h) * $\int_0^{\frac{\pi}{2}} \sin x \cdot \sin 2x \, dx$

Example 11: Integration of trig expressions leading to ln...

Remember, these are of the form: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$$\begin{aligned} \text{(a)} \quad \int \cot x \, dx \\ &= \int \frac{\cos x}{\sin x} \, dx \\ &= \ln|\sin x| + c \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \tan x \, dx \\ &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\int \frac{-\sin x}{\cos x} \, dx \\ &= -\ln|\cos x| + c \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int \frac{\cos x - \sin x}{\sin x + \cos x} \, dx \\ &= \ln|\sin x + \cos x| + c \end{aligned}$$

EXERCISE 8:

Evaluate each of the following integrals.

$$\text{a)} \quad \int \frac{\cos x}{1 + \sin x} \, dx$$

$$\text{b)} \quad \int \cot 2x \, dx$$

$$\text{c)} \quad \int \tan 3x \, dx$$

$$\text{d)} \quad \int \frac{\sin 3x}{1 + \cos 3x} \, dx$$

e) $\int \frac{\cos x}{\sqrt{1+3\sin x}} dx$

Working with square and cube functions – converting them using identities / double angles

We need to apply a ‘clever trick’ when we have **squares** of sin/cos and there is no derivative at the front.

Integrating the non-power trig function is fairly easy:

$$\int \sin 2x \cdot dx = -\frac{\cos 2x}{2} + c$$

$$\int \cos 2x \cdot dx = \frac{\sin 2x}{2} + c$$

Using a Double-Angle Formula

- $\int \sin^2 x \cdot dx$ and $\int \cos^2 x \cdot dx$ cannot be integrated without using a substitution
- the substitutions are derived from the double angle formula for $\cos 2x$
- from $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$ we can deduce the following formulae:
 - $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$
 - $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example 12:

$$\begin{aligned} \int \sin^2 x \cdot dx &= \int \frac{1}{2}(1 - \cos 2x) \cdot dx \\ &= \frac{1}{2} \int 1 \cdot dx - \frac{1}{2} \int \cos 2x \cdot dx \\ &= \frac{1}{2}x - \frac{1}{2} \cdot \frac{\sin 2x}{2} + c \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + c \end{aligned}$$

Example 13:

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$$\begin{aligned}
 \int \cos^2 x \, dx &= \int \frac{1}{2}(\cos 2x + 1) \, dx \\
 &= \frac{1}{2} \int \cos 2x \, dx + \frac{1}{2} \int 1 \, dx \\
 &= \frac{1}{2} \cdot \frac{\sin 2x}{2} + \frac{1}{2} x + c \\
 &= \frac{1}{4} \sin 2x + \frac{1}{2} x + c
 \end{aligned}$$

So what happens when we have an odd power (power function) with no derivative in front?

e.g. $\int \cos^3 x \, dx$ vs $\int \cos^3 x \cdot \sin x \, dx$

Here we must always split the cube as the product of itself and the square of itself.

Split, and then use: $\cos^2 x = 1 - \sin^2 x$

Example 14:

$$\begin{aligned}
 \int \cos^3 x \, dx &= \int \cos x \cdot \cos^2 x \, dx \\
 &= \int \cos x \cdot (1 - \sin^2 x) \, dx \\
 &= \int \cos x \, dx - \int \cos x \cdot \sin^2 x \, dx \\
 &= \sin x - \frac{1}{3} \sin^3 x + c
 \end{aligned}$$

In summary: **Convert as follows**

$$\cos^3 x = \cos x (1 - \sin^2 x)$$

$$\sin^3 x = \sin x (1 - \cos^2 x)$$

EXERCISE 9:

Evaluate each of the following integrals.

a) $\int \cos^2 2x \, dx$

b) $\int \cos^2 x \cdot \sin^3 x \, dx$

c) $\int (\cos x + \sec x)^2 \, dx$

d) $\int \frac{1}{4} \sin^2 x \, dx$

e) $\int (2\cos^2 x + \sin 3x) \, dx$

Using Compound-Angle Formulae

Special products of sin/cos – working from the formula sheet

Here we will integrate by making use of a *Suitable Trigonometric Substitution*

- the product of two trigonometric functions can be simplified using manipulations of the trigonometric compound angle formulae
- the simplification, taken directly from the formula sheet, can be substituted into the function so that the integral can be determined, using the following:

$$\circ \sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\circ \sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\circ \cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

Example 15:

$$\int \sin 4x \cdot \sin 2x \cdot dx \quad \rightarrow \quad \text{here we use } \sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

to simplify the integral.

$$= \int \frac{1}{2} [\cos(2x) - \cos(6x)] \cdot dx$$

$$= \frac{1}{2} \int \cos(2x) \cdot dx - \frac{1}{2} \int \cos(6x) \cdot dx$$

$$= \frac{1}{2} \frac{\sin(2x)}{2} - \frac{1}{2} \frac{\sin(6x)}{6} + c$$

$$= \frac{1}{4} \sin(2x) - \frac{1}{12} \sin(6x) + c$$

Example 16:

$\int \cos x \cdot \cos 2x \cdot dx \rightarrow$ here we use $\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$
to simplify the integral. Rearranging may also help...

$$\begin{aligned}
 &= \int \cos 2x \cdot \cos x \cdot dx \\
 &= \int \frac{1}{2} [\cos(x) - \cos(3x)] \cdot dx \\
 &= \frac{1}{2} \int \cos(x) \cdot dx - \frac{1}{2} \int \cos(3x) \cdot dx \\
 &= \frac{1}{2} \sin(x) - \frac{1}{2} \frac{\sin(3x)}{3} + c \\
 &= \frac{1}{4} \sin(x) - \frac{1}{6} \sin(3x) + c
 \end{aligned}$$

EXERCISE 10:

Evaluate each of the following integrals.

a) $\int \cos 7x \sin 5x \, dx$

b) $\int \cos x \cdot \cos 3x \cdot dx$

c) $\int 3 \sin 2x \sin 5x \cdot dx$

d) $\int -\sin x \cos x \, dx$

e) $\int_0^{\frac{\pi}{2}} \sin x \cdot \sin 2x \cdot dx$

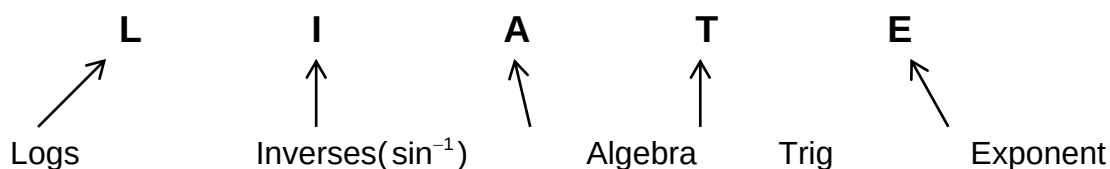
5) Integration by Parts

This method of integration is mainly used in the case of a product of 2 completely unrelated functions, that is, the one function is not the derivative of the other. This method is developed directly from the reverse of the Formula on data sheet.

$$\int f(x).g'(x)dx = f(x).g(x) - \int g(x).f'(x)dx + c$$

To use the above, we need to make one part of the original integral equal to $f(x)$ and the other part of the product equal to $g'(x)$.

Important: The part equal to $g'(x)$ must be readily integrable. How to determine which is $f(x)$ and which is $g'(x)$ use:



$f(x)$ = the first expression to appear LIATE and $g'(x)$ = second expression to appear in LIATE

Example 17:

Integrate $\int x \sin x \, dx$ by parts.

Solution:

Remember that we wish to transform the original integral:

$$\int f(x).g'(x)dx \rightarrow f(x).g(x) - \int g(x).f'(x)dx + c$$

Example 18:

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Integrate $\int (3x + 1)\cos 2x \, dx$ by parts

Solution:

From: $\int f(x).g'(x) \, dx \rightarrow f(x).g(x) - \int g(x).f'(x) \, dx + c$

Example 19:

Integrating $\ln|x|$ using integration by parts: $\rightarrow \int 1.\ln|x| \, dx$

Example 20:

Integrate $\int 2x.\ln|2x| \, dx$ by parts.

EXERCISE 11:

Evaluate each of the following integrals.

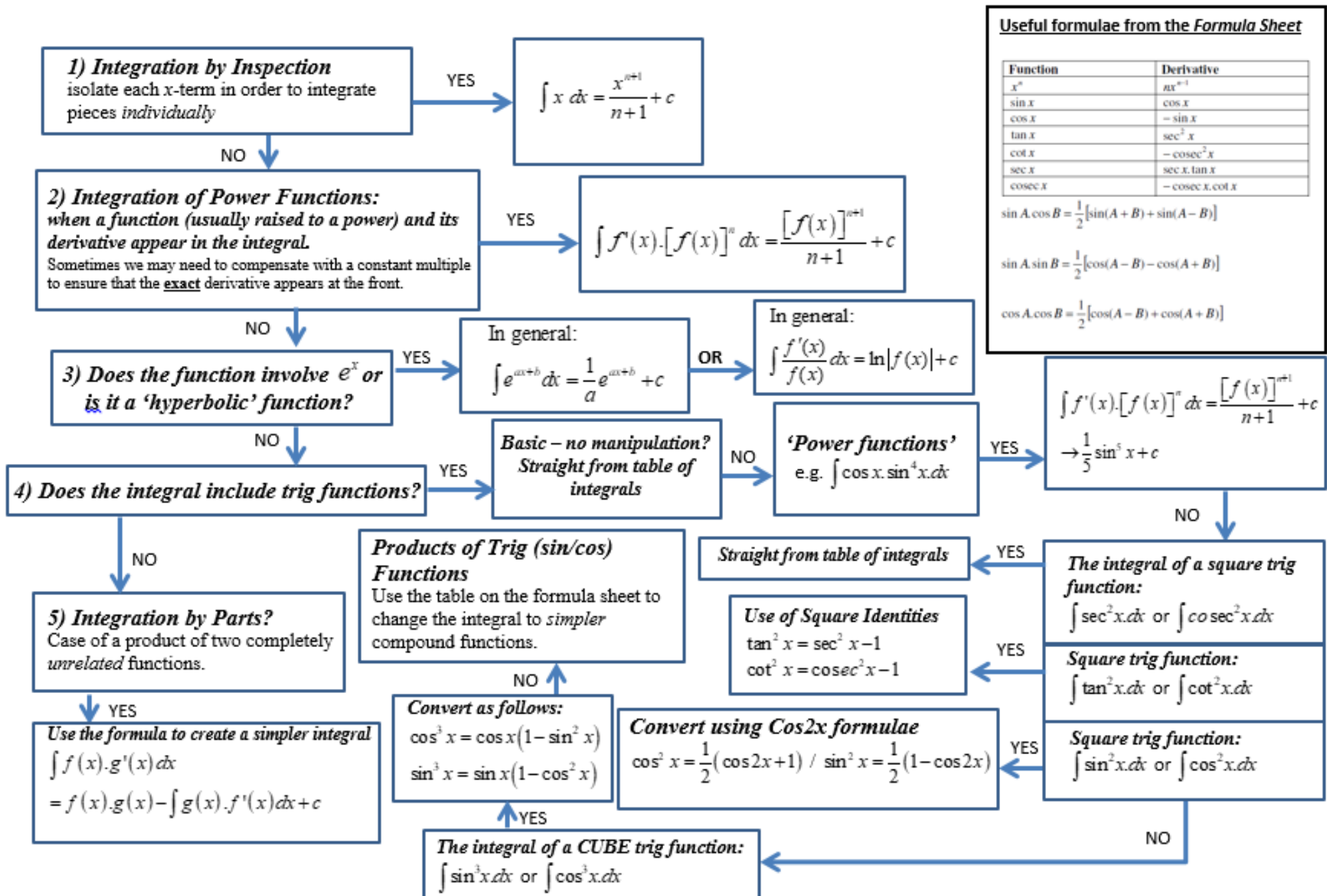
a) $\int x \cdot \cos x \, dx$

b) $\int x \cdot \sin 3x \, dx$

c) $\int x^2 \cdot \cos x \, dx$

d) $\int 3x \cdot \cos 2x \, dx$

Systematic Integration Flowchart



Useful formulae from the Formula Sheet

Function	Derivative
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$

$$\sin A \cdot \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\sin A \cdot \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cdot \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

Mixed revision exercise

EXERCISE 12:

Evaluate each of the following integrals.

a) $\int (x^3 + x^{-3}) \, dx$

b) $\int \left(x^2 + \frac{1}{x^2} - \sqrt{x} \right) dx$

c) $\int \sec^2 3x \, dx$

d) $\int (\cos 2x - \sin 5x) \, dx$

e) $\int 5 \sec 4x \cdot \tan 4x \, dx$

f) $\int (1 - 2 \sin x)^2 \, dx$

g) $\int \tan^3 x \cdot \sec^2 x \, dx$

h) $\int \sin 4x \cdot \cos 2x \, dx$

i) $\int \cos^6 x \cdot \sin x \, dx$

j) $\int 4 \cos 3x \cdot \cos 4x \, dx$

k) $\int x^2 (3x^3 + 1)^{-2} \, dx$

l) $\int x \cdot \sin\left(\frac{1}{2}x\right) \cdot dx$

m) $\int \sin 4x \cdot \sin 3x \cdot dx$

n) $\int \sqrt{16 - x^2} \, dx$ let $x = 4 \sin u$

o) $\int \cos^2 4x \, dx$

p) $\int \cos^3 x \cdot dx$

q) $\int x \cdot \sqrt{x^2 + 1} dx$

r) $\int (6x^3 - 8x + 2)^7 (9x^2 - 4) dx$

Using Integrals to Calculate Areas and Volumes

Using the Definite Integral to Solve Area Problems

Area Formed between a Graph and the x-Axis

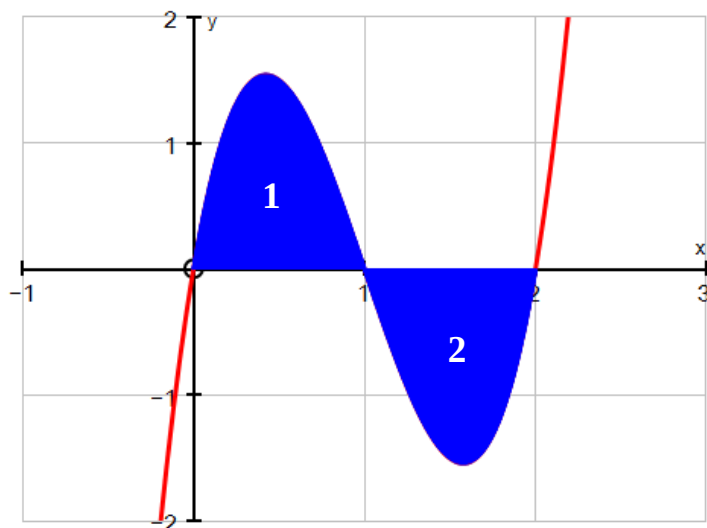
- an area will be formed between the graph, the x-axis, and some boundaries of integration
- if the graph lies on **both sides of the x-axis** within the given bounds, each part must be calculated separately and then added together
- the area below the x-axis will return a negative value. This must be calculated separately (and then we take the absolute value) in order for the areas not to 'cancel each other out'
- it is often easiest to draw a sketch before integrating the function to choose the **boundaries of integration**
- Here the x-intercepts are critical

Example 21:

Find the area enclosed by the function $f(x) = 4x(x-1)(x-2)$

Solution:

It is not too difficult sketching the function, since it is a recognisable cubic function with x-intercepts of 0, 1 and 2.



Note: Region 1 will give a positive area and Region 2 a *negative* area. This means that we must choose our boundaries of integration carefully.

Start by multiplying out the function in order to integrate nicely.

$$\text{So, } f(x) = 4x^3 - 12x^2 + 8x$$

∴ the area enclosed by the function

$$f(x) = 4x(x-1)(x-2)$$

$$\begin{aligned} & \left| \int_0^1 4x^3 - 12x^2 + 8x \, dx \right| + \left| \int_1^2 4x^3 - 12x^2 + 8x \, dx \right| \\ &= \left| \left[x^4 - 4x^3 + 4x^2 \right]_0^1 \right| + \left| \left[x^4 - 4x^3 + 4x^2 \right]_1^2 \right| \\ &= |1| + |-1| \\ &= 2 \text{ units}^2 \end{aligned}$$

It is better to take the absolute value of the area of each region, so that we don't mistakenly add positives to negatives i.e. so the areas don't 'cancel out'.

Be aware that the calculator cannot distinguish where 'regions' are positive or negative and consequently can give wrong answers.

Area Formed between Two Curves

To find the area bounded between two curves, f and g , we can follow these simple steps:

- Solve for where $f(x) = g(x)$. These are the boundaries of integration, a and b .
- Draw a rough sketch, which clearly shows where the one function is above the other. You may need to test a point, between a and b , to see which is the 'top' graph.
- Form the integral: $\int_a^b (\text{top} - \text{bottom}) \, dx$
- Simplify the 'function', and then evaluate the definite integral.
- If there are multiple points of intersection, resulting in several areas formed, the integrals for each area must be calculated separately (forming the new 'top – bottom' function each time) and added together

Example 22:

Find the area bounded between the curves $f(x) = 2 - x^2$ and $g(x) = -x$

Solution:

Find where $f(x) = g(x)$ These are our boundaries of integration

$$-x = 2 - x^2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } x = -1$$

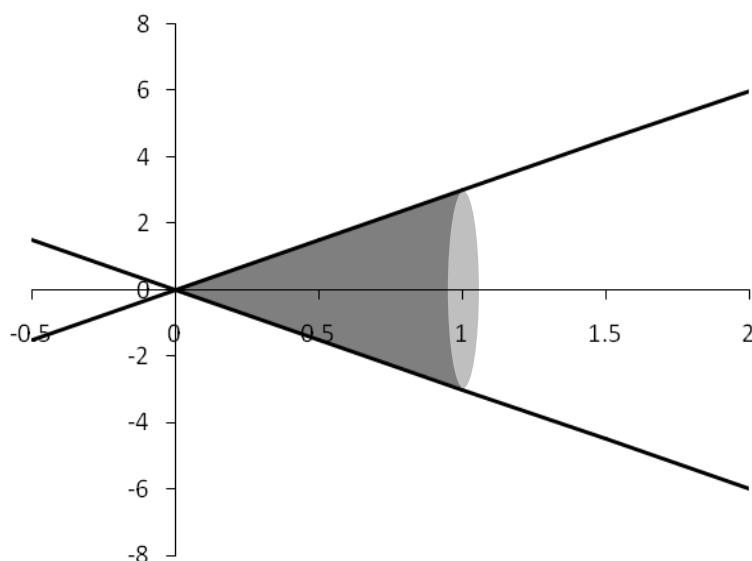
Draw a rough sketch, clearly indicating which function is top/bottom:

Evaluate the integral:

$$\begin{aligned} \int_{-1}^2 (\text{top} - \text{bottom}) \, dx &= \int_{-1}^2 (2 - x^2 - (-x)) \, dx \\ &= \int_{-1}^2 (2 - x^2 + x) \, dx \\ &= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 \\ &= \left(\frac{10}{3} \right) - \left(-\frac{7}{6} \right) \\ &= 4,5 \text{ units}^2 \end{aligned}$$

Using the Definite Integral to Solve Volumes of Solids of Revolution

- integration is used to find the volume of circular solids with varying radii i.e. a cone
- the volume of a cylinder (constant radius) is given by $V = \pi r^2 h$
- deriving the formula
 - when a shape is rotated 360° about the x-axis, the three-dimensional cone below is created



- to find this volume, slices, each dx wide with radius y , could be taken and their volumes added together
- the volume of each slice is represented by: $V = \pi y^2 dx$
- the sum of the volumes of the disks (with infinitely small dx) can be represented by

$$V = \pi \int_a^b y^2 dx$$

- generalised: $V = \pi \int_a^b f(x)^2 dx$ where:
 - $y = f(x)$ is the equation of the curve whose area is being rotated through a FULL 360°
 - a and b are the limits along the x-axis of the area being rotated
 - dx shows that the area is being rotated about the x-axis
 - note, a rotation of, say 90° , which is $\frac{1}{4}$ of the 'full' rotation, results in a

modified formula $\rightarrow V = \frac{\pi}{4} \int_a^b f(x)^2 dx$

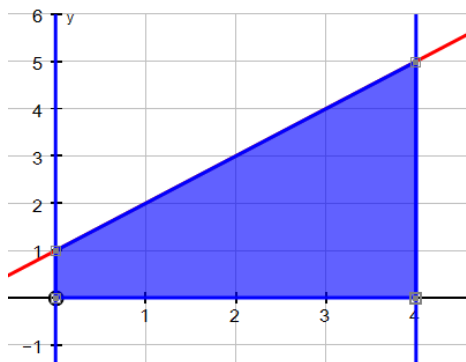
Example 23:

Find the volume of revolution between the curve $y = x + 1$, the x -axis and the lines $x = 0$ and $x = 4$, if the line is rotated 360° about the x -axis.

Solution:

Using the formula: $V = \pi \int_a^b f(x)^2 dx$

$$\begin{aligned} V &= \pi \int_0^4 (x+1)^2 dx \\ &= \pi \int_0^4 (x^2 + 2x + 1) dx \\ &= \pi \left[\frac{x^3}{3} + x^2 + x \right]_0^4 \\ &= \frac{124}{3} \pi \end{aligned}$$



EXERCISE 13:

Evaluate each of the following integrals.

a) Find the area bounded between the curves of $h(x) = x^2 - 2x + 2$ and $j(x) = -x^2 + 6$

b) Find the area bounded by the curves of $f(x) = -x^2 - 3x + 4$ and $j(x) = -2x - 2$

- c) Find the volume of revolution between the line $y = 3x$ and the x-axis, rotated 360° about the x-axis, between $x = 0$ and $x = 1$.
- d) Determine the volume of revolution of the function $x^2 + y^2 = 9$, between $x = 0$ and $x = 3$, rotated 360° about the x-axis.
- e) Determine the volume of revolution of the function $y - x^3 = 1$, between $x = 0$ and $x = 3$, rotated 90° about the x-axis.
- f) Determine the area bounded by the curve $y = 2\sin 3x \sin x$ and the x-axis, from $x = 0$ to $x = \frac{\pi}{3}$.

- g) Determine the area in the first quadrant bounded by the curves $y = \sqrt{x}$, the x -axis, and the line $y = x - 2$
- h) Determine the volume of revolution of the function $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{2}$ rotated 360° about the x -axis.

Example 24:

Find a , if $\int_2^a \frac{1}{9-2x} dx = \frac{1}{2} \ln 5$

Start by finding the indefinite integral:

$$\begin{aligned} \int \frac{1}{9-2x} dx \\ &= -\frac{1}{2} \int -2 \times \frac{1}{9-2x} dx \\ &= -\frac{1}{2} \ln|9-2x| + c \end{aligned}$$

Finding the area under the curve:

$$\begin{aligned} \int_2^a \frac{1}{9-2x} dx &= \left[-\frac{1}{2} \ln|9-2x| \right]_2^a = \frac{1}{2} \ln 5 \\ \therefore -\frac{1}{2} [\ln|9-2a| - \ln 5] &= \frac{1}{2} \ln 5 \quad \times -2 \\ \ln|9-2a| - \ln 5 &= -\ln 5 \\ \ln|9-2a| &= 0 \\ 9-2a &= 1 \\ a &= 4 \end{aligned}$$

Problem Solving – Volumes of Revolution ABOUT THE Y-AXIS

We use the formula, in a very similar way, except our boundaries of integration are with respect to the y -axis and instead of y^2 we use x^2 in the formula. This means that we must rewrite the function equation to make x the subject of the formula.

Using the formula: $V = \pi \int_a^b x^2 dy$

Example 25:

Find the volume of revolution between the line $y = 3x$, the y -axis and the lines $x = 0$ and $x = 1$, if the line is rotated 360° about the y -axis.

Solution:

Start by finding the boundaries of integration:

$$x = 0 \rightarrow y = 3(0) = 0$$

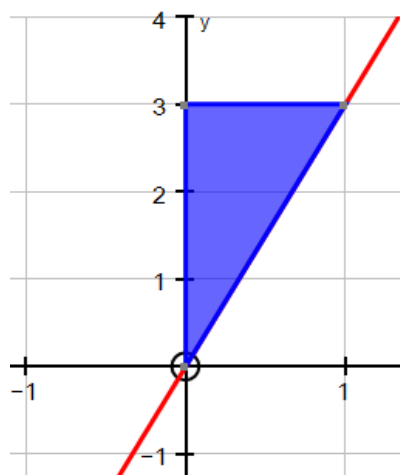
$$x = 1 \rightarrow y = 3(1) = 3$$

Make x the subject of the formula:

$$x = \frac{y}{3}$$

Integrate using the formula:

$$\begin{aligned} V &= \pi \int_a^b x^2 dy \rightarrow V = \pi \int_0^3 \left(\frac{y}{3} \right)^2 dy \\ &= \pi \int_0^3 \frac{1}{9} y^2 dy \\ &= \pi \left[\frac{1}{27} y^3 \right]_0^3 \\ &= \pi \text{ units}^2 \end{aligned}$$



MEMORANDUM TO EXERCISES

EXERCISE 1:

a) $\frac{x^3}{3} + \frac{3x^2}{2} - 2x + c$

b) $\frac{121}{15}$

c) $\frac{-1}{x} + c$

d) $x^3 - 2x - \frac{8}{x} + c$

e) $x^3 + \frac{5x^2}{2} + 8x - \frac{7}{4x^4} + c$

f) 396

EXERCISE 3:

a) $\frac{5(3x^2 - 2)^{18}}{108} + c$

b) $\frac{(7x - 3)^{10}}{70} + c$

c) $\frac{(x^2 + 1)^3}{2} + c$

d) $\frac{(5x - 3)^{13}}{65} + c$

e) $\frac{(x^3 - 2x + 1)^3}{3} + c$

f) $\frac{x^8(x + 2)^4}{4} + c$

g) $\frac{(2x^2 - 1)^5}{5} + c$

h) $\frac{(3 - x)^5}{5} + c$

i) $16(3x^3 - 4x + 1)^8 + c$

j) $\frac{(2x + 1)^{\frac{3}{2}}}{3} + c$

k)* $\frac{\sin^5 x}{5} + c$

EXERCISE 4:

a) $\frac{1}{4}e^{4x} + c$

b) $\frac{3}{4}e^{4x} + c$

c) $e^{-2x} + c$

d) $\frac{1}{4}e^{4x+3} + c$

e) $-\frac{1}{2}e^{5-2x} + c$

f) $-e^{1-2x} + c$

g) $\frac{3}{5}e^{5x+3} + c$

h) $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

EXERCISE 5:

a) $\frac{3}{2} \ln|2x - 5| + c$

b) $\ln|ex - 7| + c$

c) $2x + \ln|x - 1| + c$

d) $\frac{1}{3} \ln|2 + x^3| + c$

e) $\ln|1 + e^x| + c$

f) $-\frac{2}{3} \ln|5 - e^{3x}| + c$

g) $\frac{1}{2} \ln|x^2 - 4x + 1| + c$

h) $-\ln|2x + 1| + \ln|x - 1| + c$

i) $-\frac{2}{3} \ln|4 - 3x| + \ln|x + 1| - \frac{3}{1 + x} + c$

j) $-3 \ln|4 - x| - \ln|1 - x| - \frac{2}{1 - x} + c$

EXERCISE 6:

a) $\frac{\tan 4x}{4} + c$

b) $\frac{-\cot 2x}{2} - x + c$

c) $-2 \cos \frac{x}{2} + c$

d) $\frac{\sin 2x}{2} + \frac{\cos 5x}{5} + c$

e) $\frac{\tan 3x}{3} + c$

f) $\frac{3 \sin 5x}{5}$

g) $4 \sec \frac{x}{4} + c$

h) $\frac{1}{\sin 2x} + c$

i) 1

j) $\tan 3x - 3x + c$

EXERCISE 7:

a) $\frac{\sin^8 x}{8} + c$

b) $-\frac{\cos^4 x}{4} + c$

c) $-\frac{\cos 3x}{96} - \frac{\cos 2x}{16} - \frac{5 \cos x}{32} + c$

d) $\frac{-\cos^5 x}{5} + c$

e) $\frac{1}{6 \cos^6 x} + c$

f) $-\frac{\cot^3 x}{3} + c$

g) $\frac{-\cos x^3}{3} + c$

h) * $\frac{2}{3}$

EXERCISE 8:

a) $\frac{1}{2} \ln|\sin 2x| + c$

b) $\ln|1 + \sin x| + c$

c) $-\frac{1}{3} \ln|\cos 3x| + c$

d) $-\frac{1}{3} \ln|1 + \cos 3x| + c$

e) $\frac{2}{3} \sqrt{1 + 3 \sin x} + c$

EXERCISE 9:

Grade 12 FSM - Integration

$$a) \frac{\sin 4x}{8} + \frac{x}{2} + c$$

$$b) \frac{-\cos^3 x \sin^2 x}{5} - \frac{2\cos^3 x}{15} + c$$

$$c) \frac{\sin x \cos x}{2} + \tan x + \frac{5x}{2} + c$$

$$d) \frac{x}{8} - \frac{\sin x \cos x}{8} + c$$

$$e) \frac{-\cos 3x}{3} + \sin x \cos x + x + c$$

EXERCISE 10:

$$a) \frac{\cos 2x}{4} - \frac{\cos 12x}{24} + c$$

$$b) \frac{-\cos 4x}{8} - \frac{\cos 2x}{4} + c$$

$$c) \frac{\sin 3x}{2} - \frac{3\sin 7x}{14} + c$$

$$d) \frac{-\sin^2 x}{2} + c$$

$$e) \frac{2}{3}$$

EXERCISE 11:

$$a) x \sin x + \cos x + c$$

$$b) \frac{-x \cos 3x}{3} + \frac{\sin 3x}{9} + c$$

$$c) x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$d) \frac{3x \sin 2x}{2} + \frac{3 \cos 2x}{4} + c$$

EXERCISE 12:

$$a) \frac{x^4}{4} - \frac{1}{2x^2} + c$$

$$b) \frac{x^3}{3} - \frac{1}{x} - \frac{2x^{3/2}}{3} + c$$

$$c) \frac{\tan 3x}{3} + c$$

$$d) \frac{\sin 2x}{2} + \frac{\cos 5x}{5} + c$$

$$e) \frac{5 \sec 4x}{4} + c$$

$$f) 3x + 4 \cos x - \sin 2x + c$$

$$g) \frac{\tan^4 x}{4} + c$$

$$h) \frac{-\cos 6x}{12} - \frac{\cos 2x}{4} + c$$

$$i) -\frac{\cos^7 x}{7} + c$$

$$j) 2 \sin x + \frac{2 \sin 7x}{7} + c$$

$$k) -\frac{1}{3(3x^3 + 1)} + c$$

$$l) -2x \cos \frac{x}{2} + 4 \sin \frac{x}{2} + c$$

$$m) \frac{\sin x}{2} - \frac{\sin 7x}{14} + c$$

$$n) 8u + 4 \sin 2u + c$$

$$\text{o) } \frac{x}{2} + \frac{\sin 8x}{16} + c$$

$$\text{q) } \frac{(x^2 + 1)^{3/2}}{3} + c$$

$$\text{p) } \sin x - \frac{\sin^3 x}{3} + c$$

$$\text{r) } \frac{(6x^3 - 8x + 2)^8}{16} + c$$

EXERCISE 13:

$$\text{a) } 9 \text{ units}^2$$

$$\text{c) } 3\pi \text{ units}^3$$

$$\text{e) } \frac{4983}{14} \pi \text{ units}^3$$

$$\text{g) } \frac{10}{3} \text{ units}^2$$

$$\text{b) } \frac{125}{6} \text{ units}^2$$

$$\text{d) } 18\pi \text{ units}^3$$

$$\text{f) } \frac{3\sqrt{3}}{8} \text{ units}^2$$

$$\text{h) } \frac{\pi^2}{4} \text{ units}^3$$

CUMULATIVE REVISION EXERCISE

QUESTION 1

a) Use the Riemann Sum to determine the area under the curve of $y = -2x^2 + 18$ between the positive x and y axes.

b) Determine the integrals of the following:

i) $\int \frac{x^3 - 4}{2x^2} dx$

ii) $\int \sec(2x - 1) \cdot \tan(2x - 1) dx$

iii) $\int (\cos 5x \cdot \cos 3x) dx$

iv) $\int \frac{\sin 5x}{\operatorname{cosec} 3x} dx$

v) $\int x^2 \cos(x^3) dx$

vi) $\int x \sin^2 x dx$ (by parts)

vii) $\int (2x - 1) \sin(3x^2 - 3x + 1) dx$

c) Determine the integrals of the following:

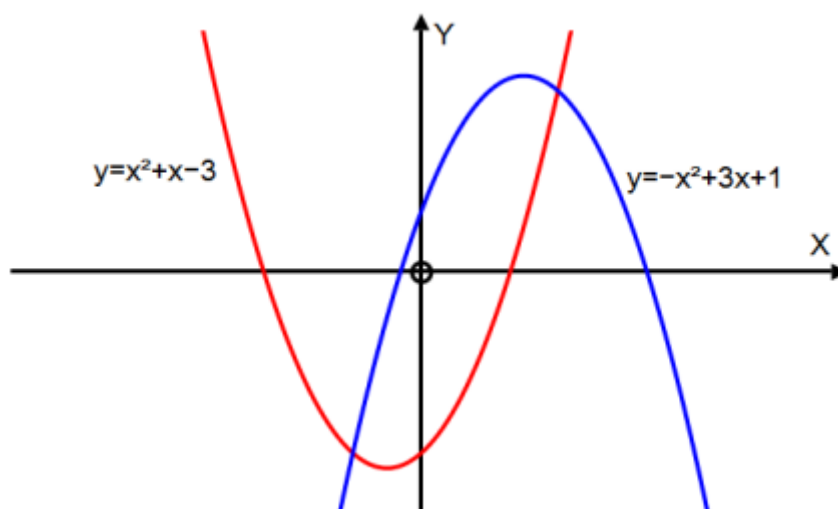
i) $\int (e^{2x} - e^{-x}) dx$

ii) $\int x \cdot e^{x^2 - 1} dx$

iii) $\int x \cdot e^x dx$

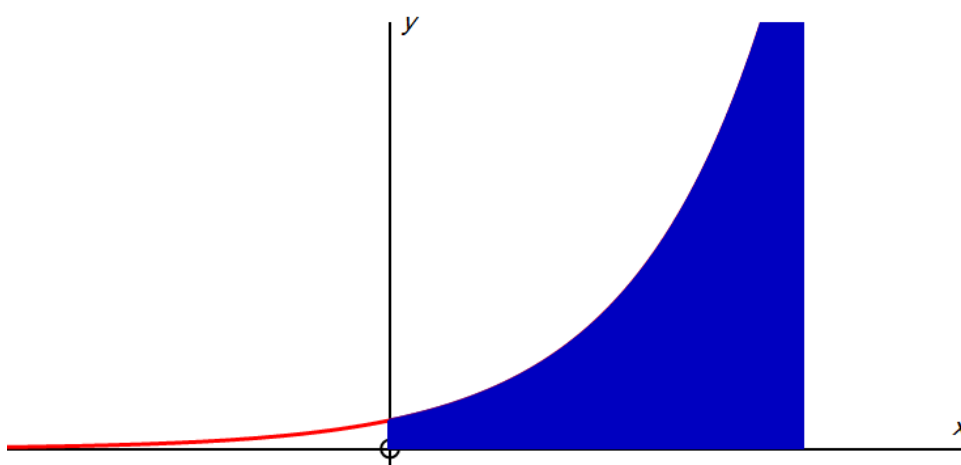
iv) $\int \frac{x}{e^{2x}} dx$

- d) i) Determine the area enclosed between the two curves shown below:



- ii) Determine the volume when the area enclosed between the x-axis and the curve $y = x^2 - 4$ is rotated about the x-axis.

- e) Find a if the area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = a$ is $\frac{1}{3}(e^6 - 1)$.



QUESTION 2

a) Given: $f(x) = \sqrt{x} + x$

i) Determine the integral: $\int f(x) dx$

ii) Given that: $\int_0^a (\sqrt{x} + x) dx = 10$ and $a \in (2; 7)$

Use Newton's method to find a , correct to 3 decimal places.

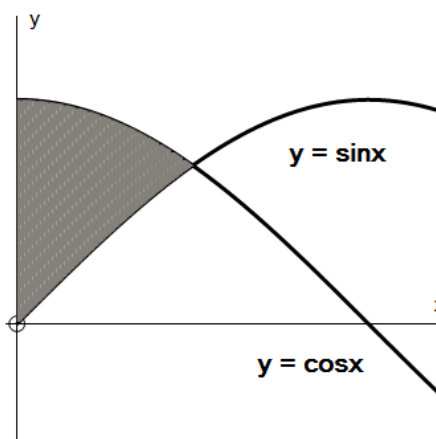
b) Integrate by parts: $\int x^2 \cos x dx$

c) Determine the integral: $\int \frac{2x}{\sqrt{5x^2 - 1}} dx$

QUESTION 3

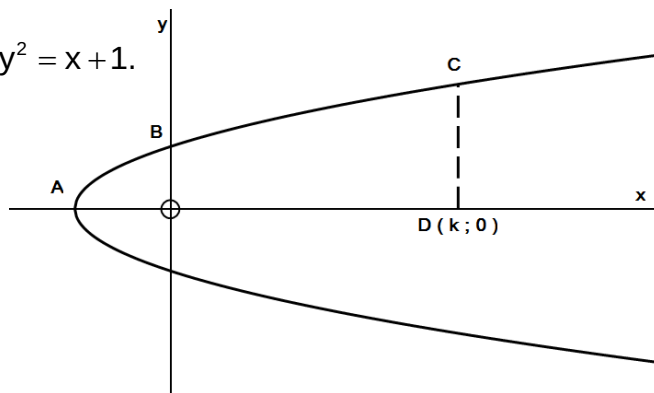
Consider the diagram alongside.

Without using a calculator, and showing all working details, find the volume of rotation if the shaded region is rotated 2π radians about the x -axis.



QUESTION 4

The diagram shows part of the curve $y^2 = x + 1$.



- Find the co-ordinates of A.
- The region AOB is rotated through 360° about the x-axis to generate a volume V . Calculate the value of V .
- The region ACD, with CD parallel to the y-axis, is rotated through 360° about the x-axis to generate a volume $9V$. Calculate the value of k .

QUESTION 5

- Write $\frac{5x-1}{2x^2+x-1}$ as a partial fraction.
- Determine: $\int \frac{5x-1}{2x^2+x-1} dx$

QUESTION 6

Find the volume of the solid of revolution formed when the curve $y = e^x$ is rotated about the x-axis from $x = 0$ to $x = 2$.

MEMORANDUM

QUESTION 1

- a) Use the Riemann Sum to determine the area under the curve of $y = -2x^2 + 18$ between the positive x and y axes.

Intercepts with positive x -axis: $0 = -2x^2 + 18$

$$2x^2 = 18 \quad \therefore x = \pm 3$$

$$\text{i.e. } \int_0^3 (-2x^2 + 18) dx \rightarrow \text{Area} = \lim_{n \rightarrow \infty} \left(\frac{b-a}{n} \right) \sum_{i=1}^n f(x_i)$$

$$\Delta x = \frac{3}{n} \quad \therefore x_i = 0 + \frac{3}{n}i$$

$$\therefore f(x_i) = -2 \left(\frac{3i}{n} \right)^2 + 18 = \frac{-18i^2}{n^2} + 18$$

$$\text{Area} = \lim_{n \rightarrow \infty} \left(\frac{3}{n} \right) \sum_{i=1}^n \left(\frac{-18i^2}{n^2} + 18 \right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{-54i^2}{n^3} + \frac{54}{n} \right) -$$

$$= \lim_{n \rightarrow \infty} \left[\frac{-54}{n^3} \sum_{i=1}^n (i^2) + \frac{54}{n} \sum_{i=1}^n (1) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{-54}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) + \frac{54}{n} (n) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\left(-18 - \frac{27}{n} - \frac{9}{n^2} \right) + 54 \right]$$

$$= -18 + 54 = 36 \text{ units}^2$$

- b) Determine the integrals of the following:

$$\text{i) } \int \frac{x^3 - 4}{2x^2} dx$$

$$= \int \frac{1}{2}x - 2x^{-2} dx = \frac{1}{4}x^2 + 2x^{-1} + c = \frac{1}{4}x^2 + \frac{2}{x} + c$$

$$\text{ii) } \int \sec(2x-1) \cdot \tan(2x-1) dx$$

$$= \frac{1}{2} \int 2(\sec(2x-1) \cdot \tan(2x-1)) dx = \frac{1}{2} \sec(2x-1) + c$$

$$\text{iii) } \int (\cos 5x \cdot \cos 3x) dx \text{ using: } \cos A \cdot \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$= \int \frac{1}{2} [\cos(5x-3x) + \cos(5x+3x)] dx = \int \frac{1}{2} \cos(2x) dx + \int \frac{1}{2} \cos(8x) dx$$

$$= \frac{1}{4} \sin 2x + \frac{1}{16} \sin 8x + c$$

$$\text{iv)} \quad \int \frac{\sin 5x}{\operatorname{cosec} 3x} dx = \int \sin 5x \cdot \sin 3x dx$$

$$\rightarrow \sin A \cdot \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$= \int \frac{1}{2} [\cos(5x - 3x) - \cos(5x + 3x)] dx = \int \frac{1}{2} \cos(2x) dx - \int \frac{1}{2} \cos(8x) dx$$

$$= \frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + c$$

$$\text{v)} \quad \int x^2 \cos(x^3) dx$$

$$= \frac{1}{3} \int 3x^2 \cos(x^3) dx = \frac{1}{3} \sin(x^3) + c$$

$$\text{vi)} \quad \int x \sin^2 x dx \quad (\text{by parts})$$

$$\text{Note: } \cos 2x = 1 - 2 \sin^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\int x \sin^2 x dx = \frac{1}{2} \int x (1 - \cos 2x) dx = \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx$$

$$= \frac{1}{4} x^2 - \frac{1}{2} \int x \cos 2x dx$$

$$\text{of the form: } \int u' \cdot v dx = uv - \int u \cdot v' dx$$

$$\text{where: } u' = \cos 2x \quad \therefore u = \frac{1}{2} \sin 2x$$

$$v = x \quad \therefore v' = 1$$

$$\int x \cos 2x dx = \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x dx$$

$$= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$$

$$\therefore \int x \sin^2 x dx = \frac{1}{4} x^2 - \frac{1}{2} \int x \cos 2x dx$$

$$= \frac{1}{4} x^2 - \frac{1}{2} \left(\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c \right)$$

$$= \frac{1}{4} x^2 - \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + c$$

$$\text{vii)} \quad \int (2x - 1) \sin(3x^2 - 3x + 1) dx$$

$$= \frac{1}{3} \int (6x - 3) \sin(3x^2 - 3x + 1) dx = -\frac{1}{3} \cos(3x^2 - 3x + 1) + c$$

c) Determine the integrals of the following:

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$$\begin{aligned}
 \text{i)} \quad & \int (e^{2x} - e^{-x}) dx \\
 &= \int e^{2x} dx - \int e^{-x} dx \\
 &= \frac{1}{2} e^{2x} + e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{ii)} \quad & \int x.e^{x^2-1} dx \\
 &= \frac{1}{2} \int 2x.e^{x^2-1} dx \\
 &= \frac{1}{2} e^{x^2-1} + c
 \end{aligned}$$

$$\text{iii)} \quad \int x.e^x dx \quad \text{using IBP: } \int f(x).g'(x) dx \rightarrow f(x).g(x) - \int g(x).f'(x) dx + c$$

$$\text{Let } f(x) = x \quad \therefore f'(x) = 1$$

$$\text{and } g'(x) = e^x \quad \therefore g(x) = e^x$$

$$\begin{aligned}
 \therefore \int x.e^x dx &= x.e^x - \int e^x dx \\
 &= x.e^x - e^x + c
 \end{aligned}$$

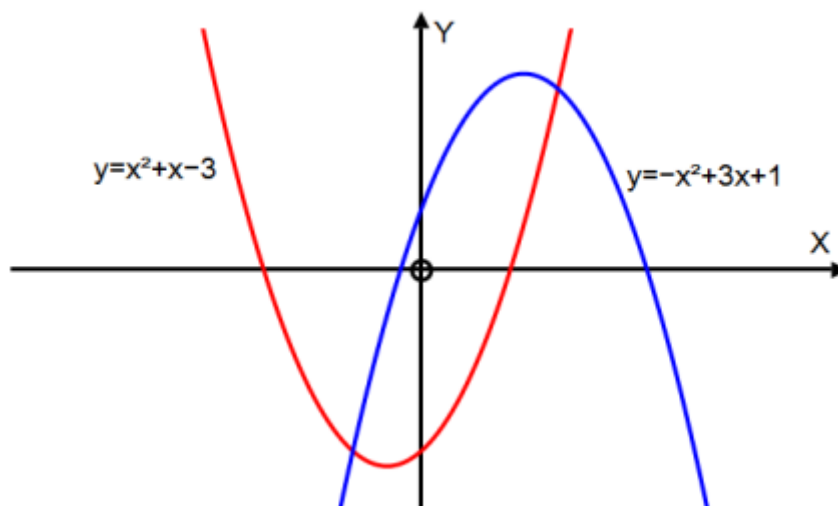
$$\begin{aligned}
 \text{iv)} \quad & \int \frac{x}{e^{2x}} dx \quad \text{using IBP: } \int f(x).g'(x) dx \rightarrow f(x).g(x) - \int g(x).f'(x) dx + c \\
 &= \int x.e^{-2x} dx
 \end{aligned}$$

$$\text{Let } f(x) = x \quad \therefore f'(x) = 1$$

$$\text{and } g'(x) = e^{-2x} \quad \therefore g(x) = -\frac{1}{2} e^{-2x}$$

$$\begin{aligned}
 \therefore \int x.e^{-2x} dx &= -\frac{1}{2} x.e^{-2x} - \int \frac{1}{2} e^{-2x} dx \\
 &= -\frac{1}{2} x.e^{-2x} + \frac{1}{4} \int (-2).e^{-2x} dx \\
 &= -\frac{1}{2} x.e^{-2x} + \frac{1}{4} e^{-2x} + c
 \end{aligned}$$

- d) i) Determine the area enclosed between the two curves shown below:



Finding points of intersection: $x^2 + x - 3 = -x^2 + 3x + 1$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } x = -1$$

Area =

$$\int_{-1}^2 \text{top} - \text{bottom} = \int_{-1}^2 (-x^2 + 3x + 1) - (x^2 + x - 3) dx$$

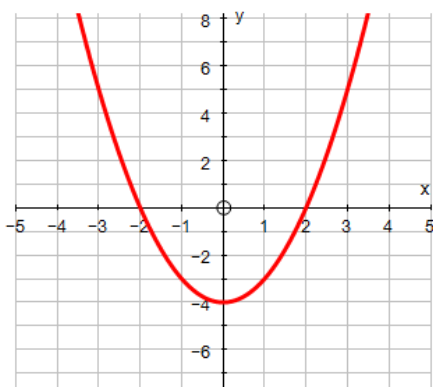
$$= \int_{-1}^2 (-2x^2 + 2x + 4) dx = \left[-\frac{2}{3}x^3 + x^2 + 4x \right]_{-1}^2$$

$$= \left(-\frac{2}{3}(2)^3 + (2)^2 + 4(2) \right) - \left(-\frac{2}{3}(-1)^3 + (-1)^2 + 4(-1) \right)$$

$$= \left(\frac{20}{3} \right) - \left(\frac{-7}{3} \right)$$

$$= 9 \text{ units}^2$$

- ii) Determine the volume when the area enclosed between the x-axis and the curve $y = x^2 - 4$ is rotated about the x-axis.



From the sketch, it can be clearly seen (or calculated), that the x-intercepts are $x = -2$ or $x = 2$.

$$\begin{aligned}
 V &= \pi \int_{-2}^2 (x^2 - 4)^2 dx = \pi \int_{-2}^2 (x^4 - 8x^2 + 16) dx \\
 &= \pi \left[\frac{x^5}{5} - \frac{8}{3}x^3 + 16x \right]_{-2}^2 \\
 &= \pi \left[\left(\frac{(2)^5}{5} - \frac{8}{3}(2)^3 + 16(2) \right) - \left(\frac{(-2)^5}{5} - \frac{8}{3}(-2)^3 + 16(-2) \right) \right] \\
 &= \frac{512}{15} \pi
 \end{aligned}$$

- e) Find a if the area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = a$ is $\frac{1}{3}(e^6 - 1)$.

$$\begin{aligned}
 \text{Area} &= \int_0^a e^{3x} dx \\
 &= \left[\frac{1}{3} e^{3x} \right]_0^a \\
 &= \frac{1}{3} e^{3a} - \frac{1}{3} e^0 \\
 &= \frac{1}{3} e^{3a} - \frac{1}{3} \\
 &= \frac{1}{3} (e^{3a} - 1) = \frac{1}{3} (e^6 - 1) \quad \text{given area} \\
 \therefore 3a &= 6 \\
 \therefore a &= 2
 \end{aligned}$$

QUESTION 2

a) Given: $f(x) = \sqrt{x} + x$; $x \in [2; 7]$

i) Determine the integral: $\int f(x) dx$

$$\int (\sqrt{x} + x) dx = \frac{2}{3} x^{\frac{3}{2}} + \frac{1}{2} x^2 + c$$

ii) Given that: $\int_0^a (\sqrt{x} + x) dx = 10$ and $a \in [2; 7]$

Use Newton's method to find a , correct to 3 decimal places.

$$\int_0^a (\sqrt{x} + x) dx = 10$$

$$\therefore \left[\frac{2}{3} x^{\frac{3}{2}} + \frac{1}{2} x^2 + C \right]_0^a = 10$$

$$\therefore \frac{2}{3} a^{\frac{3}{2}} + \frac{1}{2} a^2 - 10 = 0$$

$$\text{Using Newton Raphson, } a_{n+1} = a_n - \frac{\frac{2}{3} a_r^{\frac{3}{2}} + \frac{1}{2} a_r^2 - 10}{\sqrt{a_r} + a_r}$$

Since, $x \in [2; 7]$ let $a_0 = 5$, then

$$a_0 = 5$$

$$a_1 = 3,6244518...$$

$$a_2 = 3,4130869...$$

$$a_3 = 3,4077144...$$

$$a_4 = 3,4077109...$$

$$\therefore a = 3,408$$

b) Integrate by parts: $\int x^2 \cos x \, dx$

of the form: $\int u' \cdot v \, dx = uv - \int u \cdot v' \, dx$

$$\text{where: } u' = \cos x \quad \therefore u = \sin x$$

$$v = x^2 \quad \therefore v' = 2x$$

$$\int x^2 \cdot \cos x \cdot dx = x^2 \sin x - \int 2x \sin x \cdot dx$$

doing parts again... for $\int 2x \sin x \cdot dx \rightarrow \int u' \cdot v \, dx = uv - \int u \cdot v' \, dx$

$$\text{where: } u' = \sin x \quad \therefore u = -\cos x$$

$$v = 2x \quad \therefore v' = 2$$

$$\begin{aligned} \int 2x \sin x \cdot dx &= -2x \cos x - \int -2 \cos x \cdot dx \\ &= -2x \cos x + 2 \int \cos x \cdot dx \\ &= -2x \cos x + 2 \sin x + c \end{aligned}$$

$$\begin{aligned} \int x^2 \cdot \cos x \cdot dx &= x^2 \sin x - \int 2x \sin x \cdot dx \\ &= x^2 \sin x - (-2x \cos x + 2 \sin x + c) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + c \end{aligned}$$

c) Determine the integral: $\int \frac{2x}{\sqrt{5x^2 - 1}} \, dx$

$$\begin{aligned} &= \int 2x(5x^2 - 1)^{-\frac{1}{2}} \, dx \\ &= 2 \times \frac{1}{10} \times 2(5x^2 - 1)^{\frac{1}{2}} + c \\ &= \frac{2}{5}(5x^2 - 1)^{\frac{1}{2}} + c \end{aligned}$$

QUESTION 3

Consider the diagram alongside.

Without using a calculator, and showing all working details, find the volume of rotation if the shaded region is rotated 2π radians about the x-axis.

For the *intersection* of $y = \sin x$ and $y = \cos x$

$$\sin x = \cos x$$

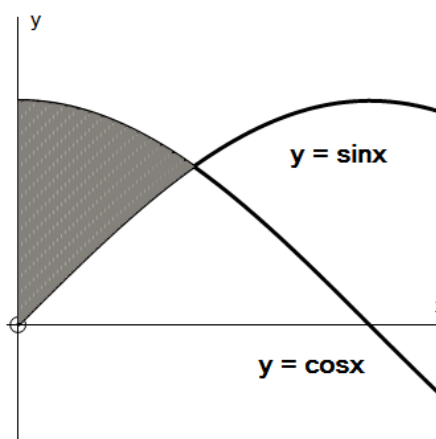
$$\therefore \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_0^{\frac{\pi}{4}} [\cos^2 x - \sin^2 x] dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \cos 2x dx = \pi \left[\frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \frac{\pi}{2}$$

**QUESTION 4**

The diagram shows part of the curve $y^2 = x + 1$:

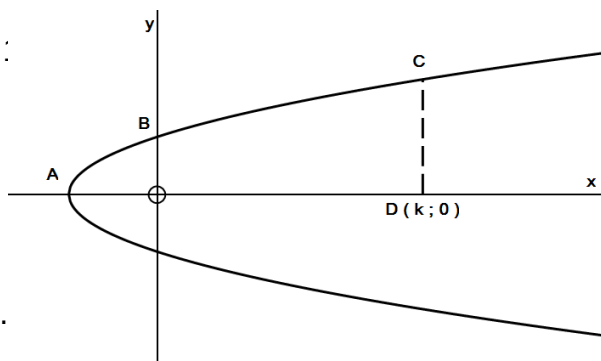
- a) Find the co-ordinates of A.

$$A (-1 ; 0)$$

- b) The region AOB is rotated through 360° about the x-axis to generate a volume V. Calculate the value of V.

$$V = \pi \int_{-1}^0 (x+1) dx$$

$$= \frac{1}{2} \pi$$



- c) The region ACD, with CD parallel to the y -axis, is rotated through 360° about the x -axis to generate a volume $9V$. Calculate the value of k .

$$9V = \pi \int_{-1}^k (x+1) dx$$

$$9 \frac{\pi}{2} = \pi \left[\frac{x^2}{2} + x \right]_{-1}^k$$

$$\frac{9}{2} = \left(\frac{k^2}{2} + k \right) - \left(\frac{1}{2} - 1 \right)$$

$$\frac{k^2}{2} + k - 4 = 0$$

$$k = 2 \text{ or } k = -4$$

$$k = 2$$

QUESTION 5

- a) Write $\frac{5x-1}{2x^2+x-1}$ as a partial fraction.

$$\frac{5x-1}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$5x-1 = A(x+1) + B(2x-1)$$

$$x = -1 \rightarrow -6 = B(-3)$$

$$\therefore B = 2$$

$$x = \frac{1}{2} \rightarrow \frac{3}{2} = A\left(\frac{3}{2}\right)$$

$$\therefore A = 1$$

$$\therefore \frac{5x-1}{2x^2+x-1} = \frac{1}{2x-1} + \frac{2}{x+1}$$

- b) Determine: $\int \frac{5x-1}{2x^2+x-1} dx$

$$\begin{aligned} \int \frac{5x-1}{2x^2+x-1} dx &= \int \frac{1}{2x-1} dx + \int \frac{2}{x+1} dx \\ &= \frac{1}{2} \int 2 \times \frac{1}{2x-1} dx + 2 \int \frac{1}{x+1} dx \\ &= \frac{1}{2} \ln|2x-1| + 2 \ln|x+1| + c \end{aligned}$$

QUESTION 6

Find the volume of the solid of revolution formed when the curve $y = e^x$ is rotated about the x -axis from $x = 0$ to $x = 2$.

Given: $y = e^x$

$$y^2 = e^{2x}$$

$$V = \pi \int_0^2 e^{2x} dx$$

$$= \pi \left[\frac{1}{2} e^{2x} \right]_0^2$$

$$= \pi \left(\frac{1}{2} e^4 - \frac{1}{2} e^0 \right)$$

$$= \frac{\pi}{2} (e^4 - 1)$$