

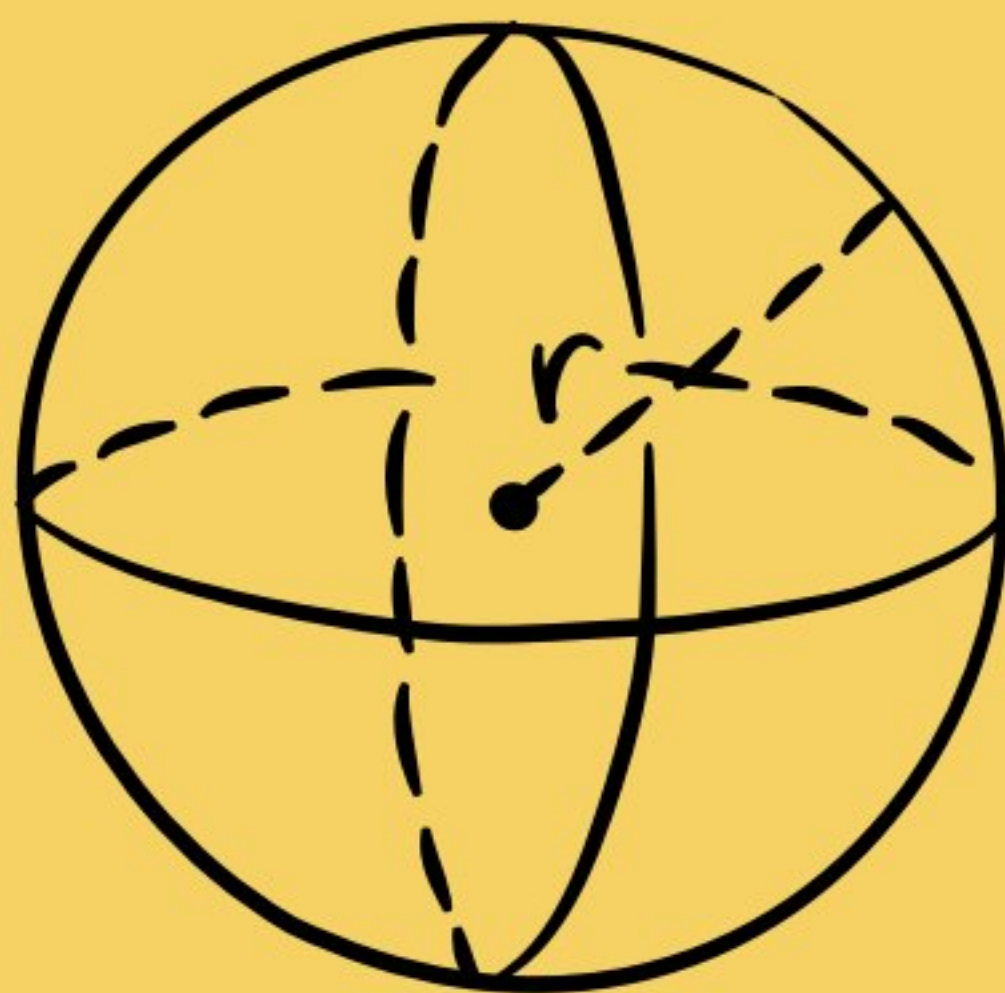
THE CHEAT SHEET

Grade

12

CAPS

MATHEMATICS: PAPER 1



$$V = \frac{4}{3} \pi r^3$$

THE CHEAT SHEET: GR. 12 MATHEMATICS PAPER 1



Miss Hanekom x



Miss Angler

1ST EDITION

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ACKNOWLEDGEMENTS

Questions taken from past NSC Examination Papers are cited individually.

Grade 12 NSC November 2022 Paper 1 pg. 84-86, 88, 98-99

Grade 12 NSC May/June 2022 Paper 1 pg. 65, 70

Grade 12 NSC November 2021 Paper 1 pg. 7-11, 63-64, 69, 97, 103

Grade 12 NSC May/June 2021 pg. 89

Grade 12 NSC November 2020 Paper 1 pg. 12, 56-57, 70, 80-81, 102

Grade 12 NSC November 2019 Paper 1 pg. 58-59, 90, 94, 95

Grade 12 NSC November 2018 Paper 1 pg. 49-51, 94, 96, 101

All questions not cited have been created by the author.

Grade 12 Mathematics Examination Guidelines, pg. 4-5

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HOW TO USE THIS STUDY GUIDE

PLEASE REMEMBER: This study guide does not replace your textbook or the instruction of your teacher. Its purpose is to make learning and revision EASY!

1. The Cheat Sheet is laid out in the same way that your final exam will be laid out.

- Keep this Cheat Sheet next to you when working through past papers for guided examples and explanations.

2.



- Use these pro-tips for exam success!
- These tips are often the reason why my students get 80%+ in their exams and finals.
- They also point out common mistakes you should be careful of when answering questions.

3.

QUESTION TYPES

Each question is also labelled with its difficulty level (Look out for either **K**, **R**, **C** or **P** next to each question!) so that you can prepare properly for the different levels of questions.

See **page 5** for a detailed explanation of the type of questions that each difficulty level could entail.

Each section is broken up into the **QUESTION TYPES** that most frequently show up in final examinations.

REMEMBER: ALWAYS USE YOUR EXAM GUIDELINE WHEN PREPARING FOR AN EXAM OR TEST!

PAPER 1: TOPICS & WEIGHTINGS

You will be writing a 3-hour paper
out of 150 marks.

The 150 marks will be made up
as follows:

Algebra, Equations & Inequalities	25 marks
Number Patterns	25 marks
Functions & Graphs	35 marks
Finance, Growth & Decay	15 marks
Differential Calculus	35 marks
Counting Principle & Probability	15 marks

The questions in your final
Paper 1 exam are also set up
in this same order.

EXAM KEY

It's a good idea to plan your
revision time per section
according to these
weightings.

25 marks = 25 min practice Qs

EXPLAINING THE DIFFICULTY LEVELS OF QUESTIONS

Your exam will include questions covering **four** difficulty levels. Each level of question will require a different level of understanding and way of answering the question. It is helpful to know which questions fall in each of the following levels:

Knowledge (20 marks) - K

Recalling information, identifying correct formulae from the formula sheet, using mathematical facts and terminology, and rounding correctly.

Routine questions (52-53 marks) - R

Proving theorems and deriving formulae, performing familiar procedures and simple calculations that could involve a few steps, identifying and using the correct formulae from the formula sheet, answering questions that are similar to those taught in class.

Complex questions (45 marks) - C

Complex calculations that require higher-order thinking to solve. Being able to make connections between different representations of information and integrating different topics.

Problem-solving (22-23 marks) - P

Problems that are unfamiliar but not necessarily difficult. Questions that are asked differently from how they were in class, but can be solved using existing knowledge on the topic.

IMPORTANT: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either **K**, **R**, **C** or **P** next to each question!

ALGEBRA, EQUATIONS & INEQUALITIES

Many people think that this topic is tested only in the first question of your final Paper 1 exam.

It is, however, tested throughout **all other topics** as well.

Being able to perform basic algebraic calculations is incredibly important if you want to achieve a good mark for your Mathematics exam!

Algebraic skills and concepts are mostly tested in the following ways:

1. **Solving linear, quadratic and exponential equations**, including making a specific variable the subject in a formula. An example of this would be in finance, where you are expected to calculate the interest rate of a compound interest investment.
2. **Solving multiple equations simultaneously**, for example, when asked to calculate the point of intersection of two functions.
3. **Solving inequalities**, for example, when asked to calculate where a cubic graph is concave up.
4. **Working with exponent and surd laws**, which is tested extensively in the Differential Calculus section, for example, when asked to find the derivative of an algebraic expression that has surds, fractions and negative exponents.

These types of questions will be dealt with later, so keep an eye out for that!

EXAM KEY

Now let's get back to **QUESTION 1** of your final exam!

This question looks the same every single year.

PRO-TIPS

You will be asked to do the following:

1. Solve a quadratic equation by **factorising**
2. Solve a quadratic equation using the **quadratic formula**
3. Solve a **quadratic inequality**
4. Solve a **surd equation**
5. Solve two equations **simultaneously**

Question Type 1: Solving a quadratic equation by factorising

You need to be able to do any of the following types of factorising:

- Taking out a common factor (including a common bracket and possibly having to do a switcharound)
- A difference of two squares
- A sum or difference of two cubes
- A trinomial
- Factorising by grouping

The most common type of factorising used in this question is **factorising a trinomial**.

When factorising a trinomial, remember to look for factors of the constant term (the term with no variable) such that the sum of/difference between these factors will be equal to the coefficient of the middle term.

Remember that the terms should always be written in the **correct order** before factorising.

$$ax^2 + bx + c = 0$$



Worked example (R)

Solve for x :

$$x^2 - 2x - 24 = 0$$

$$\begin{aligned}x^2 - 2x - 24 &= 0 \\(x + 4)(x - 6) &= 0 \\x &= -4 \text{ or } x = 6\end{aligned}$$

PRO-TIPS

If you don't want to factorise a trinomial, you can **ALWAYS** use the quadratic formula instead!

Since the constant term is negative, we need to find factors of 24 such that the difference between the factors is -2 $\therefore 4$ and 6 .
 $4 \times 6 = 24$ and $4 - 6 = -2$
 $\therefore$ One bracket will have $+4$ and the other bracket will have -6 .

Question source: NSC Grade 12 November 2021 Paper 1

Question Type 2:

Solving a quadratic equation using the quadratic formula

The quadratic formula is on the formula sheet, so you do not have to know it off by heart. You simply have to be able to identify it.

It is the very first formula on the formula sheet.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Again, it is very important that your quadratic equation is written in the **correct form** before you use the quadratic formula to solve it.

You then substitute the "a", "b" and "c" - values from the equation into the formula and use your calculator to determine the two solutions to the equation.

The first solution will be found by using the + in the formula, and the second solution will be found using the - in the formula.

Remember to round your answer correctly according to the instructions!

PRO-TIPS

A **mark** is awarded for the **substitution** step, so make sure that you don't leave it out!



Worked example (K)

Solve for x :

$$2x^2 - 3x - 3 = 0 \quad (\text{correct to TWO decimal places})$$

$$2x^2 - 3x - 3 = 0$$

$$a = 2; b = -3; c = -3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}$$

substitution
step

$$x = 2,19 \quad \text{or} \quad x = -0,69$$

Question source: NSC Grade 12 November 2021 Paper 1



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Question Type 3: Solving a quadratic inequality

We approach a quadratic inequality in a similar way to a quadratic equation, but we have to perform a few extra steps that are crucial to getting the correct answer.

How to solve a quadratic inequality:

1. Write the inequality in the **correct order**, as with quadratic equations. Make sure that the inequality has zero on the right-hand side.
2. **Factorise** the expression on the left-hand side.
3. Determine the **critical values** of the expression. Critical values are values that will make the expression **EQUAL TO ZERO** (or undefined, but that is not an issue with these questions).
4. Draw a number line with the critical values OR draw a very basic **parabola** with the critical values as x-intercepts. We are going to use the second method as it is more time-efficient.
5. Write the **answer to the inequality**, depending on whether the question was asking for where the expression must be positive or negative.

PRO-TIPS

NO marks are awarded for drawing a **number line** or parabola, so you can leave it out if you are able to visualise it.



Worked example (R)

Solve for x :

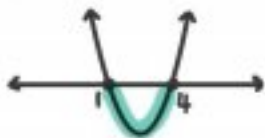
$$x^2 - 5x \leq -4$$

$$x^2 - 5x + 4 \leq 0$$

$$(x-4)(x-1) \leq 0$$

Critical values:

$$x = 4 \text{ or } x = 1$$



$$\therefore 1 \leq x \leq 4$$

Read this answer from the inside out:

x is greater than or equal to 1 AND at the same time, it's smaller than or equal to 4.

Write in the correct order: $ax^2 + bx + c \leq 0$

Factorise the trinomial on the LHS. LHS = left hand side

If $(x-4)(x-1) = 0$, the x -values would be $x = 4$ or $x = 1$
 $\therefore$ these are the critical values.

Since we are solving for x , we are not interested in the y -axis, and do not have to draw it.

We are highlighting the negative part of the parabola because of the inequality in the statement: $x^2 - 5x + 4 \leq 0$
 " ≤ 0 " means "smaller than or equal to zero" $\therefore$ negative

You can also write the answer in interval notation:
 $x \in [1; 4]$

Question source: NSC Grade 12 November 2021 Paper 1

Question Type 4: Solving a surd equation

How to solve a surd equation:

1. Isolate the surd (move all other terms to the other side of the equation).
2. Square both sides of the equation to get rid of the surd.
3. Solve the resulting equation.
4. **CHECK YOUR SOLUTIONS** by substituting them into the original surd equation. If the two sides of that equation are not equal, that means that your solution is invalid.
5. State any invalid solutions.



Worked example (R)

Solve for x :

$$\sqrt{x+28} = 2-x$$

$$\begin{aligned}\sqrt{x+28} &= 2-x \\ (\sqrt{x+28})^2 &= (2-x)^2 \\ x+28 &= 4-4x+x^2 \\ x^2-5x-24 &= 0 \\ (x-8)(x+3) &= 0 \\ x &\neq 8 \text{ or } x = -3 \\ &\text{invalid}\end{aligned}$$

The surd is already isolated here.
Square both sides.

Remember the following shortcut for squaring a binomial:
 $(a+b)^2 = a^2 + \underline{2ab} + b^2$ the middle term has to be there!

$$\begin{array}{ll}\text{Check } x = 8: \text{ LHS} = \sqrt{8+28} & \text{RHS} = 2-8 \\ & = 6 \qquad \qquad \qquad = -6\end{array}$$

$$\text{LHS} \neq \text{RHS}$$

$\therefore x = 8$ is not a valid solution

$$\begin{array}{ll}\text{Check } x = -3: \text{ LHS} = \sqrt{-3+28} & \text{RHS} = 2-(-3) \\ & = 5 \qquad \qquad \qquad = 5\end{array}$$

$$\text{LHS} = \text{RHS}$$

$\therefore x = -3$ is a valid solution

PRO-TIPS

Squaring a binomial incorrectly is known as a **BREAKDOWN** error, which will cause you to get zero.

Question source: NSC Grade 12 November 2021 Paper 1



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Question Type 5: Solving two equations simultaneously

There are two methods for solving equations simultaneously:

Method 1: SUBSTITUTION

This method involves isolating a variable in one equation, substituting that into the other equation and then solving for the other variable. You can then substitute the value found back into the first equation, and solve for the first variable.

Method 2: ELIMINATION

This method involves subtracting one equation from the other so that one of the two variables cancel out. Sometimes you have to first multiply an entire equation by a number so that it is possible for a variable to cancel out.

Worked example (R)

Solve for x and y :

$$2y = 3 + x \quad \text{and} \quad 2xy + 7 = x^2 + 4y^2$$

$$\begin{array}{l} \textcircled{1} \\ 2y = 3 + x \\ 2y - 3 = x \end{array} \quad \text{and} \quad \begin{array}{l} \textcircled{2} \\ 2xy + 7 = x^2 + 4y^2 \end{array}$$

$$\begin{aligned} 2y(2y - 3) + 7 &= (2y - 3)^2 + 4y^2 \\ 4y^2 - 6y + 7 &= 4y^2 - 12y + 9 + 4y^2 \\ -4y^2 + 6y - 2 &= 0 \\ 2y^2 - 3y + 1 &= 0 \\ (y - 1)(2y - 1) &= 0 \\ y = 1 \text{ or } y &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{If } y = 1: x &= 2(1) - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{If } y = \frac{1}{2}: x &= 2\left(\frac{1}{2}\right) - 3 \\ &= -2 \end{aligned}$$

PRO-TIPS

Elimination is **NOT** suitable for the majority of questions asked in the Paper 1 final exam, so make sure that you are able to use substitution effectively.

Identify the easiest variable to isolate.

This is the variable with a coefficient of 1

$\therefore x$ in equation 1

Now replace x in the other equation (equation 2) with the expression found for x in equation 1.

You now have an equation with only one variable that you can solve.

You now have to substitute both y -values into the expression found for x in equation 1.

Question source: NSC Grade 12 November 2021 Paper 1



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What about problem solving questions?

EXAM KEY

The **final** question of Question 1 is always a problem solving question, focussed on skills such as **simplifying exponential** expressions and working with the **nature of roots**.

Let's do some examples so that you can get a feel for the type of questions that you could expect!

REMEMBER

Each problem solving question can be solved using knowledge that you have on the topic, so **don't be afraid** to try different ways of solving the problem!

Worked example (P)

If n is the largest integer for which $n^{200} < 5^{300}$, determine the value of n .

$$\begin{aligned}n^{200} &< 5^{300} \\(n^3)^{100} &< (5^3)^{100} \\(n^3)^{100} &< (125)^{100} \\n^3 &< 125\end{aligned}$$

$$\therefore n = 11$$

We can only solve an exponential equation/inequality if we have numbers that have the same bases or the same exponents.
 $\therefore$ write both exponents in terms of the same number.
This allows us to simplify the inequality to one that is easier to solve.
 $11^3 = 121$ and $12^3 = 144$
 $\therefore$ The largest integer that n could be is 11.

Question source: NSC Grade 12 November 2020 Paper 1

Worked example (P)

For which value of k will the equation $x^2 + x = k$ have no real roots?

$$\begin{aligned}x^2 + x &= k \\x^2 + x - k &= 0\end{aligned}$$

No real roots:

$$\begin{aligned}\Delta &< 0 \\b^2 - 4ac &< 0 \\(1)^2 - 4(1)(-k) &< 0 \\1 + 4k &< 0 \\4k &< -1 \\k &< -\frac{1}{4}\end{aligned}$$

This is a quadratic equation, so we need to write the terms in the correct order, $ax^2 + bx + c = 0$.
In this case, $a = 1$, $b = 1$ and $c = -k$.
For a quadratic equation to have non-real roots, the discriminant (Δ) of the equation must be negative.
We substitute the values of a , b and c of this equation into the formula for Δ and use that to calculate the value of k .

NUMBER PATTERNS



In order to achieve maximum marks for this section, you need to:

- be able to identify the type of number pattern that is being tested
- apply the correct formulae
- be able to differentiate between "term" and "sum"
- be able to prove the sum-formulae for arithmetic and geometric series



An **ARITHMETIC** pattern is one with a **constant first difference**, which is a number that is added to each term.

This type of pattern is also called a **LINEAR** pattern.

For example, 3; 7; 11; 15; ... the constant first difference is 4



A **QUADRATIC** pattern is one with a **constant second difference**, which means that there is a constant difference among the first differences.

For example, 1; 4; 9; 16; 25; ... the first differences are 3; 5; 7; 9

the constant difference between these numbers is 2

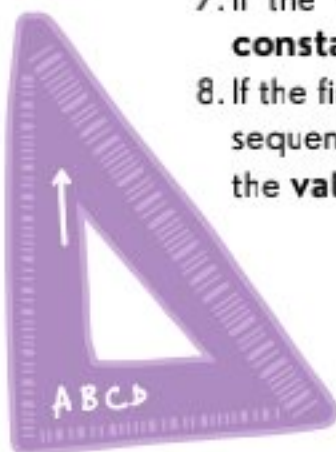


A **GEOMETRIC** pattern is one with a **constant ratio**, which is a number that each term is multiplied by.

For example, 2; 6; 18; 54; ... the constant ratio is 3

For each of these patterns, you need to be able to answer the same basic questions:

1. What is the **general** term?
2. What is the **value** of a specific term in the sequence?
3. How **many** terms are there in a **sequence**?
4. If the value of a specific term is given, which **term** in the sequence is it?
5. What is the **sum** of a certain number of terms in the series?
6. If the sum of a series is given, how **many terms** are in the **series**?
7. If the values of two terms in a sequence are given, what is the value of the **constant difference/ratio** of the sequence?
8. If the first three terms of an arithmetic or geometric sequence are given in terms of a variable, what is the **value(s)** of that **variable**?



DEFINITIONS

Sequence: a list of numbers forming a pattern, separated using a ;

Series: a list of numbers forming a pattern, added together



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Let's talk about the formulae

Most of the formulae that you might need are on the formula sheet, but you have to **KNOW** them well enough to be able to identify the correct ones.

ARITHMETIC SEQUENCES/SERIES:

To determine the general term:

$$T_n = a + (n-1)d$$

To calculate the sum of the first n terms:

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

GEOMETRIC SEQUENCES/SERIES:

To determine the general term:

$$T_n = ar^{n-1}$$

To calculate the sum of the first n terms:

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

where: a = first term

d = constant first difference

r = constant ratio

n = number of terms / term number

There is also a formula for **SUM TO INFINITY**, but we will discuss that later.

QUADRATIC PATTERNS:

To determine the general term:

$$T_n = an^2 + bn + c$$

where $a = \frac{\text{2nd difference}}{2}$

$$3a + b = T_2 - T_1$$

$$a + b + c = T_1$$

To calculate the sum of the first n terms:

There is no set formula for this.

} These formulae
are not on the
formula sheet!

How to remember which formulae go with which pattern:

- Arithmetic sequences and series have a constant **difference**, so we use the formulae with "**d**".
- Geometric sequences and series have a constant **ratio**, so we use the formulae with "**r**".
- Quadratic patterns use the formula that looks like the standard form of a quadratic equation (remember section 1!).



Quadratic number patterns

This is a Grade 11 section but it is commonly tested in Grade 12 Final Exams, so let's look at a worked example.

Worked example

Consider the quadratic number pattern: $-145; -122; -101; \dots$

- Write down the values of the next two terms.
- Determine the general term of the number pattern.
- Between which two terms of the quadratic number pattern will there be a difference of -121 ?

Worked example (K)

$$1. \quad -145; -122; -101; -82; -65$$

$$\begin{array}{ccccccc} & +23 & & +21 & & +19 & & +17 \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ & -2 & & -2 & & -2 & & -2 \end{array}$$

To get the values of the next two terms, we simply continue the pattern.
Each first difference must be 2 less than the previous one.

Worked example (R)

$$2. \quad T_n = an^2 + bn + c$$

$$a = \frac{-2}{2}$$

$$= -1$$

To calculate a , we divide the second difference by 2.

$$3a + b = T_2 - T_1$$

$$3(-1) + b = 23$$

$$-3 + b = 23$$

$$b = 26$$

Since this formula contains a and b , we have to calculate a before b . We then substitute the a -value into this formula.

$$a + b + c = T_1$$

$$-1 + 26 + c = -145$$

$$c = -170$$

Since this formula contains a , b and c , we have to calculate a and b before c . We then substitute the a - and b -values into this formula.

$$\therefore T_n = -n^2 + 26n - 170$$

Worked example (C)

$$3. \quad \underbrace{23}_{-2}; \underbrace{21}_{-2}; \underbrace{19}_{-2}; \underbrace{17}_{-2}; \dots; -121; \dots$$

$$T_n = a + (n-1)d$$

$$23 + (n-1)(-2) = -121$$

$$-2(n-1) = -144$$

$$n-1 = 72$$

$$n = 73$$

$\therefore T_{73}$ of the first differences
 $\therefore$ Between T_{73} and T_{74} of
the quadratic number pattern

The first differences form an arithmetic sequence with a constant difference of -2 .

We need to determine which term in the sequence of first differences is equal to -121 , so we make $T_n = -121$ and solve for n in the T_n - formula.

T_1 of the first differences (23) is between T_1 and T_2 of the quadratic pattern; T_2 of the first differences (21) is between T_2 and T_3 of the quadratic pattern; and so on.

Arithmetic sequences/series

Arithmetic sequences/series have a constant **FIRST DIFFERENCE**, so we use the formulae with "d".

This topic is often tested in a less straightforward way than quadratic number patterns. The terms in an arithmetic sequence/series are usually given in terms of a variable, and you then have to use the following formula to calculate the value of the variable:

$$T_3 - T_2 = T_2 - T_1$$

We know that this formula is always true for an arithmetic sequence/series because it has a **CONSTANT DIFFERENCE**, which can be calculated by **subtracting one term from the next**.

Worked example (R)

The first three terms of an arithmetic sequence are $x - 2$; $2x$; $x + 4$.

1. Determine the value of x .

$$\begin{array}{l} T_1 \quad T_2 \quad T_3 \\ x - 2; 2x; x + 4 \\ x + 4 - 2x = 2x - (x - 2) \\ -x + 4 = 2x - x + 2 \\ -2x = -2 \\ x = 1 \end{array}$$

We apply the above formula to the three terms that are given. This gives us an equation that we can use to calculate the value of x .

PRO-TIPS

Once we know what the **value of x** is, we can calculate the values of the terms and answer the next questions.

2. Write the general term of this sequence.

$$\begin{array}{lll} T_1 = x - 2 & T_2 = 2x & T_3 = x + 4 \\ = 1 - 2 & = 2(1) & = 1 + 4 \\ = -1 & = 2 & = 5 \\ \therefore a = -1 \text{ and } d = 3 \end{array}$$

$$\begin{array}{l} T_n = a + (n-1)d \\ = -1 + (n-1)(3) \\ = -1 + 3n - 3 \\ = 3n - 4 \end{array}$$

Remember to put brackets around the d -value, **ESPECIALLY** if it's a negative value. Then multiply that value into the bracket and always simplify your final answer.

3. Determine the value of the 45th term.

$$\begin{array}{l} T_{45} = 3(45) - 4 \\ = 131 \end{array}$$

Substitute $n = 45$ into the general term formula.



Let's look at some other common exam type questions and how to solve them!



Worked example (R)

Given the arithmetic sequence: 35; 28; 21; ...

Calculate which term of the sequence will have a value of -140 .

$$35; 28; 21; \dots$$

$\underbrace{\quad}_{-7} \quad \underbrace{\quad}_{-7}$

$$T_n = a + (n-1)d \quad \text{and} \quad T_n = -140$$

$$a + (n-1)d = -140$$

$$35 + (n-1)(-7) = -140$$

$$(n-1)(-7) = -175$$

$$n-1 = 25$$

$$n = 26$$

$$\therefore T_{26} = -140$$

We make the unknown term equal to T_n .

Now we substitute the values of the first term and the common difference into the T_n -formula, which will allow us to calculate n , the term number.

Worked example (R)

The 100th term of an arithmetic sequence is 512 and its common difference is 7.

Determine the first term of this sequence.

$$T_{100} = 512 \quad \text{and} \quad d = 7$$

$$a + 99d = 512$$

$$a + 99(7) = 512$$

$$a + 693 = 512$$

$$a = -181$$

It's useful to rewrite the given information in mathematical terms.

Since they gave us the value of T_{100} , we use the TERM formula and replace n with 100.

$$T_{100} = a + (100-1)d = a + 99d$$

We are now also able to replace d with the given value, which then allows us to solve for a .



Worked examples (R)



Given the finite arithmetic sequence: 5; 1; -3; ...; -83; -87



1. Calculate the number of terms in the sequence.

$$5; 1; -3; \dots; -83; -87 \quad T_n = -87$$

$\underbrace{\quad}_{-4} \quad \underbrace{\quad}_{-4}$

$$a = 5; d = -4 \text{ and } T_n = -87$$

$$a + (n-1)d = T_n$$

$$5 + (n-1)(-4) = -87$$

$$(n-1)(-4) = -92$$

$$n-1 = 23$$

$$n = 24$$

$$\therefore T_{24} = -87$$

$\therefore$ There are 24 terms in the sequence.

When asked to find the number of terms in a sequence and the value of the last term is given, we make that term equal to T_n and solve for n .

Remember to put brackets around the d -value as you are multiplying it by the $(n-1)$ bracket!

If the last term in the sequence is the 24th term, that means that there are 24 terms.

Remember to always answer the question fully in your last line.



2. Calculate the sum of all the negative numbers in the sequence.

$$-3; -7; -11; \dots; -83; -87$$

$$a = -3; d = -4 \text{ and } n = 22$$

$$S_{22} = \frac{22}{2} [2(-3) + (22-1)(-4)]$$

$$= -990$$

We now leave off the first two terms as those are positive numbers. That means that we are adding together two terms less than 24, so 22 terms.

We use the arithmetic sum formula for this calculation.

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

REMEMBER: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either of the following letters next to each question:

K - KNOWLEDGE

R - ROUTINE

C - COMPLEX

P- PROBLEM SOLVING

next to each question!



Proving the arithmetic series sum formula

The following proof is **examinable**, so make sure that you are able to do it!



Required to prove:

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

Proof:

$$\begin{array}{ccccccc} S_n = & T_1 & T_2 & T_3 & & T_{n-1} & T_n \\ & a & a+d & a+2d & + \dots & a+(n-2)d & a+(n-1)d \\ + S_n = & a+(n-1)d & a+(n-2)d & + \dots & a+2d & a+d & a \\ & T_n & T_{n-1} & & T_3 & T_2 & T_1 \end{array}$$

$$2S_n = 2a + (n-1)d + 2a + (n-1)d + \dots \text{ to } n \text{ terms}$$

$$= n[2a + (n-1)d]$$

$$\therefore S_n = \frac{n}{2} [2a + (n-1)d]$$

Write a sum of n terms using the T_n -formula, then write the terms in reverse order.

$$S_n + S_n = 2S_n$$

Terms that are underneath each other are added together,

$$\begin{array}{l} \text{eg. } a + a + (n-1)d \\ = 2a + (n-1)d \end{array}$$

$$\begin{array}{l} \text{and } a+d + a + (n-2)d \\ = 2a + (n-1)d \end{array}$$

and so on.

We are adding n terms to n terms $\therefore$ getting n solutions, and instead of writing n of the same terms, we can write it as n times that term.

All that's left to do is to divide by 2, and the proof is done!



PRO-TIPS

- In order to prove the SUM formula, you have to use the **TERM** formula for an arithmetic sequence.
- This is how every proof in Mathematics works: you use known formulae in order to prove another formula.
- **You may, however, never use the formula you are proving in your proof!**
- This is known as "circular logic" and will get you a zero straight away.



Geometric sequences/series

Geometric sequences/series have a constant **RATIO**, so we use the formulae with "r".

Similarly to arithmetic sequences/series, this topic is usually tested in a less straightforward way than quadratic number patterns.

The terms in a geometric sequence/series are often given in terms of a variable. You then have to use the following formula to calculate the value of the variable:

$$\frac{T_3}{T_2} = \frac{T_2}{T_1} \quad \text{which is the same as } T_3 \div T_2 = T_2 \div T_1$$

We know that this formula is always true for a geometric sequence/series because it has a **CONSTANT RATIO**, which can be calculated by dividing one term by the previous term.



Worked example (R)

The terms $x - 2$; x ; $x + 4$ form a geometric sequence. Calculate the value of x .

$$\begin{array}{ccc} T_1 & T_2 & T_3 \\ x-2 & ; & x & ; & x+4 \end{array}$$
$$\frac{x+4}{x} = \frac{x}{x-2}$$

$$(x+4)(x-2) = x(x)$$

$$x^2 - 2x + 4x - 8 = x^2$$

$$2x = 8$$

$$x = 4$$

We use the formula above and the values of the terms given to set up an equation that allows us to solve for x . Multiply each term by the LCD = $x(x-2)$.

PRO-TIPS

- In these types of questions, where the actual values of the terms are not given, they **HAVE** to tell you which type of sequence/series it is.
- Make sure that you use the **correct formula!**



Let's look at some other common exam type questions and how to solve them!

Worked example (R)

The first three terms of a geometric sequence are: 16; 4; 1

1. Calculate the value of the 12th term (leave your answer in simplified exponential form).

$$\begin{aligned}
 & 16; 4; 1; \dots \\
 & \quad \times \frac{1}{4} \quad \times \frac{1}{4} \\
 & a = 16 \text{ and } r = \frac{1}{4} \\
 & T_{12} = 16 \left(\frac{1}{4}\right)^{12-1} \\
 & \quad = 2^4 (2^{-2})^{11} \\
 & \quad = 2^4 \times 2^{-22} \\
 & \quad = 2^{-18} \\
 & \quad = \left(\frac{1}{2}\right)^{18}
 \end{aligned}$$

Remember that the ratio is the number that is being multiplied by
 $\therefore$ dividing by 4 is the same as multiplying by $\frac{1}{4}$

When asked to write an expression in simplified exponential form, it's best to write all of the numbers with prime bases, so write 16 as 2^4 instead of 4^2 , and write $\frac{1}{4}$ as 2^{-2} instead of 4^{-1} . Then apply the exponent laws and finally, write your answer with a positive exponent.

An easy way of finding the **ratio** is to **divide each term by the previous term**,
 eg. $r = \text{term 3 divided by term 2}$ or $r = \text{term 2 divided by term 1}$

2. Calculate the sum of the first 10 terms of the sequence.

$$\begin{aligned}
 S_{10} &= \frac{16 \left[1 - \left(\frac{1}{4}\right)^{10} \right]}{1 - \frac{1}{4}} \\
 &= 21,33
 \end{aligned}$$

We use the sum formula for geometric series.

Remember that there are two versions. You can decide to use either of the two as both will yield the same answer.



Proving the geometric series sum formula

The following proof is examinable, so make sure that you are able to do it!
Since there are two versions of this formula, there are also two versions of the proof.

Version 1:

Required to prove:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Proof:

$$\begin{array}{ccccccccccc} & T_1 & & T_2 & & T_3 & & T_4 & & \dots & & T_{n-1} & & T_n \\ S_n & = & a & + & ar & + & ar^2 & + & ar^3 & + & \dots & + & ar^{n-2} & + & ar^{n-1} \\ r \cdot S_n & = & ar & + & ar^2 & + & ar^3 & + & ar^4 & + & \dots & + & ar^{n-1} & + & ar^n \\ & & T_1 \times r & & T_2 \times r & & T_3 \times r & & T_4 \times r & & & & T_{n-1} \times r & & T_n \times r \end{array}$$

$$S_n - r \cdot S_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Write the sum to n terms using the T_n -formula: $T_n = ar^{n-1}$

Multiply each term by r and line the corresponding terms up.

Subtract the second line from the first. Most of the terms cancel out: $ar - ar = 0$; $ar^2 - ar^2 = 0$; $ar^3 - ar^3 = 0$; $ar^4 - ar^4 = 0$; $ar^{n-1} - ar^{n-1} = 0$; and so on.

The only two terms that are left are a and ar^n .

We now take out S_n as a common factor on the left, and a on the right.

All that's left to do is to divide by $(1-r)$ and the proof is done!

Version 2:

Required to prove:

$$S_n = \frac{a(r^n-1)}{r-1}$$

Proof:

$$\begin{array}{ccccccccccc} & T_1 & & T_2 & & T_3 & & T_4 & & \dots & & T_{n-1} & & T_n \\ S_n & = & a & + & ar & + & ar^2 & + & ar^3 & + & \dots & + & ar^{n-2} & + & ar^{n-1} \\ r \cdot S_n & = & ar & + & ar^2 & + & ar^3 & + & ar^4 & + & \dots & + & ar^{n-1} & + & ar^n \\ & & T_1 \times r & & T_2 \times r & & T_3 \times r & & T_4 \times r & & & & T_{n-1} \times r & & T_n \times r \end{array}$$

$$r \cdot S_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n-1)$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

PRO-TIPS

Since these two versions of the proof are so similar, **study one version** and then remember the simple **changes** in each version.

The first two lines of the proof are exactly the same.

Since we need (r^n-1) and $(r-1)$, we now have to subtract the first line from the second.

The same terms cancel out, and only a and ar^n are left, but in the reverse order.

The rest of this version of the proof is the same as above.

Sum to infinity

A geometric series that is **CONVERGING** will have a sum to infinity.

A converging geometric series is one that has a ratio that is a fraction, smaller than 1 or bigger than -1, but not equal to zero.

Therefore, a converging series is a series with $-1 < r < 1$, but with r not equal to zero.

But what does "sum to infinity" mean?

Let's look at the series that starts with 1 and where each term is half the value of the previous one.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots$$

Notice that each term is smaller than the one before it, and as we add more terms to the series, they become smaller and smaller.

This series is classified as a converging series as the ratio is a fraction between -1 and 1.

What happens when we start adding these terms together?

$$S_1 = T_1 \\ = 1$$

$$S_2 = T_1 + T_2 \\ = 1\frac{1}{2} \text{ or } 1,5$$

$$S_3 = T_1 + T_2 + T_3 \\ = 1\frac{3}{4} \text{ or } 1,75$$

$$S_4 = T_1 + T_2 + T_3 + T_4 \\ = 1\frac{7}{8} \text{ or } 1,875$$

Notice that as we add more terms, the sum is still increasing, but with a smaller amount each time.

Now let's calculate bigger sums of this series using the sum-formula.

$$S_{20} = \frac{1 \left[1 - \left(\frac{1}{2} \right)^{20} \right]}{1 - \frac{1}{2}} \\ = 1,999998093$$

$$S_{25} = \frac{1 \left[1 - \left(\frac{1}{2} \right)^{25} \right]}{1 - \frac{1}{2}} \\ = 1,9999994$$

$$S_{30} = \frac{1 \left[1 - \left(\frac{1}{2} \right)^{30} \right]}{1 - \frac{1}{2}} \\ = 1,99999998$$

Notice that as we add even more terms, it appears that the sum is getting closer and closer to 2.

We therefore say that as the number of terms tend to infinity ($n \rightarrow \infty$), the sum tends to 2 $\therefore S_{\infty} = 2$.

Luckily we don't have to do as many calculations to get the value of the sum to infinity.

We can use the following formula:

$$S_{\infty} = \frac{a}{1-r}$$

where a = first term
 r = constant ratio



Let's look at some common exam type questions and how to solve them!



Worked example (R)

For which values of x does the following infinite series converge?

$$2(x-1) + 4(x-1)^2 + 8(x-1)^3 + \dots$$

$$2(x-1) + 4(x-1)^2 + 8(x-1)^3 + \dots$$

$$r = 2(x-1)$$

To converge: $-1 < r < 1; r \neq 0$

$$-1 < 2(x-1) < 1; 2(x-1) \neq 0$$

$$-\frac{1}{2} < x-1 < \frac{1}{2}; x \neq 1$$

$$\frac{1}{2} < x < \frac{3}{2}; x \neq 1$$

First calculate the ratio. If you can't find it by inspection, you can use the formula:

$$r = T_3 \div T_2 \text{ or } r = T_2 \div T_1.$$

We now use the definition of a converging series and replace " r " with the ratio of this series.

We then have to solve for x in both parts of the definition.



Worked example (C)

A sapling (young tree) reaches a height of 118mm after one year in a greenhouse.

During the next year the height increases by 88,5mm.

In each successive year the height increases by $\frac{3}{4}$ of the previous year's increase in height.



1. Determine the increase of height of the plant during the fifth year.

Original height = 118mm

Since the ratio of $\frac{3}{4}$ is in relation to the increase in height, our series is going to consist of increases in height, not the actual height of the sapling.

$$\therefore a = 88,5\text{mm} \text{ and } r = \frac{3}{4}$$

$$T_5 = ar^{4-1}$$

$$= (88,5)\left(\frac{3}{4}\right)^3$$

$$= 37,34\text{mm}$$

To find the increase in height in the 5th year, we need to calculate the 4th term in the series

$\therefore$ use the term formula for a geometric series:

$$T_n = ar^{n-1}$$



2. Show that the plant will never reach a height of more than 472mm.

$$S_{\infty} = \frac{a}{1-r}$$

$$= \frac{88,5}{1-\frac{3}{4}}$$

$$= 354\text{mm}$$

In order to determine the height that this sapling will never reach / pass, we need to calculate the sum to infinity of the increase in height and then add that to the original height.

$\therefore$ The sapling will never reach a height of more than $118 + 354 = 472\text{mm}$.



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Sigma notation

To indicate that the numbers in a sequence must be added together (therefore form a series), we can either use the **Sn-notation**, or **sigma notation**.

Sigma notation can be used for any type of series, i.e. arithmetic, geometric, a quadratic number pattern that is added together, or even a number pattern that doesn't fall into any of these categories.

This is how to interpret a statement written in sigma notation:

$$\sum_{n=1}^{12} (3n-2)$$

12 ← the value of n in the last term
 (3n-2) ← every term in the series can be calculated using this general formula
 n=1 ← the value of n in the first term (it's not always 1!)

What should you do if you are asked to "calculate" the above statement?

1. Determine the type of series that you are working with. You can do this either by looking at the general formula or by calculating the first three terms.
2. Determine the number of terms in the series.
3. Use the correct sum formula to calculate the sum of the series.

Let's calculate the sum written above:

$$\begin{aligned} T_1 &= 3(1) - 2 = 1 \\ T_2 &= 3(2) - 2 = 4 \\ T_3 &= 3(3) - 2 = 7 \end{aligned}$$

We can calculate the values of the first three terms by substituting $n=1$, $n=2$ and $n=3$ into the formula: $T_n = 3n - 2$.

$$1 + 4 + 7 + \dots$$

+3 +3

This is an arithmetic series with $a=1$ and $d=3$.
 $\therefore$ We can use the arithmetic sum formula to calculate the sum.

$$\begin{aligned} S_{12} &= \frac{12}{2} [2(1) + (12-1)(3)] \\ &= 210 \end{aligned}$$

Number of terms = 12

PRO-TIPS

- If the series has a small number of terms, you can calculate **ALL** of the terms and add them together, instead of using a sum-formula.
- This is what you have to do in the case of a **quadratic number pattern**.

Let's look at a different example:

$$\sum_{t=0}^{23} 3\left(\frac{1}{2}\right)^t$$

We need to sub in $t=0$ for the first term!

$$T_1 = 3\left(\frac{1}{2}\right)^0 = 3$$

$$T_2 = 3\left(\frac{1}{2}\right)^1 = \frac{3}{2}$$

$$T_3 = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

This is a geometric series with $a = 3$ and $r = \frac{1}{2}$.

Since we aren't substituting $t=1$ for the first term, the number on top of the sigma symbol is not equal to the number of terms.

We now subtract the bottom number from the top number and add one.

$$\begin{aligned}\therefore \text{Number of terms} &= 23 - 0 + 1 \\ &= 24\end{aligned}$$

$$\begin{aligned}S_{24} &= \frac{3\left[1 - \left(\frac{1}{2}\right)^{24}\right]}{1 - \frac{1}{2}} \\ &= 5,999... \\ &\approx 6\end{aligned}$$

We can now use the formula for the sum of a geometric series to calculate the sum.

Here is an example of writing a series in sigma notation:

$$2 + 6 + 18 + \dots \text{ to 15 terms}$$

$\underbrace{\quad}_{\times 3} \quad \underbrace{\quad}_{\times 3}$

This is a geometric series with $a = 2$, $r = 3$ and $n = 15$.

$$\begin{aligned}T_n &= ar^{n-1} \\ &= 2(3)^{n-1}\end{aligned}$$

We have to find the general term of the series, so we use the term formula for geometric series and simplify if we can.

$$\therefore \sum_{n=1}^{15} 2(3)^{n-1}$$

When writing a series in sigma notation, always put $n=1$ underneath the symbol and the number of terms above.

PRO-TIPS

This topic is generally tested in **problem-solving questions**.



Let's look at some typical exam questions and how to solve them!

Worked example (C)

Given: $\sum_{k=1}^n 3(2)^{1-k} = 5,8125.$

Calculate the value of n .

$$\sum_{k=1}^n 3(2)^{1-k} = 5,8125$$

The sum of the first n terms of the series with the general term $T_k = 3(2)^{1-k}$ is 5,8125.

We are asked to calculate the value of n $\therefore$ the number of terms in the series. First determine which type of series it is by calculating the first two or three terms (or by looking at T_n) and then find the values of a and d/r .

$$\begin{aligned} T_1 &= 3(2)^{1-1} = 3 \\ T_2 &= 3(2)^{1-2} = \frac{3}{2} \\ T_3 &= 3(2)^{1-3} = \frac{3}{4} \end{aligned}$$

This is a geometric series with
 $a = 3$, $r = \frac{1}{2}$ and $S_n = 5,8125$.

$$\begin{aligned} S_n &= 5,8125 \\ \frac{a(1-r^n)}{1-r} &= 5,8125 \\ \frac{3[1-(\frac{1}{2})^n]}{1-\frac{1}{2}} &= 5,8125 \\ 3[1-(\frac{1}{2})^n] &= \frac{93}{32} \\ 1-(\frac{1}{2})^n &= \frac{31}{32} \\ \checkmark (\frac{1}{2})^n &= \frac{1}{32} \\ &= (\frac{1}{2})^5 \\ \therefore n &= 5 \text{ terms} \end{aligned}$$

Choose either of the two sum formulae for geometric series and substitute the a , r and S_n -values into the formula.

Now we need to solve for n .

The denominator is equal to $1 - \frac{1}{2} = \frac{1}{2}$, so first multiply both sides by $\frac{1}{2}$.

Then divide both sides by 3, and then subtract 1 from both sides. Cancel out -1 on both sides, and then write both sides with the same base of $\frac{1}{2}$.

PRO-TIPS

- Calculating the number of terms in a sequence or series is a **very common** question, not only in sigma notation, but in this entire section.
- In these types of questions, you have to **find** and **substitute** all other values in the appropriate formula, and then solve for " n " using algebraic manipulation.



Worked example (P)

Given: $\sum_{k=1}^n 3(2)^{1-k} = p$

Write down $\sum_{k=1}^n 24(2)^{-k}$ in terms of p .

In order to write the second series in terms of p , we need to rewrite $24(2)^{-k}$ so that it contains $3(2)^{1-k}$.

We can then replace it with p .

$$\begin{aligned} 24(2)^{-k} &= 8 \times 3(2)^{-k} \\ &= 4 \times 3(2)(2^{-k}) \\ &= 4 \times 3(2)^{1-k} \end{aligned}$$

$24 = 8 \times 3$, but we still need another 2 so that we can get $2^1 \times 2^{-k} = 2^{1-k}$ $\therefore$ rewrite 8 as 4×2 .

$$\begin{aligned} \therefore \sum_{k=1}^n 24(2)^{-k} &= \sum_{k=1}^n 4 \times 3(2)^{1-k} \\ &= 4 \sum_{k=1}^n 3(2)^{1-k} \\ &= 4p \end{aligned}$$

We now use the following rule:

$\sum a \times T_k = a \times \sum T_k$, if a is a constant.

Now we can replace $\sum_{k=1}^n 3(2)^{1-k}$ with p .



Worked example (C)

Calculate: $\sum_{n=2}^{\infty} 4\left(\frac{1}{2}\right)^{n+2}$

This question is asking for the sum to infinity.

Substitute $n = 2$ to determine the value of the first term.

$$\begin{aligned} T_1 &= 4\left(\frac{1}{2}\right)^{2+2} = \frac{1}{4} \\ T_2 &= 4\left(\frac{1}{2}\right)^{3+2} = \frac{1}{8} \end{aligned}$$

$a = \frac{1}{4}$ and $r = \frac{1}{2}$, which is $-1 < r < 1$ $\therefore$ this is a converging series, which has a sum to infinity.

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} \\ &= \frac{\frac{1}{4}}{1 - \frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

We now use the formula for the sum to infinity and calculate the answer.

PRO-TIPS

Many people think that sigma notation is super difficult! The most common mistake is forgetting that the **GENERAL TERM** is in the sigma notation, **NOT** the sum formula. If you don't know what to do, always start by calculating the first two/three terms and then go from there.



In summary:

If they are asking for a **TERM**, you use the **TERM FORMULA** for the type of sequence you are working with.

Arithmetic:

$$T_n = a + (n-1)d$$

Geometric:

$$T_n = ar^{n-1}$$

If they are asking for the **SUM** of a number of terms, you use the **SUM FORMULA** for the type of sequence you are working with.

Arithmetic:

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

Geometric:

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

$$S_\infty = \frac{a}{1-r}$$

If they have given you the value of a specific term in a sequence, rewrite that term using the appropriate term formula.

Eg. The 6th term of an arithmetic sequence is 12.

$$a + 5d = 12$$

If they have given you a series written in sigma notation, **determine** the **first three** terms of that series and then calculate the sum using the appropriate sum formula.

If they have given you the **sum** of a number of terms in a **series**, rewrite that sum using the appropriate sum formula.

Eg. The sum of the first 11 terms of a geometric series is 135.

$$\frac{a(1-r^{11})}{1-r} = 135$$

If they are asking you to write a **series** in **sigma** notation, first find the **general term**/formula of that series.

Then write the sigma symbol, put $n = 1$ underneath the symbol, the number of terms above the symbol, and write the **general term** next to the symbol, in **brackets**.

If they have given you a sequence/series with terms written in terms of a variable and are asking you to calculate the value(s) of the variable, use the following formulae:

Arithmetic:

$$T_3 - T_2 = T_2 - T_1$$

Geometric:

$$\frac{T_3}{T_2} = \frac{T_2}{T_1}$$

If they are asking for which **values** of a variable a certain **geometric** series will **converge**, determine the **ratio** of that series and then use the following formula to solve for the variable:

$$-1 < r < 1; r \neq 0$$

FUNCTIONS AND GRAPHS

This is one of the two biggest sections in your final Paper 1 exam!

In order to achieve maximum marks for this section, you need to be able to:

- **understand** and **apply** the definition of a function, as well as how that applies to the domain and range
- find the **equation, domain and range** of an inverse function
- **draw** a function and its inverse
- draw a **straight line, parabola, hyperbola, exponential** graph and **logarithmic** graph, as well as multiple graphs on the same set of axes
- **find** the **equation** of a straight line, parabola, hyperbola, exponential graph and logarithmic graph
- find the **points of intersection** of multiple graphs
- identify **where** one graph is **greater** than, **smaller** than, or **equal** to another graph
- identify where **two graphs** have a certain **distance** between them
- **calculate** the **length** between two points on two different graphs

Cubic graphs are also often grouped under this section, but for the purposes of this study guide, we are going to group them under the topic "Differential Calculus".



PRO-TIPS

The functions section is always tested in three parts:

1. A **mixed** functions question with a parabola and one other graph that need to be interpreted.
2. A **hyperbola** question.
3. A question about a function and its **inverse**.



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Let's summarise the different functions and some important properties of each!

When asked to draw a graph, you have to determine and label the following:

1. **All** intercepts with the x- and y-axis, if they exist.
2. **All** asymptotes
3. **All** turning points

You **ONLY** need to draw the axes of symmetry if the question specifically asks you to.

All graphs have special properties that need to be indicated. These are summarised below.

Straight lines:

$$y = mx + c$$

gradient $\rightarrow$ m
y-intercept: $(0; c)$

$$y - y_1 = m(x - x_1)$$

gradient $\rightarrow$ m
 $(x_1; y_1)$ is a point on the line

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

if $m > 0$: $\nearrow$ if $m < 0$: $\searrow$

$y = k$ a horizontal line $\parallel$ to the x-axis

$x = k$ a vertical line $\parallel$ to the y-axis

- These are the two formulae that you can use when asked to find the equation of a straight line graph.
- In both of these equations, **m** represents the gradient and you then have to substitute a point on the line into the equation in order to find the value of **c**.

Remember : if two lines are **parallel**, they have the same gradients.

If two lines are **perpendicular**, then the products of their gradients is equal to -1.

Parabolas:

Standard form: $y = ax^2 + bx + c$

y-intercept: $(0; c)$
turning pt: $x = \frac{-b}{2a}$
sub into eqn to get y-coordinate

Turning point form: $y = a(x - p)^2 + q$ turning pt = $(p; q)$

x-intercept form: $y = a(x - x_1)(x - x_2)$ x-ints: $(x_1; 0)$
 $(x_2; 0)$
turning pt: $x = \frac{x_1 + x_2}{2}$

if $a > 0$: $\cup$ if $a < 0$: $\cap$

To go from standard form to turning point form:
complete the square

To go from standard form to x-intercept form:
factorise the equation

To go from turning point or x-intercept form to standard form:
multiply the brackets out

Hyperbolas:

$$y = \frac{a}{x-p} + q$$

horizontal asymptote: $y = q$
vertical asymptote: $x = p$
Point of intersection of asymptotes: $(p; q)$
Axes of symmetry:
1. $y = x - p + q$ 2. $y = -(x - p) + q$

Exponential graphs:

$$y = a \cdot b^{x+p} + q \quad (b > 0)$$

horizontal asymptote: $y = q$
if $a > 0$: graph lies above asymptote
if $a < 0$: graph lies below asymptote

Logarithmic graphs:

These graphs are only tested as the inverses of exponential graphs.

Log laws are **NOT** in the syllabus so you will not be tested on them.

There are, however, a few things that you have to know about logs in order to answer the questions that may be asked.

Converting an exponential expression to log-form:

$$y = a^x \Leftrightarrow x = \log_a y$$

You have to be able to convert an exponential expression to log-form, and a log expression to exponential form.

The standard base of a logarithm:

$$\log x = \log_{10} x$$

If no base is written, the base of the log is 10.

Special values:

$$a^0 = 1 \Leftrightarrow \log_a 1 = 0$$

Any logarithm of 1 is equal to zero, regardless of the value of the base.

$$a^1 = a \Leftrightarrow \log_a a = 1$$

Any logarithm of a value that is the same as the base is equal to 1.

A fractional base can be changed to a whole number if the expression is negated:

$$y = \log_{\frac{1}{a}} x \Leftrightarrow y = -\log_a x$$

Even though each graph is slightly different, we use the same method to calculate the intercepts with the axes.

To find the y-intercept:

make $x = 0$

To find the x-intercept(s):

make $y = 0$ and solve the equation



Question Type 1: Hyperbolas

The standard form of a hyperbola equation can be written in two ways:

$$y = \frac{a}{x-p} + q$$

where $x = p$ is the vertical asymptote
and $y = q$ is the horizontal asymptote

OR

$$y = \frac{a}{x+p} + q$$

where $x = -p$ is the vertical asymptote
and $y = q$ is the horizontal asymptote

These are the most common questions that you can expect in your final exam:

1. Write the equations of the asymptotes of a hyperbola

A hyperbola has two asymptotes:

One of these asymptotes has the equation $x = p$, and is a vertical line that is parallel to the y-axis.

The other asymptote has the equation $y = q$, and is a horizontal line that is parallel to the x-axis.

2. Write down the coordinates of the point of intersection of the asymptotes of a hyperbola

Since the asymptotes of a hyperbola are $x = p$ and $y = q$, the point of intersection of these asymptotes is the point $(p ; q)$.

3. Write down the equations of the axes of symmetry of a hyperbola

A hyperbola has two axes of symmetry, both of which are straight lines.

One of these straight lines has a positive gradient, and can be found using the formula:
 $y = x - p + q$.

The other one of these straight lines has a negative gradient, and can be found using the formula: $y = -(x - p) + q$.

Both of these equations have to be simplified fully.

4. Draw the graph.

Make sure that you indicate the asymptotes and all the intercepts with the axes.
Label these as well, as that is what you get marks for.



Worked example (K/R)

If it is given that $f(x) = -\frac{3}{x-1} + 2$ answer the following questions:

1. Write the equations of the asymptotes of f .
2. Write the coordinates of the point of intersection of the asymptotes of f .
3. Determine the equations of the axes of symmetry of f .
4. Calculate the intercepts with the axes of f .
5. Draw the graph of f , showing all asymptotes and intercepts with the axes.



1. Horizontal asymptote: $y = 2$
Vertical asymptote: $x = 1$

Horizontal asymptote: $y = q$
Vertical asymptote: $x = p$



2. Point of intersection of asymptotes
 $= (p; q)$
 $= (1; 2)$



3. Axes of symmetry:
 $y = x - 1 + 2$ and $y = -(x - 1) + 2$
 $y = x + 1$ $y = -x + 3$

A hyperbola has two axes of symmetry,
one with a positive gradient: $y = x - p + q$
and one with a negative gradient: $y = -(x - p) + q$



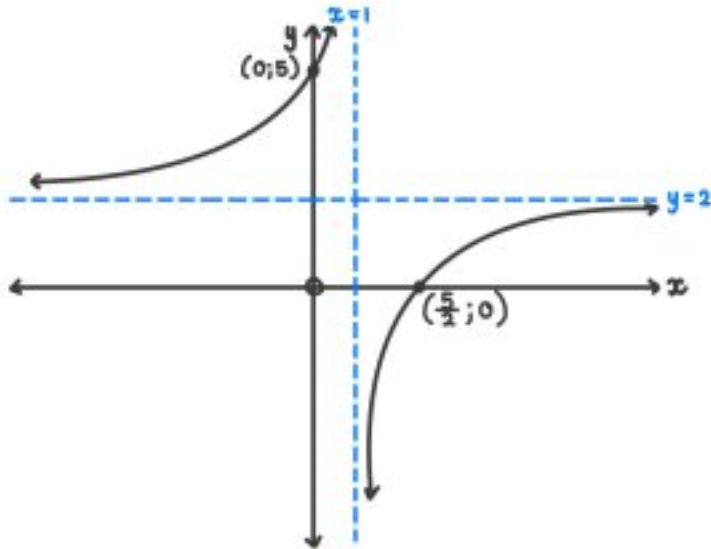
4. y -intercept: make $x = 0$
 $y = \frac{-3}{0-1} + 2$
 $= 5$
 $\therefore (0; 5)$

x -intercept: make $y = 0$
 $\frac{-3}{x-1} + 2 = 0$
 $\frac{-3}{x-1} = -2$
 $-3 = -2(x-1)$
 $\frac{3}{2} = x-1$
 $\therefore x = \frac{5}{2}$
 $\therefore (\frac{5}{2}; 0)$





5.



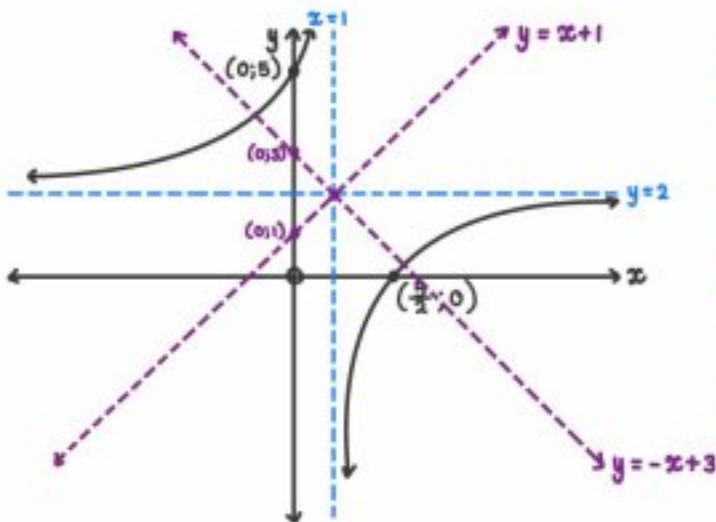
Start by drawing the asymptotes using dotted lines, and label them with their equations.

Then plot the intercepts with the axes, and draw the graph.

Pay attention to the shape! It should be a smooth curve, and the ends of the curves must slope towards the asymptotes, never away from them.



Do not draw the axes of symmetry unless the question specifically instructs you to!



If you are asked to draw the axes of symmetry, follow this shortcut:

The axes of symmetry pass through the point of intersection of the asymptotes, and their y -intercepts, which can be found simply by looking at their equations. You do NOT need to plot any other points.

Eg. $y = x+1$ passes through $(0;1)$
 $y = -x+3$ passes through $(0;3)$



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Question Type 2: Mixed graphs

The mixed graph question always consists of **two graphs drawn on the same set of axes**.

One of these graphs is always a **parabola**, and the other can be a straight line, hyperbola or exponential graph.

In order to get the best possible marks for this question, make sure that you know how to do the following:

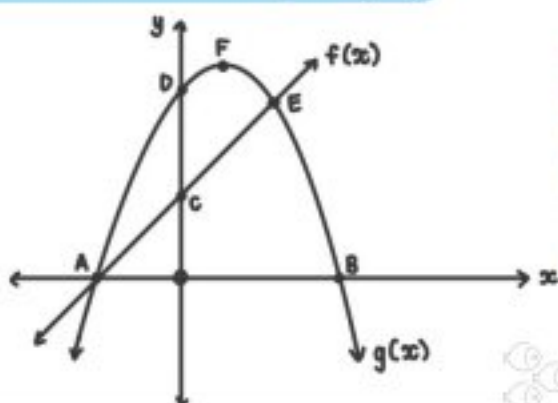
- Identify the **types** of **points** on a graph without the question pointing it out.
- **Calculate** the coordinates of the **x-intercepts** of a graph.
- **Calculate** the coordinates of the **y-intercepts** of a graph.
- Write down the **domain** and **range** of a graph.
- **Calculate** the coordinates of the **turning point** of a parabola.
- Write down the **equation** of the axis of **symmetry** of a parabola.
- Write down the **equations** of the **asymptotes** of a hyperbola.
- Write down the point of **intersection** of the asymptotes of a hyperbola.
- Determine the **equations** of the axes of **symmetry** of a hyperbola.
- **Calculate** the coordinates of the point(s) of **intersection** of two graphs.
- **Determine** the **equation** of a graph.
- **Apply transformations** to a graph, or the equation of a graph.
- **Comparing** two graphs.

Let's go through each of the points above before looking at a typical past paper question!

1. Identifying the types of points on a graph

- If a graph is given with certain points labelled with letters, you should be able to identify the types of points that are labelled as either x-intercepts, y-intercepts, turning points or points of intersection.
- If you can identify these, you will also be able to calculate the coordinates.

Worked example (K)



- A and B are the x-intercepts of $g(x)$.
- C is the y-intercept of $f(x)$.
- D is the y-intercept of $g(x)$.
- A and E are the points of intersection of $f(x)$ and $g(x)$.
- F is the turning point of $g(x)$.



2. Calculating the coordinates of the x-intercepts of a graph

- In order to calculate the coordinates of the x-intercept(s) of ANY graph, take the equation of the graph and make $y = 0$.
- This will result in an equation with x as the only variable, which you can solve.
- The coordinates of the point will be $(x ; 0)$.
- Note that if the x-axis (the line $y = 0$) is an asymptote of a graph, that graph will not have an x-intercept.



3. Calculating the coordinates of the y-intercept of a graph

- In order to calculate the coordinates of the y-intercept of ANY graph, take the equation of the graph and make $x = 0$.
- This will result in an equation with y as the only variable, which you can calculate.
- The coordinates of the point will be $(0 ; y)$.
- Note that if the y-axis (the line $x = 0$) is an asymptote of the graph, that graph will not have a y-intercept.



Worked example (R)

Calculate the coordinates of the intercepts with the axes of $y = x^2 - 5x + 6$.

x-intercept: make $y = 0$

$$0 = x^2 - 5x + 6$$

$$0 = (x-3)(x-2)$$

$$\therefore x-3=0 \quad \text{or} \quad x-2=0$$

$$x=3$$

$$x=2$$

$$\therefore (3;0)$$

$$(2;0)$$

y-intercept: make $x = 0$

$$y = (0)^2 - 5(0) + 6$$

$$= 6$$

$$\therefore (0;6)$$





4. Write the domain and range of a graph

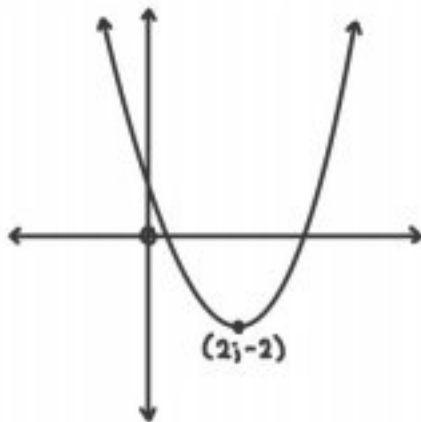
- The **domain** of a graph is the set of all x-values that exist on the graph.
- The **range** is the set of all y-values that exist on the graph.
- When you write your answers, you may use interval notation or inequality notation, but pay special attention to the notation.
- When the values are included, in other words, when the graph has a point at the highest or lowest value, we have to use the smaller than or equal to symbol (or greater than or equal to) in inequality notation.
- When the values are excluded, in other words, when the graph does not have a point at the highest or lowest value (eg. If there is an asymptote), we have to use the smaller than symbol (or the greater than symbol).
- If we are using interval notation, we use a round bracket (or) if the values are excluded, and a square bracket [or] if the values are included.



Worked example (R)



Write down the domain and range of each of the following graphs:



Domain:

$x \in \mathbb{R}$ or $x \in (-\infty; \infty)$ interval notation

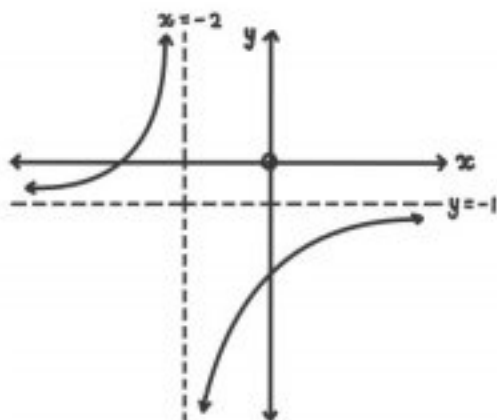
or $-\infty < x < \infty$ inequality notation

Range:

$y \in [-2; \infty)$ There is a point at $y = -2$, so we use a square bracket to indicate that the value is included.

or $-2 \leq y < \infty$ Remember to write the smaller value first.

This line indicates that $y = -2$ is included in the range.



Domain:

$x \in \mathbb{R}; x \neq -2$ We exclude the value of the asymptote,

or $-\infty < x < -2$ or $-2 < x < \infty$ as the graph will not touch that line.

Range:

$y \in \mathbb{R}; y \neq -1$

or $-\infty < y < -1$ or $-1 < y < \infty$



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5. Calculating the coordinates of the turning point of a parabola

- If the parabola equation is written in **standard form**, you can use the turning point formula to calculate the x-value. Once you have that, you have to substitute the x-value into the equation in order to get the y-value of the point.
- If the parabola equation is written in **standard form**, you can also complete the square in order to get the turning point form of the equation.
- If the parabola equation is written in **turning point form**, the turning point has coordinates (p ; q).

Worked example (R)

Calculate the coordinates of the turning point of:

$y = x^2 + 8x + 7$ Standard form: $y = ax^2 + bx + c$

$$\begin{aligned}x &= \frac{-b}{2a} && \text{This is the turning point formula. Substitute the values of } a \\ &= \frac{-8}{2(1)} && \text{and } b \text{ from the equation.} \\ &= -4\end{aligned}$$

$$\begin{aligned}\therefore y &= (-4)^2 + 8(-4) + 7 && \text{Now substitute the } x\text{-value into the equation} \\ &= -9 && \text{in order to calculate the } y\text{-coordinate of the} \\ \therefore &(-4; -9) && \text{turning point.}\end{aligned}$$

or by completing the square:

$$\begin{aligned}y &= x^2 + 8x + (4)^2 - (4)^2 + 7 \\ &= (x+4)^2 - 9\end{aligned}$$

$$\therefore (-4; -9)$$

Add and subtract $\left(\frac{b}{2}\right)^2$. This creates a perfect square trinomial of $x^2 + 8x + 16$, which can be factorized to $(x+4)(x+4)$, or $(x+4)^2$.

The last two terms simplify to -9.

The equation is now written in turning point form, $y = a(x-p)^2 + q$, and the turning point is (p;q).

6. Write the equation of the axis of symmetry of a parabola

- If the turning point of a parabola is the point (p ; q), the axis of symmetry is the straight line with the equation $x = p$.
- This is a vertical line, cutting the x-axis at $x = p$, parallel to the y-axis.

Worked example (R)

Write down the equation of the axis of symmetry of:

1. $y = (x+3)^2 - 4$
 $x = -3$

This equation is written in turning point form.

2. $y = -2x^2 + 6x - 4$

$$\begin{aligned}x &= \frac{-b}{2a} \\ &= \frac{-6}{2(-2)} \\ &= \frac{3}{2}\end{aligned}$$

Use the turning point formula.



All questions relating to hyperbolas were discussed on pages 33-35.

7. Calculate the point(s) of intersection of two graphs

- A point of intersection of two graphs is a point where the two graphs are **EQUAL**.
- The same point lies on both graphs.
- We find the x-coordinate of this point by making the two equations equal to each other, which effectively cancels out the y-variable. This leaves us with an equation with only x, which we can solve.
- Once we have the x-coordinate, we substitute that back into either one of the graph equations, which will give us the y-coordinate of the point.

NOTE: some graphs have two points of intersection, and you will sometimes have to determine which one is which based on whether the points are positive or negative.

Worked example (R)

Determine the coordinates of the point(s) of intersection of $y = x - 1$ and $y = x^2 + 5x + 3$.

$$y = x - 1 \quad \text{and} \quad y = x^2 + 5x + 3$$

$$\therefore x - 1 = x^2 + 5x + 3$$

$$0 = x^2 + 4x + 4$$

$$0 = (x + 2)(x + 2)$$

$$x = -2$$

$$y = -2 - 1 \quad \text{or} \quad y = (-2)^2 + 5(-2) + 3$$
$$= -3 \quad \quad \quad = -3$$

$\therefore$ Point of intersection: $(-2; -3)$

A point of intersection is where the two graphs are equal, so we equate the equations. We are essentially solving two equations simultaneously.

Once you have an x-value (or two), you have to substitute the value(s) into either of the two equations to calculate the corresponding y-value(s).

Always end off your answer with the coordinates. These two graphs only have one point of intersection.





8. Finding the equation of a graph

- When asked to find the equation of a graph, it is very important to remember the special properties of each graph as that will allow you to determine some of the variables in a graph equation.
- When you are left with only one variable, you usually have to substitute in a point on the graph and then solve for that variable.

Let's look at each type of graph and some examples.



Straight Line



$$y = mx + c$$

← y-intercept (0; c)
gradient: $m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$

Remember that if $AB \parallel CD$, then $m_{AB} = m_{CD}$.
If $AB \perp CD$, then $m_{AB} \times m_{CD} = -1$.

$$y - y_1 = m(x - x_1)$$

← gradient
(x_1, y_1) is a point on the straight line
This equation is quicker to work with if the y-intercept is not given.

PRO-TIPS

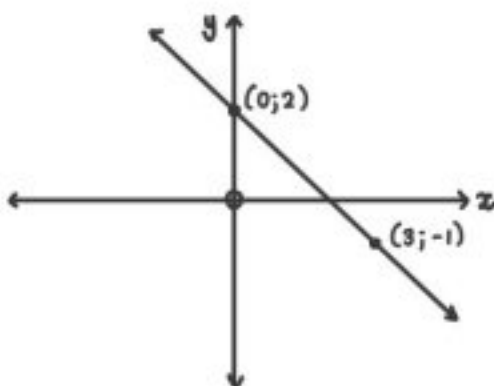
In these questions, the standard form of the equation is often given, with specific variables. You then have to find the values of the given variables.

A straight line has two standard forms, and you can decide which to use. The second form is first taught in Grade 10 Analytical Geometry.



Worked example (K)

Find the equation of the graph below.



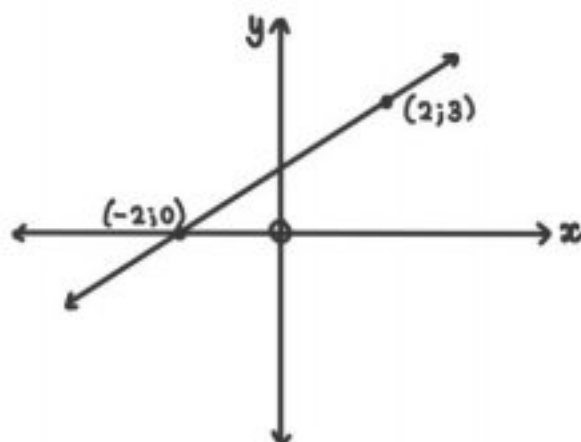
$$\begin{aligned} y &= mx + c \\ c &= 2 \\ m &= \frac{-1 - 2}{3 - 0} \\ &= -1 \\ \therefore y &= -x + 2 \end{aligned}$$

The y-intercept is given, so that is the c-value. We use the gradient-formula to calculate the value of m.



Worked example (R)

Find the equation of the graph below.



This question is different from the previous one because the y-intercept (c-value) is not given.

We therefore have to calculate m first, and then substitute in a point to find the value of c.

$$\begin{aligned} y &= mx + c \\ m &= \frac{3 - 0}{2 - (-2)} \\ &= \frac{3}{4} \\ y &= \frac{3}{4}x + c \end{aligned}$$

$$\begin{aligned} \text{Sub in } (-2; 0): \\ 0 &= \frac{3}{4}(-2) + c \\ 0 &= -\frac{3}{2} + c \\ \therefore c &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{or sub in } (2; 3): \\ 3 &= \frac{3}{4}(2) + c \\ 3 &= \frac{3}{2} + c \\ \therefore c &= \frac{3}{2} \end{aligned}$$

$$\text{so } y = \frac{3}{4}x + \frac{3}{2}$$

$$\begin{aligned} \text{OR: } y - y_1 &= m(x - x_1) \\ m &= \frac{3 - 0}{2 - (-2)} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \therefore y - y_1 &= \frac{3}{4}(x - x_1) \\ \text{Sub in } (-2; 0): \\ y - 0 &= \frac{3}{4}[x - (-2)] \\ y - 0 &= \frac{3}{4}(x + 2) \\ y &= \frac{3}{4}x + \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{or sub in } (2; 3): \\ y - 3 &= \frac{3}{4}(x - 2) \\ y - 3 &= \frac{3}{4}x - \frac{3}{2} \\ y &= \frac{3}{4}x + \frac{3}{2} \end{aligned}$$

Since the y-intercept is not given, we must calculate the gradient first.

In order to calculate the c-value, we now need to substitute either one of the two points into the straight line equation.

This allows us to solve for c.

You only have to show ONE of the four substitution calculations.



Parabola

- Standard form: $y = ax^2 + bx + c$ ↖ y-intercept: (0;c)

Only use this form if the y-intercept is given.
You will then have to substitute two other points into the equation, giving two equations with two variables, that you have to solve simultaneously.
- Turning-point form: $y = a(x-p)^2 + q$

Turning point: $(p;q)$

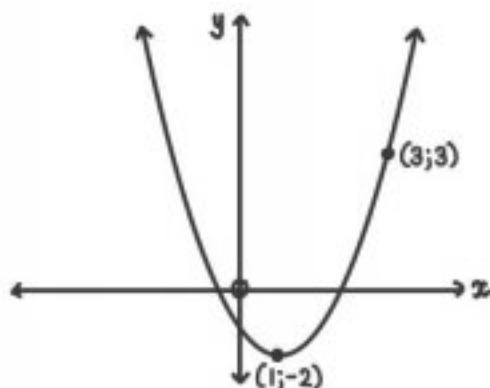
Use this formula if the turning point is given.
Replace the values of p and q in the equation with the coordinates of the turning point.
Then substitute one OTHER point (not the turning point again) into the equation and solve for a .
- x-intercept form: $y = a(x-x_1)(x-x_2)$

The x-intercepts are $(x_1;0)$ and $(x_2;0)$.
Use this formula if both x-intercepts are given.
Replace x_1 and x_2 in the equation with the x-coordinates of the x-intercepts.
Then substitute one OTHER point (not the x-intercepts) into the equation and solve for a .



Worked example (R)

Find the equation of the graph below.



$$y = a(x-p)^2 + q$$

$$= a(x-1)^2 - 2$$

Sub in $(3;3)$:

$$3 = a(3-1)^2 - 2$$

$$3 = 4a - 2$$

$$5 = 4a$$

$$\therefore a = \frac{5}{4}$$

$$\therefore y = \frac{5}{4}(x-1)^2 - 2$$

The turning point is given, so we use the turning point formula.

Replace $(p;q)$ with $(1;-2)$.

Now substitute the other point into the equation and solve for a .

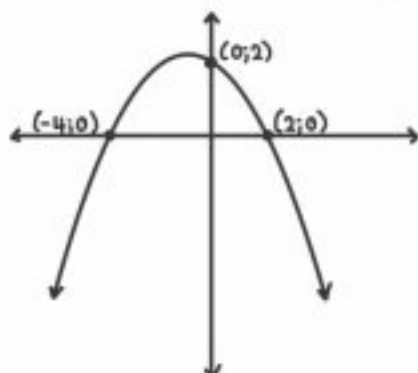


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Worked example (C)

Find the equation of the graph below and write your answer in standard form.



$$y = a(x - x_1)(x - x_2) \\ = a(x + 4)(x - 2)$$

Sub in (0; 2):

$$2 = a(0 + 4)(0 - 2)$$

$$2 = -8a$$

$$\therefore a = -\frac{1}{4}$$

$$\therefore y = -\frac{1}{4}(x + 4)(x - 2) \\ = -\frac{1}{4}(x^2 - 2x + 4x - 8) \\ = -\frac{1}{4}(x^2 + 2x - 8)$$

$$y = -\frac{1}{4}x^2 - \frac{1}{2}x + 2$$

$$\therefore b = -\frac{1}{2} \text{ and } c = 2$$

Both x -intercepts are given, so we use the x -intercept formula and replace x_1 and x_2 with the x -values of the x -intercepts.

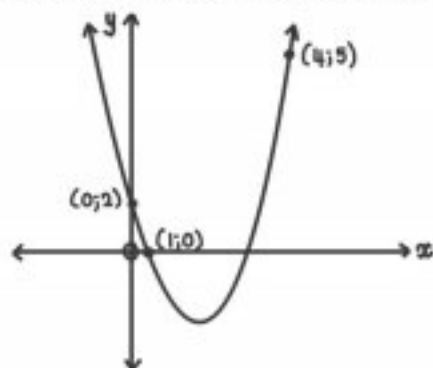
Now substitute the other point into the equation and solve for a .

Multiply the brackets out and then multiply the a -value into the brackets in order to get the equation in standard form.



Worked example (C)

Find the equation of the graph below and write your answer in standard form.



Only one x -intercept is given, and the coordinates of the turning point are not given.

$\therefore$ We have to use standard form.

$$y = ax^2 + bx + c$$

$$c = 2$$

$$\therefore y = ax^2 + bx + 2$$

The value of c is the y -value of the y -intercept.

Sub in (1; 0):

$$0 = a(1)^2 + b(1) + 2$$

$$0 = a + b + 2 \dots \textcircled{1}$$

Substitute the other two points into the equation and simplify. This gives us two equations with two variables which we can solve simultaneously.

Sub in (4; 5):

$$5 = a(4)^2 + b(4) + 2$$

$$5 = 16a + 4b + 2 \dots \textcircled{2}$$

$$\text{In } \textcircled{1}: a + b + 2 = 0$$

$$a = -b - 2$$

$$\text{Sub into } \textcircled{2}: 16a + 4b + 2 = 5$$

$$16(-b - 2) + 4b = 3$$

$$-16b - 32 + 4b = 3$$

$$-12b = 35$$

$$b = -\frac{35}{12}$$

$$\text{Sub back into } \textcircled{1}: a = -b - 2$$

$$= -\left(-\frac{35}{12}\right) - 2$$

$$= \frac{11}{12}$$

$$\therefore y = \frac{11}{12}x^2 - \frac{35}{12}x + 2$$

Simultaneous equations were explained in detail on page 11.



Hyperbola

Standard form: $y = \frac{a}{x-p} + q$

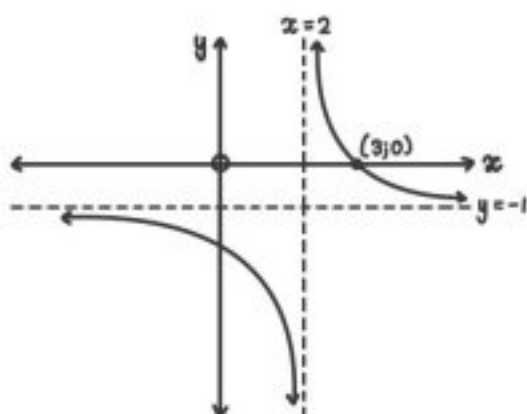
Start by replacing p with the x -value of the vertical asymptote and q with the y -value of the horizontal asymptote.

Then substitute the coordinates of a point on the graph and solve for a .



Worked example (R)

Find the equation of the graph below.



$$y = \frac{a}{x-p} + q$$

$$= \frac{a}{x-2} - 1$$

Sub in (3; 0):

$$0 = \frac{a}{3-2} - 1$$

$$1 = \frac{a}{1}$$

$$a = 1$$

$$\therefore y = \frac{1}{x-2} - 1$$

Replace p and q with the values of the asymptotes.

Then substitute the coordinates of a point on the graph into the equation and solve for a .

PRO-TIPS

LOOK OUT FOR THIS POSSIBLE TWIST ON THE QUESTION:

Instead of giving the equations of the asymptotes, they might give the coordinates of the point of intersection.

Remember that the point of intersection is $(p ; q)$.



Exponential graph

Standard form: $y = a \cdot b^{x-p} + q$
 In these questions, the value(s) of one or more of the variables is usually given.

Horizontal asymptote:
 $y = q$

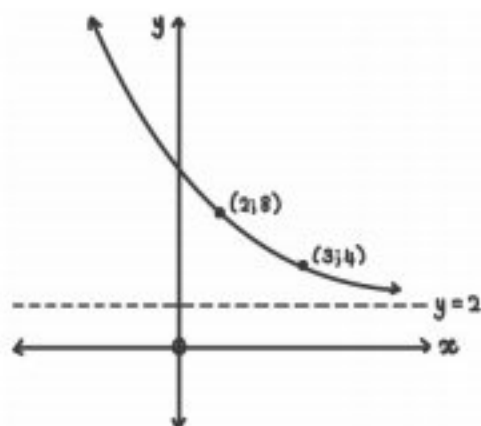
First replace q with the y -value of the horizontal asymptote. Then substitute one or more points into the equation, depending on how many variables you need to solve for.

If you have to solve for one variable, you will have to sub in one point.

If you have to solve for two variables, you will have to sub in two points and solve the resulting equations simultaneously.

Worked example (R)

If it is given that the graph below has the equation $y = 2 \cdot b^{x-p} + q$, determine the values of b , p and q .



$$\begin{aligned} y &= 2 \cdot b^{x-p} + q \\ q &= 2 \\ \therefore y &= 2 \cdot b^{x-p} + 2 \end{aligned}$$

Horizontal asymptote is $y = 2$

$$\begin{aligned} \text{Sub in } (2; 8): \\ 8 &= 2b^{2-p} + 2 \\ 6 &= 2b^{2-p} \\ 3 &= b^{2-p} \dots \textcircled{1} \end{aligned}$$

Substitute one of the given points into the equation. Since there are two variables, we cannot solve for either of them with just this one equation.

$$\begin{aligned} \text{Sub in } (3; 4): \\ 4 &= 2b^{3-p} + 2 \\ 2 &= 2b^{3-p} \\ 1 &= b^{3-p} \dots \textcircled{2} \\ b^0 &= b^{3-p} \\ \therefore 0 &= 3 - p \\ p &= 3 \end{aligned}$$

Substitute the other given point into the equation as well.

This equation has two variables but we can solve for p , because we are able to express both sides of the equation with the same base. Remember that anything to the power of zero is one!

$$\begin{aligned} \therefore \text{Sub back into } \textcircled{1}: \\ 3 &= b^{2-p} \\ 3 &= b^{2-3} \\ 3 &= b^{-1} \\ \left(\frac{1}{3}\right)^{-1} &= b^{-1} \\ \therefore b &= \frac{1}{3} \end{aligned}$$

We now use the value of p to calculate the value of b .

$$\therefore y = 2 \cdot \left(\frac{1}{3}\right)^{x-3} + 2$$

PRO-TIPS

These questions will require you to solve exponential equations.

Remember that you have to make the **bases** the **same** on both sides of the equation, after which you can equate the exponents.

Applying transformations to graphs

We have to know how to apply the following transformations:

- Translations - moving a graph a certain number of units up/down and left/right.
- Reflections - reflecting in the x-axis, the y-axis and the line $y = x$ (this is the same as finding an inverse, which will be discussed in the next section).

Here is a summary of how to apply each of these transformations to the equation of any graph in the form $y = f(x)$:

Translations:

- If a graph is shifted **m units UP**, the equation becomes: $y = f(x) + m$
- If a graph is shifted **m units DOWN**, the equation becomes: $y = f(x) - m$
- If a graph is shifted **n units to the LEFT**, the equation becomes: $y = f(x + n)$
- If a graph is shifted **n units to the RIGHT**, the equation becomes: $y = f(x - n)$

Reflections:

- If a graph is **reflected in the x-axis**, the equation becomes: $y = -f(x)$
- If a graph is **reflected in the y-axis**, the equation becomes: $y = f(-x)$
- If a graph is **reflected in the line $y = x$** , the equation becomes: $x = f(y)$. In this case, we have to make y the subject as well.
- The last point refers to **INVERSE FUNCTIONS**. More on this in the next section!



Worked example (R)

Given: $f(x) = -x^2 + 3x - 4$.



1. Write the equation of $g(x)$ if g is the graph obtained by shifting $f(x)$ 2 units up and 3 units to the left. Leave your answer in standard form.

$$\begin{aligned} g(x) &= f(x+3) + 2 \\ &= -(x+3)^2 + 3(x+3) - 4 + 2 \\ &= -(x^2 + 6x + 9) + 3x + 9 - 4 \\ &= -x^2 - 6x - 9 + 3x + 9 - 4 \\ &= -x^2 - 3x - 2 \end{aligned}$$

When a graph is shifted 3 units to the left, we must replace each " x " in the equation with " $x+3$ ".
When a graph is shifted 2 units up, we have to add 2 to the equation.



2. Write the equation of $h(x)$ if h is the graph obtained by reflecting $f(x)$ in the x-axis.

$$\begin{aligned} h(x) &= -f(x) \\ &= -(-x^2 + 3x - 4) \\ &= x^2 - 3x + 4 \end{aligned}$$

When a graph is reflected in the x-axis, the sign of y changes, which means that the entire equation is negated.



3. Write the equation of $p(x)$ if p is the graph obtained by reflecting $f(x)$ in the y-axis.

$$\begin{aligned} p(x) &= f(-x) \\ &= -(-x)^2 + 3(-x) - 4 \\ &= -x^2 - 3x - 4 \end{aligned}$$

When a graph is reflected in the y-axis, the sign of x changes, which means that we must replace every " x " in the equation with " $-x$ ".



Comparing graphs



You need to be able to answer questions such as:

- **Where is the one graph above/below the other?** For this, we need the x-values of the points of intersection of the graphs. We then read the answer off the graphs.
- **Where is the difference between the two graphs a certain number of units?** For this, we need to know what the difference between the y-values are.
- **Where is the product between the two graphs positive?** For this, we need to remember the fundamental rule of mathematics that the product of two positive numbers is positive, and the product of two negative numbers is positive. We then find x-values on the two graphs such that both graphs have positive y-values at the same time, or both graphs have negative y-values at the same time.
- **Where is the product between the two graphs negative?** For this, we remember that the product of a positive and a negative number is always negative. We then find x-values on the two graphs where the one graph has positive y-values, and the other graph has negative y-values.



In order to answer these questions, it is **important** to remember the following:

- $f(x)$ is function notation and is the same as y . Therefore, if a question is asking for the x-values where $f(x) = 0$, you have to find x-values on the graph where $y = 0$.
- Your answer **always** has to list the values of x , not y .
- You may use inequality or interval notation if your answer has to include a range of numbers, so all rules mentioned earlier about including/excluding values and how to indicate that still apply.

Now let's look at some typical exam-type questions!





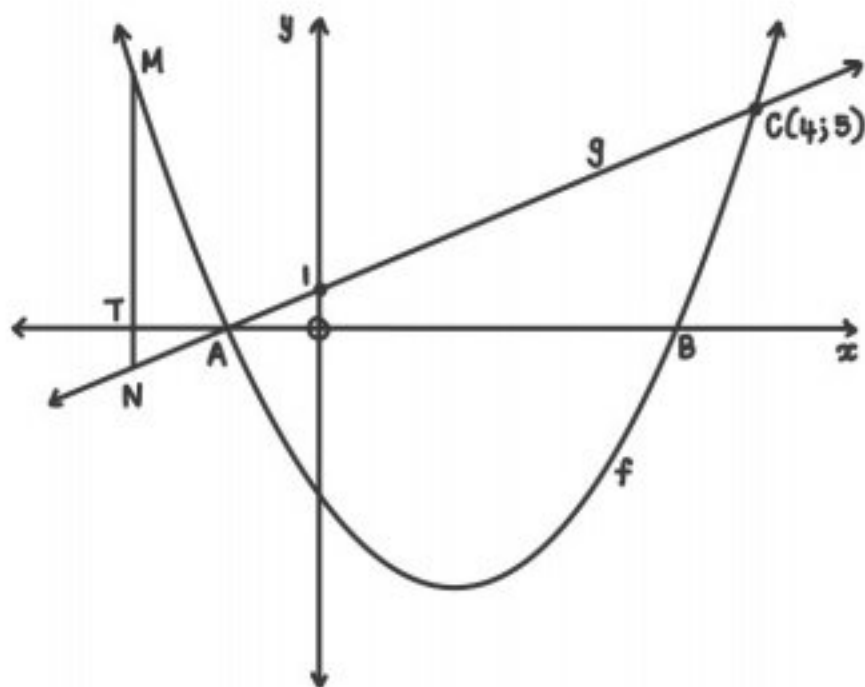
Worked example (R + C + P)

In the diagram below, A and B are the x -intercepts of the graph of $f(x) = x^2 - 2x - 3$.

A straight line, g , through A cuts f at $C(4; 5)$ and the y -axis at $(0; 1)$.

M is a point on f and N is a point on g such that MN is parallel to the y -axis.

MN cuts the x -axis at T.



Before attempting any of the questions, read through all of the given information carefully.

Fill any extra information in on the diagram and underline or highlight important information that you are unable to show on the diagram.

Worked example (R)

1. Show that $g(x) = x + 1$.

$$\begin{aligned} g: y &= mx + c \\ y\text{-int: } (0; 1) \\ \therefore c &= 1 \\ m &= \frac{1 - 5}{0 - 4} \\ &= 1 \\ \therefore y &= x + 1 \end{aligned}$$

$g(x)$ is a straight line passing through $(0; 1)$ and $C(4; 5)$. Remember that you are not allowed to use what you are trying to prove in a "show that" question! This means that you may not use $g(x) = x + 1$ in your calculations at all, it must simply be the final answer.

Finding the equation of a straight line is explained in detail on pg. 41-42.



2. Calculate the coordinates of A and B.

$$f(x) = x^2 - 2x - 3$$

x -intercepts:

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3 \text{ or } x = -1$$

$$\therefore A(-1;0) \text{ and } B(3;0)$$

To calculate the x -intercepts of any graph, make $y = 0$ and solve for x .

A has a negative x -value, and B has a positive x -value.

Worked example (C)

3. Determine the range of f .

Turning point of f :

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(-2)}{2(1)} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \therefore y &= (1)^2 - 2(1) - 3 \\ &= -4 \end{aligned}$$

The range of f consists of all y -values equal to and greater than the y -coordinate of the turning point.

We first need to calculate that value, using any of the methods discussed earlier.

The method shown here is using the turning point formula, and then substituting the x -value of the point into the equation to calculate the y -value.

$$\therefore \text{Range: } y \geq -4 \text{ or } y \in [-4; \infty)$$

4. If $MN = 6$:

4.1. Determine the length of OT if T lies on the negative x -axis. Show ALL your working.

$MN = 6$ units

MN is parallel to the y -axis, so the distance between M and N is equal to the distance between the y -value of M and the y -value of N.

Since we don't know what these y -values are, we have to use the equations of $f(x)$ and $g(x)$ as expressions for the y -values.

M lies on $f(x)$ and N lies on $g(x)$.

Since M lies above N, the distance of 6 units will be found by subtracting g (y -value of N) from f (y -value of M).

$$\therefore f(x) - g(x) = 6$$

$$x^2 - 2x - 3 - (x+1) = 6$$

$$x^2 - 2x - 3 - x - 1 - 6 = 0$$

$$x^2 - 3x - 10 = 0$$

$$(x-5)(x+2) = 0$$

$$x \neq 5 \text{ or } x = -2$$

invalid

$$\therefore x_T = -2$$

$$\therefore OT = 2 \text{ units}$$

The distance between $O(0;0)$ and $T(-2;0)$ is 2 units.



Worked example (R)

4.2. Hence, write down the coordinates of N.

$$x_N = -2$$

N has an x -value of -2 , and lies on the graph of $g(x)$.

$$y_N = g(-2)$$

Substitute $x = -2$ into the straight line equation to get the y -value.

$$= -2 + 1$$

$$= -1$$

$$\therefore N(-2; -1)$$

Worked example (C)

5. Determine the equation of the tangent to f drawn parallel to g .

$$\parallel \text{ to } g(x): m = 1$$

Parallel lines have equal gradients.

m of tangent to $f = f'(x)$ at point of contact

This is an application of calculus.

$$f'(x) = 2x - 2$$

$$\therefore 2x - 2 = 1$$

$$2x = 3$$

$$x = \frac{3}{2}$$

We can equate the expression for the gradient of the tangent (the derivative) and the actual value of the gradient in order to calculate the x -coordinate of the point of contact.

$$y = \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) - 3$$

$$= -\frac{15}{4}$$

Now substitute the x -value into the equation of $f(x)$ to calculate the y -value of the point of contact.

$$\therefore \text{Point of contact: } \left(\frac{3}{2}; -\frac{15}{4}\right)$$

$$y = x + c$$

$$\left(\frac{3}{2}; -\frac{15}{4}\right):$$

$$-\frac{15}{4} = \frac{3}{2} + c$$

$$c = -\frac{21}{4}$$

$$\therefore y = x - \frac{21}{4}$$

or

$$y - y_1 = 1(x - x_1)$$

$$\left(\frac{3}{2}; -\frac{15}{4}\right):$$

$$y + \frac{15}{4} = x - \frac{3}{2}$$

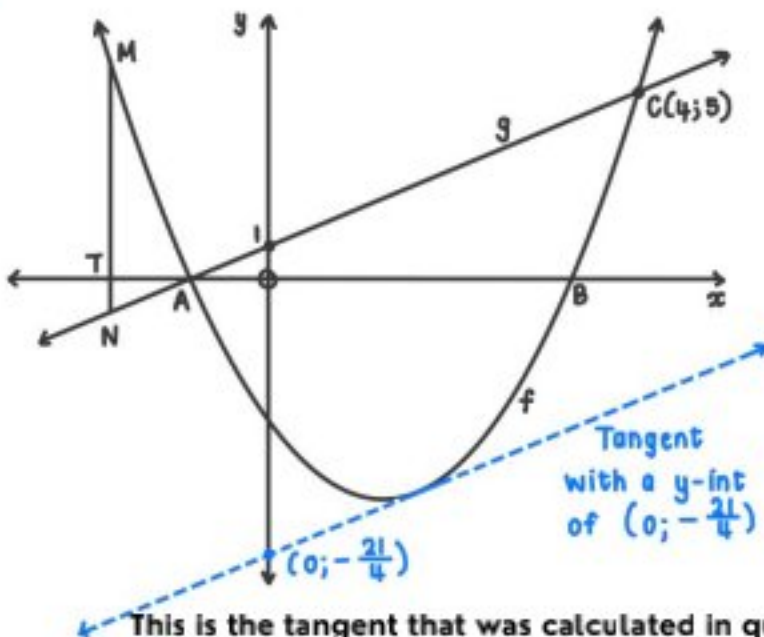
$$y = x - \frac{21}{4}$$

Substitute the point of contact into either one of the straight line equations alongside.

This allows us to calculate the equation of the tangent.

Worked example (P)

6. For which value(s) of k will $f(x) = x^2 - 2x - 3$ and $h(x) = x + k$ NOT intersect?



If $h(x) = x + k$ is BELOW the tangent drawn, the two graphs will not intersect.

Since k is the y -intercept of $h(x)$, the y -intercept will be less than $-\frac{21}{4}$ if the line is below the tangent.

$$\therefore k < -\frac{21}{4}$$

This is the tangent that was calculated in question 5.

Question source: NSC Grade 12 November 2018 Paper 1



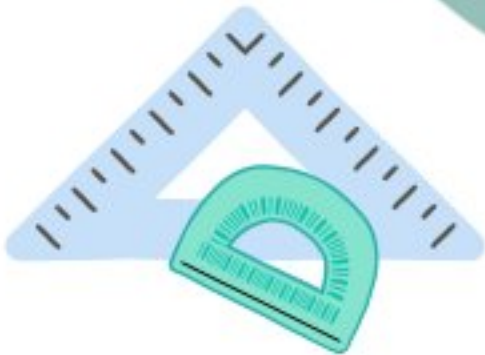
Miss Angler

Question Type 3: A function and its inverse

- A function is a relationship between two variables: x , which is the input value, and y , which is the output value, where each input value only has one output value associated with it.

This can also be described by saying that every x -value has only one y -value.

The set of input values is called the **domain**, and the set of output values is called the **range**.



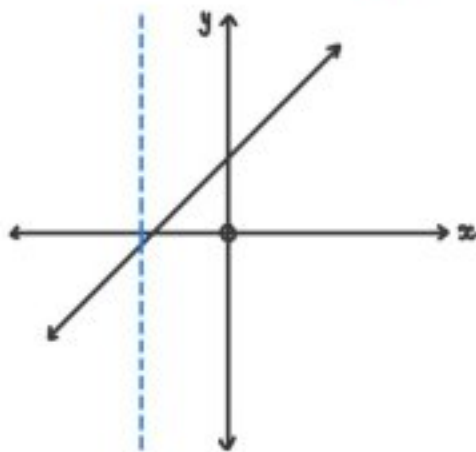
- To test whether the graph of a relationship represents a function, we use the vertical line test.

Hold a ruler parallel to the y -axis and move it across the graph, from left to right.

If the ruler crosses over the graph **more than once** at any time, that means that it is **not a function**.

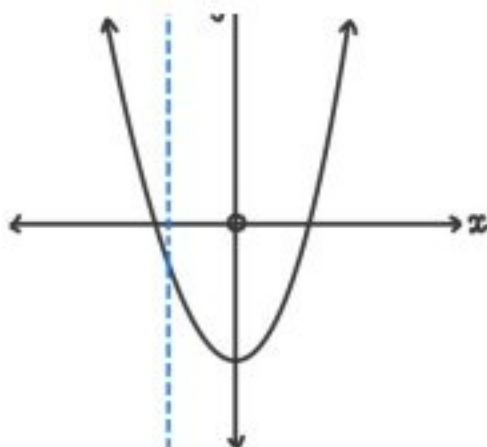
If the ruler crosses over the graph **only once** all the way from the left to the right, that means that the graph **is a function**.

Let's look at some examples of graphs and how to determine whether they are functions.



A straight line is a function because every x -value has only one y -value.

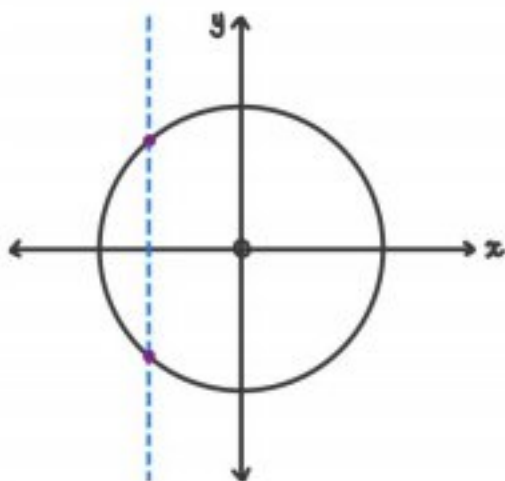
A vertical line cuts the graph only once (see dotted line as an example).



A parabola is a function because every x -value has only one y -value.

A vertical line cuts the graph only once (see dotted line as an example).





A circle is not a function because there are some x -values that have more than one (two) y -values.

A vertical line cuts the graph more than once in certain places (see dotted line as an example).

The reason why the vertical line test is used to determine whether a graph is a function, is because the vertical line represents a specific x -value.

The point of intersection of the line has that same x -value, as well as a y -value.

If there is more than one point of intersection, that means that one x -value has more than one y -value associated with it.

PRO-TIPS

You also need to be able to **determine** whether a graph is a function **without** actually having a **picture** of the graph.

There are two ways in which you can do this:

- **Visualise the graph based on the equation given.**

If you are able to recognise the type of graph from the equation, you can imagine what that type of graph will look like, and decide from there whether it would pass the **vertical line test** or not.

You do not need to know exactly how the specific graph will look, you simply need to know what the shape of the graph will be. For example, is it a straight line, parabola, hyperbola, etc?

- **Substitute specific x -values into the equation and solve for y .**

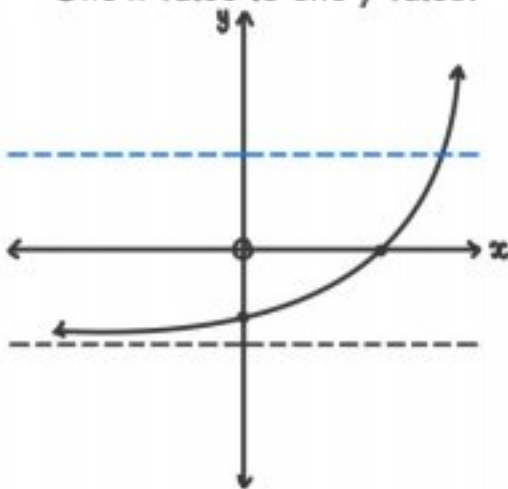
If you get multiple y -values as an answer, that means that the graph is **NOT** a function.

Make sure that you **substitute** a few x -values into the equation, as some graphs have some x -values that have only one y -value, but others that have multiple y -values associated with them.

There are different types of functions.

We use the **horizontal line test** to classify functions, which works in the same way as the vertical line test. The only difference is that you have to hold your ruler in a horizontal direction and move it from the top of the function to the bottom.

- A **one-to-one function** is a function where every x-value only has one y-value, but the reverse is also true: every y-value only has one x-value. A horizontal line will cut this type of graph only once. One x-value to one y-value.

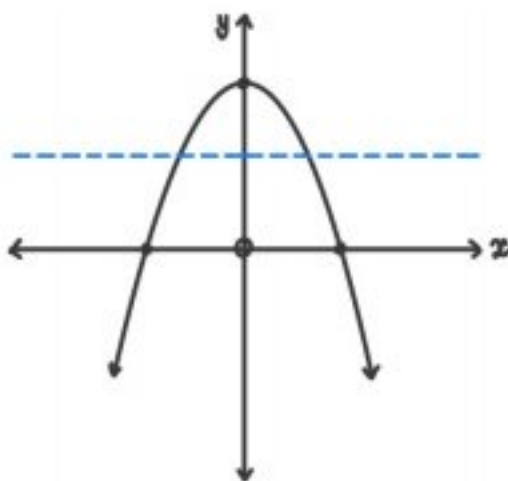


A horizontal line will only cut this exponential graph once
 $\therefore$ it is a one-to-one function.

Other graphs that are one-to-one functions are straight lines and hyperbolas.

Note that since a vertical line will also cut these only once, they are classified as functions.

- A **many-to-one function** is a function where every x-value only has one y-value (otherwise it wouldn't be a function), but multiple ("many") x-values have one specific y-value associated with it. A horizontal line will cut this type of graph more than once. Many x-values to one y-value.



A horizontal line cuts a parabola twice
 $\therefore$ it is a many-to-one functions.

A cubic graph with two unequal turning points is also a many-to-one function.

Note that since a vertical line will also cut these only once, they are classified as functions.

Let's talk about INVERSES!

An inverse of a function/relationship is the function/relationship that **reverses the process of the original**.

It is also the graph that is found by **reflecting the original graph in the line $y = x$** . Remember that when we apply this reflection to a point, the coordinates of the original point swap around.

The rule is: $(x ; y)$ becomes $(y ; x)$.

This is what happens to the graph, as well as the equation of a graph when we find the inverse.

To find the equation of the inverse:

Swap the positions of x and y in the equation of the original. Then make y the subject of the equation. **OR** reverse the operations of the original equation, as well as the order in which these operations take place.

$$\begin{aligned}\text{Original: } f(x) &= 3x - 9 & \therefore y &= 3x - 9 \\ \text{Inverse: } x &= 3y - 9 \\ x + 9 &= 3y \\ \therefore y &= \frac{1}{3}x + 3 \\ \therefore f^{-1}(x) &= \frac{1}{3}x + 3\end{aligned}$$

To find the coordinates of points on the inverse:

Swap the x and y -coordinates of points on the original graph.

$$\begin{aligned}\text{Points on } f(x): & (0; -9) \quad (1; -6) \quad (3; 0) \\ \text{Points on } f^{-1}(x): & (-9; 0) \quad (-6; 1) \quad (0; 3)\end{aligned}$$

Range and domain:

The domain of the inverse = the range of the original

The range of the inverse = the domain of the original

$$\begin{aligned}\text{Domain of original: } x &\in \mathbb{R}; x \neq 3 & \Rightarrow \text{Range of inverse: } y &\in \mathbb{R}; y \neq 3 \\ \text{Range of original: } y &\in \mathbb{R}; y \neq -1 & \Rightarrow \text{Domain of inverse: } x &\in \mathbb{R}; x \neq -1\end{aligned}$$

Restrictions on the domain and range:

If there is a restriction on the domain of the original graph, that restriction applies to the range of the inverse.

$$\text{Original: } g(x) = x^2; x \geq 0$$

$$\begin{aligned}\text{Inverse: } g^{-1}(x) &= \pm \sqrt{x}; y \geq 0 \\ \therefore g^{-1}(x) &= \sqrt{x}; x \geq 0\end{aligned}$$

Because of the nature of this expression, there is an additional restriction. If $x < 0$, this expression is non-real.



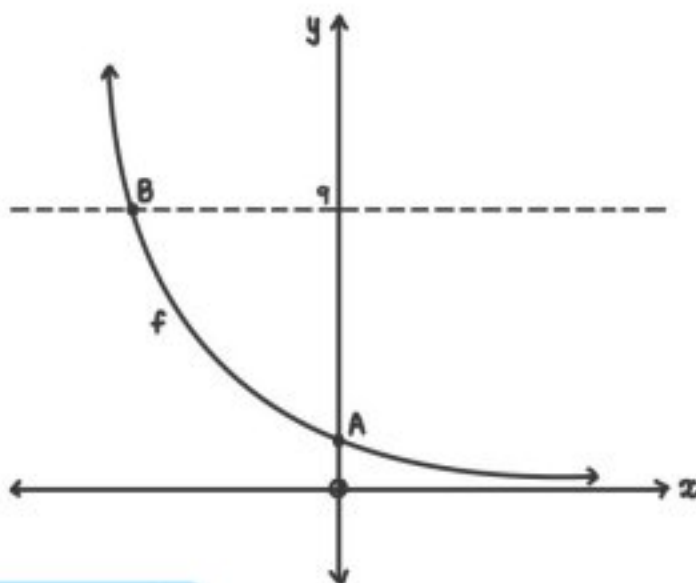


Worked example (K + R + C)

The graph of $f(x) = 3^{-x}$ is sketched below.

A is the y-intercept of f .

B is the point of intersection of f and the line $y = 9$.



Worked example (K)

1. Write down the coordinates of A.

$$\begin{aligned}
 A: & \text{y-intercept of } f \therefore \text{make } x = 0 \\
 & y = 3^{-0} \quad \text{or} \quad y = \left(\frac{1}{3}\right)^0 \\
 & \quad = 1 \quad \quad \quad = 1 \\
 \therefore & A(0; 1)
 \end{aligned}$$

To find the y-intercept of any graph:
Make $x = 0$ in the equation

Worked example (R)

2. Determine the coordinates of B.

$$\begin{aligned}
 B: & y = 9 \\
 & 3^{-x} = 9 \\
 & 3^{-x} = 3^2 \\
 \therefore & -x = 2 \\
 & x = -2 \\
 \therefore & B(-2; 9)
 \end{aligned}$$

B lies on the horizontal line $y = 9$, so will have a y-coordinate of 9. It also lies on the graph of $f(x)$, so we can calculate the x-coordinate of B by substituting $y = 9$ into the equation of f .

3. Write down the domain of f^{-1} .

$$\begin{aligned}
 & \text{Range of } f: y > 0 \text{ or } y \in (0; \infty) \\
 \therefore & \text{Domain of } f^{-1}: x > 0 \text{ or } x \in (0; \infty)
 \end{aligned}$$

The range of the original graph is the same as the domain of the inverse.



Worked example (C)

4. Describe the translation from f to $h(x) = \frac{27}{3^x}$.

$$f(x) = 3^{-x} \quad \text{to} \quad h(x) = \frac{27}{3^x}$$

$$\begin{aligned}\frac{27}{3^x} &= 27 \times 3^{-x} \\ &= 3^3 \times 3^{-x} \\ &= 3^{-x+3} \\ &= 3^{-(x-3)}\end{aligned}$$

$\therefore$ Translation of 3 units to the right

We need to rewrite the equation of $h(x)$ so that we can see which changes were made to the equation of $f(x)$.
Remember that we have to describe either a translation or a reflection.

5. Determine the values of x for which $h(x) < 1$.

$$f(x) < 1 : x > 0$$

$$\therefore h(x) < 1 : x > 3$$

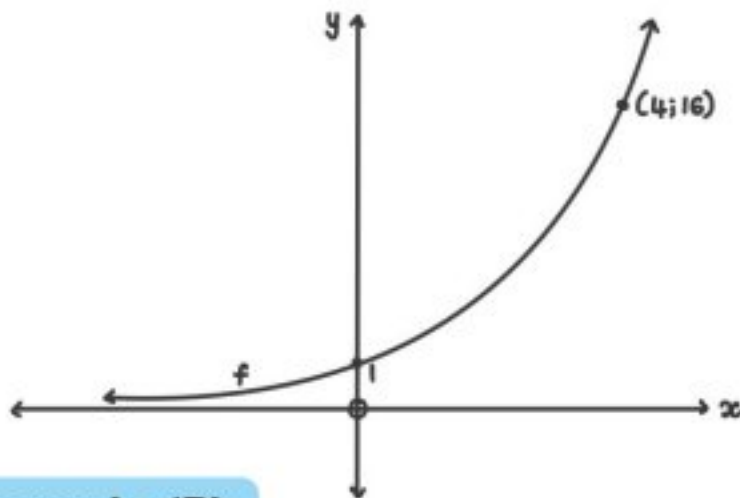
We can use the given graph of $f(x)$ to determine for which values of x the graph has y -values smaller than 1.
Since $h(x)$ is the graph of $f(x)$ shifted 3 units to the right, we can do the same with the x -value found.

Question source: NSC Grade 12 November 2020 Paper 1



Worked example (R + C)

Sketched below is the graph of $f(x) = k^x$; $k > 0$. The point $(4; 16)$ lies on f .



Worked example (R)

1. Determine the value of k .

$$y = k^x$$

$$\begin{aligned} \text{Sub in } (4; 16): 16 &= k^4 \\ 2^4 &= k^4 \\ \therefore k &= 2 \end{aligned}$$

The point $(4; 16)$ lies on the graph of $f(x)$, so we can calculate the value of k by substituting the point into the equation of $f(x)$ and then solving for k .

2. Graph g is obtained by reflecting graph f about the line $y = x$. Determine the equation of g in the form $y = \dots$

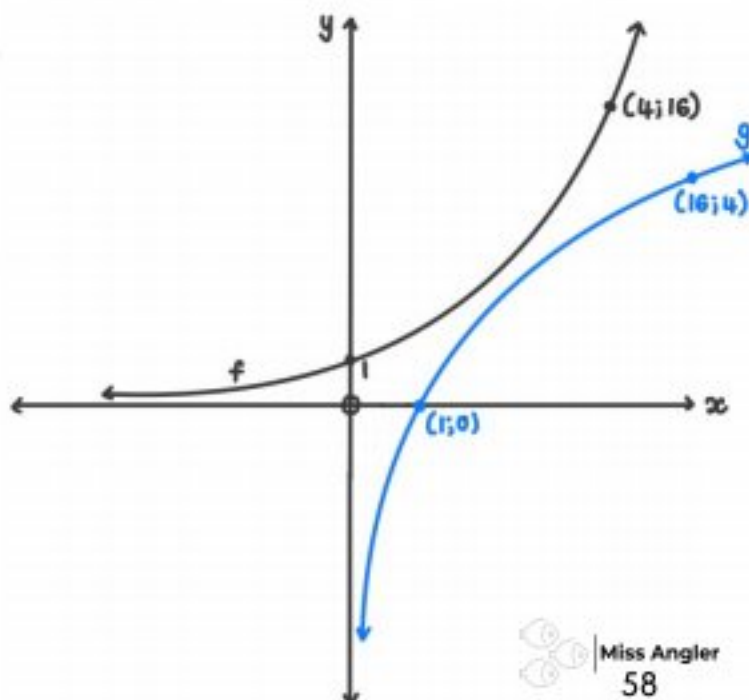
$$f(x): y = 2^x$$

$$g(x): x = 2^y$$

$$\therefore y = \log_2 x$$

Reflecting a graph in the line $y = x$ is how we obtain the inverse of that graph. Swap the positions of x and y in the original equation, and then make y the subject of the equation by converting the equation to log-form.

3. Sketch the graph of g . Indicate on your graph the coordinates of two points on g .



To draw the graph of an inverse, swap the coordinates of points on the original graph.

$$(0; 1) \rightarrow (1; 0)$$

$$(4; 16) \rightarrow (16; 4)$$

You do not have to calculate new points on the inverse.

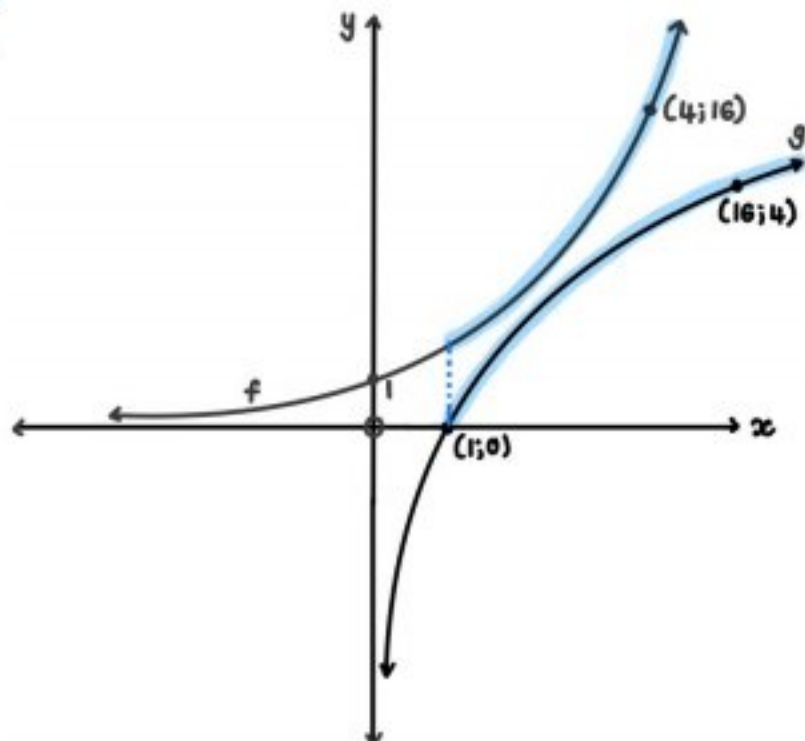
You also don't have to draw your graph to scale, but make sure that you draw a smooth curve that tends towards the y -axis.



Worked example (C)

4. Use your graph to determine the value(s) of x for which:

4.1. $f(x) \times g(x) > 0$



$$f(x) \times g(x) > 0:$$

The product of two numbers is positive if both numbers are positive, or both numbers are negative.

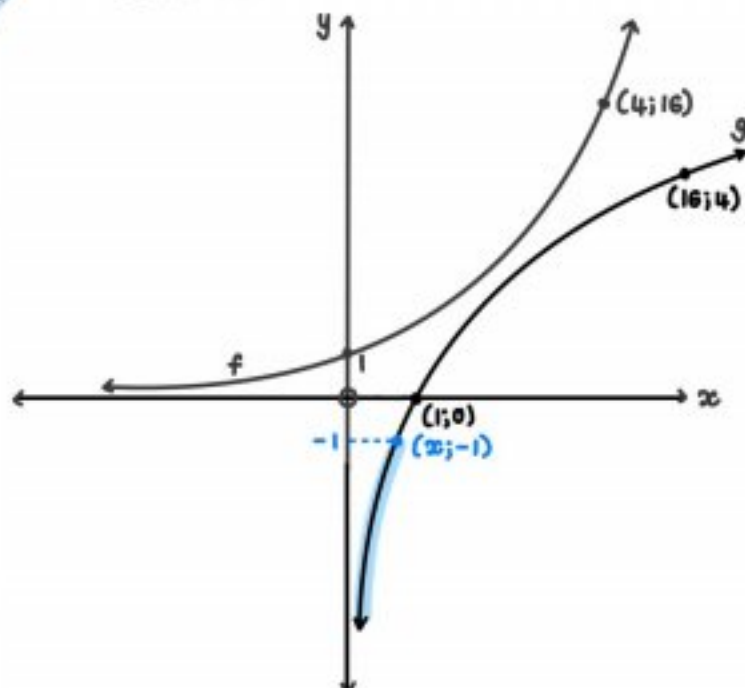
There are no sections on these graphs where they are both negative, but both graphs are positive to the right of $x=1$ (highlighted).

$$x > 1 \text{ or } x \in (1; \infty)$$

A graph is positive if it has positive y-values, so when it lies ABOVE the x-axis.

A graph is negative if it has negative y-values, so when it lies BELOW the x-axis.

4.2. $g(x) \leq -1$



$$g(x) \leq -1:$$

Even though the graph of g is not drawn to scale, we know that $y=-1$ lies below the x -axis, so we can estimate where we think it could be. This makes it easier to visualize where the graph has y-values equal to or less than -1 .

We now need to calculate the x -value of the point with $y=-1$.

$$g(x) = \log_2 x$$

$$\therefore \log_2 x = -1$$

$$2^{-1} = x$$

$$x = \frac{1}{2}$$

Rewrite in exponential form.

$$\therefore 0 < x \leq \frac{1}{2} \text{ or } x \in (0; \frac{1}{2}]$$

We have to exclude $x=0$ from the solution as it is an asymptote.

We therefore use a round bracket at 0.

Question source: NSC Grade 12 November 2019 Paper 1



Miss Angler

FINANCE



In order to achieve maximum marks for this section, you need to be able to:

- **Identify** the formula that the question requires
- **Find** the correct **formula** on the formula sheet or combine the correct formulae to suit the question
- **Substitute** the correct values into the formula
- Answer "**time line**" questions where multiple deposits and/or withdrawals are taking place
- Use your **calculator** effectively to calculate the final answer to a question.

Let's look at a summary of the formulae that you need to be able to work with:

Simple interest / growth:

$$A = P(1 + i \times n)$$

Compound interest / growth:

$$A = P(1 + i)^n$$

Simple decay / straight line depreciation:

$$A = P(1 - i \times n)$$

Compound decay / reducing balance depreciation:

$$A = P(1 - i)^n$$

where A = accumulated (final) amount this amount includes interest

P = original amount invested or borrowed this amount does not include interest

i = interest rate (as a percentage)

n = number of time periods

Future value annuities

$$F = \frac{x[(1+i)^n - 1]}{i}$$

where F = future value of investment

x = regular payment

i = interest rate (as a percentage)

n = number of payments

Present value annuities

$$P = \frac{x[1 - (1+i)^{-n}]}{i}$$

where P = present value of loan

Converting interest rates: From nominal to effective

$$1 + i_{\text{eff}} = \left(1 + \frac{i}{m}\right)^m$$

where i_{eff} = effective (annual) interest rate

$\frac{i}{m}$ = nominal interest rate

m = number of interest calculations per year

From effective to nominal

$$\left(1 + \frac{i}{m}\right)^m = 1 + i_{\text{eff}}$$

Worked example (R)

Convert an interest rate of 14% p.a. compounded monthly to an effective interest rate.

$$1 + i_{\text{eff}} = \left(1 + \frac{i}{m}\right)^m$$

$$\begin{aligned} i_{\text{eff}} &= \left(1 + \frac{0,14}{12}\right)^{12} - 1 \\ &= 0,14934... \\ &= 14,93\% \text{ p.a.} \end{aligned}$$

Since we are calculating an effective interest rate, we write the side of the formula with i_{eff} first.

14% p.a. compounded monthly = $\frac{0,14}{12}$ or $\frac{14}{1200}$ ($\frac{14}{100} \div 12$)

Multiply this by 100 to get the answer as an interest rate.

Calculate a quarterly interest rate that will provide the same return on investment as a rate of 10,5% effective.

$$\left(1 + \frac{i}{m}\right)^m = 1 + i_{\text{eff}}$$

$$\left(1 + \frac{i}{4}\right)^4 = 1 + 0,105$$

$$1 + \frac{i}{4} = \sqrt[4]{1,105}$$

$$\begin{aligned} \frac{i}{4} &= 1,0252... \\ \frac{i}{4} &= 0,0252... \end{aligned}$$

$$i = 0,1011...$$

$$= 10,11\% \text{ p.a.}$$

Since we are calculating a nominal (quarterly) interest rate, we write the side of the formula with $\frac{i}{m}$ first.

Quarterly = 4 times a year $\therefore m = 4$

10,5% p.a. effective = 0,105 or $\frac{10,5}{100}$

Now solve for i .

Multiply this by 100 to get the answer as an interest rate.

Balance on loan after m payments

$$= P(1+i)^m - \frac{x[(1+i)^m - 1]}{i}$$

$$\text{or } \frac{x[1 - (1+i)^{-(n-m)}]}{i}$$

where P = loan amount

x = value of regular payment

m = number of payments made

n = total number of payments to be made

$\therefore n - m$ = number of payments remaining

PRO-TIPS

The formulae on this page are **NOT** on the formula sheet, so make sure to memorise them!





Dealing with deferred payments

Deferred payments are what we call it when you take out a loan but don't make the first payment at the end of the first time period. The first payment is only made a number of time periods later.

If the first payment of a loan is only made "m" time periods after the loan was taken out, follow these steps when answering questions:

1. Add (m - 1) time periods' worth of interest to the loan amount, using the compound growth formula.
2. Use the present value annuity formula to calculate what you are asked for, but make sure to decrease the number of payments by (m - 1). You also have to use the answer to step 1 as the new present value of the loan.



The final payment on a loan

The final payment on a loan is generally smaller than the other payments. Follow these steps to calculate the value of this payment.



1. Calculate the balance on the loan after the last full payment. This will always be one less than the total number of payments.
2. Now add one time periods' worth of interest to the balance on the loan, using the compound growth formula.



Calculating the amount of capital versus interest paid on a loan

If a loan has been paid in full, the **CAPITAL** amount paid is the same as the original loan amount.

The **INTEREST** amount paid is the difference between the total amount paid and the original loan amount.

Interest = total amount paid - original loan amount

If the loan has not yet been paid in full, it will require more working, follow these steps:

1. Determine how much money has actually been paid.

Amount paid = regular payment amount x number of payments

2. Determine the **balance on the loan**. This is the CAPITAL amount that still has to be paid.

3. Determine the CAPITAL amount paid.

Capital amount paid = original loan amount - balance on loan

4. The INTEREST amount is the difference between the actual amount paid (calculated in step 1) and the capital amount paid (calculated in step 3).

Interest amount paid = actual amount paid - capital amount paid



Let's look at some typical exam questions!



Worked example (R)

A farmer bought a tractor for R980 000. The value of the tractor depreciates annually at a rate of 9,2% p.a. on the reducing balance method. Calculate the book value of the tractor after 7 years.

$$\begin{aligned} A &= P(1 - i)^n \\ &= 980\,000(1 - 0,092)^7 \\ &= R498\,685,82 \end{aligned}$$

Make sure that you are using the correct formula!
9,2% = 0,092 or $\frac{9,2}{100}$



Worked example (C)

How many years will it take for an amount of R75 000 to accrue to R116 253,50 in an account earning interest of 6,8% p.a. compounded quarterly?

$$\begin{aligned} A &= P(1 + i)^n \\ 116\,253,50 &= 75\,000\left(1 + \frac{0,068}{4}\right)^{4n} \end{aligned}$$

We make this equal to $4n$ because the number of payments is equal to $4 \times$ number of years.

$$\begin{aligned} 1,55004... &= \left(1 + \frac{0,068}{4}\right)^{4n} \\ 4n &= \log_{\left(1 + \frac{0,068}{4}\right)} 1,55004... \\ &= 25,999... \\ n &= 6,5 \text{ years} \end{aligned}$$

6,8% p.a. compounded quarterly = $\frac{0,068}{4}$ or $\frac{6,8}{400}$
We need to solve for n , which is in the exponent.
Since we cannot make the bases the same, we have to rewrite the equation in log form.
Remember: if $a^m = n$, then $m = \log_a n$



Worked example (C)

Thabo wanted to save R450 000 as a deposit to buy a house on 30 June 2018.

1. He deposited a fixed amount of money at the end of every month into an account earning interest of 8,35% p.a. compounded monthly. His first deposit was made on 31 July 2013 and his 60th deposit on 30 June 2018.

Calculate the amount he deposited monthly.

$$\begin{aligned} F &= \frac{x[(1 + i)^n - 1]}{i} \\ 450\,000 &= \frac{x\left[\left(1 + \frac{0,0835}{12}\right)^{60} - 1\right]}{\frac{0,0835}{12}} \\ 3\,131,25 &= x\left[\left(1 + \frac{0,0835}{12}\right)^{60} - 1\right] \\ \therefore x &= R6\,068,69 \end{aligned}$$

Since this is an investment, and Thabo is saving up to reach a specific amount in the future, we use the formula for future value annuities.



2. Thabo bought a house costing R1 500 000 and used his savings as the deposit. He obtained a home loan for the balance of the purchase price at an interest rate of 12% p.a. compounded monthly over 25 years. He made his first monthly instalment of R11 058,85 towards the loan on 31 July 2018.

2.1. What will the balance outstanding on the loan be on 30 June 2039, 21 years after the loan was granted?

Remember that there are two ways to calculate the balance on a loan.

Method 1 uses the number of payments already made.

Method 2 uses the number of payments remaining.

Method 1:

$$\begin{aligned} &\text{Balance after 21 years (21 x 12 = 252 payments)} \\ &= 1\,050\,000 \left(1 + \frac{0,12}{12}\right)^{252} - \frac{11\,058,85 \left[\left(1 + \frac{0,12}{12}\right)^{252} - 1\right]}{\frac{0,12}{12}} \end{aligned} \quad \leftarrow \text{This is the formula for future value annuities.}$$

$$= R419\,952,39$$

Method 2:

$$\begin{aligned} &\text{Balance with 4 years remaining (4 x 12 = 48 payments)} \\ &= \frac{11\,058,85 \left[1 - \left(1 + \frac{0,12}{12}\right)^{-48}\right]}{\frac{0,12}{12}} \end{aligned} \quad \leftarrow \text{This is the formula for present value annuities.}$$

$$= R419\,948,32$$



- 2.2. Calculate the interest Thabo will have paid over the first 21 years of the loan.

For this question, we will use the balance calculated using Method 1.

$$\begin{aligned} &\text{Actual amount paid} \\ &= \text{monthly payment} \times 252 \\ &= 11\,058,85 \times 252 \\ &= R2\,786\,830,20 \end{aligned}$$

$$\begin{aligned} &\text{Balance on loan (capital amount still owing)} \\ &= R419\,952,39 \end{aligned}$$

If you had not calculated this in an earlier question, you would have to do the calculation here.

$$\begin{aligned} &\text{Capital amount paid} \\ &= \text{original loan amount} - \text{balance on loan} \\ &= 1\,050\,000 - 419\,952,39 \\ &= R630\,047,61 \end{aligned}$$

$$\begin{aligned} &\text{Interest amount paid} \\ &= \text{actual amount paid} - \text{capital amount paid} \\ &= 2\,786\,830,20 - 630\,047,61 \\ &= R2\,156\,782,59 \end{aligned}$$

Question source: NSC Grade 12 Mathematics Paper 1 2021





Worked example (R + C)

A company purchased machinery for R500 000. After 5 years, the machinery was sold for R180 000 and new machinery was bought.



1. Calculate the rate of depreciation of the old machinery over the 5 years, using the reducing balance method.

$$A = P(1 - i)^n$$

$$180\,000 = 500\,000(1 - i)^5$$

$$0,36 = (1 - i)^5$$

$$\sqrt[5]{0,36} = 1 - i$$

$$0,8151... = 1 - i$$

$$-0,184... = -i$$

$$i = 0,184...$$

$$= 18,4\% \text{ p.a.}$$

Use the formula for compound decay and substitute the given values into the formula.
Now solve for i .

Only round off in your final answer!



2. The rate of inflation of the new machinery is 6,3% p.a. over the 5 years. What will the new machinery cost at the end of 5 years?

$$A = P(1 + i)^n$$

$$= 500\,000(1 + 0,063)^5$$

$$= R678\,635,11$$

Inflation is a form of compound growth, so you have to use this formula.



3. The company set up a sinking fund and made the first payment into this fund on the day the old machinery was bought. The last payment was made three months before the new machinery was purchased at the end of the 5 years. The interest earned on the sinking fund was 10,25% p.a. compounded monthly. The money from the sinking fund and the R180 000 from the sale of the old machinery was used to pay for the new machinery.

Calculate the monthly payment into the sinking fund.

First payment made at the **beginning of the first month**
 $\therefore$ add an extra interest bracket to the formula

If the last payment was made at the beginning of the last month (so one month before the end of the 5 years), there would have been $5 \times 12 = 60$ payments.

The last payment is made **three months before the end**
 $\therefore$ 2 months earlier, so there are **58 payments**.

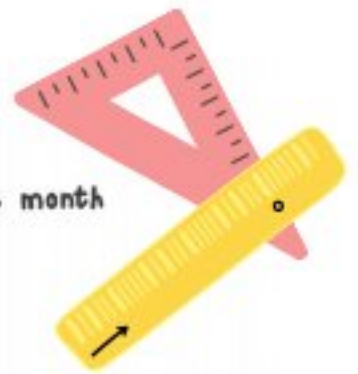
$$F = \frac{x[(1 + i)^n - 1]}{i}$$

$$678\,635,11 - 180\,000 = \frac{x\left[\left(1 + \frac{0,1025}{12}\right)^{58} - 1\right]}{\frac{0,1025}{12}}$$

$$4\,259,174... = x\left[\left(1 + \frac{0,1025}{12}\right)^{58} - 1\right]$$

$$\therefore x = R6\,678,62$$

A sinking fund is a type of investment set up in order to be able to replace an item in the future, so we use the future value formula. The amount that is needed to replace the machinery (the future value) is the difference between the cost of the new machinery and the R180 000 from the sale of the old machinery.



Question source: NSC Grade 12 Mathematics Paper 1 May/June 2022

CALCULUS



In order to achieve maximum marks for this section, you need to be able to:

- **calculate** the derivative of an expression using **first principles**
- **calculate** the derivative of an expression using the **power rule**
- **calculate** the **second derivative** of an expression
- **draw** a cubic graph
- **interpret** a cubic graph
- **calculate** the equation of a **tangent** to a curve at a given point
- answer questions about a **cubic graph** if the derivative of that graph is given
- calculate the **maximum** or **minimum** value using derivatives
- calculate the **rate of change** of a scenario at a given time
- answer questions about **displacement, velocity** and **acceleration**

PRO-TIPS

The calculus section is always tested in four parts:

1. A first principles question
2. A power rule question
3. A section focussing on cubic graphs (usually multiple questions)
4. An application question, either on optimisation or rates of change



REMEMBER: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either:

K - KNOWLEDGE

R - ROUTINE

C - COMPLEX

P- PROBLEM SOLVING

next to each question!



Miss Angler

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Question type 1: Finding the derivative by first principles

Use the following formula to answer this question:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This question always counts 5 or 6 marks, and is classified as routine because you simply have to follow the same steps every time.



Worked example (R)

Determine the derivative of $f(x) = 3x^2$ by first principles.

$$f(x) = 3x^2$$

$$f(x+h) = 3(x+h)^2$$

$$= 3(x^2 + 2xh + h^2)$$

$$= 3x^2 + 6xh + 3h^2$$

First determine an expression for $f(x+h)$ by replacing x in the original function with $(x+h)$. Multiply the brackets out and simplify the expression.

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - (3x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(6x + 3h)}{h} \\ &= \lim_{h \rightarrow 0} (6x + 3h) \\ &= 6x \end{aligned}$$

This formula is given on the formula sheet.

Substitute the expressions for $f(x)$ and $f(x+h)$ into the formula.

Simplify by subtracting like terms.

Take out h as a common factor.

Now cancel out the h in the numerator and denominator.

If h tends to zero, then $3xh$ also tends to zero.

PRO-TIPS

- **Incorrect notation** will cost you a mark in this question!
- Firstly, you have to write "**limit as h tends to zero**" in every single step, until you actually make h equal to zero and get your final answer.
- Secondly, once you have cancelled out the h 's, you have to put **brackets** around the expression **after** the limit symbol.



Question type 2: The Power Rule

The power rule is very easy to apply.

However, you have to have good knowledge of **Grade 11 exponent and surd laws**, as well as a good understanding of **general algebra concepts**.



Let's recap the laws of exponents:

1. $a^m \times a^n = a^{m+n}$

eg. $2^m \times 2^{3m} = 2^{4m}$

The base of your answer is the same as the bases that are being multiplied.

2. $a^m \div a^n = a^{m-n}$

eg. $3^{4x} \div 3^{3x} = 3^x$

The base of your answer is the same as the bases that are being divided.

3. $(a^m)^n = a^{mn}$

eg. $(2^6)^3 = 2^{18}$

4. $(ab)^m = a^m b^m$

eg. $(3x^2y)^3 = 27x^6y^3$

5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
eg. $\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$

6. $a^m \times b^m = (ab)^m$

eg. $2^x \times 4^x = 8^x$

If two (or more) powers have the same exponents, you can multiply the bases together and keep that same exponent.

7. $a^m \div b^m = \left(\frac{a}{b}\right)^m$

eg. $12^x \div 4^x = 3^x$

If two powers have the same exponents, you can divide the bases and keep that same exponent.

8. $a^0 = 1$

eg. $(3x^2y)^0 = 1$
 $3x^2y^0 = 3x^2$

Working with negative exponents is very important, so let's look at some examples:

Rule: $a^{-n} = \frac{1}{a^n}$

eg. $3a^{-2} = \frac{3}{a^2}$

$$(3a)^{-2} = \frac{1}{(3a)^2}$$

$$= \frac{1}{9a^2}$$

$$\frac{a^{-4}}{3} = \frac{1}{3a^4}$$

$$\left(\frac{a}{3}\right)^{-2} = \frac{a^{-2}}{3^{-2}}$$

$$= \frac{3^2}{a^2}$$

$$= \frac{9}{a^2}$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

eg. $\sqrt{x} = x^{\frac{1}{2}}$

$$\sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$\sqrt{x^3} = x^{\frac{3}{2}}$$

You also have to be able to write a surd expression in exponential form in order to apply the power rule:



The Power Rule

If $f(x) = ax^n$, where a and n are constants, then $f'(x) = n \cdot ax^{n-1}$.

∴ Multiply the coefficient of x by the exponent, and then subtract 1 from the exponent.

PRO-TIPS

You **CANNOT** apply the power rule to a fraction with a variable in the denominator, or to unsimplified brackets. You **first** need to **rewrite** the expression in the correct form.

Let's look at some typical exam questions!

Worked example (R)

Determine the derivative of $y = 4x^5 - 6x^4 + 3x$.

$y = 4x^5 - 6x^4 + 3x$ Since each term is in the form ax^n , we can apply the product rule.

$$\begin{aligned}\frac{dy}{dx} &= 5 \cdot 4x^{5-1} - 4 \cdot 6x^{4-1} + 1 \cdot 3x^{1-1} \\ &= 20x^4 - 24x^3 + 3\end{aligned}$$

You do not have to show this step. You can do the multiplication and subtraction straight away.

Worked example (R)

Determine :

$$D_x \left[-\frac{\sqrt[3]{x}}{2} + \left(\frac{1}{3x} \right)^2 \right]$$

$$\begin{aligned}D_x \left[-\frac{\sqrt[3]{x}}{2} + \left(\frac{1}{3x} \right)^2 \right] \\ &= D_x \left[-\frac{1}{2} \sqrt[3]{x} + \frac{1}{9x^2} \right] \\ &= D_x \left[-\frac{1}{2} x^{\frac{1}{3}} + \frac{1}{9} x^{-2} \right] \\ &= -\frac{1}{6} x^{-\frac{2}{3}} - \frac{2}{9} x^{-3} \\ &= -\frac{1}{6x^{\frac{2}{3}}} - \frac{2}{9x^3}\end{aligned}$$

This notation is used very often. The " D_x " means "derivative with respect to x " of the expression in brackets.

Before we can apply the power rule, we have to use exponent and surd laws to write each term in the form ax^n .

You have to keep writing " $D_x[...]$ " in every line before applying the power rule.

Make sure to write your answer with positive exponents. You don't have to write your answer in surd form.

Question source: Grade 12 NSC November 2021 Mathematics Paper 1



Worked example (R)

Determine:

$$D_x \left[\frac{-6\sqrt[3]{x}+2}{x^4} \right]$$

$$D_x \left[\frac{-6\sqrt[3]{x}+2}{x^4} \right]$$
$$= D_x \left[\frac{-6x^{\frac{1}{3}}}{x^4} + \frac{2}{x^4} \right]$$

$$= D_x \left[-6x^{\frac{1}{3}-4} + 2x^{-4} \right]$$
$$= D_x \left[-6x^{-\frac{11}{3}} + 2x^{-4} \right]$$
$$= 22x^{-\frac{14}{3}} - 8x^{-5}$$
$$= \frac{22}{x^{\frac{14}{3}}} - \frac{8}{x^5}$$

This expression is not derivative-ready as it's a fraction with a variable in the exponent.

Write the surd in exponential form and split the numerator up into two terms. You can only do this if there is only one term in the denominator.

Then apply exponent laws to get rid of the variable in the denominator.

You can then apply the power rule.

Leave your answer with positive exponents.

Question source: Grade 12 NSC May/June 2022 Mathematics Paper 1



Worked example (R)

Determine $f'(x)$ if $f(x) = \frac{4x^2-9}{4x+6}$.

$$f(x) = \frac{4x^2-9}{4x+6}$$
$$= \frac{(2x+3)(2x-3)}{2(2x+3)}$$
$$= \frac{2x-3}{2}$$
$$= \frac{2x}{2} - \frac{3}{2}$$
$$= x - \frac{3}{2}$$

$$\therefore f'(x) = 1$$

This expression is not derivative-ready as there is a variable in the denominator. Since the denominator has two terms, we have to simplify this fraction by factorising.

Now that there is one term in the denominator, we can split the fraction up and further simplify if possible.

We can now apply the power rule.

The derivative of a constant is always zero, because $\frac{3}{2} = \frac{3}{2}x^0$, and when you apply the power rule, you get $0 \times \frac{3}{2}x^{0-1}$, which is zero.



Question source: Grade 12 NSC November 2020 Mathematics Paper 1



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Question type 3: Cubic Graphs

Drawing a cubic graph

Standard form:

$$y = ax^3 + bx^2 + cx + d$$

y-intercept:
(0; d)

The sign of a influences the shape.

$a > 0$:

or

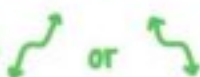
$a < 0$:

or

- x -intercepts: make $y = 0$ and solve for x
 $ax^3 + bx^2 + cx + d = 0$
- Turning point(s): make $\frac{dy}{dx} = 0$ and solve for x
substitute x -value(s) into the equation to find the y -coordinate(s)
- Point of inflection: make $\frac{d^2y}{dx^2} = 0$ and solve for x
substitute x -value(s) into the equation to find the y -coordinate(s)

The shape of a cubic graph with two distinct (unequal) turning points is different from that of a cubic graph with only one distinct turning point:

Two TPs: 

One TP: 

Two TPs: the PoI is exactly halfway between them

One TP: the PoI is the same point

PRO-TIPS

- Just like with other graphs, you should **always** plot all points of interest, specifically the x - and y -intercepts, and the turning points.
- You **only** need to plot the point of **inflection** if the question specifically asks you to.

"PoI" is an abbreviation for "point of inflection".

Worked example (C)

Draw the graph of $f(x) = x^3 - 6x^2 + 9x - 4$, showing all intercepts with the axes, turning points, and inflection points.

Start by calculating the intercepts with the axes:

$$f(x) = x^3 - 6x^2 + 9x - 4 \quad \leftarrow \text{y-intercept: } (0; -4)$$

x-intercept(s): make $y = 0$

$$\begin{aligned} x^3 - 6x^2 + 9x - 4 &= 0 \\ (x-1)(x^2 - 5x + 4) &= 0 \\ &\quad \begin{array}{c} \text{---} x^2 \text{---} \\ \text{---} 5x^2 \text{---} \end{array} \\ \text{and } -x^2 - 5x^2 &= -6x^2 \end{aligned}$$

This is a cubic equation.

To get the first bracket, substitute x -values into the equation until you get one that makes the equation true ($= 0$). Start with $x = 1$, then $x = -1$, $x = 2$, $x = -2$, and so on.

If x_1 is the first value that works, then $(x - x_1)$ is the first bracket.

We then use the first bracket to determine what will be in the second bracket.

First terms: what must we multiply x by to get x^3 ?

$$x \times x^2 = x^3 \quad \therefore x^2 \text{ is the first term of the second bracket.}$$

Last terms: what must we multiply -1 by to get -4 ?

$$-1 \times 4 = -4 \quad \therefore +4 \text{ is the last term of the second bracket.}$$

Middle term: look at the x^2 -term in the cubic equation, $-6x^2$.

If we were to multiply out the two brackets, we would get an x^2 -term by multiplying the last term in bracket 1 by the first term in bracket 2.

$$-1 \times x^2 = -x^2$$

We need $-6x^2$, so we need another $-5x^2$.

The other product that will result in an x^2 -term is the first term in bracket 1 and the middle term in bracket 2.

What must we multiply x by to get $-5x^2$?

$$x \times (-5x) = -5x^2 \quad \therefore -5x \text{ is the middle term of the second bracket.}$$

$$(x-1)(x^2 - 5x + 4) = 0$$

$$(x-1)(x-1)(x-4) = 0$$

$$x = 1 \quad \text{or} \quad x = 4$$

$$\therefore (1; 0) \quad (4; 0)$$

Now factorise the second bracket as well.

If $x-1=0$, then $x=1$, and if $x-4=0$, then $x=4$.

Notice that there are two $(x-1)$ brackets.

We can call the x -intercept $(1; 0)$ a **DOUBLE** x -intercept, which means that it is also a turning point.

Next, calculate the coordinates of the turning point(s):

Turning point(s): $f'(x) = 0$

$$3x^2 - 12x + 9 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$$x = 3 \text{ or } x = 1$$

$$y = -4 \quad y = 0$$

$$\therefore (3; -4) \quad (1; 0)$$

↖ double x-intercept

Find the derivative of the original equation and solve for x . You can do this by factorising or by using the quadratic formula.

Now substitute these values into the equation to calculate the y -coordinates of each turning point.

Lastly, if the question specifies, calculate the coordinates of the point of inflection:

Point of inflection: $f''(x) = 0$

$$6x - 12 = 0$$

$$6x = 12$$

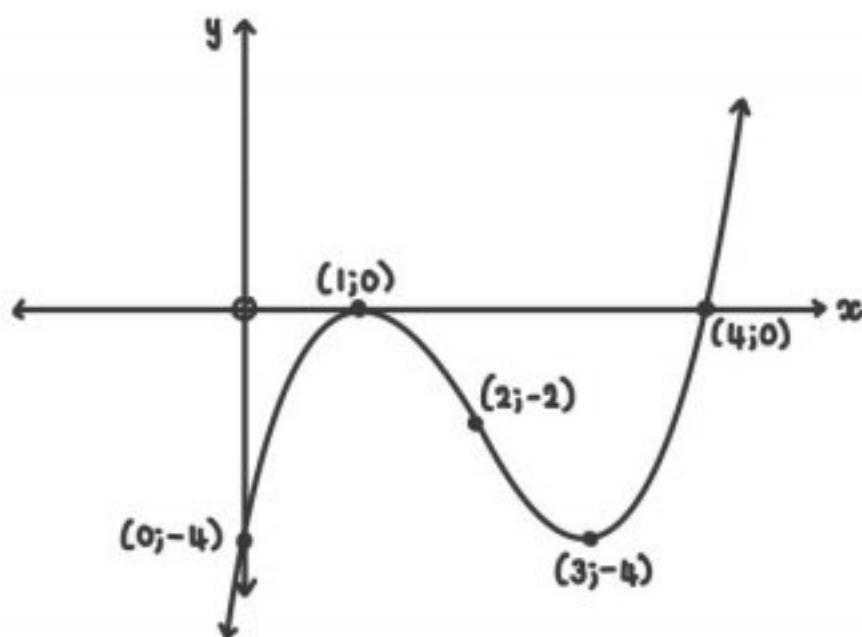
$$x = 2$$

$$y = -2$$

$$\therefore (2; -2)$$

This is one method for finding the point of inflection. The other method is to find the point exactly halfway between the two turning points, $(1; 0)$ and $(3; -4)$, which is $(2; -2)$.

You can now draw the graph. You do not have to draw it to scale but you do have to label each point that you are plotting with both coordinates:



PRO-TIPS

If you are **struggling** to visualise the graph, draw a double circle around the **turning points** as that will remind you that the graph has to turn at those points.



Worked example (C)

Draw the graph of $g(x) = -x^3 + 6x^2 - 12x + 16$, showing all intercepts with the axes, turning points, and inflection points.

Start by calculating the intercepts with the axes:

$$g(x) = -x^3 + 6x^2 - 12x + 16 \quad \leftarrow \text{y-intercept: } (0; 16)$$

x -intercept(s): make $y = 0$

$$-x^3 + 6x^2 - 12x + 16 = 0$$

$$x^3 - 6x^2 + 12x - 16 = 0$$

$$(x-4)(x^2-2x+4)=0$$

$$\begin{aligned} & -4x^2 \\ & -2x^2 \\ & -4x^2 - 2x^2 \\ & = -6x^2 \end{aligned}$$

The first x -value that makes the equation true is $x=4$ $\therefore$ the first bracket is $(x-4)$.

The second bracket has no solution.

$\Delta = b^2 - 4ac = (-2)^2 - 4(1)(4) = -12$, and we know that if $\Delta < 0$, the equation has non-real roots $\therefore$ no real solution.

$x = 4$ or no real solution There is only one x -intercept.
 $\therefore (4; 0)$

Next, calculate the coordinates of the turning point(s):

Turning point(s): $q'(x) = 0$

$$-3x^2 + 12x - 12 = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

$x = 2$

$y = 8$

$$\therefore (2; 8)$$

Make sure to find the derivative of the ORIGINAL function, not a simplified version of it.

Since we are getting two $(x-2)$ brackets, we only have one turning point. It is a double turning point, so it will be the point of inflection as well.

Remember to substitute $x=2$ into the original equation to get the y -coordinate.

Lastly, if the question specifies, calculate the coordinates of the point of inflection:

Point of inflection: $g''(x) = 0$

$$-6x + 12 = 0$$

$$-6x = -12$$

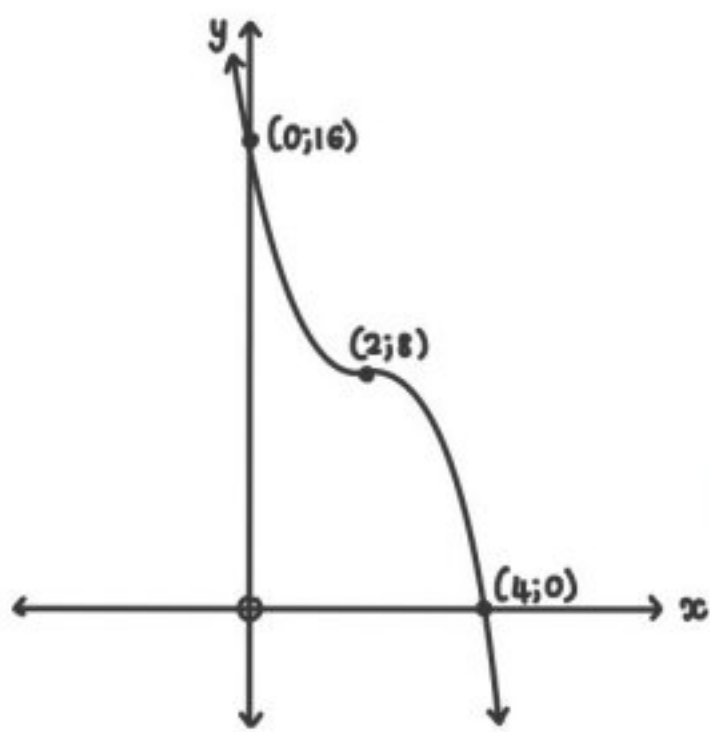
$x = 2$

$y = 8$

$$\therefore (2; 8)$$

As predicted, the point of inflection is also the turning point. You don't have to do these calculations if you realized this during the turning point calculation.

You can now draw the graph. You do not have to draw it to scale but you do have to label each point that you are plotting with both coordinates:



PRO-TIPS

Since this graph **only** has **one stationary** point (turning point and point of inflection), we draw a short **flat section** on either side of that point to indicate that the concavity changes at that point.



Discussing first and second derivatives of cubic graph equations

First derivative

The **first derivative** of any equation is equal to the **gradient of that equation at a specific point**.

It is therefore directly related to whether the corresponding graph is increasing, decreasing or has a stationary point (a turning point where the derivative is equal to zero).

Let's think of a straight line:

If the gradient (m) is positive, that line is increasing.

If the gradient (m) is negative, that line is decreasing.

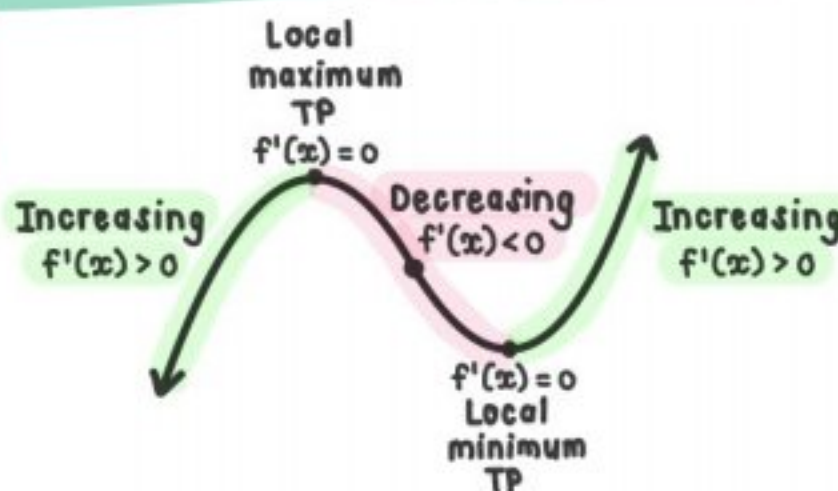
The same concept can be applied to any graph.

In the case of a cubic graph, as shown below, in the section(s) where the **first derivative (m) is positive**, the graph is **increasing**.

In the section(s) where the **first derivative (m) is negative**, the graph is **decreasing**.

Where the **first derivative (m) is equal to zero**, the graph has a **turning point**.

If the graph has two turning points, the one with a higher y -coordinate is called the "local maximum turning point", and the one with a lower y -coordinate is called the "local minimum turning point".



To test whether a turning point is a **local maximum or minimum**, you can use the **first derivative test**:

If the first derivative is positive on the left-hand side of the point, and negative on the right-hand side, it is a **LOCAL MAXIMUM TURNING POINT**.

If the first derivative is negative on the left-hand side of the point, and positive on the right-hand side, it is a **LOCAL MINIMUM TURNING POINT**.

Second derivative

The **second derivative** of a function is related to the **concavity** of that function.

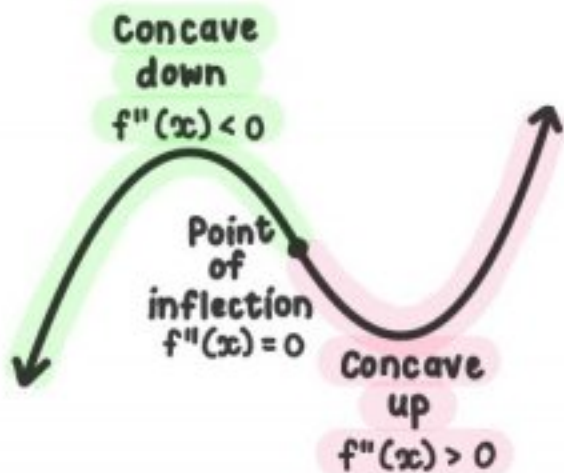
A function is **concave up** where the **second derivative of that function is positive**.

A function is **concave down** where the **second derivative of that function is negative**.

A function has a **point of inflection** where the **second derivative of that function is equal to zero**.

A point of inflection is the point where a graph **changes its concavity**, which means that the graph is concave up on the one side of the point and concave down on the other side.

If a graph has only one turning point, that point will also be the point of inflection. In order to show that, make sure that you draw a short flat section on either side of the point of inflection.



To remember:

Concave up like a **CUP**

Concave down like a **FROWN**

There is therefore another way of testing whether a turning point is a **local maximum or minimum turning point**:

If the **second derivative is positive** at the x-value of the turning point, that means that the graph is **concave up**, so it would be a **local minimum turning point**.

If the **second derivative is negative** at the x-value of the turning point, that means that the graph is **concave down**, so it would be a **local maximum turning point**.

Comparing the graphs of a cubic, its derivative and its second derivative

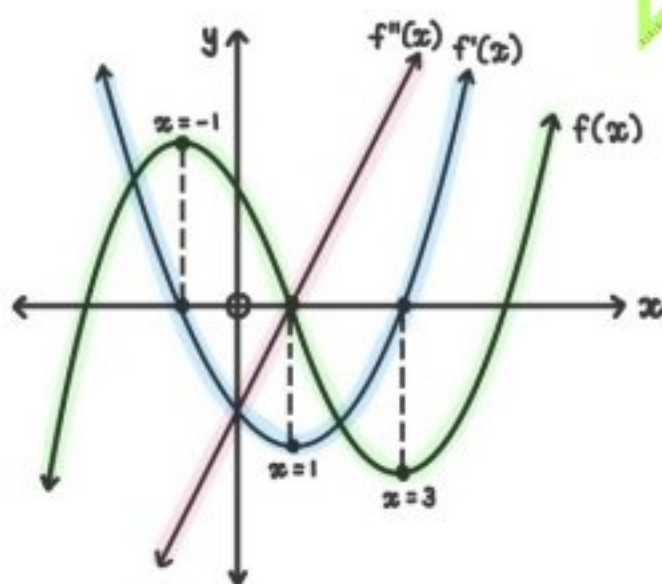
Note the following:

If $f(x)$ is a cubic function, then $f'(x)$ is a quadratic function (parabola) and $f''(x)$ is a linear function (straight line).

You have to be able to relate the three functions to each other, and to answer questions about all three even if only one of the functions are given.

Here is a summary of how the three graphs are related to each other:

	Cubic	Parabola	Straight line
$f(x) = 0$	x -intercepts		
$f(x) > 0$	positive		
$f(x) < 0$	negative		
$f'(x) = 0$	turning points	x -intercepts	
$f'(x) > 0$	increasing	positive	
$f'(x) < 0$	decreasing	negative	
$f''(x) = 0$	pt of inflection	turning point	x -intercept
$f''(x) > 0$	concave up	increasing	positive
$f''(x) < 0$	concave down	decreasing	negative



From the graphs drawn alongside, we can see that:

- The cubic graph has turning points at $x = -1$ and $x = 3$, where $f'(x) = 0$. This is also where the parabola has its x -intercepts.
- The cubic graph has a positive shape. That means that the parabola also has a positive shape, and the straight line also has a positive gradient.
- The cubic graph has a point of inflection at $x = 1$, where $f''(x) = 0$. This is also where the parabola has a turning point (remember that we can find the turning point of a graph by making its derivative equal to zero). This is also where the graph of $f''(x)$, the straight line, has its x -intercept, where $f''(x) = 0$.
- The **cubic** graph is **increasing** where the **parabola** is **positive** (above the x -axis), where $x < -1$ or $x > 3$.
- The **cubic** graph is **decreasing** where the **parabola** is **negative** (below the x -axis), where $-1 < x < 3$.
- The **cubic** graph is **concave down** where the **parabola** is **decreasing** and where the **straight** line is **negative** (below the x -axis).
- The **cubic** graph is **concave up** where the **parabola** is **increasing** and where the **straight** line is **positive** (above the x -axis).

Finding the equation of a tangent to a curve

A tangent is a line that touches a curve at only one point, called the point of contact.

There are two types of questions that are frequently asked in the final exams:

1. Finding the equation of a tangent to a curve at a given point.
2. Finding the equation of the tangent to a curve with maximum/minimum gradient.

When finding the equation of a tangent to a curve, and the equation of the curve is given, follow these steps:

1. Find the **derivative** of the equation of the curve. This will give you the gradient of the tangent to the curve at any point.
2. Now **substitute** the x-value of the point of contact into the derivative. This will give you the actual gradient of the specific tangent.
3. Now substitute the x- and y-coordinates of the point of contact into the straight line equation in order to find the full equation.



Worked example (C)

Find the equation of the tangent to $f(x) = 2x^3 + 3x^2 - 12x$ at the point where $x = 2$.

$$f(x) = 2x^3 + 3x^2 - 12x$$

$$f'(x) = 6x^2 + 6x - 12$$

$$m = f'(2)$$

$$= 6(2)^2 + 6(2) - 12$$

$$= 24$$

$$\therefore y = 24x + c$$

$$\text{Sub in } (2; 4):$$

$$4 = 24(2) + c$$

$$c = -44$$

$$\therefore y = 24x - 44$$

First find the derivative of the function, then substitute the x-coordinate of the point of contact into the derivative to get the value of the gradient of the tangent.

$$\text{or } y - y_1 = 24(x - x_1)$$

$$\text{Sub in } (2; 4):$$

$$y - 4 = 24(x - 2)$$

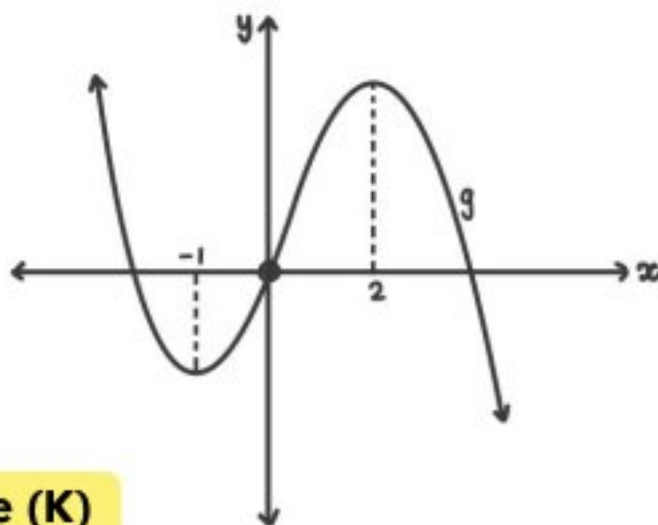
$$y = 24x - 44$$

Now substitute the point of contact into either one of the straight line formulae and write the equation of the tangent.

- The tangent to a curve with the **maximum/minimum gradient** is the tangent at the **point of inflection**.
- If a tangent is drawn at this point, that tangent will have the **steepest possible gradient**.
- For an increasing line (a line with a positive gradient), the steepest gradient will be the highest value, in other words, the steepest gradient will be the maximum gradient.
- For a decreasing line (a line with a negative gradient), the steepest gradient will be the smallest negative value, in other words, the steepest gradient will be the minimum gradient.
- In these questions, the **point of contact will not be given**. You will have to remember that the point of contact is the **point of inflection**.

Worked example (K + C + P)

The graph of $g(x) = ax^3 + bx^2 + cx$, a cubic function having a y-intercept of 0, is drawn below. The x-coordinates of the turning points of g are -1 and 2 .



Worked example (K)

1. For which values of x will g increase?

$$-1 < x < 2$$

$$\text{or } -1 \leq x \leq 2$$

This graph is increasing between the turning points.

If the question simply says "increasing", you can decide to either include or exclude the turning points. If the question said "strictly increasing", we would have to EXCLUDE the turning points.

2. Write down the x-coordinate of the point of inflection.

$$x = \frac{-1+2}{2}$$

$$= \frac{1}{2}$$

Since we don't have the equation of the graph, we can only find the x -value halfway between the turning points.

3. For which values of x will g be concave down?

$$x > \frac{1}{2}$$

This graph is concave down on the right-hand side of the point of inflection.

Worked example (C)

4. If $g'(x) = -6x^2 + 6x + 12$, determine the equation of g .

$$g'(x) = -6x^2 + 6x + 12$$

$$g'(x) = 3ax^2 + 2bx + c$$

$$3a = -6 \quad \text{and} \quad 2b = 6 \quad \text{and} \quad c = 12$$

$$a = -2 \quad \quad b = 3$$

$$\therefore g(x) = -2x^3 + 3x^2 + 12x$$

The derivative of g is given, and we can also find the derivative of the equation of $g(x) = ax^3 + bx^2 + cx$.

These two expressions for $g'(x)$ are equal, so we can equate the coefficients of the same terms to solve for a , b and c .



Worked example (P)



5. Determine the equation of the tangent to g that has the maximum gradient. Write your answer in the form $y = mx + c$.

$$\begin{aligned} g'(x) &= -6x^2 + 6x + 12 \\ m &= g'\left(\frac{1}{2}\right) \\ &= -6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 12 \\ &= \frac{27}{2} \end{aligned}$$

The gradient of the tangent is the derivative of the curve at the point of contact.

The tangent with the maximum gradient is at the point of inflection, which is at $x = \frac{1}{2}$.

$$\begin{aligned} \text{Point of contact: } x &= \frac{1}{2} \\ y &= -2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) \\ &= \frac{13}{2} \\ \therefore \left(\frac{1}{2}; \frac{13}{2}\right) \end{aligned}$$

We don't have the y -coordinate of the point of inflection, so we have to substitute $x = \frac{1}{2}$ into the equation of the cubic graph.

$$\begin{aligned} y &= \frac{27}{2}x + c \\ \text{Sub in } \left(\frac{1}{2}; \frac{13}{2}\right): \\ \frac{13}{2} &= \frac{27}{2}\left(\frac{1}{2}\right) + c \\ c &= -\frac{1}{4} \\ \therefore y &= \frac{27}{2}x - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \text{or } y - y_1 &= \frac{27}{2}(x - x_1) \\ \text{Sub in } \left(\frac{1}{2}; \frac{13}{2}\right): \\ y - \frac{13}{2} &= \frac{27}{2}\left(x - \frac{1}{2}\right) \\ y &= \frac{27}{2}x - \frac{1}{4} \end{aligned}$$

Now substitute the point of contact into either of the straight line equations to find the equation of the tangent.

Question source: Grade 12 NSC November 2020 Paper 1

Finding the equation of a cubic graph

There are two different forms of a cubic graph equation, standard form and x-intercept form.

If the x-intercepts are given, you need to use the x-intercept form.

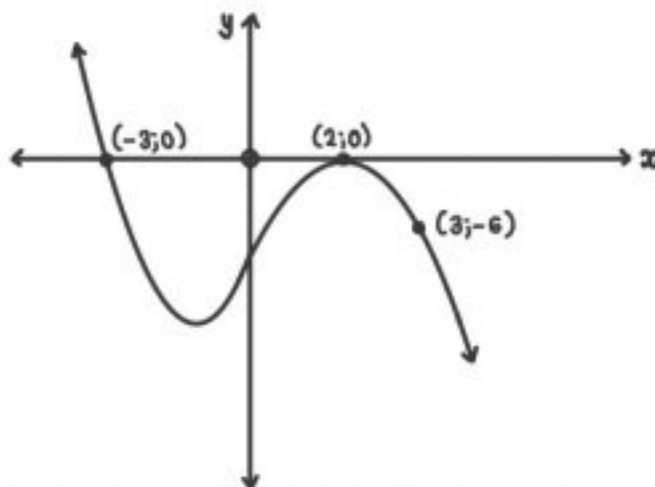
Your final answer should always be left in standard form.

Standard form: $y = ax^3 + bx^2 + cx + d$ ← y-intercept: $(0; d)$

x-intercept form: $y = a(x - x_1)(x - x_2)(x - x_3)$
 $(x_1; 0)$, $(x_2; 0)$ and $(x_3; 0)$ are x-intercepts

Worked example (R)

Determine the equation of the cubic function shown below, if it has x-intercepts of $(-3; 0)$ and $(2; 0)$ and passes through the point $(3; -6)$.



$$y = a(x - x_1)(x - x_2)(x - x_3)$$

$$= a(x + 3)(x - 2)(x - 2)$$

Both x-intercepts are given, so we use x-intercept form.
 Since $(2; 0)$ is also a turning point, it is a double x-intercept
 $\therefore$ we will have two $(x - 2)$ brackets.

Sub in $(3; -6)$:

$$-6 = a(3 + 3)(3 - 2)(3 - 2)$$

$$-6 = 6a$$

$$\therefore a = -1$$

Now substitute the other point into the equation and solve for a .

$$\therefore y = -(x + 3)(x - 2)(x - 2)$$

$$= -(x + 3)(x^2 - 4x + 4)$$

$$= -x^3 + 4x^2 - 4x - 3x^2 + 12x - 12$$

$$y = -x^3 + x^2 + 8x - 12$$

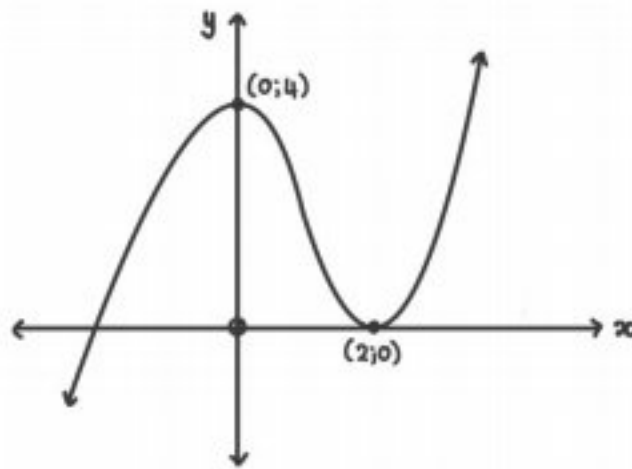
Make sure that you multiply the brackets out so that you can leave your answer in standard form.





Worked example (C)

$y = x^3 + bx^2 + cx + d$ has turning points at $(0; 4)$ and $(2; 0)$.
Determine the values of b , c and d .



$$\begin{aligned} \text{y-int: } (0; 4) \quad \therefore d &= 4 \\ \therefore y &= x^3 + bx^2 + cx + 4 \end{aligned}$$

The y-coordinate of the y-intercept is the d-value.

$$\begin{aligned} \text{At turning points: } \frac{dy}{dx} &= 0 \\ 3x^2 + 2bx + c &= 0 \end{aligned}$$

We have one x-intercept only, so we can't use x-intercept form. Since we have both turning points, we can use the derivative. Remember that the derivative is equal to zero at the turning points.

$$\begin{aligned} \text{Sub in } (0; 4): 3(0)^2 + 2b(0) + c &= 0 \\ c &= 0 \end{aligned}$$

$$\therefore y = x^3 + bx^2 + 4$$

$$\begin{aligned} \text{Sub in } (2; 0): 0 &= (2)^3 + b(2)^2 + 4 \\ -12 &= 4b \\ b &= -3 \end{aligned}$$

$$\therefore y = x^3 - 3x^2 + 4$$

Now substitute the coordinates of the turning points into the "derivative = 0" formula. In this case it wasn't necessary, but you might have to solve equations simultaneously. We will look at this in the exam type questions.



Let's look at some mixed exam-type questions!

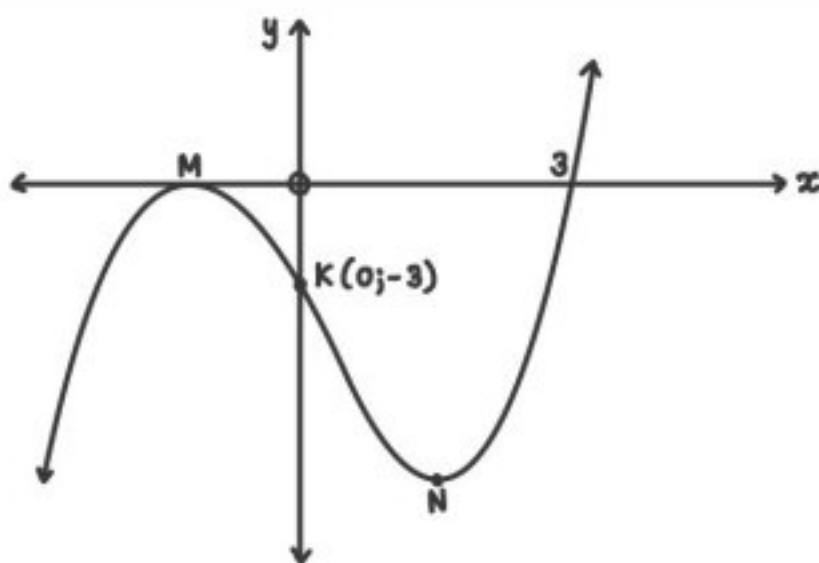
Worked example (P + R + C)

Sketched below is the graph of $f(x) = x^3 + ax^2 + bx + c$.

The x -intercepts of f are at $(3; 0)$ and M , where M lies on the negative x -axis.

$K(0; -3)$ is the y -intercept of f .

M and N are the turning points of f .



Worked example (P)

1. Show that the equation of f is given by $f(x) = x^3 - x^2 - 5x - 3$.

$$f(x) = x^3 + ax^2 + bx + c$$

x -intercept form:

$$y = (x - k)^2(x - 3)$$

Sub in $(0; -3)$:

$$-3 = (0 - k)^2(0 - 3)$$

$$-3 = -3k^2$$

$$\therefore k^2 = 1$$

$$k = \pm 1$$

M has a negative x -value

$$\therefore M(-1; 0)$$

$$\therefore y = (x + 1)^2(x - 3)$$

$$= (x^2 + 2x + 1)(x - 3)$$

$$= x^3 - 3x^2 + 2x^2 - 6x + x - 3$$

$$f(x) = x^3 - x^2 - 5x - 3$$

Two x -intercepts are given, one of which is a turning point, so a double x -intercept.

We can use the x -intercept form and introduce another variable to represent that x -intercept. Now substitute the other point given, point K , and solve for that variable.

We are getting two possible solutions for k , but since it is given that M has a negative x -value, we need to select $x = -1$.

Now substitute that x -intercept into the x -intercept form and multiply the brackets out.



Worked example (R)

Remember that even if you weren't able to answer question 1, you can still use the given equation for f in the following questions!

2. Calculate the coordinates of N.

N: turning point

$$f'(x) = 0:$$

$$3x^2 - 2x - 5 = 0$$

$$(x+1)(3x-5) = 0$$

$$x = -1 \text{ or } x = \frac{5}{3}$$

$$N: x = \frac{5}{3}$$

$$y = \left(\frac{5}{3}\right)^3 - \left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right) - 3$$

$$\therefore N\left(\frac{5}{3}; -\frac{256}{27}\right)$$

N is a turning point so we have to make the derivative of the equation equal to zero and solve for x .

The x -value of M is -1 , so the x -value of N is $\frac{5}{3}$.

Substitute that x -value into the equation to get the y -coordinate of the point.

3. For which values of x will $f(x) < 0$?

$$f(x) < 0:$$

$$x < 3; x \neq -1$$

The graph is negative where it lies BELOW the x -axis.

We have to exclude point M, where $x = -1$, as the graph is EQUAL to zero at that point.

4. For which values of x will f be increasing?

f increasing:

$$x < -1 \text{ or } x > \frac{5}{3}$$

$$\text{OR } x \leq -1 \text{ or } x \geq \frac{5}{3}$$

The graph is increasing where it has a positive slope.

You can decide to include or exclude the turning points.

5. For which values of x will f be concave up?

f concave up:

$$\text{Point of inflection at } f''(x) = 0:$$

$$6x - 2 = 0$$

$$6x = 2$$

$$x = \frac{1}{3}$$

$$\therefore x > \frac{1}{3}$$

This graph is concave up on the right-hand side of the point of inflection, so we first have to calculate the x -value of the point of inflection.



Worked example (C)



6. Determine the maximum vertical distance between the graphs of f and f' in the interval $-1 < x < 0$.

Distance between graphs

$$\begin{aligned} &= x^3 - x^2 - 5x - 3 - (3x^2 - 2x - 5) \\ &= x^3 - x^2 - 5x - 3 - 3x^2 + 2x + 5 \\ &= x^3 - 4x^2 - 3x + 2 \end{aligned}$$

Maximum distance: $D' = 0$

$$3x^2 - 8x - 3 = 0$$

$$(x-3)(3x+1) = 0$$

$$x = 3 \text{ or } x = -\frac{1}{3}$$

$$\text{but } -1 < x < 0 \therefore x = -\frac{1}{3}$$

$\therefore$ Maximum distance

$$\begin{aligned} &= \left(-\frac{1}{3}\right)^3 - 4\left(-\frac{1}{3}\right)^2 - 3\left(-\frac{1}{3}\right) + 2 \\ &= \frac{68}{27} \text{ units} \\ &\text{or } 2,52 \text{ units} \end{aligned}$$

The distance between two graphs can be found by subtracting the equation of the one graph from the other.

To maximize/minimize a quantity, make the derivative of the equation of that quantity equal to zero. Then solve for x .

Choose the correct x -value based on the condition given.

Now substitute the correct x -value into the formula for the distance between the two graphs to determine the maximum distance.

This question is an application question, under the topic of **Optimisation**.

We will look at more of these types of questions in the next section.

Question source: Grade 12 NSC November 2022 Paper 1



Miss Angler

86

Question type 4: Applications of Calculus

There are three types of application questions that can be asked:

1. Optimisation (maximum or minimum values)
2. Rate of change
3. Calculus of motion

These three types of application questions are often mixed together, and are also mostly tested in unseen/problem-solving contexts, which makes it quite difficult to prepare for.

We will simply summarise the basics and then work through a few examples so that you can see how this section has been tested in the past.

Optimisation

Optimisation is the process of using derivatives to find the maximum or minimum values of a certain quantity.

These questions often require you to do the following:

1. **Interpret** given information to **determine** a **formula** for the quantity that is being optimised.
2. Make the **derivative** of this formula equal to **zero** and solve for the variable in the formula.
3. **Substitute** the value of this **variable** into the original equation to determine the actual maximum or minimum value of the quantity that you are working with.

Rate of change

The rate of change of any quantity with respect to another, is equal to the derivative of that quantity.

For example, the rate at which the volume of a tank of water is changing over time, is equal to the derivative of the volume with respect to time. This means that, in the formula for volume, the variable will be "t", for time.

To find the rate of change at a specific time, you would have to substitute that specific time into the derivative expression.

PRO-TIPS

- Sometimes the first part of the question is a "show that" question, which is meant to help you.
- The examiner is essentially giving you the formula in the question, so that, even if you are unable to get to that formula, you are still able to use it in the next part of the question.

!! This section is not often tested in final exams.

Calculus of motion

Any questions relating to displacement (distance from the starting point), velocity (speed) and acceleration are classified as "calculus of motion" questions.

Displacement is calculated using a formula that is given in the question.

Velocity is the first derivative of the displacement formula.

Acceleration is the first derivative of the velocity formula, or the second derivative of the displacement formula.

Even if you are finding a negative value for any of these, you do not have to explain why the answer is negative.

Worked example (P)

Given $f(x) = x^2$.

Determine the minimum distance between the point $(10; 2)$ and a point on f .

To calculate the distance between two points, we need the coordinates of those points and the distance formula.

Since they didn't give us the coordinates of the second point, we have to work with a generalised expression for the coordinates of any point on the function $f(x) = x^2$.

If the x -value of any point on this function is x , the y -value will be x^2 , as that is the equation.

Distance between $(10; 2)$ and $(x; f(x))$
 $= (x; x^2)$

$$\begin{aligned} D &= \sqrt{(10-x)^2 + (2-x^2)^2} \\ &= \sqrt{100 - 20x + x^2 + 4 - 4x^2 + x^4} \\ &= \sqrt{x^4 - 3x^2 - 20x + 104} \\ D^2 &= x^4 - 3x^2 - 20x + 104 \end{aligned}$$

We now write a formula for the distance between these two points using the distance formula.

Since we are going to find the derivative of this expression, we need to square both sides to get rid of the square root.

We can do this because the x -value that will result in the minimum $(\text{distance})^2$ will also result in the minimum distance.

Minimum $D^2 : (D^2)' = 0$

$$4x^3 + 6x - 20 = 0$$

$$2x^3 - 3x - 10 = 0$$

$$(x-2)(2x^2 + 4x + 5) = 0$$

$$x = 2 \quad \text{or} \quad \Delta = 4^2 - 4(2)(5)$$

$$= -24$$

$\therefore$ no real solution

For maximum/minimum: make the derivative of the expression for distance = 0 and solve for x .

$\therefore$ Minimum distance

$$= \sqrt{2^4 - 3(2)^2 - 20(2) + 104}$$

$$= 2\sqrt{17} \text{ units or } 8,25 \text{ units}$$

Then substitute that x -value into the distance formula.

Question source: NSC Grade 12 November 2022 Paper 1

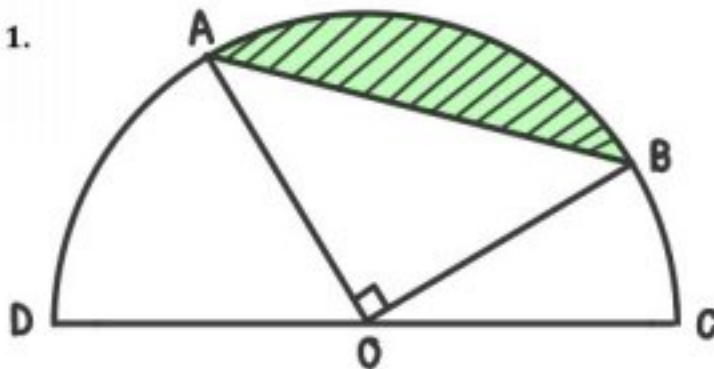


Worked example (P)

O is the centre of a semicircle passing through A, B, C and D.

The radius of the semicircle is $(x - x^2)$ units for $0 < x < 1$.

$\triangle AOB$ is right-angled at O.



1. Show that the area of the shaded part is given by:

$$\text{Area} = \left(\frac{\pi-2}{4}\right)(x^4 - 2x^3 + x^2)$$

$$r = (x - x^2) \text{ units ; } 0 < x < 1$$

$$\begin{aligned} \text{Shaded area} &= \text{area of segment} - \text{area of } \triangle \\ &= \frac{1}{4} \pi r^2 - \frac{1}{2} (r)(r) \\ &= \frac{1}{4} \pi r^2 - \frac{1}{2} r^2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{4} r^2 (\pi - 2) \\ &= \frac{1}{4} (x - x^2)^2 (\pi - 2) \\ &= \frac{\pi - 2}{4} (x - x^2)^2 \\ &= \left(\frac{\pi - 2}{4}\right)(x^4 - 2x^3 + x^2) \end{aligned}$$

The sector of the semicircle that includes the shaded area is $\frac{1}{4}$ of a full circle as 90° is $\frac{1}{4}$ of a full 360° revolution $\therefore \frac{1}{4} \times \pi r^2$

The area of a $\triangle$ is $\frac{1}{2} \times \text{base} \times \perp \text{ height}$. The base and $\perp$ height are both equal to the radius.

Replace r with the given radius, $x - x^2$, and take out $\frac{1}{4}$ as a common factor.

Then rewrite the expression so that it is equal to the given expression.

2. Determine the value of x for which the shaded area will be a maximum.

Maximum area where $A' = 0$:

Remember that $\frac{\pi-2}{4}$ is a number. We could have multiplied it into the bracket in the formula for area, then derived each term separately, before then dividing each term by the common factor.

We would still have been left with the derived expression in brackets.

The quicker method is to leave the constant $\frac{\pi-2}{4}$ outside the brackets and simply find the derivative of the terms in brackets.

$$\left(\frac{\pi-2}{4}\right)(4x^3 - 6x^2 + 2x) = 0$$

$$4x^3 - 6x^2 + 2x = 0$$

$$2x^3 - 3x^2 + x = 0$$

$$x(2x^2 - 3x + 1) = 0$$

$$x(2x-1)(x-1) = 0$$

$$x = 0 \text{ or } x = \frac{1}{2} \text{ or } x = 1$$

$$\text{but } 0 < x < 1 \therefore x = \frac{1}{2}$$

Divide each term by 2 to simplify the equation.

Now factorize the equation and determine the three possible solutions.

Since we were told that x is between 0 and 1, but not equal to 0 or 1, we have to exclude those values from our solutions.

Question source: NSC Grade 12 May/June 2021 Paper 1



Worked example (C)

After flying a short distance, an insect came to rest on a wall. Thereafter the insect started crawling on the wall. The path that the insect crawled can be described by:

$h(t) = (t - 6)(-2t^2 + 3t - 6)$, where h is the height (in cm) above the floor and t is the time (in minutes) since the insect started crawling.



1. At what height above the floor did the insect start to crawl?

$$h(t) = (t - 6)(-2t^2 + 3t - 6)$$

$$\begin{aligned} h(0) &= (0 - 6)[-2(0)^2 + 3(0) - 6] \\ &= 36 \text{ cm} \end{aligned}$$

When the insect just starts crawling, no time has passed $\therefore$ make $t = 0$.

This is also called the initial displacement / height.



2. How many times did the insect reach the floor?

$$h(t) = 0$$

$$(t - 6)(-2t^2 + 3t - 6) = 0$$

$$t = 6 \quad \text{or} \quad -2t^2 + 3t - 6 = 0$$

$$\begin{aligned} \Delta &= (3)^2 - 4(-2)(-6) \\ &= -39 \end{aligned}$$

$\therefore$ no real solution

One solution

$\therefore$ The insect reached the floor once.

When the insect reaches the floor, the height above the floor is 0cm.

$\therefore$ Make the height equal to zero and solve for t .



3. Determine the maximum height that the insect reached above the floor.

$$h(t) = (t - 6)(-2t^2 + 3t - 6)$$

$$= -2t^3 + 3t^2 - 6t + 12t^2 - 18t + 36$$

$$= -2t^3 + 15t^2 - 24t + 36$$

This expression is not derivative-ready.

Multiply the brackets out first and simplify the expression so that we can apply the power rule.

$$\text{Maximum height: } h'(t) = 0$$

$$-6t^2 + 30t - 24 = 0$$

$$t^2 - 5t + 4 = 0$$

$$(t - 4)(t - 1) = 0$$

$$t = 4 \quad \text{or} \quad t = 1$$

For maximum/minimum: make the derivative of the expression for height = 0 and solve for t .

$$\begin{aligned} h(4) &= -2(4)^3 + 15(4)^2 - 24(4) + 36 \\ &= 52 \text{ cm} \end{aligned}$$

$$\begin{aligned} h(1) &= -2(1)^3 + 15(1)^2 - 24(1) + 36 \\ &= 25 \text{ cm} \end{aligned}$$

$\therefore$ Maximum height = 52cm

Logic should tell us that if the insect is walking up the wall, it will be higher up after 4 seconds than 1 second.

If you are not sure, you can substitute both values into the formula for height and then select the maximum height.

Question source: NSC Grade 12 November 2019 Paper 1



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COUNTING PRINCIPLE & PROBABILITY

In order to achieve maximum marks for this section, you need to be able to:

- Use the **counting principle** to calculate numbers of combinations, as well as the probability that certain combinations will occur.
- Use the **basic formula for probability** to calculate the probabilities of single or compound events.
- Be able to differentiate between **"and"** and **"or"** questions.
- Use the **probability identity** to calculate probabilities, and to determine whether events are independent and/or mutually exclusive.
- Complete **Venn diagrams** using given information and calculate probabilities using those Venn diagrams.

Probability is the last section in your exam, and also one of the smallest sections.

It is very difficult to fit all of the content into 15 marks, so this section is usually split up into two questions:

1. A question testing **basic probability knowledge**, using the **probability identity** or **Venn diagrams**.
2. A question testing the **counting principle**.

Let's look at the basic knowledge that you need to have and be able to apply for this section!

Calculating the probability of a specific event occurring:

If we are calculating the probability of event A occurring, we must use this formula:

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{number of outcomes in event A}}{\text{number of outcomes in sample space}}$$

eg. If we are rolling a six-sided die, what is the probability of the die landing on an odd number?

There are 3 outcomes in the event (1; 3; 5). This is $n(A) = 3$.

There are 6 outcomes in total (1; 2; 3; 4; 5; 6). This is $n(S) = 6$.

The probability is therefore 3 over 6, or 0,5, or 50%.

Applying the Probability Identity

The Probability Identity is an equation that is always true, which is why it's called an "identity".

This equation provides a method for calculating either the **NUMBER** of items in the union of two events, or the **PROBABILITY** of randomly selecting an item in the union of two events.

The union of two events is the same as the "**OR**" section, in other words, the probability of **either one OR the other event occurring**.

This identity also provides a method for calculating any of the other values in the identity, if the other three values are known.

The Probability Identity is as follows:

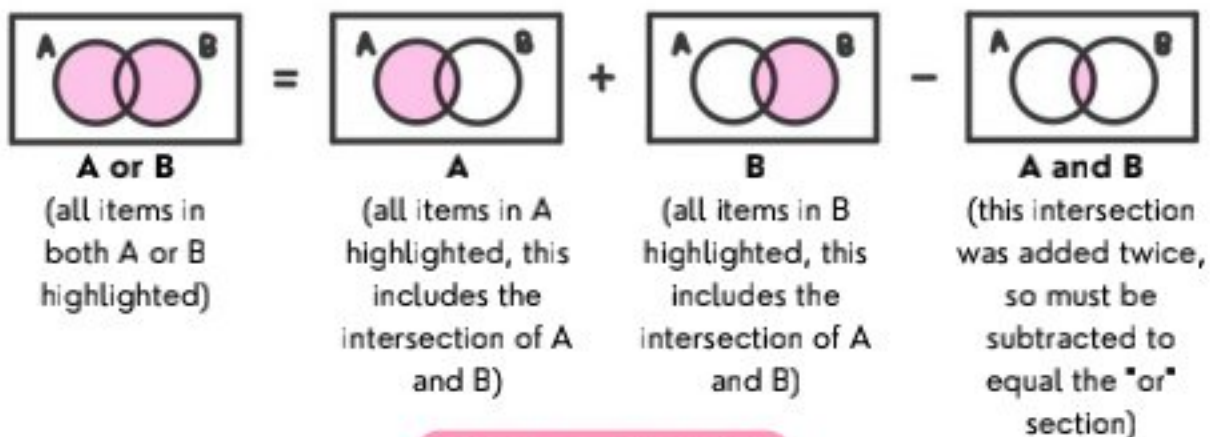
NUMBERS

$$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$$

PROBABILITIES

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

To make sense of this identity, let's look at the visualisation of the identity in the Venn diagrams below.



PRO-TIPS

If a question states **more** than **one probability** value in decimal form, you are most likely going to have to use the probability identity.

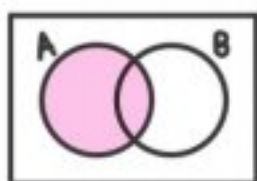
Working with Venn diagrams

You need to be able to do the following types of Venn diagram questions:

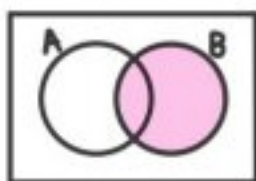
- **Complete** a Venn diagram using given information
- **Calculate** missing information from a partially completed Venn diagram
- **Interpret** a Venn diagram to determine whether two events are mutually **exclusive** or **independent**
- Do **probability calculations** based on a completed Venn diagram

In order to do these questions, you have to know how to **describe certain sections** of a Venn diagram.

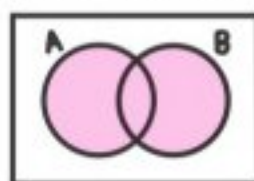
Here are some examples of shaded sections along with their descriptions.



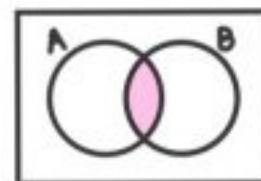
A



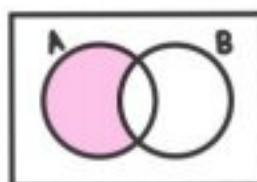
B



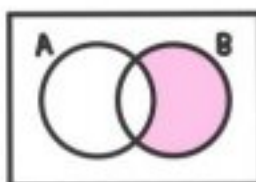
A or B



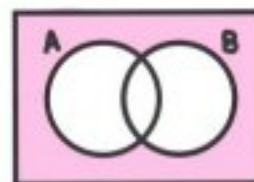
A and B



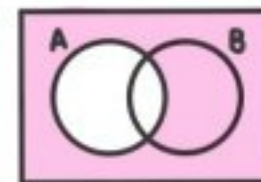
A only



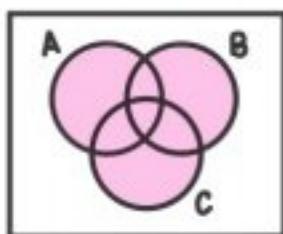
B only



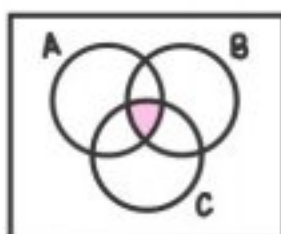
Neither A nor B



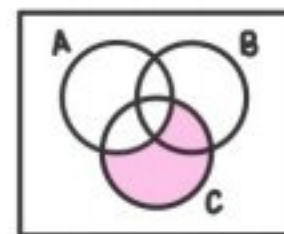
Not A



A or B or C



A and B and C



B and C but not A

PRO-TIPS

Instead of writing $P(\text{not } A)$, you can write $P(A')$.

The dash on the outside of the bracket negates everything inside the bracket.

Note the difference:

$$P(\text{not } A \text{ or } B) = P(A' \text{ or } B)$$

$$P(\text{neither } A \text{ nor } B) = P(A \text{ or } B)'$$

Now let's look at some examples of questions that require this knowledge!



Question Type 1:

Probability basics, Probability Identity and Venn diagrams

Worked example (R)

The school library is open from Monday to Thursday. Anna and Ben both studied in the school library one day this week. If the chance of studying any day in the week is equally likely, calculate the probability that Anna and Ben studied on:

1. The same day

$$\begin{aligned} P(\text{same day}) &= P(\text{Mon and Mon}) + P(\text{Tues and Tues}) + P(\text{Wed and Wed}) + P(\text{Thurs and Thursday}) \\ &= \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) \\ &= \frac{4}{16} \\ &= \frac{1}{4} \end{aligned}$$

"and" = multiply
"or" = add

Instead of calculating the probability using multiplication, you can think of the problem in this way:

$$\begin{aligned} P(\text{same day}) &= \frac{4}{16} \\ &= \frac{1}{4} \end{aligned}$$

Mon and Mon; Tues and Tues; Wed and Wed; Thurs and Thurs
There is a total of 16 combinations of study days, eg.
Anna Mon and Ben Mon, Anna Mon and Ben Tues, Anna Mon and Ben Wed, and so on.

2. Consecutive days

$$\begin{aligned} P(\text{consecutive days}) &= \frac{6}{16} \\ &= \frac{3}{8} \end{aligned}$$

Anna Mon and Ben Tues; Anna Tues and Ben Wed; Anna Wed and Ben Thurs;
Ben Mon and Anna Tues; Ben Tues and Anna Wed; Ben Wed and Anna Thurs
16 total combinations

Question source: NSC Grade 12 November 2019 Paper 1

PRO-TIPS

Mutually exclusive events are events that cannot occur at the same time, so the two events **CANNOT** overlap.

Worked example (K)

Given: $P(A) = 0,45$; $P(B) = y$ and $P(A \text{ or } B) = 0,74$.

Determine the value(s) of y if A and B are mutually exclusive.

Mutually exclusive $\therefore P(A \text{ and } B) = 0$

$$P(A \text{ or } B) = P(A) + P(B)$$

$$0,74 = 0,45 + y$$

$$y = 0,29$$

Question source: NSC Grade 12 November 2018 Paper 1



Worked example (C)

Events A and B are independent. $P(A) = 0,4$ and $P(B) = 0,25$.

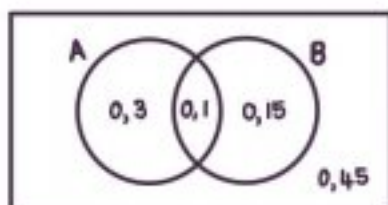


1. Represent the given information on a Venn diagram. Indicate on the Venn diagram the probabilities associated with each region.

$$P(A) = 0,4 \quad \text{and} \quad P(B) = 0,25$$

Independent events

$$\begin{aligned} \therefore P(A \text{ and } B) &= P(A) \times P(B) \\ &= 0,4 \times 0,25 \\ &= 0,1 \end{aligned}$$



In order to draw a Venn diagram, we need to know if the two events overlap.

We can use this formula to calculate the probability of A and B.

$$P(A \text{ and } B) = 0,1 \quad \text{overlap}$$

$P(A \text{ only})$

$$\begin{aligned} &= P(A) - P(A \text{ and } B) \\ &= 0,4 - 0,1 \\ &= 0,3 \end{aligned}$$

$P(B \text{ only})$

$$\begin{aligned} &= P(B) - P(A \text{ and } B) \\ &= 0,25 - 0,1 \\ &= 0,15 \end{aligned}$$

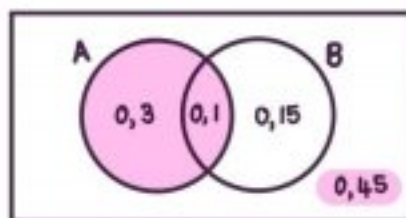
$$P(\text{neither A nor B}) / P(A \text{ or } B)^c$$

$$\begin{aligned} &= 1 - 0,3 - 0,1 - 0,15 \\ &= 0,45 \end{aligned}$$

If two events are **independent**, then the following is true:
 $P(A \text{ and } B) = P(A) \times P(B)$



2. Determine $P(A \text{ or NOT } B)$.



$$\begin{aligned} P(A \text{ or not } B) &= 0,4 + 0,45 \\ &= 0,85 \end{aligned}$$

PRO-TIPS

We have to include **all** the values in A, as well as all the values in **NOT** B. These values are all highlighted in the Venn diagram.

Question source: NSC Grade 12 November 2019 Paper 1



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Worked example (C)

An organization decided to distribute gift bags of sweets to a Grade R class at a certain school.

There is a mystery gift in exactly $\frac{1}{4}$ of the total number of bags.

Each learner in the class may randomly select two gift bags of sweets, one after the other.

The probability that a learner selects two bags of sweets with a mystery gift inside is $\frac{7}{118}$.

Calculate the number of gift bags with a mystery gift inside.

$$\begin{aligned} P(\text{gift and gift}) &= P(\text{gift at first draw}) \times P(\text{gift at second draw}) \\ &= \frac{1}{4} \times P(\text{gift at second draw}) \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{4} \times P(\text{gift at second draw}) &= \frac{7}{118} \\ P(\text{gift at second draw}) &= \frac{14}{59} \end{aligned}$$

$$\therefore P(\text{gift at first draw}) = \frac{15}{60}$$

$\therefore$ There are 15 bags with a mystery gift inside.

We are told that there is a gift in $\frac{1}{4}$ of the bags $\therefore$ the probability of choosing a gift bag in the first draw is $\frac{1}{4}$.

We can use the combined probability to calculate the probability of choosing a gift bag in the second draw.

If there are 14 gift bags out of 59 bags for the second draw, there would have been 15 gift bags out of 60 bags in total at the beginning.

Question source: Grade 12 NSC November 2018 Paper 1

PRO-TIPS

This question seems very complicated at first, as is the case with many probability questions.

Make sure that you are able to identify the **relevance** of the information given, and then try to think about how the information can be used to get to the answer.





Worked example (C)

A and B are independent events. It is further given that:

$$P(A \text{ and } B) = 0,3 \text{ and } P(\text{only } B) = 0,2.$$



1. Are A and B mutually exclusive? Motivate your answer.

$$P(A \text{ and } B) = 0,3 \text{ and } P(\text{only } B) = 0,2$$

A and B are not mutually exclusive.

$$P(A \text{ and } B) \neq 0$$

Two events are mutually exclusive if they cannot occur at the same time.

This means that $P(A \text{ and } B) = 0$.



2. Determine $P(\text{only } A)$.

$$P(A \text{ and } B) = 0,3 \text{ and } P(\text{only } B) = 0,2$$

$$\therefore P(B) = 0,3 + 0,2$$

$$= 0,5$$

A and B are independent events

$$\therefore P(A \text{ and } B) = P(A) \times P(B)$$

$$0,3 = P(A) \times 0,5$$

$$P(A) = 0,6$$

$P(A \text{ only})$ can be found by subtracting $P(A \text{ and } B)$ from $P(A)$.

To get $P(A)$, we first need to find $P(B)$ so that we can then use the fact that A and B are independent events.

$$P(B) = P(\text{only } B) + P(A \text{ and } B)$$

$$\therefore P(A \text{ only}) = P(A) - P(A \text{ and } B)$$

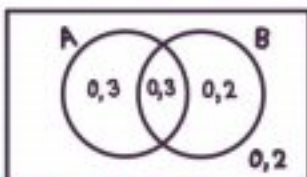
$$= 0,6 - 0,3$$

$$= 0,3$$



3. Determine $P(\text{not } A \text{ or not } B)$.

If you are struggling to visualise a question, it can help to draw a Venn diagram.



$$P(\text{not } A \text{ or not } B)$$

$$= P(\text{not } A) + P(\text{not } B) - P(\text{not } A \text{ and not } B)$$

$$= (0,2 + 0,2) + (0,3 + 0,2) - (0,2)$$

neither A nor B
 $\therefore$ the probability outside the two circles

"Not A" is made up of "B only" and "neither A nor B".

"Not B" is made up of "A only" and "neither A nor B".

Question source: NSC Grade 12 November 2021 Paper 1

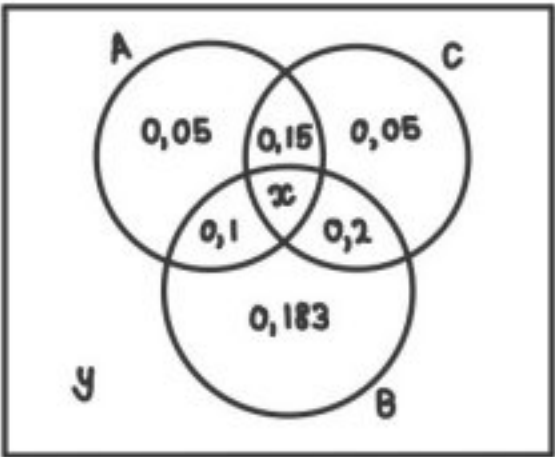


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Worked example (R)

A, B and C are three events. The probabilities of these events (or any combination of them) occurring is given in the Venn diagram below.



If it is given that the probability that at least one of the events will occur is 0,893, calculate the value of:

1. y , the probability that none of the events will occur

$$P(\text{at least one event}) = 0,893$$
$$1 - y = 0,893$$
$$y = 0,107$$

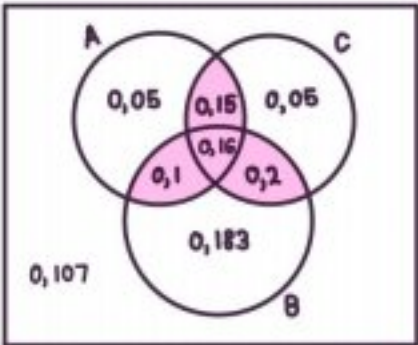
$$P(\text{at least one event}) = 1 - P(\text{none of the events})$$
$$= 1 - y$$

2. x , the probability that all three events will occur

$$x = 1 - 0,05 - 0,15 - 0,05 - 0,1 - 0,2 - 0,183 - 0,107$$
$$= 0,16$$

If only one value in the Venn diagram is unknown, that value can be calculated by subtracting all the other values from 1.

3. Determine the probability that at least two of the events will take place.



Let's redraw the Venn diagram and fill in the values that we have already calculated.

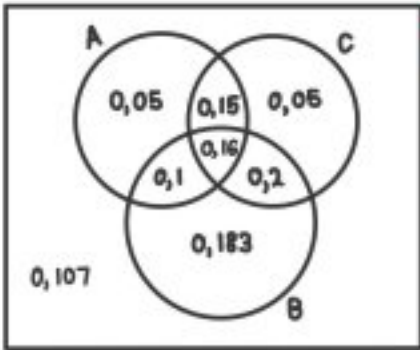
$$P(\text{at least two events})$$
$$= 0,15 + 0,1 + 0,2 + 0,16$$
$$= 0,61$$

"At least two events" include the sections where two events overlap, as well as the section where all three events overlap.





4. Are events B and C independent? Justify your answer.



Two events are independent if $P(B \text{ and } C) = P(B) \times P(C)$.

$$P(B \text{ and } C) = 0,16 + 0,2 \\ = 0,36$$

$$P(B) \times P(C) = 0,643 \times 0,56 \\ = 0,36$$

$$\therefore P(B \text{ and } C) = P(B) \times P(C)$$

$\therefore$ B and C are independent events

These are the probabilities in the section where B and C overlap.

$$P(B) = 0,1 + 0,16 + 0,2 + 0,163$$

$$P(C) = 0,15 + 0,05 + 0,16 + 0,2$$

Question source: Grade 12 NSC November 2022 Paper 1

REMEMBER: WORKED EXAMPLE KEY

This study guide will highlight which questions are classified under each difficulty level.

Look out for either:

K - KNOWLEDGE

R - ROUTINE

C - COMPLEX

P- PROBLEM SOLVING

next to each question!



Question Type 2: The Fundamental Counting Principle

The Fundamental Counting Principle is a rule that we use to calculate the total number of possible outcomes for a certain event.

If there are m ways of doing something, and n ways of doing another thing, then there are $m \times n$ ways to do both of these things.

Eg. if you have 3 pairs of pants, 4 shirts and 2 pairs of shoes, you will have $3 \times 4 \times 2 = 24$ possible outfits (combinations of pants, shirts and shoes) to choose from.

When using the Fundamental Counting Principle, it is useful to use the SLOT METHOD.

1. Identify how many slots there are and draw a line to represent each slot.

Eg. if you are told to make a 3-digit code, there are 3 slots - each slot represents a digit. In the example above about outfits, there were also 3 slots - pants, shirts and shoes. If you are told to arrange 6 people in a row, there are 6 slots - each slot represents a seat in the row.

2. Identify the number of options available for each slot.

Eg. In the example about outfits above, there were 3 options for slot 1 (pants), 4 options for slot 2 (shirts) and 2 options for slot 3 (shoes).

You would have to consider any specific conditions set on the scenario in this step, for example, if you are told that the first digit of a code may not be a 0, as that will influence the number of options available for that slot.

3. Multiply the number of options of all slots together.

This will give you the total number of outcomes for the specific event.

You might have to consider the following:

- Whether items may be repeated or not (this is important when creating codes, passwords, telephone numbers, etc.)
- When certain items have to be grouped together, and therefore form a set or unit.
- When asked to calculate probability instead of the number of items, remember the formula for probability that we have worked with before.



Let's look at some typical exam-type questions and how to solve them!



Worked example (R)

Given the digits: 3; 4; 5; 6; 7; 8 and 9.

1. Calculate how many unique 5-digit codes can be formed using the digits above, if the digits may be repeated.

3 ; 4 ; 5 ; 6 ; 7 ; 8 ; 9 : 7 digits to choose from

1 1 1 1 1

Since the digits can be repeated, any of the 7 digits can be used in all five slots of the codes.

$$\therefore 7^5 = 16\,807 \text{ codes}$$

2. Calculate how many unique 5-digit codes can be formed using the digits above, if the digits may not be repeated.

1 6 5 4 3

The digits may not be repeated.

The digit used in the first slot may not be used again $\therefore$ there are only 6 digits to choose from for the second slot.

The first and second digits may now not be used again, so there are only 5 digits to choose from for the third slot, and so on.

$$\therefore 7 \times 6 \times 5 \times 4 \times 3 = 2\,520 \text{ codes}$$

3. How many unique 3-digit codes can be formed using the above digits, if:

- Digits may be repeated
- The code is greater than 400 but less than 600
- The code is divisible by 5

First slot: 4 or 5 (if it's a 6, it would not be "less than 600")

$\therefore 2$ options

Second slot: any digit $\therefore 7$ options

Third slot: 0 or 5 (the entire code must be divisible by 5)

$\therefore 2$ options

$$\underline{1} \quad \underline{1} \quad \underline{2} \quad \therefore 2 \times 7 \times 2 = 28 \text{ codes}$$

Question source: NSC Grade 12 November 2018 Paper 1



Worked example (R)

In a certain country, 10-digit telephone numbers with the following format were introduced.

Format	Area code	Exchange code	Number
Number of digits	3 digits	3 digits	4 digits
Example	901	544	1230

Digits may be repeated.

1. How many possible 10-digit telephone numbers could be formed?

$\underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10} \ \underline{10}$ The digits may be repeated, so there are 10 options
 $\therefore 10^{10}$ or 10 000 000 000 codes for each of the 10 slots.

2. Certain restrictions were placed on the groups of digits:

- Area code: must be 3 digits and the first digit can NOT be 0 or 1
- Exchange code: must be 3 digits and the first and second digits can NOT be 0 or 1
- Number: must be 4 digits and the first digit MUST be a 0 or 1

2.1. How many valid 10-digit telephone numbers could be formed by applying the given restrictions?

Area code: $\underline{8} \ \underline{10} \ \underline{10}$ The first digit can't be 0 or 1, so there are only 8 options for the first slot.

Exchange code: $\underline{8} \ \underline{8} \ \underline{10}$ The first and second digits can't be 0 or 1, so there are only 8 options for those slots.

Number: $\underline{2} \ \underline{10} \ \underline{10} \ \underline{10}$ The first digit must be 0 or 1, so those are the only two options for that slot.

$$\therefore 8^3 \times 2 \times 10^6 \\ = 1\,024\,000\,000 \text{ codes}$$

2.2. Determine the probability that any randomly chosen 10-digit telephone number would be a valid phone number.

$$\begin{aligned} P(\text{valid phone number}) &= \frac{\text{no. of valid phone numbers}}{\text{no. of possible 10-digit codes}} \\ &= \frac{8^3 \times 2 \times 10^6}{10^{10}} \quad \text{or} \quad \frac{1\,024\,000\,000}{10\,000\,000\,000} \\ &= \frac{64}{625} \quad \text{or} \quad 0,1024 \end{aligned}$$

Question source: NSC Grade 12 November 2020 Paper 1



Worked example (R)

A teacher has 5 different poetry books, 4 different dramas and 3 different novels. She must arrange these 12 books from left to right on a shelf.



1. Write down the probability that a novel will be the first book placed on the shelf.

There are 12 books and 3 of them are novels

$$\begin{aligned} P(\text{novel first}) &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$



2. Calculate the number of different ways these 12 books can be placed on the shelf if any book can be placed in any position.

$$\begin{array}{cccccccccccc} 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 12! & = & 479 & 001 & 600 \end{array}$$

We see factorial notation here for the first time.

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 3 \times 2 \times 1$$

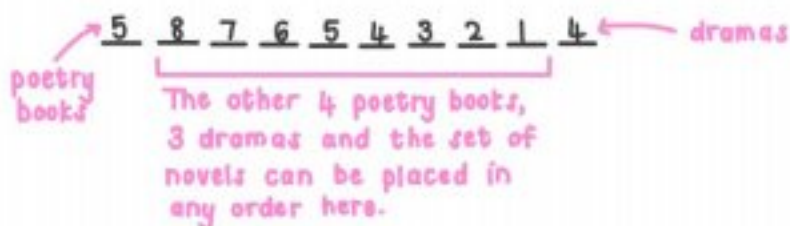


3. Calculate the probability that a poetry book is placed in the first position, the three novels are placed next to each other and a drama is placed in the last position.

The three novels cannot be separated, and now form a unit/set.

This unit can be placed in any position on the shelf, along with the other books.

There are 5 poetry books, 4 dramas and one set of novels $\therefore$ 10 moveable books/units.



$$P(\text{poetry first, drama last, novels together})$$

$$= \frac{5 \times 8! \times 4 \times 3!}{12!}$$

$$= \frac{1}{99}$$

The three novels form a set, but the three books can still be arranged in 3! different ways within the set.

$$\begin{array}{ccc} 3 & 2 & 1 \end{array} \therefore 3 \times 2 \times 1 = 3!$$

Question source: NSC Grade 12 November 2021 Paper 1



Miss Angler
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INFORMATION SHEET

Familiarise yourself with the placement of the formulae that are given on the information/formula sheet by using this labelled formula sheet.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{quadratic formula}$$

$$A = P(1 + ni)$$

simple growth

$$A = P(1 - ni)$$

simple decay/straight line depreciation

$$A = P(1 - i)^n$$

compound decay/reducing balance depreciation

$$A = P(1 + i)^n$$

compound growth

$$T_n = a + (n-1)d$$

arithmetic: term formula

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

arithmetic: sum formula

$$T_n = ar^{n-1}$$

geometric: term formula

$$S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1$$

geometric: sum formula

$$S_\infty = \frac{a}{1 - r}; -1 < r < 1$$

sum to infinity

$$F = \frac{x[(1+i)^n - 1]}{i}$$

future value annuities

$$P = \frac{x[1 - (1+i)^{-n}]}{i}$$

present value annuities

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{derivative by first principles}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

distance formula

$$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

midpoint formula

$$y = mx + c \quad y - y_1 = m(x - x_1)$$

straight line equations

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

gradient formula

$$m = \tan \theta$$

angle of inclination

$$(x-a)^2 + (y-b)^2 = r^2 \quad \text{equation of a circle}$$

$$\text{In } \triangle ABC: \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

sine rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

cosine rule

$$\text{area } \triangle ABC = \frac{1}{2} ab \sin C$$

area rule

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

trig compound angle identities

$$\cos 2\alpha = \begin{cases} \cos^2 \alpha - \sin^2 \alpha \\ 1 - 2 \sin^2 \alpha \\ 2 \cos^2 \alpha - 1 \end{cases}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad \text{trig double angle identities}$$

$$\bar{x} = \frac{\sum x}{n} \quad \text{mean of a set of data values}$$

$$\sigma^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \quad \text{variance of a set of data values}$$

$$P(A) = \frac{n(A)}{n(S)} \quad \text{probability of event A}$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

probability identity

$$\hat{y} = a + bx \quad \text{equation of least squares regression line}$$

$$b = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} \quad \text{gradient of least squares regression line}$$

