

GRADE 12 THEOREM BOOKLET PAPER 1



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PAPER 2

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THE SUM OF ARITHMETIC SERIES

Required to Prove: $S_n = \frac{n}{2} [2a + (n-1)d]$

Proof:

$$S_n = T_1 + T_2 + T_3 + \dots + T_{n-1} + T_n$$

$$S_n = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l \dots \textcircled{1}$$

$$S_n = l + (l - d) + (l - 2d) + \dots + (a + 2d) + (a + d) + a \dots \textcircled{2}$$

$$\textcircled{1} + \textcircled{2} : 2S_n = \underbrace{(a + l) + (a + l) + (a + l) + \dots + (a + l) + (a + l) + (a + l)}_{\text{for } n \text{ terms}}$$

$$\therefore 2S_n = n \times (a + l)$$

$$\therefore S_n = \frac{n}{2} (a + l)$$

But $l = a + (n-1)d$

$$\therefore S_n = \frac{n}{2} (a + l) = \frac{n}{2} [2a + (n-1)d]$$

THE SUM OF GEOMETRIC SERIES

Required to Prove: $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \text{ for } r \neq 1$

Proof:

$$S_n = T_1 + T_2 + T_3 + \dots + T_{n-1} + T_n$$

$$S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \dots \textcircled{1}$$

$$r \times S_n = ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n \dots \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} : S_n - rS_n = a - ar^n$$

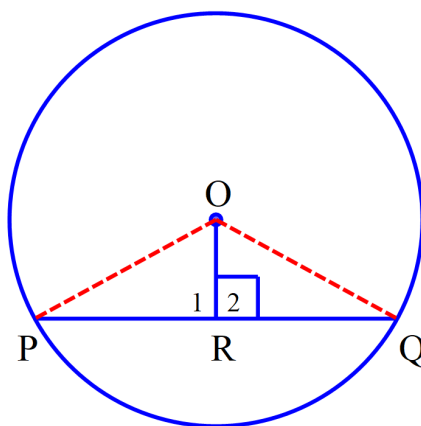
$$\therefore S_n(1-r) = a(1-r^n)$$

$$\therefore S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a(r^n-1)}{(r-1)} \text{ for } r \neq 1$$

THEOREM 1

The line drawn from the centre of a circle, perpendicular to a chord, bisects the chord.

Given: Circle with centre O and chord PQ. $OR \perp PQ$ with R on PQ.



Required to Prove: $PR = RQ$

Construction: OP and OQ

Proof:

In $\triangle OPR$ and $\triangle OQR$

$$1) \quad \hat{R}_1 = \hat{R}_2 = 90^\circ \quad (OR \perp PQ)$$

$$2) \quad OP = OQ \quad (\text{radii})$$

3) OR is common

$$\therefore \triangle OPR \equiv \triangle OQR \quad (\text{RHS})$$

$$PR = RQ \quad (\triangle OPR \equiv \triangle OQR)$$

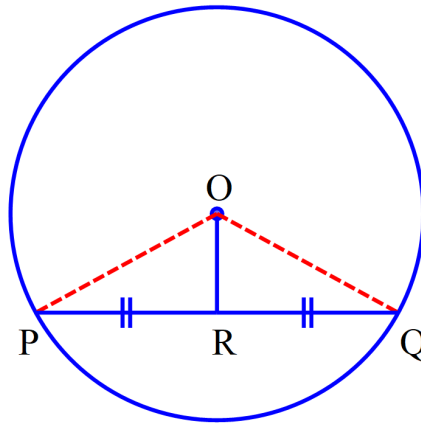
$\Rightarrow OR$ bisects PQ

The converse theorem states that the line drawn from the centre of a circle to the midpoint of a chord will be perpendicular to the chord.

THEOREM 1 (CONVERSE)

The line drawn from the centre of a circle that bisects a chord, is perpendicular to the chord

Given: Circle with centre O and chord PQ. $PR = RQ$ with R on PQ.



Required to Prove: $OR \perp PQ$

Construction: OP and OQ

Proof:

In $\triangle OPR$ and $\triangle OQR$

1) $OP = OQ$ (equal radii)

2) $PR = RQ$ (given)

3) OR is common

$\therefore \triangle OPR \equiv \triangle OQR$ (SSS)

$\therefore \angle ORP = \angle ORQ$

and $\angle ORP + \angle ORQ = 180^\circ$ ($\angle$ on str. line)

$\therefore \angle ORP = \angle ORQ = 90^\circ$

$\Rightarrow OR \perp PQ$

THEOREM 3

The angle subtended by an arc at the centre of the circle is double the size of the angle subtended by the same arc at any point on the circumference of the circle.

Given: A, B and C are 3 points on the circle with centre O.

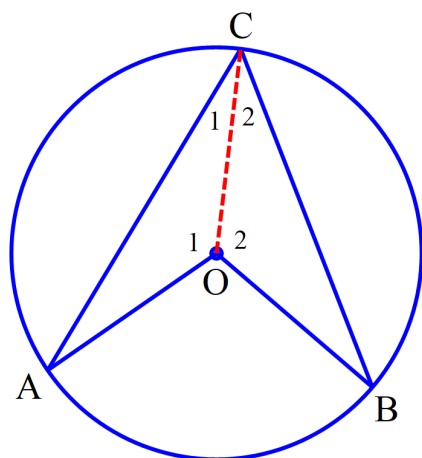


FIGURE 1

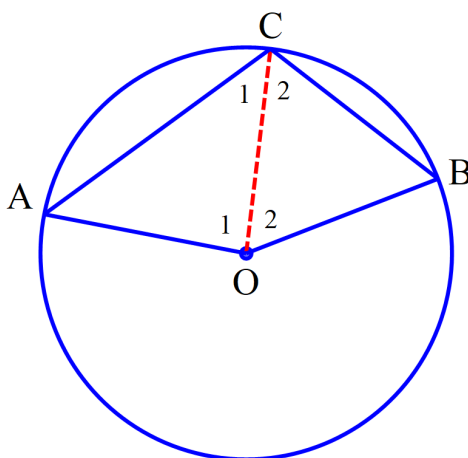


FIGURE 2

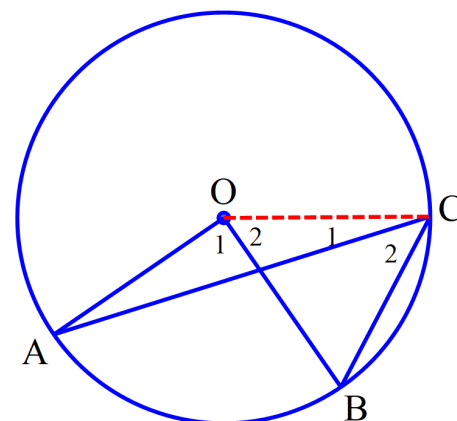


FIGURE 3

Required to Prove: In FIGURES 1 & 3 prove $\hat{AOB} = 2\hat{ACB}$
In FIGURE 2 prove **reflex** $\hat{AOB} = 2\hat{ACB}$

Construction: CO (AO = CO = OB)

Proof: FIGURE 1 and FIGURE 2

Let $\hat{C}_1 = x$ and $\hat{C}_2 = y$

$\Rightarrow \hat{A} = x$ and $\hat{B} = y$ ($\angle$'s opp = sides, radii =)

$\hat{AOC} = 180^\circ - 2x$ ($\angle$ sum ΔAOC)

$\hat{BOC} = 180^\circ - 2y$ ($\angle$ sum ΔBOC)

FIGURE 1: $\hat{AOB} = 2(x + y) = 2\hat{ACB}$ (sum of $\angle$'s around a point)

FIGURE 2: Reflex $\hat{AOB} = 2(x + y) = 2\hat{ACB}$ (sum of $\angle$'s around a point)

Proof: FIGURE 3

Let $\hat{C}_1 = x$ and $\hat{C}_2 = y$

$\hat{A} = x$ and $\hat{B} = x + y$ ($\angle$'s opp = sides, radii =)

$\hat{AOC} = 180^\circ - 2x$ ($\angle$ sum of ΔAOC)

$\hat{BOC} = 180^\circ - (2x + 2y)$ ($\angle$ sum of ΔBOC)

$\hat{AOB} = \hat{AOC} - \hat{BOC} = 2y = 2\hat{ACB}$

This theorem does not have a converse.

THEOREM 5

The opposite angles of a cyclic quadrilateral are supplementary.

Given: D, E, F and G are 4 points on the circle with centre O.

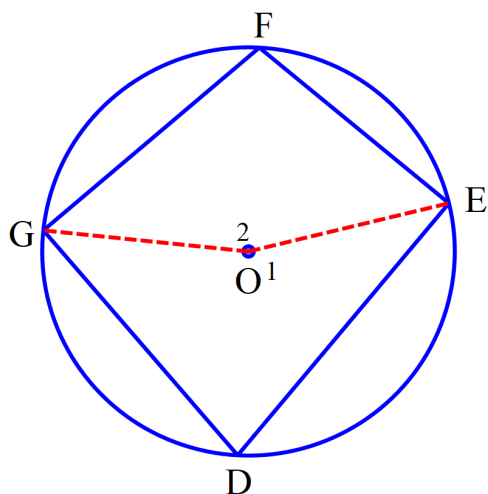


FIGURE 1

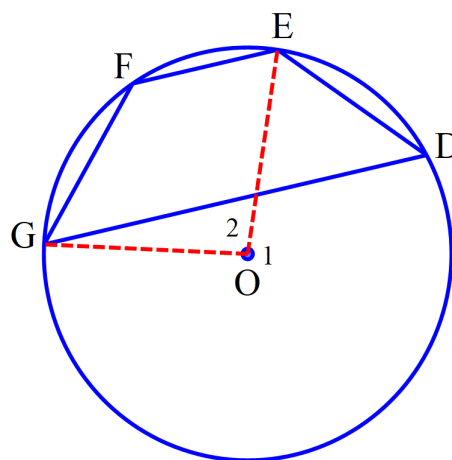


FIGURE 2

Required to Prove: $\hat{D} + \hat{F} = 180^\circ$ and $\hat{D} + \hat{F} = 180^\circ$

Construction: EO and GO

Proof: The proof is the same for FIGURE 1 and FIGURE 2

Let $\hat{D} = x$

$\hat{O}_2 = 2x$ ($\angle$ at centre = $2 \times \angle$ at circumference)

$\hat{O}_1 = 360^\circ - 2x$ ($\angle$ s in a revolution)

$\hat{F} = 180^\circ - x$ ($\angle$ at centre = $2 \times \angle$ at circumference)

$\hat{D} + \hat{F} = 180^\circ$

$\hat{F} + \hat{D} = 180^\circ$ ($\angle$ sum quad)

The converse theorem states that a quadrilateral will be cyclic if its opposite angles are supplementary.

THEOREM 7

The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle subtended by the chord in the alternate segment.

Given: A, B and C are points on the circle with centre O. DA is a tangent to the circle at A.

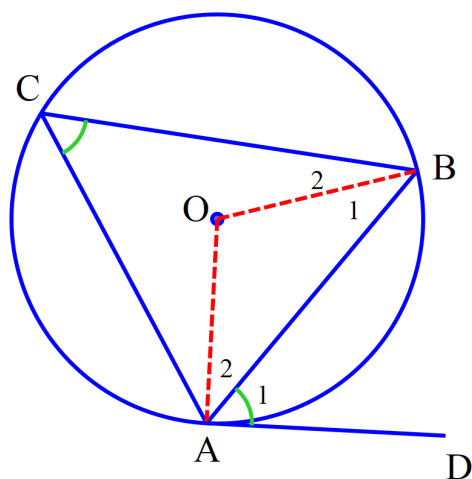


FIGURE 1

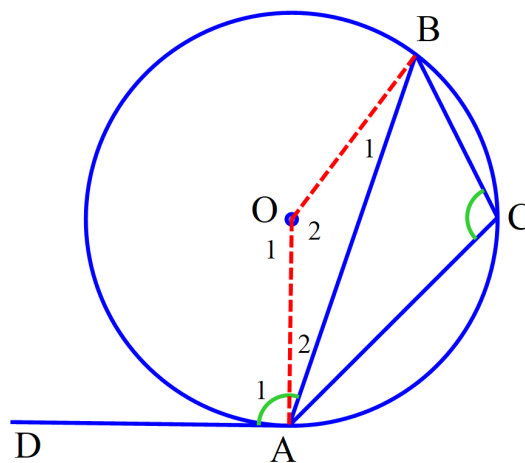


FIGURE 2

Required to Prove: In FIGURE 1, acute $\hat{B}AD = \hat{C}$
In FIGURE 2, obtuse $\hat{B}AD = \hat{C}$

Construction: OA and OB

Proof: FIGURE 1

$$\text{Let } \hat{C} = x$$

$$\hat{O} = 2x \quad (\angle \text{ at centre} = 2x \angle \text{ at circumference})$$

$$\hat{A}_2 = \hat{B}_1 \quad (\angle \text{'s opp} = \text{sides, radii} =)$$

$$\hat{A}_2 = 90^\circ - x \quad (\angle \text{ sum in } \triangle)$$

$$\hat{O}AD = 90^\circ \quad (\text{rad} \perp \text{tan})$$

$$\hat{B}AD = x = \hat{C}$$

Proof: FIGURE 1

$$\text{Let } \hat{C} = x$$

$$\text{Reflex } \hat{O}_1 = 2x \quad (\angle \text{ at centre} = 2x \angle \text{ at circumference})$$

$$\hat{O}_2 = 360^\circ - 2x \quad (\text{sum of } \angle \text{'s around a point})$$

$$\hat{A}_2 = \hat{B}_1 \quad (\angle \text{'s opp} = \text{sides, radii} =)$$

$$\hat{A}_2 = x - 90^\circ \quad (\angle \text{ sum in } \triangle)$$

$$\hat{O}AD = 90^\circ \quad (\text{rad} \perp \text{tan})$$

$$\hat{B}AD = x = \hat{C}$$

In FIGURE 1: Acute $\hat{B}AD = \hat{C}$

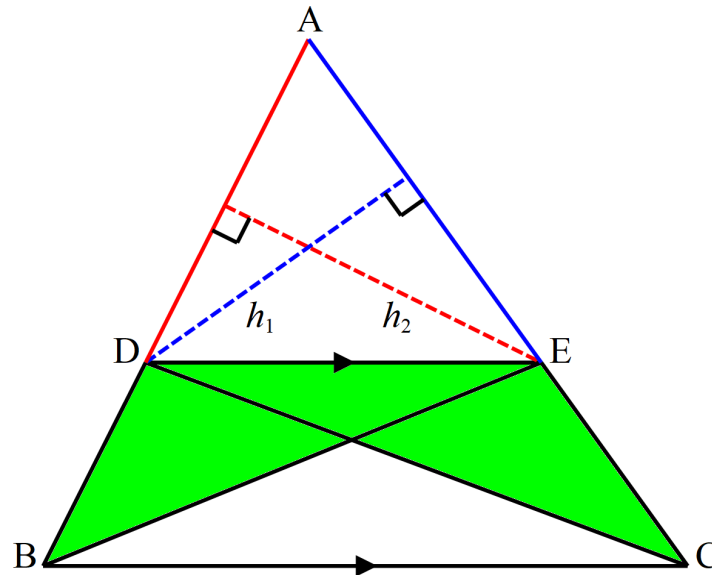
In FIGURE 2: Obtuse $\hat{B}AD = \hat{C}$

The converse theorem states that if the angle between a line and a chord equals the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

THEOREM

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

Given: $\triangle ABC$ with $DE \parallel BC$, such that D lies on AB and E lies on AC.



Construction: $h_1 \perp AE$ and $h_2 \perp AD$

Proof:

$$\frac{\text{Area } \triangle ADE}{\text{Area } \triangle DBE} = \frac{\frac{1}{2} AD \times h_2}{\frac{1}{2} DB \times h_2} = \frac{AD}{DB} \quad \text{common vertex E, same height } h_2$$

$$\frac{\text{Area } \triangle AED}{\text{Area } \triangle ECD} = \frac{\frac{1}{2} AE \times h_1}{\frac{1}{2} EC \times h_1} = \frac{AE}{EC} \quad \text{common vertex D, same height } h_1$$

$$\text{Area } \triangle DBE = \text{Area } \triangle ECD \quad \text{common base DE, same height, } DE \parallel BC$$

$$\frac{\text{Area } \triangle ADE}{\text{Area } \triangle DBE} = \frac{\text{Area } \triangle AED}{\text{Area } \triangle ECD} \quad \text{Area } \triangle ADE \text{ is common and } \text{Area } \triangle DBE = \text{Area } \triangle ECD$$

$$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$$

Using the same sketch, we can also prove that:

$$\frac{AD}{AB} = \frac{AE}{AC}$$

Proof:

$$\frac{\text{Area } \triangle ADE}{\text{Area } \triangle ABE} = \frac{\frac{1}{2} AD \times h_2}{\frac{1}{2} AB \times h_2} = \frac{AD}{AB} \quad \text{common vertex E, same height } h_2$$

$$\frac{\text{Area } \triangle AED}{\text{Area } \triangle ACD} = \frac{\frac{1}{2} AE \times h_1}{\frac{1}{2} AC \times h_1} = \frac{AE}{AC} \quad \text{common vertex D, same height } h_1$$

$$\text{Area } \triangle DBE = \text{Area } \triangle ECD \quad \text{common base DE, same height, } DE \parallel BC$$

$$\text{Area } \triangle ABE = \text{Area } \triangle ACD \quad \text{Area } \triangle ADE \text{ is common and } \text{Area } \triangle DBE = \text{Area } \triangle ECD$$

$$\frac{\text{Area } \triangle ADE}{\text{Area } \triangle ABE} = \frac{\text{Area } \triangle AED}{\text{Area } \triangle ACD} \quad \text{Area } \triangle ADE \text{ is common and}$$

$$\text{Area } \triangle ABE = \text{Area } \triangle ACD$$

$$\Rightarrow \frac{AD}{AB} = \frac{AE}{AC}$$

Finally, it can also be proved that:

$$\frac{AB}{DB} = \frac{AC}{EC}$$

Proof:

$$\frac{\text{Area } \triangle ABE}{\text{Area } \triangle DBE} = \frac{\frac{1}{2} AB \times h_2}{\frac{1}{2} DB \times h_2} = \frac{AB}{DB} \quad \text{common vertex E, same height } h_2$$

$$\frac{\text{Area } \triangle ACD}{\text{Area } \triangle ECD} = \frac{\frac{1}{2} AC \times h_1}{\frac{1}{2} EC \times h_1} = \frac{AC}{EC} \quad \text{common vertex D, same height } h_1$$

$$\text{Area } \triangle DBE = \text{Area } \triangle ECD \quad \text{common base DE, same height, } DE \parallel BC$$

$$\text{Area } \triangle ABE = \text{Area } \triangle ACD \quad \text{Area } \triangle ADE \text{ is common and } \text{Area } \triangle DBE = \text{Area } \triangle ECD$$

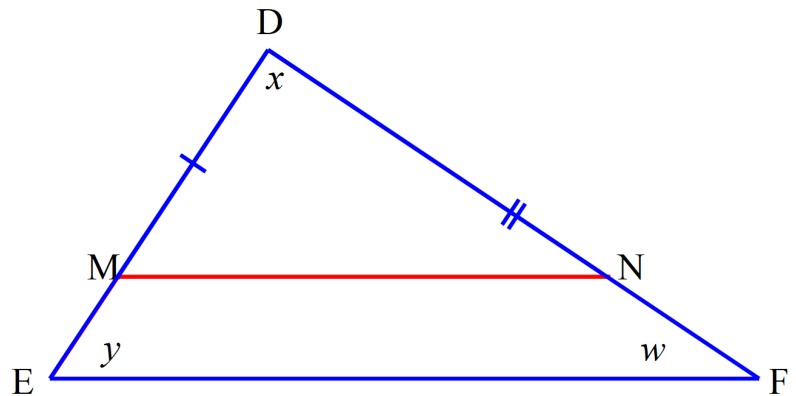
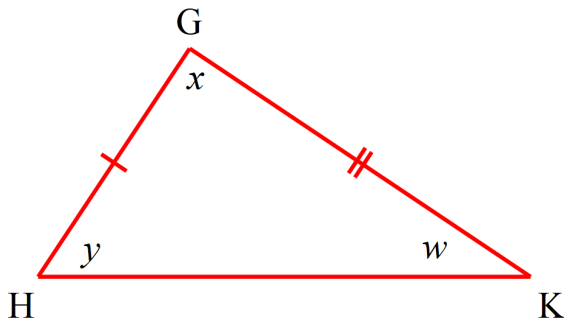
$$\frac{\text{Area } \triangle ABE}{\text{Area } \triangle DBE} = \frac{\text{Area } \triangle ACD}{\text{Area } \triangle ECD} \quad \text{Area } \triangle ADE \text{ is common and } \text{Area } \triangle ABE = \text{Area } \triangle ACD$$

$$\Rightarrow \frac{AB}{DB} = \frac{AC}{EC}$$

THEOREM

If two triangles are equiangular, their corresponding sides are in proportion.

Given: $\triangle DEF$ and $\triangle GHK$ with $\hat{D} = \hat{G}$, $\hat{E} = \hat{H}$ and $\hat{F} = \hat{K}$



Required to Prove: $\frac{DE}{GH} = \frac{DF}{GK} = \frac{EF}{HK}$

Construction: On DE and DF mark points M and N so that $DM = GH$ and $DN = GK$. Join MN.

Proof:

In $\triangle DMN$ and $\triangle GHK$

1) $DM = GH$ (construction)

2) $DN = GK$ (construction)

3) $\hat{D} = \hat{G}$ (given)

$\therefore \triangle DMN \equiv \triangle GHK$ (SAS)

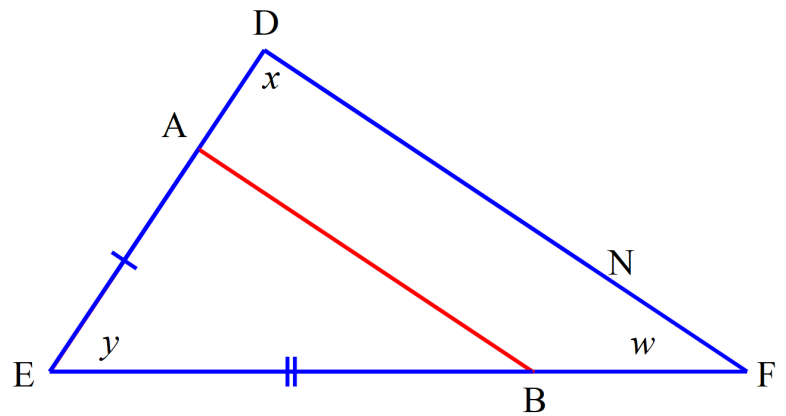
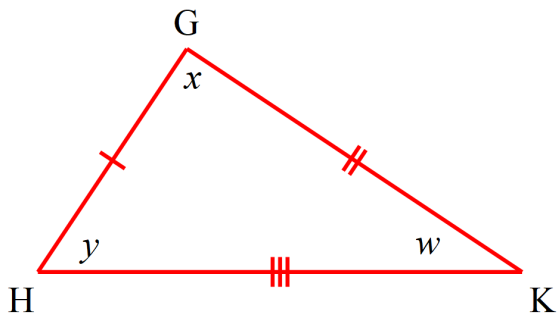
$\therefore \hat{DMN} = \hat{GHK}$ (congruency)
 $= \hat{E}$ (given)

$\therefore MN \parallel EF$ (corresponding $\angle$ s =)

$\therefore \frac{DE}{DM} = \frac{DF}{DN}$ (Proportionality Theorem $MN \parallel EF$)

But $DM = GH$ and $DN = GK$ (construction)

$\therefore \frac{DE}{GH} = \frac{DF}{GK}$



Similarly, by marking off points A and B on ED and EF respectively so that

$EA = HG$ and $EB = HK$, it can be proved that $\frac{DE}{GH} = \frac{EF}{HK}$

$$\therefore \frac{DE}{GH} = \frac{DF}{GK} = \frac{EF}{HK}$$

$\Rightarrow \triangle DEF \sim \triangle GHK$ corresponding angles equal and corresponding sides in proportion

TRIGONOMETRIC COMPOUND FORMULAE

1. Derivation of $\cos(\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ from $\cos(\alpha - \beta)$

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) \\ &= \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta)\end{aligned}$$

But $\cos(-\beta) = \cos \beta$ and $\sin(-\beta) = -\sin \beta$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

2. Derivation of $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ from $\cos(\alpha - \beta)$

$$\begin{aligned}\sin(\alpha + \beta) &= \cos[90^\circ - (\alpha + \beta)] \\ &= \cos[(90^\circ - \alpha) - \beta] \\ &= \cos(90^\circ - \alpha) \cos \beta + \sin(90^\circ - \alpha) \sin \beta\end{aligned}$$

But $\cos(90^\circ - \alpha) = \sin \alpha$ and $\sin(90^\circ - \alpha) = \cos \alpha$

$$\therefore \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

3. Derivation of $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ from $\cos(\alpha - \beta)$

$$\begin{aligned}\sin(\alpha - \beta) &= \cos[90^\circ - (\alpha - \beta)] \\ &= \cos[(90^\circ - \alpha) - (-\beta)] \\ &= \cos(90^\circ - \alpha) \cos(-\beta) + \sin(90^\circ - \alpha) \sin(-\beta)\end{aligned}$$

But $\cos(90^\circ - \alpha) = \sin \alpha$; $\sin(90^\circ - \alpha) = \cos \alpha$ and $\cos(-\beta) = \cos \beta$; $\sin(-\beta) = -\sin \beta$

$$\therefore \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

4. Derivation of $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ from $\sin(\alpha + \beta)$

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\ &= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)\end{aligned}$$

But $\cos(-\beta) = \cos \beta$ and $\sin(-\beta) = -\sin \beta$

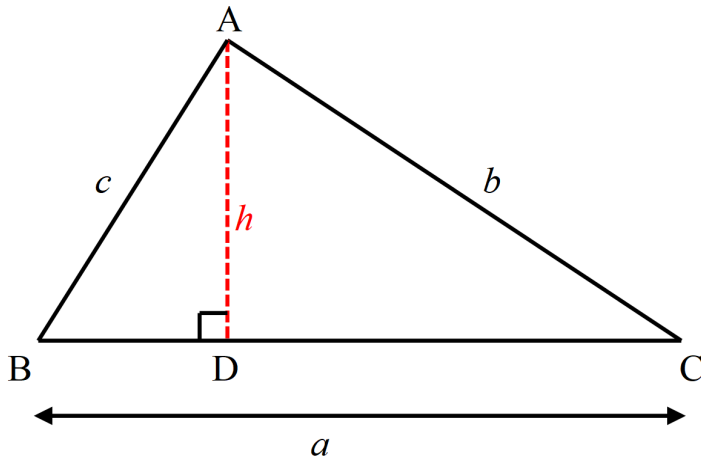
$$\therefore \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

etc...

SIN RULE and its proof

Required to Prove: In any $\triangle ABC$: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ OR $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Proof:



Acute-angled triangle

Let $AD = h$
= height of $\triangle ABC$ with base BC

$$\sin B = \frac{h}{c} \quad \text{and} \quad \sin C = \frac{h}{b}$$

$$\therefore h = c \sin B \quad \text{and} \quad h = b \sin C$$

Equate h on both sides:

$$\therefore c \sin B = b \sin C$$

Divide both sides by bc :

$$\therefore \frac{c \sin B}{bc} = \frac{b \sin C}{bc}$$

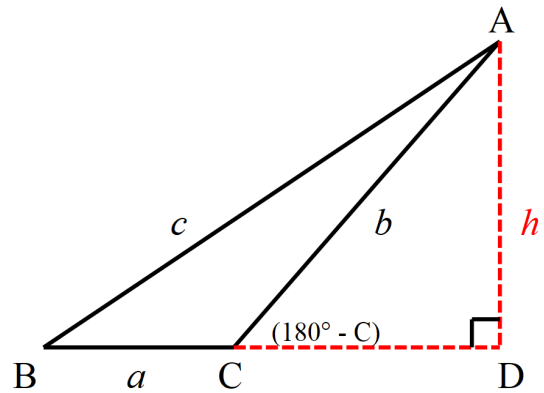
$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let $CE = h$
= height of $\triangle ABC$ with base AB

Repeat the steps above to get:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Obtuse-angled triangle

Let $AD = h$
= height of $\triangle ABC$ with base BC

$$\sin B = \frac{h}{c} \quad \text{and} \quad \sin(180^\circ - C) = \frac{h}{b}$$

$$\text{but } \sin(180^\circ - C) = \sin C$$

$$\therefore h = c \sin B \quad \text{and} \quad h = b \sin C$$

Equate h on both sides:

$$\therefore c \sin B = b \sin C$$

Divide both sides by bc :

$$\therefore \frac{c \sin B}{bc} = \frac{b \sin C}{bc}$$

$$\therefore \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let $AD = h$
= height of $\triangle ABC$ with base AB

Repeat the steps above to get:

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{which is the same as } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

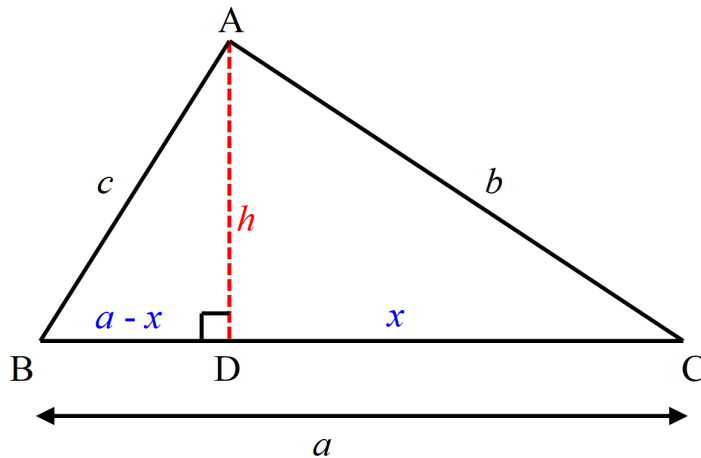
COSINE RULE and its proof

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Required to Prove: In any $\triangle ABC$: $b^2 = a^2 + c^2 - 2ac \cos B$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof:



Acute-angled triangle

Let $AD = h$
= height of $\triangle ABC$ with base BC

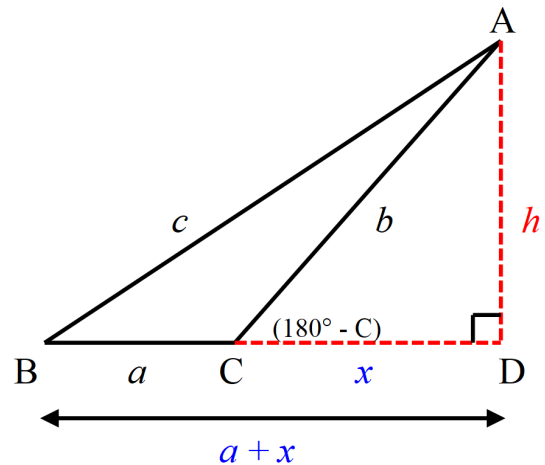
Apply Pythagoras' Theorem to $\triangle ABD$
to find c^2

$$\begin{aligned} c^2 &= h^2 + (a - x)^2 \\ &= h^2 + a^2 - 2ax + x^2 \\ &= a^2 + (h^2 + x^2) - 2ax \\ &= a^2 + b^2 - 2ax \end{aligned}$$

$$b^2 = h^2 + x^2 \quad (\text{Pythagoras in } \triangle ADC)$$

$$\begin{aligned} \frac{x}{b} &= \cos C \\ \therefore x &= b \cos C \end{aligned}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



Obtuse-angled triangle

Let $AD = h$
= height of $\triangle ABC$ with base BC

Apply Pythagoras' Theorem to $\triangle ABD$
to find c^2

$$\begin{aligned} c^2 &= h^2 + (a + x)^2 \\ &= h^2 + a^2 + 2ax + x^2 \\ &= a^2 + (h^2 + x^2) + 2ax \\ &= a^2 + b^2 + 2ax \end{aligned}$$

$$b^2 = h^2 + x^2 \quad (\text{Pythagoras in } \triangle ADC)$$

$$\begin{aligned} \frac{x}{b} &= \cos(180^\circ - C) \\ \therefore x &= -b \cos C \end{aligned}$$

$$\cos(180^\circ - C) = -\cos C$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

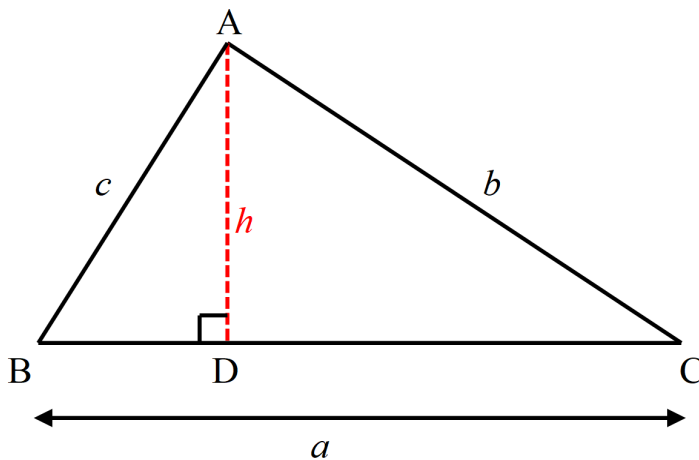
AREA RULE and its proof

$$Area = \frac{1}{2}ab \sin C$$

Required to Prove: In any $\triangle ABC$: $Area = \frac{1}{2}ac \sin B$

$$Area = \frac{1}{2}bc \sin A$$

Proof:



Acute-angled triangle

Let $AD = h$
 = height of $\triangle ABC$ with base BC

$$Area \triangle ABC = \frac{1}{2}a \times h$$

$$\frac{h}{c} = \sin B \quad OR \quad \frac{h}{b} = \sin C$$

$$\therefore h = c \sin B \quad OR \quad h = b \sin C$$

Substitute h

$$Area = \frac{1}{2}ac \sin B \quad OR \quad A = \frac{1}{2}ab \sin C$$

Let $CE = h$ = height of $\triangle ABC$ with base AB
 Repeat the steps above to get:

$$Area = \frac{1}{2}ac \sin B \quad OR \quad A = \frac{1}{2}bc \sin A$$

