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A comprehensive guide to Grade 11 Mathematics

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COVERS THE FUNDAMENTALS OF GRADE 11 MATHEMATICS AND COMPLEMENTS THE MATERIAL IN ANY CLASS TEXT

GRADE 11

KEVIN SMITH

SUITABLE FOR IEB AND NATIONAL CURRICULUM STUDENTS

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ABOUT THE AUTHOR

KEVIN SMITH

Kevin Smith is a young Electrical Engineer who is passionate about Education, specifically Maths and Science. He has been giving extra lessons in Maths and Science for the past 8 years and is the founder of The Lab Maths and Science Academy which is attended by students across Johannesburg.

As an extra lesson teacher, Kevin has established the critical areas where students struggle and has written the **Maths Handbook and Study Guides** with these areas in mind.
The books are written in a clear, simple, visual and logical manner. The colour coding facilitates explanations, definitions, formulas, recaps of previous work, hints and ideas.
They are easy to read, easy to understand and easy to apply what has been learnt. They work in conjunction with all other Maths Books. They are welcome additions to the **Handbook and Study Guide** series.

Kevin's objective is for the **Maths Handbook and Study Guides** to demystify Maths and help students to reach their potential in this challenging subject. The subtitles of the books are 'Maths made Easy' and this is what he aims to do. Kevin ensures that his work is up to date at all times and that it is suitable for IEB and National Curriculum students.

Beryl Lutrin – Berlut Books
Author of the **Afrikaans Handbook and Study Guide**
Co-author of the **English Handbook and Study Guide**
December 2011

Acknowledgments

A very special thanks to my past Grade 11 and 12 teachers, Mrs Nadia Wilmans and Mrs Christine Geere, for making Mathematics come alive in their outstanding Maths classes and for Their contribution towards the information contained within this textbook.

I feel that Professor Rex van Olst also needs a special thank you for his advice, feedback and encouragement whilst the idea of this textbook was still in the developmental stages.

I would also like to extend my thanks and appreciation to Beryl Lutrin for her invaluable advice which has helped to make this book what it is today.

I would like to thank my family for their continual support and encouragement. My mother Vera Smith deserves a special mention for her assistance in the graphic design of this textbook. I could not have done it without her.

Thank you to my proof readers and advisors Mr Ari Levin and Doctor Belinda Huntley for helping to ensure that the information contained in this book is accurate and CAPS compliant

Lastly, I would like to dedicate this book to all of my past and present students, wishing them the best of luck in Mathematics and in life.

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TIPS ON HOW TO STUDY WITH THIS STUDY GUIDE

1. **Read slowly**. A mathematics book does not flow like a novel - even missing a single word could cause a misunderstanding in a fundamental concept.

2. Take note of the **colours** used. If something is in colour, it is there for a reason.

3. The **environment** in which you study is extremely important. **A quiet room with no distractions** is conducive to learning. You will get very little out of a mathematics book if you cannot concentrate.

4. **Ensure that each sentence or line** of an equation **is understood** before proceeding.

5. **Do not leave out anything**. Each example illustrates a specific concept and these are rarely repeated. There is very little chance of understanding missed information by reading further in the textbook.

6. All **symbols** in mathematics have a very specific meaning. If there is any uncertainty about the meaning of a term or symbol, do not set it aside, look it up or ask a teacher to explain it to you. The **glossary** at the back of this book will explain most of the important terms you need to know.

7. Always study with a **pencil and paper**. Work out proofs and derivations yourself, using the textbook as a guide. **After reading an example, cover it up and attempt the problem yourself, taking note of any errors you made.**

8. **Mathematics is a hands-on subject**. A sportsman doesn't become a professional by watching on the sidelines. He/she **practises** for hours every day. In the same way, you cannot become a mathematician by watching others do maths. **Practice, practice** and more **practice** is the only way to truly master the subject.

9. Your brain also needs to **rest**. Do not study for hours at a time without a break. Effective studying requires concentration. If you are tired take a **nap**.

10. When **preparing** for an exam or test, make sure that you do as many **past papers** as you can. If you are unsure how to answer a question, try to find the method in this book before looking at the answer. This will prove very satisfying and build your confidence. Before your exam, practise at least two papers under exam conditions and within the allocated time.

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CHAPTER 1 – THE NUMBER SYSTEM, EXPONENTS AND SURDS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
3	**A. The Number System**	4	• Take note that there exist numbers other than those on the real number line, the so-called non-real numbers. It is possible to square certain non-real numbers and obtain negative real numbers as answers. • Apply the laws of exponents to expressions involving rational exponents. • Add, subtract, multiply and divide simple surds.
3	• Natural Numbers		
3	• Whole Numbers		
3	• Integers		
3	• Rational Numbers		
3	• Irrational Numbers		
3	• Non Real / Imaginary Numbers		
4	**B. Definitions and Laws of Exponents and Surds**		
4	• Definitions and Laws Revision		
5	• Grade 10 Revision		
6	• Addition and Subtraction of Fractions with Variable Exponents		
7	• Simplification of Surds		
10	• Simple Equations involving Surds		
10	• Steps to solving Surd Equations		
11	• Equations with Rational Exponents		

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A. THE NUMBER SYSTEM

The real number system consists of many different classifications.
This section recaps on the types of numbers that you have learnt so far and introduces the concept of non-real/ imaginary numbers.

NATURAL NUMBERS

$\{1; 2; 3; 4; ...\}$
Natural numbers start at 1 and increase in ones up to infinity (∞).

The symbol for a set of natural numbers is: $\mathbb{N}$

WHOLE NUMBERS

$\{0; 1; 2; 3; 4; ...\}$
Whole numbers start at 0 and increase in ones up to infinity (∞).

The symbol for a set of whole numbers is: $\mathbb{N}_0$

INTEGERS

$\{...; -4; -3; -2; -1; 0; 1; 2; 3; 4; ...\}$
Integers extend whole numbers to include negative numbers.
Integers start at $-\infty$ and increase in ones up to $+\infty$.

The symbol for a set of integers is: $\mathbb{Z}$

RATIONAL NUMBERS

Examples of rational numbers are: $\frac{3}{4}; 0,\dot{3}; 0,75$

Rational numbers are numbers which can be written as a **fraction** (where the numerator and denominator are integers) or as a **recurring decimal** or as a **decimal that terminates**. In order for a fraction to be a rational number, the numerator and denominator have to be integers and the denominator cannot be equal to zero.

The symbol for a set of rational numbers is: $\mathbb{Q}$

IRRATIONAL NUMBERS

Examples of irrational numbers are: $\pi; \sqrt{2}; \sqrt[3]{9}$; 2,35408...
Irrational numbers **cannot be written as a fraction** or with **terminating or recurring decimals**. In other words irrational numbers are non-recurring and non-terminating. Examples of **irrational numbers** are **pi** and the **square root of numbers that are not perfect squares** or **cube roots of numbers that are not perfect cubes**.

The symbol for a set of irrational numbers is: $\mathbb{Q}'$

NON REAL / IMAGINARY NUMBERS

Examples of non real numbers are: $\sqrt{-2}; \sqrt[4]{-9}; \sqrt[12]{-25}$
A non real number occurs when you take the even root of a negative number. Non-real numbers are known as complex numbers.

The odd root of a negative number has a real solution. For example: $\sqrt[3]{-8} = -2$



Complete Exercise 1 on page 247.

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B. DEFINITIONS AND LAWS OF EXPONENTS AND SURDS

Notation: $x^{\overset{\text{Exponent}}{a}}$ (x = Base)

The table below summarises the laws and definitions of exponents and surds that you have learnt so far.

DEFINITIONS AND LAWS REVISION



Definition/Law	Example	Explanation
1. $x^0 = 1$	$2^0 \times x^0 = 1 \times 1 = 1$	Any base raised to the power of zero is equal to 1. ($x \neq 0$ as 0^0 is undefined)
2. $x^{-m} = \frac{1}{x^m}$	$2x^{-3} = \frac{2}{x^3}$ also $\frac{2}{x^{-3}} = 2x^3$	A base raised to a negative exponent is equal to its reciprocal raised to the same positive exponent. (once again $x \neq 0$)
3. $x^m \times x^n = x^{m+n}$	$x^2 \times x^3 \times x = x^{2+3+1} = x^6$	When multiplying like bases keep the base and add the exponents.
4. $\frac{x^m}{x^n} = x^{m-n}$	$\frac{8x^7}{4x^3} = 2x^{7-3} = 2x^4$	When dividing like bases keep the base and subtract the exponents. (numbers divide normally)
5. $(x^m)^n = x^{m \times n}$	$(2a^3b)^3 = 2^3 a^{3\times3} b^{1\times3} = 8a^9b^3$	When raising exponents to a power, keep the base and multiply the exponents.
6. $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{x^2}{y}\right)^3 = \frac{x^6}{y^3}$	When a fraction is raised to an exponent, the numerator and denominator must be raised to that exponent.
7. $x^{\frac{m}{n}} = \sqrt[n]{x^m}$	$27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9$	If the exponent is a fraction, the base is raised to the numerator and rooted to the denominator. ($x > 0$ and $n > 0$)
8. $\sqrt[n]{xy} = \sqrt[n]{x} \times \sqrt[n]{y}$	$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$	When surds are multiplied they can be split apart and rooted individually.
9. $\sqrt[mn]{x} = \sqrt[m]{\sqrt[n]{x}}$	$\sqrt[6]{64} = \sqrt[3\times2]{64} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$	When taking the root of a root, it is the same as taking the single root to the product of both roots.



All of the above laws and definitions also apply in reverse.

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GRADE 10 REVISION 

Example 1: Simplify $16^{-\frac{1}{4}}+(-10)^0$

$16^{-\frac{1}{4}}+(-10)^0$ → split base into its prime factors and apply law 1

$=\left(2^4\right)^{-\frac{1}{4}}+1$ → apply law 5

$=2^{-1}+1$ → apply law 2

$=\frac{1}{2}+1$

$=\frac{3}{2}$

Example 2: Simplify $\frac{10^{2x}.3^{2x}}{25^x.6^{2x-1}}$

$\frac{10^{2x}.3^{2x}}{25^x.6^{2x-1}}$ → split bases into their prime factors

$=\frac{(5.2)^{2x}.3^{2x}}{\left(5^2\right)^x.(3.2)^{2x-1}}$ → apply law 5

$=\frac{5^{2x}.2^{2x}.3^{2x}}{5^{2x}.3^{2x-1}.2^{2x-1}}$ → apply laws 3 and 4

$=5^{2x-2x}.3^{2x-(2x-1)}.2^{2x-(2x-1)}$

$=5^0.3^{2x-2x+1}.2^{2x-2x+1}$ → apply law 1

$=1.3^1.2^1$

$=6$

Example 3: Simplify $\frac{9^{2x+1}.2^{2x+2}}{3^{3x-2}.12^{x+1}}$

$\frac{9^{2x+1}.2^{2x+2}}{3^{3x-2}.12^{x+1}}$ → split bases into their prime factors

$=\frac{\left(3^2\right)^{2x+1}.2^{2x+2}}{3^{3x-2}.\left(3.2^2\right)^{x+1}}$ → apply law 5

$=\frac{3^{4x+2}.2^{2x+2}}{3^{3x-2}.3^{x+1}.2^{2x+2}}$ → apply laws 3 and 4

$=3^{4x+2-(3x-2+x+1)}.2^{2x+2-(2x+2)}$

$=3^{4x+2-3x+2-x-1}.2^{2x+2-2x-2}$

$=3^3.2^0$

$=27$

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ADDITION AND SUBTRACTION OF FRACTIONS WITH VARIABLE EXPONENTS

During addition and subtraction of fractions with variable exponents, law 3 on pg. 4 is generally used in reverse to split apart exponents. In most cases factorisation can then be used to simplify the expression.

Example 1: Simplify $\dfrac{5^{1+2x}-3.5^{2x}}{25^{x}.4}$

$\dfrac{5^{1+2x}-3.5^{2x}}{25^{x}.4}$ → split bases apart using law 3 in reverse

$=\dfrac{5^{1}.5^{2x}-3.5^{2x}}{\left(5^{2}\right)^{x}.4}$ → with addition and subtraction always look for a HCF

$=\dfrac{5^{2x}(5-3)}{5^{2x}.4}$

$=\dfrac{2}{4}$

$=\dfrac{1}{2}$

Example 2: Simplify $\dfrac{3^{2x+2}+3^{2x+1}}{3^{2x+2}-3^{2x+1}}$

$\dfrac{3^{2x+2}+3^{2x+1}}{3^{2x+2}-3^{2x+1}}$ → do not cancel over a '±' sign, split bases apart using law 3 in reverse

$=\dfrac{3^{2x}.3^{2}+3^{2x}.3^{1}}{3^{2x}.3^{2}-3^{2x}.3^{1}}$ → with addition and subtraction always look for a HCF

$=\dfrac{3^{2x}\left(3^{2}+3\right)}{3^{2x}\left(3^{2}-3\right)}$

$=\dfrac{12}{6}$

$=2$



Complete Exercise 2 on page 247.

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SIMPLIFICATION OF SURDS

A surd is the root of a number that would result in an irrational number.
For example, $\sqrt{2}$ is a surd as the answer is irrational. However, $\sqrt{4}$ would not be referred to as a surd as it has an answer which is rational (2).
All the laws on page 4 for rational exponents also apply to surds.

Example 1: Simplify the following, writing all answers in simplest surd form:

a) $\sqrt{72}$

b) $\sqrt{98}-\sqrt{32}-\sqrt{18}$

c) $\sqrt{12}+\sqrt{48}-\sqrt{75}$

d) $\dfrac{6+\sqrt{18}}{3}$

a) $\sqrt{72}$ → break apart into a perfect square and another number
$=\sqrt{36\times 2}$ → perfect square, apply law 8
$=\sqrt{36}\times\sqrt{2}$
$=6\sqrt{2}$

b) $\sqrt{98}-\sqrt{32}-\sqrt{18}$ → break apart into a perfect square and another number
$=\sqrt{49\times 2}-\sqrt{16\times 2}-\sqrt{9\times 2}$ → perfect square, apply law 8
$=7\sqrt{2}-4\sqrt{2}-3\sqrt{2}$
$=0\sqrt{2}$
$=0$

c) $\sqrt{12}+\sqrt{48}-\sqrt{75}$ → break apart into a perfect square and another number
$=\sqrt{4\times 3}+\sqrt{16\times 3}-\sqrt{25\times 3}$ → perfect square, apply law 8
$=2\sqrt{3}+4\sqrt{3}-5\sqrt{3}$
$=\sqrt{3}$

d) $\dfrac{6+\sqrt{18}}{3}$ → break apart into a perfect square and another number
$=\dfrac{6+\sqrt{9\times 2}}{3}$ → perfect square, apply law 8
$=\dfrac{6+3\sqrt{2}}{3}$ → take out a HCF of 3
$=\dfrac{3\left(2+\sqrt{2}\right)}{3}$
$=2+\sqrt{2}$

It is important to remember the following properties of surds:
1) $\sqrt{x}\geq 0;\ x\geq 0$
2) $\sqrt{x}\times\sqrt{x}=x\ ;\ x\geq 0$

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Example 2: Simplify $\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)$ without the use of a calculator.

$\left(\sqrt{7}+2\right)\left(\sqrt{7}-2\right)$ → difference of squares $(a+b)(a-b)=a^2-b^2$

$=\left(\sqrt{7}\right)^2-(2)^2$

$=7-4$

$=3$

Example 3: Simplify $\left(2\sqrt{2}+\sqrt{3}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ without the use of a calculator.

$\left(2\sqrt{2}+\sqrt{3}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ → difference of squares $(a+b)(a-b)=a^2-b^2$

$=\left(2\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2$

$=8-3$

$=5$

Example 4: Simplify $\left(3\sqrt{3}-\sqrt{8}\right)^2$ without the use of a calculator.

$\left(3\sqrt{3}-\sqrt{8}\right)^2$ → perfect square, don't forget the middle term: $(a-b)^2=a^2-2ab+b^2$

$=27-6\sqrt{24}+8$ → $2\times3\sqrt{3}\times\sqrt{8}=6\sqrt{3\times8}=6\sqrt{24}$

$=35-6\sqrt{4\times6}$ → $6\times\sqrt{4\times6}=6\times2\sqrt{6}=12\sqrt{6}$

$=35-12\sqrt{6}$

Example 5: Simplify $\left(2\sqrt{2}-6\right)\left(\sqrt{8}+6\right)$ without the use of a calculator.

$\left(2\sqrt{2}-6\right)\left(\sqrt{8}+6\right)$ → multiply out

$=2\sqrt{16}+12\sqrt{2}-6\sqrt{8}-36$ → $-6\sqrt{8}=-6\sqrt{4\times2}=-12\sqrt{2}$

$=8+12\sqrt{2}-12\sqrt{2}-36$

$=-28$

Example 6: Simplify $\sqrt[3]{64x^6}-\sqrt[4]{81x^8}+\sqrt{25x^4}$ without the use of a calculator.

$\sqrt[3]{64x^6}-\sqrt[4]{81x^8}+\sqrt{25x^4}$ → rewrite using law 7

$=\left(64x^6\right)^{\frac{1}{3}}-\left(81x^8\right)^{\frac{1}{4}}+\left(25x^4\right)^{\frac{1}{2}}$ → split numbers into their prime factors

$=\left(4^3x^6\right)^{\frac{1}{3}}-\left(3^4x^8\right)^{\frac{1}{4}}+\left(5^2x^4\right)^{\frac{1}{2}}$ → use law 5

$=4x^2-3x^2+5x^2$

$=6x^2$

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Example 7: Simplify $\dfrac{6\sqrt{2x^8}-2\sqrt{8x^8}}{\sqrt{8x^8}}$ **without the use of a calculator.**

$\dfrac{6\sqrt{2x^8}-2\sqrt{8x^8}}{\sqrt{8x^8}}$ → split numbers into a perfect square and another number

$=\dfrac{6\sqrt{2x^8}-2\sqrt{4\times2\times x^8}}{\sqrt{4\times2\times x^8}}$ → perfect squares use law 8

$=\dfrac{6x^4\sqrt{2}-4x^4\sqrt{2}}{2x^4\sqrt{2}}$

$=\dfrac{2x^4\sqrt{2}}{2x^4\sqrt{2}}$

$=1$



Recap: Rationalising the denominator

It is often useful to write fractions with rational numbers in the denominator. When a surd is present in a denominator it is possible to write the denominator as a rational number by multiplying the numerator and denominator by the surd.

Example: $\dfrac{1}{\sqrt{x}}=\dfrac{1}{\sqrt{x}}\times\dfrac{\sqrt{x}}{\sqrt{x}}=\dfrac{\sqrt{x}}{\sqrt{x}\sqrt{x}}=\dfrac{\sqrt{x}}{x}$

Example 8: Prove that $\dfrac{2\sqrt{a}}{a}+\dfrac{2b}{\sqrt{a}}=\dfrac{2\sqrt{a}(1+b)}{a}$ **without the use of a calculator.**

$\text{LHS}=\dfrac{2\sqrt{a}}{a}+\dfrac{2b}{\sqrt{a}}$ → LCD of $a\sqrt{a}$

$=\dfrac{2\sqrt{a}\left(\sqrt{a}\right)+2ab}{a\sqrt{a}}$

$=\dfrac{2a+2ab}{a\sqrt{a}}$ → take out a highest common factor of $2a$

$=\dfrac{2\cancel{a}(1+b)}{\cancel{a}\sqrt{a}}$

$=\dfrac{2(1+b)}{\sqrt{a}}$ → rationalise the denominator by multiplying by 1 in the form of $\dfrac{\sqrt{a}}{\sqrt{a}}$

$=\dfrac{2(1+b)}{\sqrt{a}}\times\dfrac{\sqrt{a}}{\sqrt{a}}$

$=\dfrac{2\sqrt{a}(1+b)}{a}=\text{RHS}$



Complete Exercise 3 on pages 247 and 248.

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SIMPLE EQUATIONS INVOLVING SURDS

In Grade 11 you are expected to be able to solve equations with surds.
Simple surd equations can be solved by inspection or by squaring both sides of the equation as shown in the examples below.

Whenever dealing with surd equations you have to check your answers by substituting your answer into the original equation.

STEPS TO SOLVING SURD EQUATIONS

1. **Isolate the surd.**
2. **Square both sides.**
3. **Solve for x.**
4. **Check your answer.**

Example 1: Determine the value of x if $\sqrt{x+2}-3=0$.

$\sqrt{x+2}-3=0 \rightarrow$ isolate the surd

$\therefore \sqrt{x+2}=3 \rightarrow$ square both sides and note that $\left(\sqrt{a}\right)^2=a$

$\left(\sqrt{x+2}\right)^2=(3)^2$

$\therefore x+2=9$

$\therefore x=7$

Check: $\sqrt{7+2}-3=0 \rightarrow$ answer is valid

Example 2: Determine the value of x if $\sqrt{x-5}+4=8$.

$\sqrt{x-5}+4=8 \rightarrow$ isolate the surd

$\therefore \sqrt{x-5}=4 \rightarrow$ square both sides

$\therefore \left(\sqrt{x-5}\right)^2=(4)^2$

$\therefore x-5=16$

$\therefore x=21$

Check: $\sqrt{21-5}+4=8 \rightarrow$ answer is valid

Example 3: Determine the value of x if $3\sqrt{40-x}-5=13$.

$3\sqrt{40-x}-5=13 \rightarrow$ isolate the surd

$\therefore 3\sqrt{40-x}=18 \rightarrow$ square both sides

$\therefore \left(3\sqrt{40-x}\right)^2=(18)^2$

$\therefore 9(40-x)=324 \rightarrow$ don't forget that the 3 also had to be squared

$\therefore 360-9x=324$

$\therefore -9x=-36$

$\therefore x=4$

Check: $3\sqrt{40-4}-5=13 \rightarrow$ answer is valid



Complete Exercise 4 on page 248.

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EQUATIONS WITH RATIONAL EXPONENTS

In Grade 11 you will deal with equations which have rational exponents (exponents with fractions). Simple exponential equations are also shown in this section. These will be explored in more detail further on in the year.

If an equation is in the form of : $x^{\frac{p}{q}} = a$

- **There will be a positive and negative solution if** p **is even and** q **is odd.**
- **There will be 1 solution if** p **is odd.**

Illustrative Example: Solve for x **if** $x^{\frac{2}{3}} - 2^2 = 0$.

There are two methods that can be used to solve the above equation:

Method 1:

$x^{\frac{2}{3}} - 2^2 = 0 \rightarrow$ difference of squares

$\therefore \left(x^{\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} + 2\right) = 0$

$\therefore x^{\frac{1}{3}} = 2$ or $x^{\frac{1}{3}} = -2 \rightarrow$ raise both sides to the power of 3

$\therefore \left(x^{\frac{1}{3}}\right)^3 = (2)^3$ or $\left(x^{\frac{1}{3}}\right)^3 = (-2)^3$

$\therefore x = 8$ or $x = -8$

Method 2:

$x^{\frac{2}{3}} = 2^2 \rightarrow p$ is even and q is odd, $\therefore$ there will be a positive and negative solution

$\therefore \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \pm\left(2^2\right)^{\frac{3}{2}} \rightarrow$ both sides were raised to the power of $\frac{3}{2}$

$\therefore x = \pm 2^3 = \pm 8$

Example 1: Solve for x **if** $x^{\frac{2}{3}} = 2^4$.

$x^{\frac{2}{3}} = 2^4 \rightarrow$ raise both sides to the power of $\frac{3}{2}$

$\therefore \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = \pm\left(2^4\right)^{\frac{3}{2}} \rightarrow$ don't forget the '$\pm$' as p is even and q is odd

$\therefore x = \pm 2^6 = \pm 64 \rightarrow 4 \times \frac{3}{2} = 6$

Example 2: Solve for x **if** $x^{\frac{3}{2}} = 125$

$x^{\frac{3}{2}} = 125 \rightarrow$ convert 125 to a prime base

$\therefore x^{\frac{3}{2}} = 5^3 \rightarrow$ raise both sides to the power of $\frac{2}{3}$

$\therefore \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(5^3\right)^{\frac{2}{3}} \rightarrow$ only 1 solution (no '$\pm$') as p is odd

$\therefore x = 5^2 = 25$

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Example 3: Solve for x if $(x-3)^{2/3}=9$.

$(x-3)^{2/3}=9 \rightarrow$ convert 9 to a prime base

$\therefore (x-3)^{2/3}=3^2 \rightarrow$ raise both sides to the power of $3/2$

$\therefore \left[(x-3)^{2/3}\right]^{3/2}=\pm\left(3^2\right)^{3/2} \rightarrow$ don't forget the '$\pm$' as p is even and q is odd

$\therefore x-3=27$ or $x-3=-27$

$\therefore x=30$ or $x=-24$

Example 4: Solve for x if $3^{x/2}=9$.

$3^{x/2}=9 \rightarrow$ convert 9 to a prime base

$\therefore 3^{x/2}=3^2 \rightarrow$ if the bases are the same the exponents have to be equal

$\therefore \frac{x}{2}=2$

$\therefore x=4$

Example 5: Solve for x if $x^{4/5}=\frac{2^8}{3^8}$

$x^{4/5}=\frac{2^8}{3^8} \rightarrow$ raise both sides to the power of $5/4$

$\therefore \left(x^{4/5}\right)^{5/4}=\pm\left(\frac{2^8}{3^8}\right)^{5/4} \rightarrow$ don't forget the '$\pm$'

$\therefore x=\frac{2^{10}}{3^{10}} \rightarrow 8\times\frac{5}{4}=10$

Example 6: Solve for x if $4.3^{x/2}=9.2^{x/2}$

$4.3^{x/2}=9.2^{x/2} \rightarrow$ isolate the bases with variables (divide both sides by $2^{x/2}$ and 4)

$\therefore \frac{3^{x/2}}{2^{x/2}}=\frac{9}{4} \rightarrow$ convert 9 and 4 into their prime bases

$\therefore \frac{3^{x/2}}{2^{x/2}}=\frac{3^2}{2^2}$

$\therefore \left(\frac{3}{2}\right)^{x/2}=\left(\frac{3}{2}\right)^2 \rightarrow$ as the bases are the same the exponents are equal

$\therefore \frac{x}{2}=2 \rightarrow$ multiply both sides by 2

$\therefore x=4$



Complete Exercise 5 and the mixed exercise on pages 248 and 249.

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CHAPTER 2 – EQUATIONS AND INEQUALITIES

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
15	**A. Grade 10 Revision**	4	• Factorise expressions.
15	• Factorisation		• Solve quadratic equations
17	• Factorisation by Grouping		• Solve quadratic inequalities in one variable and interpret the solution graphically
18	• Factorisation of the Sum and Difference of Cubes		• Solve equations in two unknowns, one of which is linear the other quadratic, algebraically or graphically.
19	• Rules of Fractions		• Complete the square on quadratic expressions and equations
20	**B. Quadratic Equations**		• Solve equations using the quadratic formula
20	• Steps to Solving Quadratic Equations		• Determine the nature of roots
21	• Equations with Fractions		
23	• Equations with substitution (K-method)		
25	• Surd Equations which lead to Quadratic Equations		
25	• Steps to Solving Surd Equations which lead to Quadratic Equations		
28	• Exponential Equations		
30	**C. Simultaneous Equations**		
32	**D. Completing the Square**		
32	• Completing the square on a quadratic expression		
34	• Completing the square on a quadratic equation		
36	• Applications of completing the square		
38	**E. The Quadratic Formula**		
38	• Derivation of the Quadratic Formula		
39	• Nature of Roots		
40	• The Discriminant		
41	• Determining a Quadratic Equation from the Roots of an Equation		

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CHAPTER 2 – EQUATIONS AND INEQUALITIES

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
42	**F. Inequalities**		
42	• Revision of Number Lines and Notation		
42	• Revision of Linear Inequalities		
43	• Quadratic Inequalities		
45	• Inequalities with Repeated Factors		
46	• Unusual Inequalities		
47	• Inequalities which do not Factorise		
48	• Mixed Example		
49	**G. Mathematical modelling**		
49	• Steps to Solving Word Problems		
49	• Number and Age Problems (Revision of Grade 10)		
50	• Shopping problems		
51	• Speed, Distance and Time Problems		
52	• Area Problems		
53	• Rate of Flow Problems		

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A. GRADE 10 REVISION

FACTORISATION

In this section you will revise the factorisation that you learnt in Grade 10.

Factorisation by taking out a Highest Common Factor

Example 1: Factorise x^2-2x.

$x^2-2x \rightarrow x$ is common to both terms

$=x(x-2)$

Example 2: Factorise $2x(x-2)-3(x-2)$.

$2x(x-2)-3(x-2) \rightarrow (x-2)$ is common to both terms

$=(x-2)(2x-3)$

Factorisation of the Difference of Squares

When factorising the difference of two squares, rewrite the two squares as the products of their roots.

$$x^2-a^2=\left(\sqrt{x^2}-\sqrt{a^2}\right)\left(\sqrt{x^2}+\sqrt{a^2}\right)=(x-a)(x+a)$$

Only use the difference of two squares when the squares are separated by a minus sign.

Example 1: Factorise x^2-4.

x^2-4

$=(x-2)(x+2)$

Example 2: Factorise $27x^2-12$.

$27x^2-12$

$=3\left(9x^2-4\right) \rightarrow$ always look for a HCF first

$=3(3x-2)(3x+2)$

Example 3: Factorise $81x^4-16$.

$81x^4-16$

$=\left(9x^2+4\right)\left(9x^2-4\right) \rightarrow$ this can be factorised further

$=\left(9x^2+4\right)(3x-2)(3x+2)$

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Factorisation of Trinomials

The general form a of a quadratic trinomial is $ax^2 \pm bx \pm c$.
There are three types of trinomials that you will be expected to factorise:

1. **Trinomials of the form $x^2 \pm bx + c \rightarrow (a = 1)$.**
2. **Trinomials of the form $x^2 \pm bx - c \rightarrow (a = 1)$.**
3. **Trinomials of the form $ax^2 \pm bx \pm c \rightarrow (a \neq 1)$.**

1. Factorising Trinomials in the form of $x^2 \pm bx + c$.

$x^2 \pm bx + c$ → if the sign before the constant is a '+' then you know both signs in the brackets will be the same

→ both signs in the brackets will be the same as the sign of the coefficient of x

Illustrative Example 1: Factorise $x^2 + 8x + 15$.

$x^2 + 8x + 15$

Both signs will be the same because the coefficient of 15 (the constant) is a '+'.
Both signs will be '+' as the coefficient of x is a '+'.

$(x + \square)(x + \square)$

In order to find the missing numbers you look for the factors of 15 (the constant) that will add up to 8, the coefficient of x.

Factors of 15 are: $15 \quad \boxed{5}$ → $5 + 3 = 8$
$\quad\quad\quad\quad\quad 1 \quad \boxed{3}$

$\therefore x^2 + 8x + 15 = (x + 5)(x + 3)$

2. Factorising Trinomials in the form of $x^2 \pm bx - c$.

$x^2 \pm bx + c$ → if the sign before the constant is a '–' then you know there will be opposite signs in the brackets

→ the larger of the two factors will get the sign of the coefficient of x

Illustrative Example 2: Factorise $x^2 + 2x - 15$.

$x^2 + 2x - 15$

The signs will be opposite because the coefficient of 15 (the constant) is a '–'.
The larger factor will get a '+' as the coefficient of x is a '+'.

$(x + \square)(x - \square)$

In order to find the missing numbers you look for the factors of 15 (the constant) where the difference between the factors will be equal to 2, the coefficient of x.

Factors of 15 are: $15 \quad \boxed{5}$ → $5 - 3 = 2$
$\quad\quad\quad\quad\quad 1 \quad \boxed{3}$

$\therefore x^2 + 2x - 15 = (x - 3)(x + 5)$ → 5 gets a '+' sign as it is the larger factor

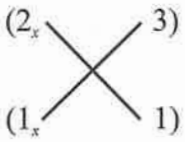
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3. Factorising Trinomials in the form of $ax^2 \pm bx - c$.

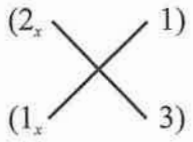
When the coefficient (a) of x^2 is not equal to one, the cross method is used to factorise the expression. This is illustrated in the example below.

Illustrative Example 1: Factorise $2x^2 + 7x + 3$.

In this method you draw a cross with the factors of 2 (a) on the left side and the factors of 3 (the constant) on the right side. You then multiply diagonally across the cross. The factors which add or subtract to equal 7 (the coefficient of x) will be the correct factors.



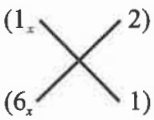
$2\times1=2$ and $3\times1=3$
Incorrect as you cannot make 7 with a 2 and a 3



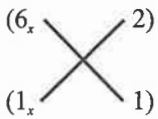
$2\times3=6$ and $1\times1=1$
correct: $6+1=7$

$\therefore 2x^2+7x+3=(2x+1)(x+3)$

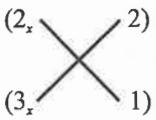
Illustrative Example 2: Factorise $6x^2 - x - 2$.



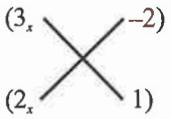
$1\times1=1$ and $6\times2=12$
Incorrect as you cannot make -1 with a 1 and 12



$6\times1=6$ and $1\times2=2$
Incorrect as you cannot make -1 with a 6 and 2



$2\times1=2$ and $3\times2=6$
Incorrect as you cannot make -1 with a 2 and 6



$3\times1=3$ and $2\times2=4$
Correct: you can get -1 with a 3 and 4. However the 4 must be negative.

$\therefore 6x^2-x-2=(3x-2)(2x+1)$



Always look for a HCF before using the cross method.

FACTORISATION BY GROUPING

The idea behind grouping is to group terms together and then, if possible, take out an HCF in order to obtain a common bracket.

Example 1: Factorise $6x^2 - 2ax - 3xy + ay$.

$6x^2-2ax-3xy+ay$
$=3x(2x-y)-a(2x-y)$ → group terms to obtain a common bracket
$=(2x-y)(3x-a)$



Grouping is generally used if there are more than 3 terms in an expression.

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FACTORISATION OF THE SUM AND DIFFERENCE OF CUBES

The sum and difference of perfect cubes always factorise into the form shown below:

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

1st bracket : cube root each term and keep the same sign

2nd bracket : square the first term

change the sign and multiply both terms together

square the last term (the sign of the last term is always positive)

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

The sum and difference of cubes always factorises into a binomial and a trinomial.

Example 1: Factorise $27x^3 + y^3$

$27x^3 + y^3$

$= (3x + y)(9x^2 - 3xy + y^2)$

Example 2: Factorise $54x^3 - 16$

$54x^3 - 16 \rightarrow$ always look for a HCF when factorising (2 is common to both terms)

$= 2(27x^3 - 8)$

$= 2(3x-2)(9x^2 + 6x + 4) \rightarrow$ you cannot drop the 2

Example 3: Factorise $(x-y)^3 - 64x^3$

$(x-y)^3 - 64x^3 \rightarrow$ difference of two perfect cubes

$= [(x-y) - 4x][(x-y)^2 + 4x(x-y) + 16x^2]$

$= (x - y - 4x)(x^2 - 2xy + y^2 + 4x^2 - 4xy + 16x^2) \rightarrow$ gather like terms

$= (-3x - y)(21x^2 - 6xy + y^2)$

In factorisation of the sum and difference of two cubes the trinomial (in the second bracket) will never factorise further in the real number system.

Complete Exercise 1 number 1 on page 258.

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RULES OF FRACTIONS

	Rule	Explanation
1	$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$	**When multiplying fractions, multiply the numerators and the denominators.**
	Restrictions: $b\neq 0$; $d\neq 0$	**Factorise numerators and denominators if possible and try to cancel first.**
2	$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$	**When dividing fractions, invert whatever is after the divide sign and change the operation from division to multiplication.**
	Restrictions: $b\neq 0$; $c\neq 0$; $d\neq 0$	**(This is known as the tip and times rule.)**
3	$\frac{a}{b}\pm\frac{c}{b}=\frac{a\pm c}{b}$	**Fractions can only be added (or subtracted) if they have the same denominators. If the denominators are the same, keep the denominator and add the numerators together.**
	Restrictions: $b\neq 0$	

Example 1: Simplify $\frac{x^2+5x-6}{x^2+2x-15}\div\frac{x^2+8x+12}{x^2+7x+10}$.

$\frac{x^2+5x-6}{x^2+2x-15}\div\frac{x^2+8x+12}{x^2+7x+10}$ → factorise numerators and denominators

$=\frac{(x+6)(x-1)}{(x+5)(x-3)}\div\frac{(x+6)(x+2)}{(x+5)(x+2)}$

$=\frac{(x+6)(x-1)}{(x+5)(x-3)}\times\frac{(x+5)}{(x+6)}$ → 'tip and times'

$=\frac{x-1}{x-3}$

Example 2: Simplify $x+5+\frac{9}{x-3}+\frac{3x}{3-x}$.

$x+5+\frac{9}{x-3}+\frac{3x}{3-x}$ → take out a '$-$' to make a LCD of $x-3$

$=x+5+\frac{9}{x-3}-\frac{3x}{x-3}$ → LCD $=x-3$

$=\frac{(x+5)(x-3)+9-3x}{x-3}$

$=\frac{x^2+2x-15+9-3x}{x-3}$

$=\frac{x^2-x-6}{x-3}$

$=\frac{(x-3)(x+2)}{x-3}$

$=x+2$



Complete Exercise 1 number 2 on page 258.

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B. QUADRATIC EQUATIONS

A quadratic equation is an equation of the form: $ax^2+bx+c=0$

In most cases a quadratic equation will have two solutions.

STEPS TO SOLVING QUADRATIC EQUATIONS

1. **Make the right hand side or left hand side equal to zero.**
2. **Factorise (always look for a common factor first).**
3. **Solve for x.**

The x-values of the solutions to a quadratic equation are known as roots or zeros. These represent the x-intercepts of a function.

Example 1: Solve for x if $x^2-16=0$.

$x^2-16=0$

$\therefore (x-4)(x+4)=0$

$\therefore x-4=0$ or $x+4=0$ → it is not necessary to show this step

$\therefore x=4$ or $x=-4$

Example 2: Solve for x if $x^2-16x=0$.

$x^2-16x=0$ → take out a HCF of x

$\therefore x(x-16)=0$ → do not divide by x as x could be equal to 0

$\therefore x=0$ or $x-16=0$ → it is not necessary to show this step

$\therefore x=0$ or $x=16$

Example 3: Solve for the roots of $x(x-9)=-18$.

$x(x-9)=-18$ → LHS or RHS must be equal to 0

$\therefore x^2-9x+18=0$ → factorise

$\therefore (x-6)(x-3)=0$

$\therefore x-6=0$ or $x-3=0$ → it is not necessary to show this step

$\therefore x=6$ or $x=3$

Example 4: Solve for x if $-5x^2-5x+30=0$.

$-5x^2-5x+30=0$ → take out a HCF of -5

$\therefore -5(x^2+x-6)=0$ → divide both sides by -5

$\therefore x^2+x-6=0$

$\therefore (x+3)(x-2)=0$

$\therefore x=-3$ or $x=2$

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Example 5: Solve for x if $x^3 - 2x^2 + x = 0$.

$x^3 - 2x^2 + x = 0$

$\therefore\ x(x^2 - 2x + 1) = 0$ → always look for a HCF first

$\therefore\ x(x-1)(x-1) = 0$ → do not divide by x as x could be equal to 0

$\therefore\ x = 0 \quad or \quad x - 1 = 0$

$\therefore\ x = 0 \ or \ x = 1$



Complete Exercise 2 on page 258.

EQUATIONS WITH FRACTIONS

Example 1: Solve for x if $\frac{1}{x+1} = 1 - \frac{2x}{x-1}$.

$\frac{1}{x+1} = 1 - \frac{2x}{x-1}$

LCD: $(x-1)(x+1)$ $\quad x \neq 1$ *and* $x \neq -1$ → you have to put in restrictions

$\therefore 1(x-1) = 1(x-1)(x+1) - 2x(x+1)$ → multiply both sides by the LCD

$\therefore x - 1 = x^2 - 1 - 2x^2 - 2x$

$\therefore x - 1 = -x^2 - 2x - 1$ → quadratic equation, make one side equal 0

$\therefore x^2 + 3x = 0$ → take out a HCF of x

$\therefore x(x+3) = 0$

$\therefore x = 0 \quad or \quad x = -3$

Example 2: Solve for x if $\frac{x^2 - 1}{x+1} = 2$.

$\frac{x^2 - 1}{x+1} = 2$

LCD: $(x+1)$ $\quad x \neq -1$ → you have to put in restrictions

$\therefore x^2 - 1 = 2(x+1)$ → multiply both sides by the LCD

$\therefore x^2 - 1 = 2x + 2$ → quadratic equation, make one side equal 0

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x-3)(x+1) = 0$

$\therefore x = 3 \quad or \quad x = -1$

but from the restriction above $x \neq -1$

$\therefore x = 3$ only

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Example 3: Solve for x if $\frac{2x}{x+3}-\frac{x}{x-3}=\frac{3}{3-x}$.

$\frac{2x}{x+3}-\frac{x}{x-3}=\frac{3}{3-x}$

$\therefore \frac{2x}{x+3}-\frac{x}{x-3}=-\frac{3}{x-3}$ → take out a '−' to make a common denominator of $x-3$

LCD: $(x-3)(x+3)$ $x\neq\pm3$ → you have to put in restrictions

$\therefore 2x(x-3)-x(x+3)=-3(x+3)$ → both sides were multiplied by the LCD

$\therefore 2x^2-6x-x^2-3x=-3x-9$ → don't forget to distribute the '−'

$\therefore x^2-6x+9=0$

$\therefore (x-3)(x-3)=0$ → this could also be written as $(x-3)^2$

$\therefore x=3$

but $x\neq3$ from the above restrictions

$\therefore$ No solution

Example 4: Solve for x if $\frac{3x+6}{x^2-x-2}+\frac{4}{2-x}=\frac{-4}{x^2+4x+3}$.

$\frac{3x+6}{x^2-x-2}+\frac{4}{2-x}=\frac{-4}{x^2+4x+3}$ → factorise non-linear denominators

$\therefore \frac{3x+6}{(x-2)(x+1)}-\frac{4}{x-2}=\frac{-4}{(x+3)(x+1)}$ → take out a '−' as you did in example 3 above

LCD: $(x+3)(x+1)(x-2)$, $x\neq-3$ and $x\neq-1$ and $x\neq2$ → put in restrictions

$\therefore (3x+6)(x+3)-4(x+3)(x+1)=-4(x-2)$ → both sides were multiplied by the LCD

$\therefore 3x^2+15x+18-4\left(x^2+4x+3\right)=-4x+8$

$\therefore 3x^2+15x+18-4x^2-16x-12=-4x+8$ → don't forget to distribute the '−'

$\therefore -x^2+3x-2=0$

$\therefore x^2-3x+2=0$

$\therefore (x-1)(x-2)=0$

$\therefore x=1$ or $x=2$

but $x\neq2$ from the above restrictions

$\therefore x=1$ only



You can only multiply both sides by the LCD (commonly known as dropping the LCD) if you are dealing with an equation. Always put your restrictions in and check that none of your answers are equal to your restrictions.



Complete Exercise 3 on page 258.

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EQUATIONS WITH SUBSTITUTION (K-METHOD)

When an equation has an expression which is repeated, it is often simpler to replace the repeated expression with a single variable.
The variable that is normally used is k, hence the name k-method.

Example 1: Solve for x if $(x^2-3x)^2-8(x^2-3x)-20=0$.

$(x^2-3x)^2-8(x^2-3x)-20=0 \rightarrow$ repeated expression, use the k-method

Let $k=x^2-3x$

$\therefore k^2-8k-20=0$

$\therefore (k-10)(k+2)=0$

$\therefore k=10$ or $k=-2$

Now substitute $k=x^2-3x$ back into your solutions for k

$\therefore x^2-3x=10$ or $x^2-3x=-2$

$\therefore x^2-3x-10=0$ or $x^2-3x+2=0$

$\therefore (x-5)(x+2)=0$ or $(x-2)(x-1)=0$

$\therefore x=5$ or $x=-2$ or $x=2$ or $x=1$

Example 2: Solve for x if $\frac{50}{x^2+1}+5=x^2+1$.

$\frac{50}{x^2+1}+5=x^2+1 \rightarrow$ repeated expression, use the k-method

Let $k=x^2+1$

$\therefore \frac{50}{k}+5=k$

LCD $=k$, $k\neq 0, \therefore x^2+1\neq 0 \rightarrow$ you have to put in restrictions

$\therefore 50+5k=k^2$

$\therefore k^2-5k-50=0$

$\therefore (k-10)(k+5)=0$

$\therefore k=10$ or $k=-5$

Now substitute $k=x^2+1$ back into your solutions for k

$\therefore x^2+1=10$ or $x^2+1=-5$

$\therefore x^2=9$ or $x^2=-6 \rightarrow$ no real solutions as you cannot square root -6

$\therefore x=\pm 3 \rightarrow$ don't forget the '$\pm$' and remember that this is two answers, $+3$ and -3



If you obtain an x^4 when simplifying an equation, the k-method is the most likely method to use provided that you cannot take out a HCF.

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Example 3: Solve for x if $x^2-4x=17-\dfrac{60}{x^2-4x}$.

$x^2-4x=17-\dfrac{60}{x^2-4x}$ → repeated expression, use the k-method

Let $k=x^2-4x$

$\therefore k=17-\dfrac{60}{k}$

LCD $=k$, $k\neq 0, \therefore x^2-4x\neq 0$ $\therefore x\neq 0$ and $x\neq 4$ → you have to put in restrictions

$\therefore k^2=17k-60$

$\therefore k^2-17k+60=0$

$\therefore (k-5)(k-12)=0$

$\therefore k=5$ or $k=12$

Now substitute $k=x^2-4x$ back into your solutions for k

$\therefore x^2-4x=5$ or $x^2-4x=12$

$\therefore x^2-4x-5=0$ or $x^2-4x-12=0$

$\therefore (x-5)(x+1)=0$ or $(x-6)(x+2)=0$

$\therefore x=5$ or $x=-1$ or $x=6$ or $x=-2$

Example 4: Solve for x if $2x^2-4x=36-\dfrac{270}{3x^2-6x}$.

$2x^2-4x=36-\dfrac{270}{3x^2-6x}$ → take out a HCF to obtain the repeated expression

$\therefore 2\left(x^2-2x\right)=36-\dfrac{270}{3\left(x^2-2x\right)}$ → repeated expression, use the k-method

Let $k=x^2-2x$

$\therefore 2k=36-\dfrac{270}{3k}$

LCD $=3k$, $k\neq 0, \therefore x^2-2x\neq 0$ $\therefore x\neq 0$ and $x\neq 2$ → you have to put in restrictions

$\therefore 6k^2=108k-270$

$\therefore 6k^2-108k+270=0$ → take out a HCF of 6

$\therefore 6\left(k^2-18k+45\right)=0$ → divide both sides by 6 and factorise

$\therefore (k-15)(k-3)=0$

$\therefore k=15$ or $k=3$

Now substitute $k=x^2-2x$ back into your solutions for k

$\therefore x^2-2x=15$ or $x^2-2x=3$

$\therefore x^2-2x-15=0$ or $x^2-2x-3=0$

$\therefore (x-5)(x+3)=0$ or $(x-3)(x+1)=0$

$\therefore x=5$ or $x=-3$ or $x=3$ or $x=-1$



Complete Exercise 4 on page 259.

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SURD EQUATIONS WHICH LEAD TO QUADRATIC EQUATIONS
(SOLVING EQUATIONS BY SQUARING BOTH SIDES)

In Chapter 1 you dealt with simple linear surd equations (pg. 10). You will now deal with surd equations which lead to a quadratic equation. It is important to note that in this case you may introduce an answer which is not a solution to the original surd equation.

For example consider the case if you solve for x in the equation $\sqrt{x+2}=x$:

$\sqrt{x+2}=x \rightarrow$ square both sides (line 1: original surd equation)

$\therefore x+2=x^2$ (line 2: new quadratic equation)

$\therefore x^2-x-2=0$

$\therefore (x-2)(x+1)=0$

$\therefore x=2$ or $x=-1 \rightarrow$ N/A

Two solutions have been obtained for x. Both solutions are valid for line 2 of the above equation. However, if the solutions are substituted back into the original equation (line 1) it can be seen that $x=-1$ is not a solution as it will cause the square root of a number to be negative which is not possible.

For this reason it is essential that you always check your answers by substituting your solutions back into the original equation when dealing with surd equations.

STEPS TO SOLVING SURD EQUATIONS WHICH LEAD TO QUADRATIC EQUATIONS

1. **Isolate the surd.**
2. **Square both sides.**
3. **Solve for x as you would in a normal equation.**
4. **Check your answers**

Example 1: Solve for x if $\sqrt{2x+8}-x=0$.

$\sqrt{2x+8}-x=0 \rightarrow$ isolate the surd

$\therefore \sqrt{2x+8}=x \rightarrow$ square both sides

$\therefore 2x+8=x^2$

$\therefore x^2-2x-8=0$

$\therefore (x-4)(x+2)=0$

$\therefore x=4$ or $x=-2 \rightarrow$ N/A see check below

$\therefore x=4$ only

Check

$x=4$

$\text{LHS}=\sqrt{2(4)+8}-4$

$=0$

$=\text{RHS}$

$\therefore x=4$ is a solution

$x=-2$

$\text{LHS}=\sqrt{2(-2)+8}-(-2)$

$=4$

$\neq \text{RHS}$

$\therefore x \neq -2$



It is not necessary to show your check, only which answer is not applicable.

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Example 2: Solve for x if $3\sqrt{x+4}-x=0$.

$3\sqrt{x+4}-x=0 \rightarrow$ isolate the surd

$\therefore 3\sqrt{x+4}=x \rightarrow$ square both sides, don't forget to square the 3 as well

$\therefore 9(x+4)=x^2$

$\therefore 9x+36=x^2$

$\therefore x^2-9x-36=0$

$\therefore (x-12)(x+3)=0$

$\therefore x=12$ or $x=-3 \rightarrow$ N/A see check below

$\therefore x=12$ only

Check

$x=12$

$\text{LHS}=3\sqrt{(12)+4}-12$

$=0$

$=\text{RHS}$

$\therefore x=12$ is a solution

$x=-3$

$\text{LHS}=3\sqrt{(-3+4)}-(-3)$

$=6$

$\neq \text{RHS}$

$\therefore x \neq -3$

Example 3: Solve for x if $2\sqrt{x-1}+6=2x$.

$2\sqrt{x-1}+6=2x \rightarrow$ isolate the surd

$\therefore 2\sqrt{x-1}=2x-6 \rightarrow$ square both sides, don't forget to square the 2 as well

$\therefore 4(x-1)=(2x-6)^2$

$\therefore 4x-4=4x^2-24x+36 \rightarrow$ don't forget the middle term

$\therefore 4x^2-28x+40=0 \rightarrow$ take out a HCF of 4 and factorise

$\therefore 4\left(x^2-7x+10\right)=0 \rightarrow$ divide both sides by 4 and factorise

$\therefore (x-5)(x-2)=0$

$\therefore x=5$ or $x=2 \rightarrow$ N/A see check below

$\therefore x=5$ only



Recap:

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

Check

$x=5$

$\text{LHS}=2\sqrt{(5)-1}+6$

$=10$

$\text{RHS}=2(5)$

$=10$

$=\text{LHS}$

$\therefore x=5$ is a solution

$x=2$

$\text{LHS}=2\sqrt{(2)-1}+6$

$=8$

$\text{RHS}=2(2)$

$=4$

$\neq \text{LHS}$

$\therefore x \neq 2$

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Example 4: Solve for x if $2\sqrt{x+1}+1=\dfrac{20}{2\sqrt{x+1}}$.

$2\sqrt{x+1}+1=\dfrac{20}{2\sqrt{x+1}}$ → repeated expression, use k-method

Let $k=2\sqrt{x+1}$

$\therefore k+1=\dfrac{20}{k}$

LCD $=k$, $k\neq 0$ → you have to put in restrictions

$\therefore k^2+k=20$

$\therefore k^2+k-20=0$

$\therefore (k-4)(k+5)=0$

$\therefore k=4$ or $k=-5$

Now substitute $k=2\sqrt{x+1}$ back into your solutions for k

$\therefore 2\sqrt{x+1}=4$ or $2\sqrt{x+1}=-5$ → N/A a square root cannot be a negative number

$\therefore 4(x+1)=16$

$\therefore 4x+4=16$

$\therefore 4x=12$

$\therefore x=3$

Check

$x=3$

$\text{LHS}=2\sqrt{(3)+1}+1$

$=5$

$\text{RHS}=\dfrac{20}{2\sqrt{(3)+1}}$

$=5$

$=\text{LHS}$

$\therefore x=3$ is a solution



Complete Exercise 5 on page 259.

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EXPONENTIAL EQUATIONS

An exponential equation is an equation in which the variable you are solving for is an exponent. When solving exponential equations we use the property that:

$$\text{if } a^x = a^y, \text{ then } x = y$$

It is also possible for exponential equations to lead to quadratic equations.

Example 1: Solve for x if $27^x = \frac{1}{9}$.

$27^x = \frac{1}{9}$

$\therefore \left(3^3\right)^x = 3^{-2}$ → ensure that you have the same base on both sides of the equation

$\therefore 3^{3x} = 3^{-2}$

$\therefore 3x = -2$

$\therefore x = -\frac{2}{3}$

Example 2: Solve for x if $125^x.25^{x-2} = 1$.

$125^x.25^{x-2} = 1$

$\therefore \left(5^3\right)^x \left(5^2\right)^{x-2} = 5^0$ → write numbers as their prime bases, $5^0 = 1$ (law/definition 1)

$\therefore 5^{3x}.5^{2x-4} = 5^0$ → don't forget to distribute the 2

$\therefore 5^{3x+2x-4} = 5^0$

$\therefore 3x + 2x - 4 = 0$

$\therefore 5x = 4$

$\therefore x = \frac{4}{5}$

Example 3: Solve for x if $3^{x+2} + 3^{x+3} - 3^x = 315$.

$3^{x+2} + 3^{x+3} - 3^x = 315$ → split bases apart using law 3 in reverse

$\therefore 3^x.3^2 + 3^x.3^3 - 3^x = 315$ → take out a HCF of 3^x

$\therefore 3^x\left(3^2 + 3^3 - 1\right) = 315$

$\therefore 3^x\,(35) = 315$ → divide both sides by 35

$\therefore 3^x = 9$

$\therefore 3^x = 3^2$

$\therefore x = 2$

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Example 4: Solve for x if $2^{2x} - 2^x - 2 = 0$.

$2^{2x} - 2^x - 2 = 0 \rightarrow 2^{2x} = \left(2^x\right)^2$, use the k-method

Let $k = 2^x$

$\therefore k^2 - k - 2 = 0 \rightarrow$ factorise

$\therefore (k-2)(k+1) = 0$

$\therefore k = 2$ or $k = -1$

Now substitute $k = 2^x$ back into your solutions for k

$\therefore 2^x = 2^1$ or $2^x = -1 \rightarrow$ N/A an exponent cannot cause a number to be negative

$\therefore x = 1$

Example 5: Solve for x if $3^{2x} - 10.3^x + 9 = 0$.

$3^{2x} - 10.3^x + 9 = 0 \rightarrow 3^{2x} = \left(3^x\right)^2$, use the k-method

Let $k = 3^x$

$\therefore k^2 - 10k + 9 = 0 \rightarrow$ factorise

$\therefore (k-9)(k-1) = 0$

$\therefore k = 9$ or $k = 1$

Now substitute $k = 3^x$ back into your solutions for k

$\therefore 3^x = 9$ or $3^x = 1 \rightarrow$ get the same bases on both sides of each equation

$\therefore 3^x = 3^2$ or $3^x = 3^0 \rightarrow$ any number raised to the power of 0 is equal to 1

$\therefore x = 2$ or $x = 0$

Example 6: Solve for x if $2^{2x+1} - 9.2^x + 4 = 0$.

$2^{2x+1} - 9.2^x + 4 = 0 \rightarrow$ split base apart using law 3 in reverse

$\therefore 2^{2x}.2^1 - 9.2^x + 4 = 0 \rightarrow 2^{2x} = \left(2^x\right)^2$, use the k-method

Let $k = 2^x$

$\therefore 2k^2 - 9k + 4 = 0 \rightarrow$ factorise

$\therefore (2k-1)(k-4) = 0$

$\therefore k = \frac{1}{2}$ or $k = 4$

Now substitute $k = 2^x$ back into your solutions for k

$\therefore 2^x = \frac{1}{2}$ or $2^x = 4 \rightarrow$ get the same bases on both sides of each equation

$\therefore 2^x = 2^{-1}$ or $2^x = 2^2$

$\therefore x = -1$ or $x = 2$



Complete Exercise 6 on page 259.

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C. SIMULTANEOUS EQUATIONS

In previous grades you have dealt with linear simultaneous equations.
In Grade 11 you will deal with simultaneous equations where one is linear and the other is quadratic. The method of substitution is used to solve these types of problems.

Example 1: Solve for x and y if $x+2y=1$ and $x^2+4y^2=41$.

$x+2y=1 \rightarrow [1]$

$x^2+4y^2=41 \rightarrow [2]$

from [1]:

$x=1-2y \rightarrow [3]$

Substitute [3] into [2]

$\therefore (1-2y)^2+4y^2=41$

$\therefore 1-4y+4y^2+4y^2-41=0$ → don't forget the middle term

$\therefore 8y^2-4y-40=0$ → take out a HCF of 4

$\therefore 4\left(2y^2-y-10\right)=0$ → divide both sides by 4

$\therefore 2y^2-y-10=0$ → factorise

$\therefore (y+2)(2y-5)=0$

$\therefore y=-2 \quad \text{or} \quad y=\frac{5}{2} \rightarrow [4]$

Substitute [4] into [3]

$\therefore x=1-2(-2) \qquad \text{or} \qquad x=1-2\left(\frac{5}{2}\right)$

$\therefore x=5 \qquad \text{or} \qquad x=-4$

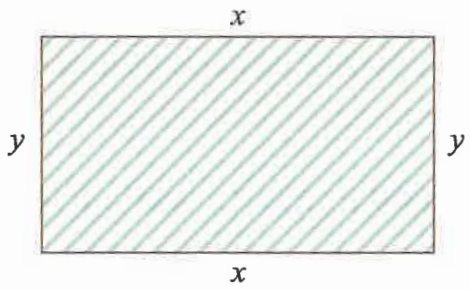
$\therefore x=5 \text{ and } y=-2 \quad \text{or} \quad x=-4 \text{ and } y=\frac{5}{2}$



The solution to simultaneous equations represents the points of intersection of two graphs.

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Example 2: **A rectangular garden has a length of x m and a breadth of y m. It has an area of 20 m^2 and a perimeter of 18 m. Determine the values of x and y.**



Perimeter of a rectangle = 2×Length + 2×breadth

Area of a rectangle = length×breadth

$2x+2y=18 \rightarrow [1]$ (perimeter)

$xy=20 \rightarrow [2]$ (area)

from [1]:

$2x=18-2y \rightarrow$ divide both sides by 2

$\therefore x=9-y \rightarrow [3]$

Substitute [3] into [2]

$\therefore (9-y)y=20$

$\therefore 9y-y^2=20$

$\therefore y^2-9y+20=0$

$\therefore (y-5)(y-4)=0$

$\therefore y=5 \text{ or } y=4 \rightarrow [4]$

Substitute [4] into [3]

$\therefore x=9-5 \text{ or } x=9-4$

$\therefore x=4 \text{ or } x=5$

$\therefore x=4\text{m} \text{ and } y=5\text{m} \text{ or } x=5\text{m} \text{ and } y=4\text{m}$



Complete Exercise 7 on page 259.

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D. COMPLETING THE SQUARE

There are two cases in which you will be expected to complete the square:
1. **You will have to complete the square on a quadratic expression.**
2. **You will have to complete the square on a quadratic equation.**

1. COMPLETING THE SQUARE ON A QUADRATIC EXPRESSION

Completing the square on a quadratic expression is a technique used in order to express a quadratic expression in the form of $a(x \pm p)^2 + q$.

Steps to completing the square on a quadratic expression:

1. Take out the coefficient of x^2 if it is not equal to 1.
2. Add and immediately subtract (half the coefficient of x)^2.
3. Factorise the perfect square trinomial and distribute the coefficient of x^2.

Example 1: Complete the square on $x^2 - 6x + 7$.

$x^2 - 6x + 7 \rightarrow$ add and subtract $\left(\frac{1}{2} \times -6\right)^2 = (-3)^2$

$= \left[x^2 - 6x + (-3)^2\right] - (-3)^2 + 7 \rightarrow$ this is a perfect square trinomial

$= (x-3)^2 - 9 + 7 \rightarrow$ this is always half the coefficient of x

$= (x-3)^2 - 2$

Example 2: Complete the square on $x^2 + 4x - 8$.

$x^2 + 4x - 8 \rightarrow$ add and subtract $\left(\frac{1}{2} \times 4\right)^2 = (2)^2$

$= \left[x^2 + 4x + (2)^2\right] - (2)^2 - 8 \rightarrow$ this is a perfect square trinomial

$= (x+2)^2 - 4 - 8 \rightarrow$ this is always half the coefficient of x

$= (x+2)^2 - 12$

Example 3: Complete the square on $x^2 - 5x - 8$.

$x^2 - 5x - 8 \rightarrow$ add and subtract $\left(\frac{1}{2} \times -5\right)^2 = \left(-\frac{5}{2}\right)^2$

$= \left[x^2 - 5x + \left(-\frac{5}{2}\right)^2\right] - \left(-\frac{5}{2}\right)^2 - 8 \rightarrow$ this is a perfect square trinomial

$= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} - 8 \rightarrow$ this is always half the coefficient of x

$= \left(x - \frac{5}{2}\right)^2 - \frac{57}{4}$

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Example 4: Complete the square on $2x^2+6x-6$.

$2x^2+6x-6 \rightarrow$ take out the coefficient of x^2

$2\left[x^2+3x-3\right] \rightarrow$ add and subtract $\left(\frac{1}{2}\times 3\right)^2=\left(\frac{3}{2}\right)^2$

$=2\left(\left[x^2+3x+\left(\frac{3}{2}\right)^2\right]-\left(\frac{3}{2}\right)^2-3\right) \rightarrow$ this is a perfect square trinomial

$=2\left[\left(x+\frac{3}{2}\right)^2-\frac{9}{4}-3\right] \rightarrow$ this is always half the coefficient of x

$=2\left[\left(x+\frac{3}{2}\right)^2-\frac{21}{4}\right] \rightarrow$ distribute 2 $\left(\text{the coefficient of } x^2\right)$ back in

$=2\left(x+\frac{3}{2}\right)^2-\frac{21}{2}$

Example 5: Complete the square on $3x^2-5x-9$.

$3x^2-5x-9 \rightarrow$ take out the coefficient of x^2

$3\left[x^2-\frac{5}{3}x-3\right] \rightarrow$ add and subtract $\left(\frac{1}{2}\times-\frac{5}{3}\right)^2=\left(-\frac{5}{6}\right)^2$

$=3\left(\left[x^2-\frac{5}{3}x+\left(-\frac{5}{6}\right)^2\right]-\left(-\frac{5}{6}\right)^2-3\right) \rightarrow$ this is a perfect square trinomial

$=3\left[\left(x-\frac{5}{6}\right)^2-\frac{25}{36}-3\right] \rightarrow$ this is always half the coefficient of x

$=3\left[\left(x-\frac{5}{6}\right)^2-\frac{133}{36}\right] \rightarrow$ distribute 3 $\left(\text{the coefficient of } x^2\right)$ back in

$=3\left(x-\frac{5}{6}\right)^2-\frac{133}{12}$



Complete Exercise 8 number 1 on page 260.

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2. COMPLETING THE SQUARE ON A QUADRATIC EQUATION

Up until now you have only dealt with quadratic equations which have rational roots (solutions). However, it is common for quadratic equations to also have irrational roots. It is difficult to solve these types of equations by factorisation. A method of solving an equation which does not factorise is to complete the square on the equation.

Steps to completing the square on a quadratic expression:

1. Move the constant to the right hand side (RHS).
2. Divide both sides by the coefficient of x^2 if it is not equal to 1.
3. Add (half the coefficient of x)^2 to both sides of the equation.
4. Solve for x.

Example 1: Solve for x if $x^2+12x-9=0$. Write answers in simplest surd form.

$x^2+12x-9=0 \rightarrow$ move the constant to the RHS

$\therefore x^2+12x=9 \rightarrow$ add $\left(\frac{1}{2}\times 12\right)^2=(6)^2$ to both sides

$\therefore \left[x^2+12x+(6)^2\right]=9+(6)^2 \rightarrow$ this is a perfect square trinomial

$\therefore (x+6)^2=45 \rightarrow$ this is always half the coefficient of x, square root both sides

$\therefore x+6=\pm\sqrt{45} \rightarrow$ don't forget the '$\pm$'

$\therefore x=-6\pm 3\sqrt{5} \rightarrow \sqrt{45}=\sqrt{9\times 5}=3\sqrt{5}$

$\therefore x=-6+3\sqrt{5}$ or $x=-6-3\sqrt{5}$

Example 2: Solve for x correct to one decimal place if $3x^2-15x-21=0$.

$3x^2-15x-21=0 \rightarrow$ move the constant to the RHS

$\therefore 3x^2-15x=21 \rightarrow$ divide both sides by 3 (the coefficient of x^2)

$\therefore x^2-5x=7 \rightarrow$ add $\left(\frac{1}{2}\times -5\right)^2=\left(-\frac{5}{2}\right)^2$ to both sides

$\therefore \left[x^2-5x+\left(-\frac{5}{2}\right)^2\right]=7+\left(-\frac{5}{2}\right)^2 \rightarrow$ this is a perfect square trinomial

$\therefore \left(x-\frac{5}{2}\right)^2=\frac{53}{4} \rightarrow$ this is always half the coefficient of x, square root both sides

$\therefore x-\frac{5}{2}=\pm\frac{\sqrt{53}}{2} \rightarrow$ don't forget the '$\pm$'

$\therefore x=\frac{5\pm\sqrt{53}}{2}$

$\therefore x=6{,}1$ or $x=-1{,}1 \rightarrow$ correct to one decimal place

Leave your answer in simplest surd form unless the question asks you to determine a solution to a certain number of decimal places.

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Example 3: Solve for x if $4x^2+12x-8=0$.

$4x^2+12x-8=0$ → move the constant to the RHS

$\therefore 4x^2+12x=8$ → divide both sides by 4 (the coefficient of x^2)

$\therefore x^2+3x=2$ → add $\left(\frac{1}{2}\times 3\right)^2=\left(\frac{3}{2}\right)^2$ to both sides

$\therefore \left[x^2+3x+\left(\frac{3}{2}\right)^2\right]=2+\left(\frac{3}{2}\right)^2$ → this is a perfect square trinomial

$\therefore \left(x+\frac{3}{2}\right)^2=\frac{17}{4}$ → this is always half the coefficient of x, square root both sides

$\therefore x+\frac{3}{2}=\pm\frac{\sqrt{17}}{2}$ → don't forget the '$\pm$'

$\therefore x=\frac{-3\pm\sqrt{17}}{2}$

$\therefore x=\frac{-3+\sqrt{17}}{2}$ or $x=\frac{-3-\sqrt{17}}{2}$

Example 4: Solve for x if $-2x^2+8x+12=0$.

$-2x^2+8x+12=0$ → move the constant to the RHS

$\therefore -2x^2+8x=-12$ → divide both sides by -2 (the coefficient of x^2)

$\therefore x^2-4x=6$ → add $\left(\frac{1}{2}\times -4\right)^2=(-2)^2$ to both sides

$\therefore \left[x^2-4x+(-2)^2\right]=6+(-2)^2$ → this is a perfect square trinomial

$\therefore (x-2)^2=10$ → this is always half the coefficient of x, square root both sides

$\therefore x-2=\pm\sqrt{10}$ → don't forget the '$\pm$'

$\therefore x=2+\sqrt{10}$ or $x=2-\sqrt{10}$

Example 5: Solve for x if $x^2-8x+17=0$.

$x^2-8x+17=0$ → move the constant to the RHS

$\therefore x^2-8x=-17$ → add $\left(\frac{1}{2}\times -8\right)^2=(-4)^2$ to both sides

$\therefore \left[x^2-8x+(-4)^2\right]=-17+(-4)^2$ → this is a perfect square trinomial

$\therefore (x-4)^2=-1$ → this is always half the coefficient of x, square root both sides

$\therefore x-4=\pm\sqrt{-1}$ → you cannot square '$-$' in the real number system

$\therefore$ No real solution



Complete Exercise 8 number 2 on page 260.

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APPLICATIONS OF COMPLETING THE SQUARE

Example 1: Determine the minimum value of $x^2-8x+11$.

$x^2-8x+11 \rightarrow$ add and subtract $\left(\frac{1}{2}\times -8\right)^2=(-4)^2$

$=\left[x^2-8x+(-4)^2\right]-(-4)^2+11 \rightarrow$ this is a perfect square trinomial

$=(x-4)^2-5$

$(x-4)^2$ is always greater than or equal to 0.

$\therefore$ The smallest value that $(x-4)^2$ can ever be is 0

$\therefore$ The minimum value of the expression is $0-5$

$\therefore$ The minimum value of $x^2-8x+11$ is -5.

Example 2: Determine the maximum value of $-3x^2-9x+18$.

$-3x^2-9x+18 \rightarrow$ take out the coefficient of x^2

$=-3\left(x^2+3x-6\right) \rightarrow$ add and subtract $\left(\frac{1}{2}\times 3\right)^2=\left(\frac{3}{2}\right)^2$

$=-3\left(\left[x^2+3x+\left(\frac{3}{2}\right)^2\right]-\left(\frac{3}{2}\right)^2-6\right) \rightarrow$ this is a perfect square trinomial

$=-3\left[\left(x+\frac{3}{2}\right)^2-\frac{33}{4}\right]$

$=-3\left(x+\frac{3}{2}\right)^2+\frac{99}{4}$

$\left(x-\frac{3}{2}\right)^2$ is always greater than or equal to 0.

$\therefore -3\left(x-\frac{3}{2}\right)^2$ is always less than or equal to 0

$\therefore$ The greatest value that $-3\left(x-\frac{3}{2}\right)^2$ can ever be is 0

$\therefore$ The maximum value of the expression is $0+\frac{99}{4}$

$\therefore$ The maximum value of $-3x^2-9x+18$ is $\frac{99}{4}$.

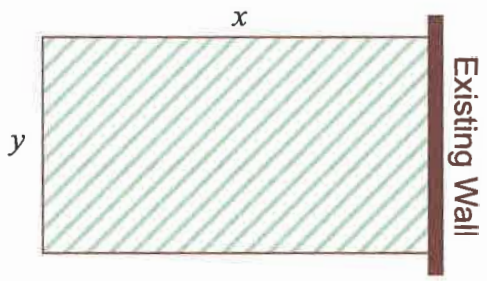
In general, if an expression is in the form $a(x\pm p)^2+q$, the expression will:

- **have a minimum value of q if a is greater than 0 (positive)**
- **have a maximum value of q if a is less than 0 (negative)**

This will be explored in more detail when we cover the parabola later on in the year.

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Example 3: **A farmer wishes to make a rectangular garden against an existing wall. If he has 12 metres of fencing with which to make a garden:**



a) Determine an expression for the area of the garden in terms of x.
b) What is the maximum area that the garden can have?

a) $2x+y=12 \rightarrow$ perimeter is 12 m, an existing wall is used for one side

$\therefore y=12-2x \rightarrow [1]$

$A=xy \rightarrow [2]$

Substitute [1] into [2]

$\therefore A=x(12-2x)$

$\therefore A=12x-2x^2$

b) In order to determine the maximum area, complete the square on the expression for area:

$A=12x-2x^2 \rightarrow$ take out the coefficient of x^2

$=-2\left(x^2-6x\right) \rightarrow$ add and subtract $\left(\frac{1}{2}\times -6\right)^2=(-3)^2$

$=-2\left(\left[x^2-6x+(-3)^2\right]-(-3)^2\right) \rightarrow$ this is a perfect square trinomial

$=-2\left[(x-3)^2-9\right]$

$=-2(x-3)^2+18 \rightarrow a<0$ and $q>0$ $(a=-2$ and $q=18)$

$\therefore$ The maximum area the garden can have is $18\,m^2$.



Complete Exercise 9 on page 260.

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E. THE QUADRATIC FORMULA

The general form of a quadratic equation is $ax^2+bx+c=0$. By completing the square on the general form of a quadratic equation an equation to solve for the roots of any quadratic equation can be obtained. This is known as the quadratic formula and is shown below.

The Quadratic Formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

DERIVATION OF THE QUADRATIC FORMULA

Complete the square on $ax^2+bx+c=0$.

$ax^2+bx+c=0 \rightarrow$ move the constant to the RHS

$\therefore ax^2+bx=-c \rightarrow$ divide both sides by a (the coefficient of x^2)

$\therefore x^2+\frac{b}{a}x=-\frac{c}{a} \rightarrow$ add $\left(\frac{1}{2}\times\frac{b}{a}\right)^2=\left(\frac{b}{2a}\right)^2$ to both sides

$\therefore \left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2 \rightarrow$ this is a perfect square trinomial

$\therefore \left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2} \rightarrow$ LCD of $4a^2$

$\therefore \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \rightarrow$ square root both sides

$\therefore x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a} \rightarrow$ don't forget the '$\pm$'

$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

Example: **Solve for x if $2x^2-5x-8=0$. Leave answers in simplest surd form.**

$2x^2-5x-8=0 \rightarrow$ use the quadratic formula as the expression does not factorise

$\therefore x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-8)}}{2(2)} \rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=\frac{5\pm\sqrt{89}}{4}$

$\therefore x=\frac{5+\sqrt{89}}{4}$ or $x=\frac{5-\sqrt{89}}{4}$



Complete Exercise 10 number 1 on page 260.

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NATURE OF ROOTS

In Grade 11 you will be expected to be able to interpret and determine the nature of roots of a quadratic equation.

Example 1: Given that $(x^2-5)(2x^2+3x-2)=0$ solve for x if:

a) $x \in \mathbb{Z}$ **b)** $x \in \mathbb{R}$
c) $x \in \mathbb{Q}$ **d)** $x \in \mathbb{Q}'$

Before starting to answer the questions, it is recommended that you solve for all of the possible solutions to the equation:

$(x^2-5)(2x^2+3x-2)=0 \rightarrow$ difference of squares, factorise normally

$\therefore (x-\sqrt{5})(x+\sqrt{5})(2x-1)(x+2)=0$

$\therefore x=\sqrt{5}$ or $x=-\sqrt{5}$ or $x=\frac{1}{2}$ or $x=-2$

These are all the solutions to x. You now have to select the solutions which fit the criteria of each question.

a) $x=-2 \rightarrow -2$ is the only integer

b) $x=\sqrt{5}$ or $x=-\sqrt{5}$ or $x=\frac{1}{2}$ or $x=-2 \rightarrow$ all of the above solutions are real

c) $x=\frac{1}{2}$ or $x=-2 \rightarrow$ both solutions are rational

d) $x=\sqrt{5}$ or $x=-\sqrt{5}$

Example 2: Show that the roots of $x^2-2x-7=0$ are irrational.

$x^2-2x-7=0 \rightarrow$ cannot factorise, use the quadratic formula

$\therefore x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-7)}}{2(1)} \rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=\frac{2\pm\sqrt{32}}{2} \rightarrow \sqrt{32}=\sqrt{16\times 2}=4\sqrt{2}$

$\therefore x=\frac{2\pm 4\sqrt{2}}{2} \rightarrow$ take out a HCF of 2

$\therefore x=\frac{2(1\pm 2\sqrt{2})}{2}$

$\therefore x=1\pm 2\sqrt{2} \rightarrow$ the solution contains a surd

$\therefore$ The roots are irrational



Complete Exercise 10 number 2 and 3 on page 260.

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THE DISCRIMINANT

In the previous section it was established that the roots of any quadratic equation can be found using the quadratic formula below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By examining the quadratic formula it can be seen that the nature of roots relies on the value of $b^2 - 4ac$ as this expression has to be square rooted.
It follows that if:

- $b^2 - 4ac > 0$ **the roots of the equation will be real.**
- $b^2 - 4ac < 0$ **the roots of the equation will be non-real.**
- $b^2 - 4ac = 0$ **there will be two roots which are equal (the same).**

The expression $b^2 - 4ac$ is known as the discriminant and is denoted by the Greek letter Δ (delta). This allows us to determine the nature of roots of an equation without actually solving for the roots of an equation.

Value of discriminant	$\Delta > 0$ and is a perfect square	$\Delta > 0$ and is not a perfect square	$\Delta = 0$	$\Delta < 0$
Nature of Roots	**Two real, unequal, rational roots**	**Two real, unequal, irrational roots**	**Two real, equal, rational roots**	**Roots are non-real**

Example: **Determine the nature roots of the following equations without solving the equations.**

a) $x^2 + 8x - 3 = 0$

b) $4x^2 + 12x + 9 = 0$

c) $-2x^2 - 4x + 6 = 0$

d) $4x^2 + 3x + 2 = 0$

a) $\Delta = b^2 - 4ac$
$\therefore \Delta = 8^2 - 4(1)(-3)$
$\therefore \Delta = 76$
$\therefore$ Roots are real, unequal and irrational.

b) $\Delta = b^2 - 4ac$
$\therefore \Delta = (12)^2 - 4(4)(9)$
$\therefore \Delta = 0$
$\therefore$ Roots are real, equal and rational.

c) $\Delta = b^2 - 4ac$
$\therefore \Delta = (-4)^2 - 4(-2)(6)$
$\therefore \Delta = 64 \rightarrow 64$ is a perfect square
$\therefore$ Roots are real, unequal and rational.

d) $\Delta = b^2 - 4ac$
$\therefore \Delta = (3)^2 - 4(4)(2)$
$\therefore \Delta = -23 \rightarrow \Delta < 0$
$\therefore$ Roots are non-real.



Complete Exercise 10 number 4 on page 260.

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DETERMINING A QUADRATIC EQUATION FROM THE ROOTS OF AN EQUATION

If a quadratic expression is in the form of ax^2+bx+c the roots of the expression are the values for x that would make the expression equal to 0.

If an equation has roots of $\frac{a}{b}$ and $\frac{p}{q}$ the quadratic equation will always be $(bx-a)(qx-p)=0$ in its simplest form.

Example 1: Determine the values of b and c if the equation $2x^2+bx+c=0$ has roots of –2 and 3. Assume that b and c are integers.

$(x+2)(x-3)=0$

$\therefore x^2-x-6=0$

This would be the equation in its simplest form. As the coefficient of x^2 is 2 in the question, both sides of the equation have to be multiplied by 2.

$\therefore 2x^2-2x-12=0$

$\therefore b=-2$ and $c=-12$

Example 2: Determine the values of a and c if the equation $ax^2+27x+c=0$ has roots of $-\frac{3}{2}$ and $\frac{3}{5}$. Assume that a and c are integers.

$(2x+3)(5x-3)=0$

$\therefore 10x^2+9x-9=0$

This would be the equation in its simplest form. As the coefficient of x is 27 in the question, both sides of the equation have to be multiplied by 3.

$\therefore 30x^2+27x-27=0$

$\therefore a=30$ and $c=-27$

Example 3: Determine the value of p if 3 is a root of the equation $x^2+px-3=0$. In addition, determine the other root.

If 3 is a root of the equation, substituting $x=3$ into the equation will make the equation equal to 0.

$(3)^2+p(3)-3=0$

$\therefore 9+3p-3=0$

$\therefore 3p=-6$

$\therefore p=-2$

$\therefore$ The equation is $x^2-2x-3=0$

$\therefore (x-3)(x+1)=0$

$\therefore x=3$ or $x=-1 \rightarrow$ one of the roots must be 3 or you have made a mistake

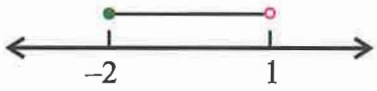
$\therefore$ The other root is -1.

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F. INEQUALITIES

REVISION OF NUMBER LINES AND NOTATION

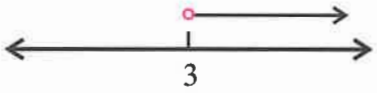
Example 1: Determine the solution to x shown on the number line below:



Interval notation : $x \in [-2;1) \rightarrow x$ is always $\in \mathbb{R}$ in interval notaion

Set - builder notation : $\{x : -2 \leq x < 1\}$

Example 2: Determine the solution to x shown on the number line below:



Interval notation : $x \in (3;\infty)$

Set - builder notation : $\{x : x > 3\}$

Example 3: Determine the solution to x shown on the number line below:



Interval notation : $x \in (-\infty;-4]$ or $x \in (3;\infty)$

Set - builder notation : $\{x : x \leq -4\}$ or $\{x : x > 3\}$



Recap:

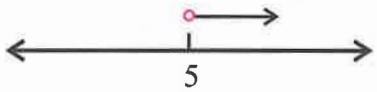
On a number line, an open circle means that the number is not included whilst a filled circle means that it is.

REVISION OF LINEAR INEQUALITIES

Example 1: Solve for x if $x - 5 > 0$ and illustrate your solution on a number line.

$x - 5 > 0$

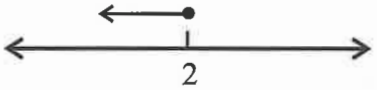
$\therefore x > 5$



Example 2: Solve for x if $x - 2 \leq 0$ and illustrate your solution on a number line.

$x - 2 \leq 0$

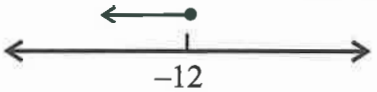
$\therefore x \leq 2$



Example 3: Solve for x if $-x - 12 \geq 0$ and illustrate your solution on a number line.

$-x - 12 \geq 0$

$\therefore -x \geq 12$



$\therefore x \leq -12 \rightarrow$ change the direction of the inequality (see recap below)



Recap:

When multiplying or dividing both sides by a negative number, the inequality sign has to change direction.

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QUADRATIC INEQUALITIES

When dealing with quadratic inequalities you have to solve for the critical values of the inequality. Critical values are values of the variable that would make the inequality equal to 0 or the denominator of an inequality equal to 0.

In most cases the solution to inequalities will lie between, or on either side of the critical values. This is illustrated in the examples below.

Example 1: Solve for x if $(x+2)(x-5)>0$

$(x+2)(x-5)>0$

$\therefore$ Critical values: $x=-2$ or $x=5$



$\therefore x<-2$ or $x>5$

This can also be written as $x\in(-\infty;-2)$ or $(5;\infty)$

Number line method and explanation:

- Place the critical values on a number line.
- Substitute a value less than -2 into the factorised expression, for example $x=-3$:
  $(-3+2)(-3-5)=+8$
  This means that if x is less than -2 the expression will be positive.
- Substitute a value between -2 and 5 into the factorised expression, for example $x=0$:
  $(0+2)(0-5)=-10$
  This means that if x is between -2 and 5 the expression will be negative.
- Substitute a value greater than 5 into the factorised expression, for example $x=6$:
  $(6+2)(6-5)=+8$
  This means that if x is greater than 5 the expression will be positive.
- You can then label which areas on the number line are positive or negative. In general, **unless there are repeated factors**, the number line will follow the trend of '+' '–' '+' or '–' '+' '–'.

It is important to understand inequality notation:

- **$x>0$ reads x is greater than 0 ($x\geq 0$ reads x is greater than or equal to 0)**
- **$x<0$ reads x is less than 0 ($x\leq 0$ reads x is less than or equal to 0)**

If an expression has to be greater than 0 you are looking for the areas which are positive on the number line.

If an expression has to be less than 0 you are looking for the areas which are negative on the number line.

If an expression is greater (or less) than or equal to 0, then the critical values are also solutions to the inequality and are represented by closed dots on top of the critical values on the number line.

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Example 2: Solve for x if $x^2 - x \leq 12$

$x^2 - x \leq 12$

$\therefore x^2 - x - 12 \leq 0$

$\therefore (x-4)(x+3) \leq 0$

$\therefore$ Critical values: $x = 4$ or $x = -3$



$\therefore -3 \leq x \leq 4$

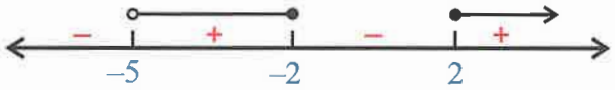
This can also be written as $x \in [-3;4]$

Example 3: Solve for x if $\dfrac{x^2-4}{x+5} \geq 0$

$\dfrac{x^2-4}{x+5} \geq 0 \rightarrow$ do not multiply both sides by $x+5$, difference of squares

$\therefore \dfrac{(x-2)(x+2)}{x+5} \geq 0$

$\therefore$ Critical values: $x = \pm 2$ or $x = -5$



$\therefore -5 < x \leq -2$ or $x \geq 2 \rightarrow x$ cannot be equal to 5 (see the Edison below)

This can also be written as $x \in (-5;-2]$ or $[2;\infty)$

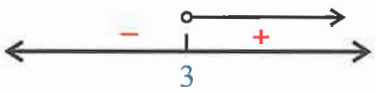


A value for x that makes the denominator of a fraction equal to 0 is also a critical value. However, the denominator can never be equal to zero. Therefore, there will always be an open dot on top of critical values that come from the denominator. Never multiply both sides by a denominator with a variable in it, as the denominator could be negative.

Example 4: Solve for x if $\dfrac{x^2+16}{x-3} \geq 0$

$\dfrac{x^2+16}{x-3} \geq 0 \rightarrow$ as x^2+16 cannot factorise there are no critical values for it

$\therefore$ Critical value: $x = 3$



$\therefore x > 3$

This can also be written as $x \in (3;\infty)$

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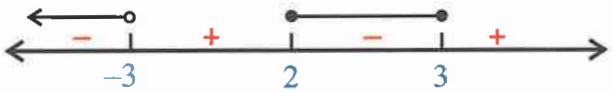
Example 5: Solve for x if $\frac{-x^2+5x-6}{x+3} \geq 0$

$\frac{-x^2+5x-6}{x+3} \geq 0 \rightarrow$ take out a '$-$'

$\therefore \frac{-(x^2-5x+6)}{x+3} \geq 0 \rightarrow$ multiply by -1 and change the direction of the inequality sign

$\therefore \frac{(x-2)(x-3)}{x+3} \leq 0$

$\therefore$ Critical values: $x=-3$ or $x=2$ or $x=3$



$\therefore x<-3$ or $2 \leq x \leq 3$

This can also be written as $x \in (-\infty;-3)$ or $[2;3]$



Recap:

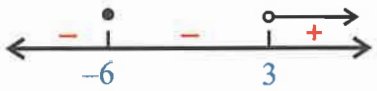
When multiplying or dividing both sides by a negative number, the inequality sign has to change direction.

INEQUALITIES WITH REPEATED FACTORS

Example 1: Solve for x if $\frac{(x+6)^2}{x-3} \geq 0$

$\frac{(x+6)^2}{x-3} \geq 0$

$\therefore$ Critical values: $x=-6$ or $x=3$



$\therefore x>3$ or $x=-6$

This can also be written as $x \in (3;\infty)$ or $x=-6$

Example 2: Solve for x if $\frac{x^2-6x+9}{x+2} \geq 0$

$\frac{x^2-6x+9}{x+2} \geq 0$

$\therefore \frac{(x-3)^2}{x+2} \geq 0$

$\therefore$ Critical values: $x=-2$ or $x=3$



$\therefore x>-2$

This can also be written as $x \in (-2;\infty)$



Complete Exercise 11 number 1 on pages 260.

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UNUSUAL INEQUALITIES

Example 1: Solve for x if $x^2+1\geq 0$

$x^2+1\geq 0$

$\therefore x\in\mathbb{R}$

Any number squared is always greater than or equal to 0. Therefore, adding one to a squared number will always be greater than 0, no matter what the value of x is.

Example 2: Solve for x if $x^2+1<0$

$x^2+1<0$

$\therefore$ No Solution

Example 3: Solve for x if $(x-2)^2\leq 0$

$(x-2)^2\leq 0$

$\therefore x=2$

The square of a number can never be negative. Therefore the only solution to the above inequality is where the inequality is equal to 0.

Example 4: Solve for x if $(x-2)^2>0$

$(x-2)^2>0$

$\therefore x\in\mathbb{R}$ but $x\neq 2\rightarrow$ as the inequality cannot equal 0

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INEQUALITIES WHICH DO NOT FACTORISE

If an inequality does not factorise you can use the quadratic formula (pg. 38) to obtain the critical values.

Example 1: Solve for x if $x^2+4x-11\leq 0$

$x^2+4x-11\leq 0 \rightarrow$ does not factorise, use the quadratic formula

$\therefore$ Critical values: $x=\dfrac{-(4)\pm\sqrt{(4)^2-4(1)(-11)}}{2(1)} \rightarrow x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\therefore x=-2-\sqrt{15}$ or $x=-2+\sqrt{15}$



$\therefore -2-\sqrt{15}\leq x\leq -2+\sqrt{15}$

This can also be written as $x\in\left[-2-\sqrt{15};-2+\sqrt{15}\right]$

If it is not possible to obtain critical values for an inequality you have to complete the square to obtain the maximum or minimum value (pg. 36).

Example 2: Solve for x if $x^2+4x+11<0$

$x^2+4x+11<0$

CV: $x=\dfrac{-4\pm\sqrt{-28}}{8}$ $\rightarrow$ no solution to critical values, therefore complete the square

$\therefore\left(x^2+4x+(2)^2\right)-(2)^2+11<0 \rightarrow \left(\dfrac{1}{2}\times 4\right)^2=(2)^2$ was added and subtracted

$\therefore (x+2)^2+7<0 \rightarrow$ this will always be $\dfrac{1}{2}$ the coefficient of x

as you have seen previously, this inequality has a minimum value of 7 $\rightarrow a>0$ and $q>0$

$\therefore$ the inequality is always greater than or equal to 7.

$\therefore$ There is no solution

Example 3: Solve for x if $x^2-6x+14>0$

$x^2-6x+14>0$

CV: $x=\dfrac{6\pm\sqrt{-20}}{2}$ $\rightarrow$ no solution to critical values, therefore complete the square

$\therefore\left[x^2-6x+(-3)^2\right]-(-3)^2+14>0 \rightarrow \left(\dfrac{1}{2}\times 4\right)^2=(2)^2$ was added and subtracted

$\therefore (x-3)^2+5>0 \rightarrow$ this will always be $\dfrac{1}{2}$ the coefficient of x

As you have seen previously, this inequality has a minimum value of 5 $\rightarrow a>0$ and $q>0$

$\therefore$ the inequality is always greater than or equal to 5.

$\therefore x\in\mathbb{R}$

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MIXED EXAMPLE

Example: **For what values of x are the roots of the expression $\frac{\sqrt{x^2-x-6}}{x+2}$:**

a) Real.
b) Non-real.
c) Undefined.

a) For real roots $x^2-x-6\geq 0$ and $x\neq -2$

$\therefore (x-3)(x+2)\geq 0$

$\therefore$ Critical values: $x=3$ or $x=-2$



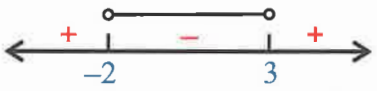
$\therefore x<-2$ or $x\geq 3 \rightarrow x\neq 2$ as it would cause a 0 in the denominator

This can also be written as $x\in(-\infty;-2)$ or $x=[3;\infty)$

b) For non-real roots $x^2-x-6<0$

$\therefore (x-3)(x+2)<0$

$\therefore$ from the above number line:



$-2<x<3$

This can also be written as $x\in(-2;3)$

c) For roots to be undefined division by 0 has to occur.

$\therefore x+2=0$

$\therefore x=-2$

It is important to understand the difference between roots which are non-real and roots which are undefined.

A non-real root occurs when you take an even root of a negative number. In this case the solution is non-real. This means that a graph would not have an x-intercept,

Roots are undefined when division by 0 occurs (a zero in the denominator). In this case the roots are undefined and there is no solution to the inequality or equation. In the case a graph would have a vertical asymptote.

Complete Exercise 11 numbers 2,3 and 4 on page 261.

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G. MATHEMATICAL MODELLING

In problems involving mathematical modelling you will be given a problem in words and be expected to solve it by deriving a mathematical equation. It will often be useful to use tables in solving these types of problems.

STEPS TO SOLVING WORD PROBLEMS

1. **Identify what you are asked to find and call it x.**
2. **Only work in one unit, eg. kg and g or minutes and seconds.**
3. **When you have two expressions in an equation make sure you know which one is larger.**
4. **Make sure you understand what operation the sum is asking you to perform, eg. The sum of is '+' and the product of is 'x'.**
5. **Draw a table or sketch if it is possible.**
6. **Make sure that you state the answer clearly.**

NUMBER AND AGE PROBLEMS (REVISION OF GRADE 10)

Example 1: Sipho is 10 years older than his brother Thabo. In 5 years time Sipho will be twice as old as Thabo. How old is Thabo?

Let Thabo's age $= x$

$\therefore$ Sipho's age $= x+10$

In 5 years time Thabo's age is $x+5$ and Sipho's age is $x+15 \rightarrow (x+10+5)$

$\therefore x+15 = 2(x+5) \rightarrow$ Sipho is twice as old as Thabo in 5 years time

$\therefore x+15 = 2x+10$

$\therefore -x = -5$

$\therefore x = 5$

$\therefore$ Thabo is 5 years old.

Example 2: The product of two consecutive even whole numbers is 168. What are the numbers?

Let the first number $= x$

$\therefore$ The second number $= x+2 \rightarrow$ consecutive even numbers increase in twos

$\therefore x(x+2) = 168 \rightarrow$ the product of the two numbers

$\therefore x^2+2x-168 = 0 \rightarrow$ if you cannot factorise this use the quadratic formula

$\therefore (x-12)(x+14) = 0$

$\therefore x = 12$ or $x = -14 \rightarrow$ N/A as -14 is not a whole number

$\therefore$ The numbers are 12 and 14.

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SHOPPING PROBLEMS

It is useful to draw a table with these type of problems using the triangle below. It is important to note that when drawing the table you have a **variable column** (usually what you are looking for), a **constant column** (given in the question) and a **formula column**. The **formula column** is filled in using the relevant formula and information from the other two columns **not** from information in the question. You then use the **formula column** and any extra information given in the question in order to make an equation.



$$\text{Total Cost} = \text{Price per Item} \times \text{Number of Items}$$

$$\text{Price per Item} = \frac{\text{Total Cost}}{\text{Number of Items}} \qquad \text{Number of Items} = \frac{\text{Total Cost}}{\text{Price per Item}}$$

Example:

A grocery store orders a certain number of avocados at a cost of R600. When they are unpacked it is found that 20 are bruised and cannot be sold. By selling the remaining avocados for R4 more than they were purchased for, a total profit of R200 is made. How many avocados were originally purchased and what was the price of each avocado?

Situation	Number of avocados	Total Price	Price per Avocado
Bought	x	600	$\frac{600}{x}$
Sold	$x-20$	800	$\frac{800}{x-20}$

Now use the extra information (R4 profit per avocado) to create an equation.

Remember that Profit = Selling Price – Cost Price.

$\frac{800}{x-20} - \frac{600}{x} = 4 \rightarrow x \neq 0$ and $x \neq 20$ as this would cause a 0 in the denominator

$\therefore 800x - 600(x-20) = 4x(x-20) \rightarrow$ multiply each term by the LCD: $x(x-20)$

$\therefore 800x - 600x + 12\,000 = 4x^2 - 80x$

$\therefore 4x^2 - 280x - 12\,000 = 0$

$\therefore 4(x^2 - 70x - 3000) = 0$

$\therefore (x-100)(x+30) = 0$

$\therefore x = 100$ or $x = -30 \rightarrow$ N/A, as there cannot be a negative number of avocados

$\therefore$ 100 avocados were bought at R6 each



Complete Exercise 12 on page 261.

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SPEED, DISTANCE AND TIME PROBLEMS

Most speed, distance and time problems are solved using a table in the same way as you have done previously in shopping problems and using the triangle below.



$\text{Distance} = \text{Speed} \times \text{Time}$ $\quad$ $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$ $\quad$ $\text{Time} = \frac{\text{Distance}}{\text{Speed}}$

Example:

A salesman travels 360 km from Johannesburg to Nelspruit. He travels 30 km/h faster on his return journey due to less traffic. By travelling faster he saves 1 hour on the return journey. Determine the speed he returned at.

Situation	Speed (km/h)	Distance (km)	Time (h)
Original Journey	x	360	$\frac{360}{x}$
Return Journey	$x+30$	360	$\frac{360}{x+30}$

Use the extra information (1 hour saved on the return journey) to create an equation.

Remember that Time Saved = Longer Time – Shorter Time.

$\frac{360}{x} - \frac{360}{x+30} = 1 \rightarrow x \neq 0$ and $x \neq -30$ as this would cause a 0 in the denominator

$\therefore 360(x+30) - 360(x) = 1x(x+30) \rightarrow$ multiply each term by the LCD: $x(x+30)$

$\therefore 360x + 10800 - 360x = x^2 + 30x$

$\therefore x^2 + 30x - 10800 = 0 \rightarrow$ if you cannot factorise this use the quadratic formula

$\therefore (x-90)(x+120) = 0$

$\therefore x = 90$ or $x = -120 \rightarrow$ N/A, as you cannot have a negative speed

$\therefore$ He returned at 120 km/h $\rightarrow$ 90+30=120

Complete Exercise 13 on page 261.

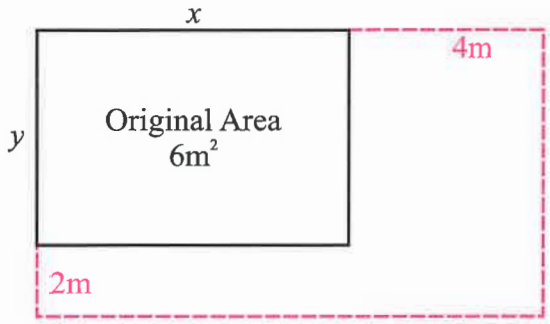
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AREA PROBLEMS

In most area problems you will have to read the problem and assign variables to the different lengths given in the problem according to the type of shape. In the majority of these problems simultaneous equations are required (pg. 30).

Example:

Mr Cassim wishes to enlarge his kitchen which is currently 6 m^2. If he increases the length of the kitchen by 4 m and the width by 2 m the area of the new kitchen will be 5 times that of the original. What are the possible dimensions of the original kitchen?



Formulas for rectangles:

Area of a rectangle = length × breadth

Perimetre of a rectangle = 2 × length + 2 × breadth

Area of original kitchen: $xy = 6$

$\therefore y = \frac{6}{x} \rightarrow [1]$

Area of the new kitchen: $(x+4)(y+2)$

$\therefore (x+4)(y+2) = 5(6) \rightarrow$ area of the new kitchen = five times the area of the old kitchen

$\therefore xy + 2x + 4y + 8 = 30 \rightarrow [2]$

Substitute [1] into [2]:

$\therefore x\left(\frac{6}{x}\right) + 2x + 4\left(\frac{6}{x}\right) + 8 = 30$

$\therefore 6 + 2x + \frac{24}{x} + 8 = 30$

$\therefore 6x + 2x^2 + 24 + 8x = 30x$

$\therefore 2x^2 - 16x + 24 = 0 \rightarrow$ divide by the coefficient of x^2 (2)

$\therefore x^2 - 8x + 12 = 0$

$\therefore (x-6)(x-2) = 0$

$\therefore x = 6$ or $x = 2 \rightarrow [3]$

Substitute [3] into [1]:

$\therefore$ if $x = 6$ then $y = 1$ and if $x = 2$ then $y = 3$

$\therefore$ The original dimensions were: **6m × 1m** or **2m × 3m**



Complete Exercise 14 on page 261.

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RATE OF FLOW PROBLEMS

Rate of flow problems deal with objects or people completing a specific task in a certain time. In general, the problems work with inverse proportion as the more objects or people you add, the less time a task will take to complete.

Example 1:

Every time a store receives new stock, the items have to be marked and entered into the store's computer. Sarah and Tumi are responsible for this job. On her own Sarah can finish the job in 12 days. Tumi is more experienced and can finish the job in half of Sarah's time. How long would it take for Sarah and Tumi to complete the job working together?

Let x be the number of days they take working together.

$\therefore$ Working together they can do $\frac{1}{x}$ portion of the job in one day.

$\therefore$ In one day Sarah does $\frac{1}{12}$ of the job and Tumi does $\frac{1}{6}$ of the job.

$\therefore \frac{1}{12}+\frac{1}{6}=\frac{1}{x} \rightarrow$ LCD $= 12x$, $x \neq 0$ as this would cause a 0 in the denominator

$\therefore x+2x=12 \rightarrow$ both sides were mulitplied by the LCD of $12x$

$\therefore 3x=12$

$\therefore x=4$

$\therefore$ It would take them 4 days if they worked together.

Example 2:

A pool has two pipes entering it. When operating together they fill the pool in twenty-four minutes. Operating independently, one pipe fills the pool twenty minutes faster than the other one can on its own. How long does each pipe take to fill the pool individually?

Let x be the time it takes the slower pipe to fill the pool
$\therefore x-20$ is the time it takes the faster pipe to fill the pool

$\therefore \frac{1}{x}+\frac{1}{x-20}=\frac{1}{24} \rightarrow \text{LCD}=24x(x-20), \quad x \neq 0 \text{ and } x \neq 20$

$\therefore 24(x-20)+24x=x(x-20) \rightarrow$ both sides were multiplied the LCD of $24x(x-20)$

$\therefore 24x-480+24x=x^2-20x$

$\therefore x^2-68x+480=0$

$\therefore (x-60)(x-8)=0$

$\therefore x=60 \text{ or } x=8$

$x-20$ is the time it takes the faster pipe to fill the pool.

If $x=8$ the faster pipe will fill the pool in $8-20=-16$ minutes, which is not possible.

$\therefore x \neq 8$

$\therefore$ It takes the faster pipe 40 min to fill the pool, and the slower pipe 60 min.



Complete Exercise 15 on page 262 and the Mixed Exercise on pages 262 and 263.

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CHAPTER 3 – NUMBER PATTERNS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
55 58	**A. Arithmetic (linear) Sequences** **B. Quadratic Sequences**	2	• Work with and determine the general term of arithmetic number patterns. • Investigate number patterns leading to those where there is a constant second difference between consecutive terms, and the general term is therefore quadratic.

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A. ARITHMETIC (LINEAR) SEQUENCES

You have already dealt with linear sequences in Grade 10. In a linear sequence there is a constant difference between successive terms. In Grade 11 a formal equation to solving the n^{th} term of a linear sequence will now be introduced.

It is important to understand the notation used in number patterns.
The **first term** of a sequence is denoted as T_1, the **second** as T_2, the **third** term as T_3 and so on. The **last term** (or n^{th} term) of a sequence is denoted as T_n.

If a sequence is arithmetic, the difference between any two consecutive terms in the sequence will be the same. In other words: $T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = ... = T_n - T_{n-1}$

Derivation of the Arithmetic Sequence Formula

Let the first term of an arithmetic sequence equal a and the difference between successive terms equal d.

$\therefore T_1 = a$

$T_2 = a + 1d \rightarrow 1 = 2 - 1$

$T_3 = a + 2d \rightarrow 2 = 3 - 1$

$\therefore T_n = a + (n-1)d$

Arithmetic Sequence Formula

$$T_n = a + (n-1)d$$

Where : $a \rightarrow$ **The first term of a sequence**

$d \rightarrow$ **The difference between two successive terms** $(T_2 - T_1, T_3 - T_2, ...)$

$n \rightarrow$ **The position of the n^{th} term in the sequence (ie : Term 1, Term 2, ...)**

Example 1: Given the sequence 12;6;0;–6;...

a) Determine a general term for the n^{th} term of the sequence.
b) Determine the 50^{th} term of the sequence.
c) Which term of the sequence is equal to –198?

a) Ensure that the sequence is arithmetic: $T_2 - T_1 = T_3 - T_2$

$T_2 - T_1 = 6 - 12 = -6$ and $T_3 - T_2 = 0 - 6 = -6 \rightarrow$ sequence is linear

$d = -6$

$a = 12$

$T_n = a + (n-1)d$

$\therefore T_n = 12 + (n-1)(-6)$

b) $T_{50} = 12 + (50-1)(-6)$

$= -282$

c) $T_n = -198$

$\therefore -198 = 12 + (n-1)(-6)$

$\therefore -210 = (n-1)(-6)$

$\therefore n - 1 = 35$

$\therefore n = 36 \rightarrow$ the 36^{th} term is equal to -198 $(T_{36} = -198)$

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Example 2: In a linear sequence the 3rd term is –2 and the 10th term is 19. Determine an expression for the nth term of the sequence.

$T_3 = -2$ and $T_{10} = 19$

$\therefore -2 = a + (3-1)d$ $\quad\quad$ $\therefore 19 = a + (10-1)d$

$\therefore -2 = a + 2d \rightarrow \boxed{1}$ $\quad\quad$ $\therefore 19 = a + 9d \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\therefore 21 = 7d$

$\therefore d = 3 \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$

$\therefore -2 = a + 2(3)$

$\therefore a = -8$

$\therefore T_n = -8 + (n-1)(3)$

Example 3: The sequence $2x+10; x+14; x+8; \ldots$ is linear:

a) Determine the value of x.

b) Determine an expression for the nth term of the sequence.

a) As the sequence is linear: $T_2 - T_1 = T_3 - T_2$

$\therefore x+14-(2x+10) = x+8-(x+14)$

$\therefore x+14-2x-10 = x+8-x-14$

$\therefore x = 10$

b) As $x = 10$ the sequence is: $30; 24; 18; \ldots$

$T_n = a + (n-1)d$

$\therefore a = 30$ and $d = -6$

$\therefore T_n = 30 + (n-1)(-6)$

Example 4: Determine the general term of the sequence $\frac{3}{2}; \frac{5}{9}; \frac{7}{16}; \frac{9}{23}; \ldots$ in its simplest form.

In this example the sequence is not linear. However it can be seen that the numerators and denominators form a linear sequence individually.

Numerator sequence: $3; 5; 7; 9; \ldots$

$T_n = a + (n-1)d$

$\therefore a = 3$ and $d = 2$

$\therefore T_n = 3 + (n-1)(2)$

$\therefore T_n = 2n+1$

Denominator: sequence: $2; 9; 16; 23; \ldots$

$T_n = a + (n-1)d$

$\therefore a = 2$ and $d = 7$

$\therefore T_n = 2 + (n-1)(7)$

$\therefore T_n = 7n-5$

You now can give an expression for the entire sequence

$\therefore T_n = \frac{2n+1}{7n-5}$

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Example 5: **Romy is training for a cycling race. She begins by cycling 40km per week. Every week she increases the distance she cycles by 12,5km. After 7 weeks she can no longer increase the distance she cycles. What is the maximum distance Romy can cycle?**

Sequence: $40;52,5;65;77,5;...$

$T_n = a+(n-1)d$

$\therefore a = 40$ and $d = 12,5$

$\therefore T_7 = 40+(7-1)(12,5)$

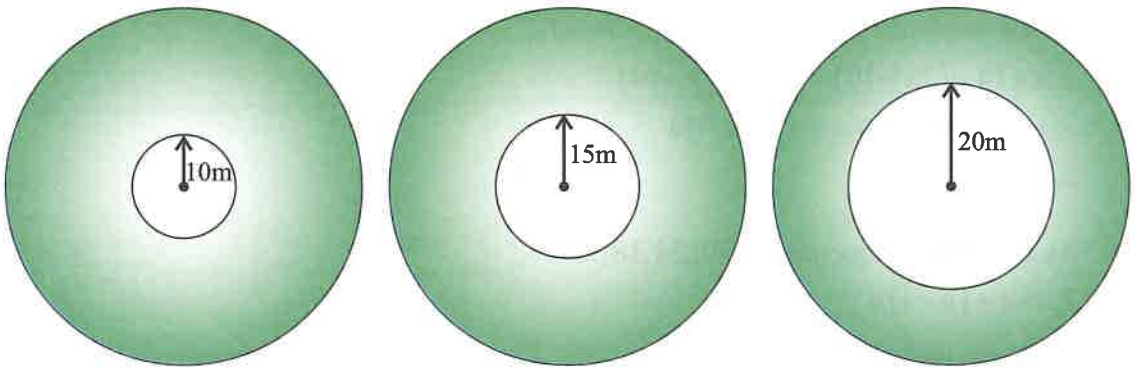
$\therefore T_7 = 115\,km$

$\therefore$ Romy's maximum distance is 115km

Example 6: **A farmer grows his crops in a circular area of his land. He harvests his crops by driving his tractor in circles. The radius of his first circle is 10m. The radius of each subsequent circle that he makes increases by 5m and he completes 50 circles to harvest his entire crop.**

a) What is the radius of his final circle?

b) What is the area of the farmer's land? Give your answer correct to the nearest square metre.



a) Sequence: $10;15;20;...$

$T_n = a+(n-1)d$

$\therefore a = 10$ and $d = 5$

$\therefore T_{50} = 10+(49)(5)$ → the 50^{th} circle is the farmer's final circle

$\therefore T_{50} = 255\,m$

b) $A_{\bigcirc} = \pi r^2$

$= \pi(255)^2$ → the radius of the final circle is the radius of the farmer's land

$= 65025\pi$

$= 204282\,m^2$



Complete Exercise 1 and 2 on page 285.

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B. QUADRATIC SEQUENCES

In a quadratic sequence the second difference is constant. For example, in the following sequence, the first difference is not constant $(T_2 - T_1 \neq T_3 - T_2)$.
However, if we calculate the difference of the first difference in the same manner, it will be constant.

$$0 \quad 5 \quad 14 \quad 27$$
$$5 \quad 9 \quad 13 \rightarrow \text{first difference}$$
$$4 \quad 4 \rightarrow \text{second difference}$$

The general term of a quadratic sequence has the same form as a quadratic expression.
The general term for the nth term of a quadratic sequence is: $T_n = an^2 + bn + c$

Quadratic Sequence Formula

$$T_n = an^2 + bn + c$$

Where : $a \rightarrow$ **Half the second difference** $(2a = 2^{nd}$ **difference**$)$

$b \rightarrow$ **A constant**

$c \rightarrow$ **A constant**

$n \rightarrow$ **The position of the nth term in the sequence (ie : Term 1, Term 2, ...)**

By substituting 1, 2, 3 and 4 into n in the general formula for a quadratic sequence, the first 4 terms of a general quadratic sequence can be obtained as shown below:

$$\overset{n=1}{a+b+c} \; ; \; \overset{n=2}{4a+2b+c} \; ; \; \overset{n=3}{9a+3b+c} \; ; \; \overset{n=4}{16a+4b+c}$$
$$3a+b \qquad 5a+b \qquad 7a+b \rightarrow \text{first difference}$$
$$2a \qquad 2a \rightarrow \text{second difference}$$

From the above general sequence it can clearly be seen that the second difference is equal to twice the coefficient of x^2. In other words $2a$ = **second difference**.

Steps to writing an expression for the nth term of a quadratic sequence:

1. **Calculate the first and second difference.**
   If the first difference is constant the sequence is linear.
2. **Calculate a: $2a$ = second difference** $\left(a = \frac{\text{second difference}}{2}\right)$
3. **Solve for b and c simultaneously by substituting two terms into the general equation.**

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Example 1: For the sequence 8 ; 20 ; 38 ; 62 ; ... determine:

a) An algebraic expression that represents the n^{th} term of the sequence.

b) The 20^{th} term of the sequence.

a) Calculate the first and second difference:

$8 \searrow\swarrow 20 \searrow\swarrow 38 \searrow\swarrow 62$

$12 \searrow\swarrow 18 \searrow\swarrow 24 \rightarrow$ first difference

$6 \quad 6 \rightarrow$ constant second difference (sequence is quadratic)

$\therefore Tn = an^2 + bn + c$

Determine the value of a:

$2a = 6$

$\therefore a = 3$

$\therefore T_n = 3n^2 + bn + c$

Solve for b and c by substituting in two terms and using simultaneous equations:

Substitute $T_1 = 8$: *Substitute* $T_2 = 20$:

$8 = 3(1)^2 + b(1) + c$ $\quad$ $20 = 3(2)^2 + b(2) + c$

$\therefore 5 = b + c \rightarrow \boxed{1}$ $\quad$ $8 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$8 = 2b + c$

$-(5 = b + c)$

$3 = b + 0$

$\therefore b = 3 \rightarrow \boxed{3}$

Subsitute $\boxed{3}$ into $\boxed{1}$

$5 = 3 + c$

$\therefore 2 = c$

$\therefore b = 3$ and $c = 2$

$\therefore T_n = 3n^2 + 3n + 2$

b) $T_{20} = 3(20)^2 + 3(20) + 2$

$= 1262$



Using elimination (subtracting one equation from the other) is the easiest method of solving simultaneous equations involving quadratic number patterns.

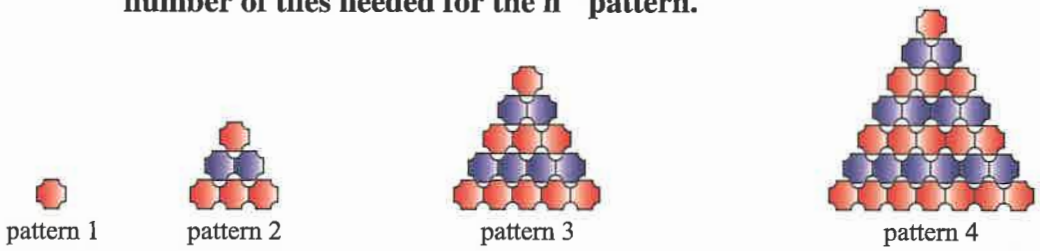
Your calculator can also solve for the general term of a quadratic or linear sequence using the statistics mode (see calculator shortcuts on pg. 231)



Complete Exercise 3 on page 285.

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Example 2: **Using tiles, a design student makes the pattern shown below. Determine a formula that the student can use to determine the number of tiles needed for the n^{th} pattern.**



$1 \searrow\swarrow 6 \searrow\swarrow 15 \searrow\swarrow 28$

$5 \searrow\swarrow 9 \searrow\swarrow 13 \rightarrow$ first difference

$4 \quad 4 \rightarrow$ constant second difference (sequence is quadratic)

$\therefore 2a = 4$

$\therefore a = 2$

$\therefore T_n = 2n^2 + bn + c$

Substitute $T_1 = 1$:

$1 = 2(1)^2 + b(1) + c$

$\therefore -1 = b + c \rightarrow \boxed{1}$

Substitute $T_2 = 6$:

$6 = 2(2)^2 + b(2) + c$

$-2 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\therefore -1 = b$

Subsitute $\boxed{3}$ into $\boxed{1}$

$-1 = -1 + c$

$\therefore 0 = c$

$\therefore T_n = 2n^2 - n$

Example 3: **The second term of a quadratic sequence is equal to 1. The third term is equal to –6 and the fifth term is equal to –14.**

a) Determine the second difference of the quadratic sequence.

b) Hence, determine the first term of the quadratic sequence.

a) The sequence is: $x;1;-6;y;-14$

$x \;;\; 1 \;;\; -6 \;;\; y \;;\; -14$

$1-x \quad -7 \quad y+6 \quad -14-y \rightarrow$ first difference

$-8+x \quad y+13 \quad -2y-20 \rightarrow$ second difference

$-2y - 20 = y + 13 \rightarrow$ the second difference of a quadratic sequence is constant

$\therefore -3y = 33$

$\therefore y = -11$

$\therefore$ The second difference is equal to $2 \rightarrow [-11+13=2]$ or $[-2(-11)-20=2]$

b) As the second difference of a quadratic sequence is constant:

$-8 + x = 2$

$\therefore x = 10$

$\therefore$ The first term of the sequence is 10.

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Example 4: **In the first stage of a rugby tournament each team plays each other once. For the tournament to take place there has to be at least 3 teams in the tournament. How many games will be played if there are:**

a) 3 teams in the tournament? **b) 4 teams in the tournament?**

c) 6 teams in the tournament? **d) n teams in the tournament?**

a) If there are 3 teams in the tournament, say Team **A**, **B** and **C**:
Team **A** will play Teams **B** and **C** $\rightarrow$ 2 games played

This leaves Team **B** to still play Team **C** $\rightarrow$ 1 game played

$\therefore$ Total games played $= 2+1=3$

b) If there are 4 teams in the tournament, say Team **A**, **B**, **C** and **D**:
Team **A** will play Teams **B**, **C** and **D** $\rightarrow$ 3 games played

This leaves Team **B** to still play Teams **C** and **D** $\rightarrow$ 2 games played

Finally Team **C** will play Team **D** $\rightarrow$ 1 game played, all teams have played each other

$\therefore$ Total games played $= 3+2+1=6$

c) By now you should be able to see a pattern forming for the number of games played:
3 teams: $2+1=3$

4 teams: $3+2+1=6$

5 teams: $4+3+2+1=10$

6 teams: $5+4+3+2+1=15$ games played

d) Sequence: $3;6;10;15;21;...;n$

$n=3 \quad n=4 \quad n=5 \quad n=6$

$3 \searrow\swarrow 6 \searrow\swarrow 10 \searrow\swarrow 15 \rightarrow$ the sequence starts on term 3

$3 \searrow\swarrow 4 \searrow\swarrow 5 \rightarrow$ first difference

$1 \qquad 1 \rightarrow$ constant second difference (sequence is quadratic)

$\therefore T_n = an^2 + bn + c$

$2a = 1$

$\therefore a = \frac{1}{2}$

$\therefore T_n = \frac{1}{2}n^2 + bn + c$

Substitute $T_3 = 3 : \rightarrow n = 3$

$3 = \frac{1}{2}(3)^2 + b(3) + c$

$\therefore -\frac{3}{2} = 3b + c \rightarrow \boxed{1}$

Substitute $T_4 = 6 : \rightarrow n = 4$

$6 = \frac{1}{2}(4)^2 + b(4) + c$

$-2 = 4b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\therefore b = -\frac{1}{2} \rightarrow \boxed{3}$

Subsitute $\boxed{3}$ into $\boxed{2}$

$-2 = 4\left(-\frac{1}{2}\right) + c$

$\therefore 0 = c$

$\therefore T_n = \frac{1}{2}n^2 - \frac{1}{2}n \; ; \; n \geq 3 \rightarrow$ there have to be at least 3 teams

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Example 5: At a dinner party every guest shakes hands with each other. How many handshakes will take place if there are:

a) 2 guests? **b) 3 guests?**

c) 4 guests? **d) 5 guests?**

e) 6 guests? **f) n guests?**

g) 100 guests?

a) If there are **2** guests there will be **1 handshake**.

b) If there are **3** guests there will be **3 handshakes**.

c) If there are **4** guests there will be **6 handshakes**.

d) If there are **5** guests there will be **10 handshakes**.

e) If there are **6** guests there will be **15 handshakes**.

f) The sequence is: $1;3;6;10;15;...;n$

$n=2 \quad n=3 \quad n=4 \quad n=5$

$1 \searrow\swarrow 3 \searrow\swarrow 6 \searrow\swarrow 10 \rightarrow$ the sequence starts on term 2

$2 \searrow\swarrow 3 \searrow\swarrow 4 \rightarrow$ first difference

$1 \quad 1 \rightarrow$ constant second difference (sequence is quadratic)

$\therefore Tn = an^2 + bn + c$

$2a = 1$

$\therefore a = \frac{1}{2}$

$\therefore T_n = \frac{1}{2}n^2 + bn + c$

Substitute $T_2 = 1 : \rightarrow n = 2$

$1 = \frac{1}{2}(2)^2 + b(2) + c$

$\therefore -1 = 2b + c \rightarrow \boxed{1}$

Substitute $T_3 = 3 : \rightarrow n = 3$

$3 = \frac{1}{2}(3)^2 + b(3) + c$

$-\frac{3}{2} = 3b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\therefore b = -\frac{1}{2} \rightarrow \boxed{3}$

Subsitute $\boxed{3}$ into $\boxed{1}$

$-1 = 2\left(-\frac{1}{2}\right) + c$

$\therefore 0 = c$

$\therefore T_n = \frac{1}{2}n^2 - \frac{1}{2}n; \quad n \geq 2 \rightarrow$ there have to be at least 2 guests to shake hands

g) $T_{100} = \frac{1}{2}(100)^2 - \frac{1}{2}(100)$

$= 4950$ handshakes

Complete Mixed Exercise on pages 285 and 286.

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CHAPTER 4
ANALYTICAL GEOMETRY

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
64	**A. Summary of Analytical Formulas**	9	Use a Cartesian co-ordinate system to derive and apply: • The equation of a line through two given points • The equation of a line through one point and parallel or perpendicular to a given line • The inclination of a line
65	**B. Properties of Quadrilaterals**		
65	• Quadrilaterals		
67	**C. Grade 10 Revision of Midpoint, Distance and Gradient Formulas**		
69	**D. Finding The Equation of a Straight Line**		
70	**E. Angle of Inclination of a Straight Line**		
70	• Angle of Inclination when the gradient of a line is positive ($m > 0$)		
70	• Angle of Inclination when the gradient of a line is negative ($m < 0$)		
71	• Using the Angle of Inclination to determine the angle between two lines		

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A. SUMMARY OF ANALYTICAL FORMULAS

Average Gradient:

The average gradient of a line passing through points A and B is: $m_{AB} = \frac{y_b - y_a}{x_b - x_a}$

Distance Formula:

The distance between any two points A and B is:

$AB = \sqrt{(y_b - y_a)^2 + (x_b - x_a)^2}$ → also written as $AB^2 = (y_b - y_a)^2 + (x_b - x_a)^2$

Midpoint Formula:

The coordinates of the midpoint of any line segment AB are: $\left(\frac{x_a + x_b}{2}; \frac{y_a + y_b}{2}\right)$

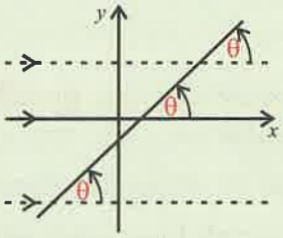
Angle of Inclination:

The angle of inclination is often denoted as θ.

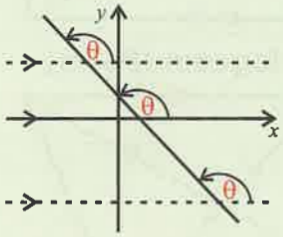
The gradient of a line (m) is equal to the tangent of the angle of inclination (θ):

$\therefore \tan\theta = m$ where $\theta \in [0°;180°]$ → m is the gradient of a line

$m > 0$; $\theta \in [0°;90°]$ $\quad$ $m < 0$; $\theta \in [90°;180°]$



θ is acute (less than 90°)



θ is obtuse (greater than 90°)

Equation of a straight line:

$y = mx + c$ or $y - y_1 = m(x - x_1)$

Equation of a horizontal line:

$y = k$ → where k is a constant

Equation of a vertical line:

$x = k$ → where k is a constant

Properties of perpendicular (⊥) lines:

The product of the gradients of perpendicular lines is –1:

$m_1 \times m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

Properties of parallel (//) lines:

The gradients of parallel lines are equal: $m_1 = m_2$

Collinear points:

If points are collinear it means that they lie on the same straight line.

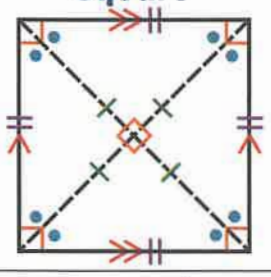
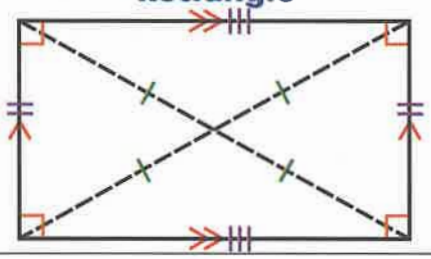
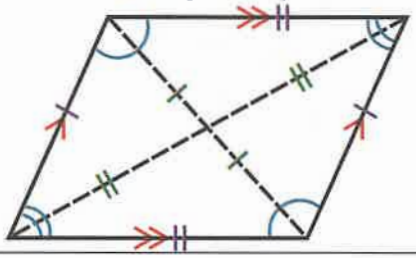
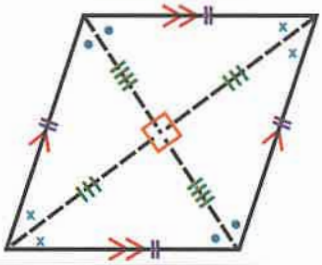
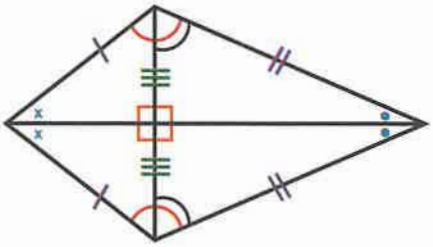
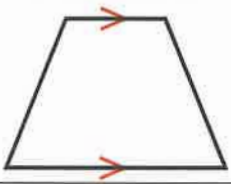
If points A, B and C are collinear: $m_{AB} = m_{BC}$

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B. PROPERTIES OF QUADRILATERALS

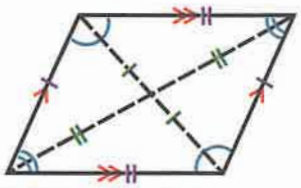
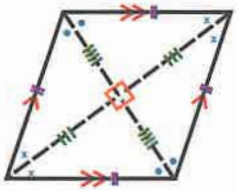
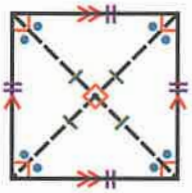
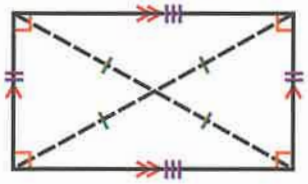
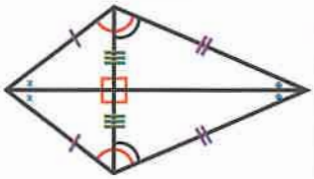
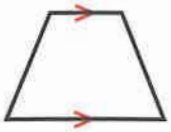
QUADRILATERALS

A quadrilateral is defined as a closed four-sided figure whose interior angles add up to 360°. The table below shows some common quadrilaterals and their properties.

Quadrilateral	Properties
Square 	• All **sides** are **equal** in length • Opposite sides are **parallel** • Corner **angles** equal **90°** • Diagonals are **equal** and **bisect** each other at **90°** • Diagonals **bisect** the corner **angles**
Rectangle 	• Opposite sides are **parallel** and **equal** in length • Corner **angles equal 90°** • **Diagonals** are **equal** and **bisect** each other
Parallelogram (Parm) 	• Opposite sides are **parallel** and **equal** in length • Opposite angles are **equal** • **Diagonals bisect** each other
Rhombus 	• All **sides** are **equal** in length • Opposite sides are **parallel** • Opposite angles are **equal** • **Diagonals bisect** each other at **90°** • Diagonals bisect the corner **angles**
Kite 	• Two pairs of **adjacent sides** are **equal** • A single pair of **opposite angles are equal** • Diagonals intersect each other at **90°** • One diagonal **bisects** the corner **angles** • Shorter diagonal is **bisected** by the longer diagonal
Trapezium 	• One pair of **opposite sides** are **parallel**

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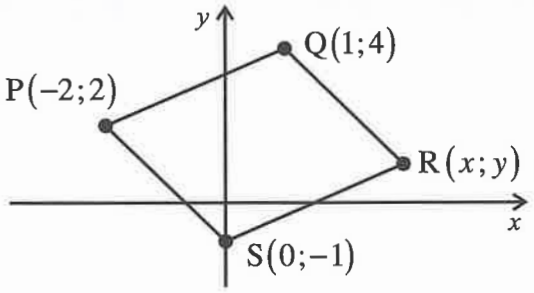
The table below shows the properties needed to prove that a quadrilateral is a specific shape.

Quadrilateral	Properties to prove
Parallelogram (parm) 	Prove any **one** of the following: 1) **Both pairs of opposite sides parallel.** 2) **Both pairs of opposite sides equal.** 3) **Both pairs of opposite angles equal.** 4) **One pair of opposite sides equal and parallel.** 5) **Diagonals bisect each other. (They have the same midpoint.)**
Rhombus 	Prove the quadrilateral is a parallelogram **and one** of the following: 1) **One pair of adjacent sides equal.** 2) **Diagonals bisect at 90^{o}.**
Square 	Prove any **one** of the following: 1) **The quadrilateral is a parm with a corner angle equal to 90^{o} and a pair of adjacent sides equal.** 2) **The quadrilateral is a rhombus with a corner angle equal to 90^{o}.** 3) **The quadrilateral is a rhombus with equal diagonals.**
Rectangle 	Prove the quadrilateral is a parallelogram **and one** of the following: 1) **The parm has a corner angle equal to 90^{o}.** 2) **The parm has diagonals that are equal in length.**
Kite 	Prove that: 1) **Each pair of adjacent sides equal in length.**
Trapezium 	Prove that: 1) **One pair of opposite sides are parallel.**

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C. GRADE 10 REVISION OF MIDPOINT, DISTANCE AND GRADIENT FORMULAS

Example 1: Given that P,Q,R and S are vertices of the parallelogram shown below, determine the coordinates of R.



As PQRS is a parallelogram the diagonals bisect each other. This means that diagonals PR and QS have the same midpoint.

$$M_{QS} = \left(\frac{x_q + x_s}{2}; \frac{y_q + y_s}{2}\right) \qquad M_{PR} = \left(\frac{x_p + x_r}{2}; \frac{y_p + y_r}{2}\right)$$

$$= \left(\frac{1+0}{2}; \frac{4-1}{2}\right) \qquad = \left(\frac{x-2}{2}; \frac{y+2}{2}\right)$$

$$= \left(\frac{1}{2}; \frac{3}{2}\right)$$

$\therefore \frac{x-2}{2} = \frac{1}{2}$ and $\frac{y+2}{2} = \frac{3}{2}$ → diags of a parm have the same midpoint

$\therefore x-2=1$ and $y+2=3$

$\therefore x=3$ and $y=1$

$\therefore$ R is the point $(3;1)$.

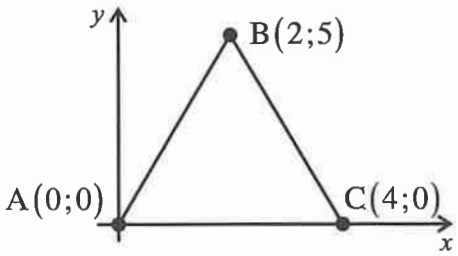
Example 2: The vertices of Triangle ABC have the following coordinates: A(0;0);B(2;5);C(4;0):

a) Determine the lengths of the 3 sides of the triangle, leaving your answer in simplest surd form where necessary.

b) What type of triangle is triangle ABC?

a) It is often useful to draw a rough sketch of the triangle:

$$AB = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$$
$$= \sqrt{(2-0)^2 + (5-0)^2}$$
$$= \sqrt{29} \text{ units}$$



$$BC = \sqrt{(x_c - x_b)^2 + (y_c - y_b)^2}$$
$$= \sqrt{(4-2)^2 + (0-5)^2}$$
$$= \sqrt{29} \text{ units}$$

$AC = 4$ → this can be seen from the diagram as AC is a horizontal line on the x-axis

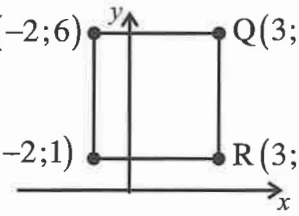
b) Triangle ABC is an isosceles triangle as $AB = BC$.

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Example 3: Given the points $P(-2;6)$; $Q(3;6)$; $R(3;1)$ and $S(-2;1)$:

a) Plot the points on the Cartesian plane.
b) Calculate the length of PR and QS.
c) Determine the midpoint of PR and QS.
d) Determine the gradient of PR and QS.
e) What does your answer in d) tell you about lines PR and QS?
f) Discuss what type of quadrilateral PQRS is, giving reasons.
g) If E is the point $(5;-1)$, are the points P,R, and E collinear?

a) P(−2;6) Q(3;6) S(−2;1) R(3;1)



b) $PR = \sqrt{(x_r - x_p)^2 + (y_r - y_p)^2}$

$= \sqrt{(3+2)^2 + (1-6)^2}$

$= \sqrt{50} = 5\sqrt{2}$ units

$QS = \sqrt{(x_s - x_q)^2 + (y_s - y_q)^2}$

$= \sqrt{(-2-3)^2 + (1-6)^2}$

$= \sqrt{50} = 5\sqrt{2}$ units

c) $M_{PR} = \left(\frac{x_p + x_r}{2}; \frac{y_p + y_r}{2}\right)$

$= \left(\frac{-2+3}{2}; \frac{6+1}{2}\right)$

$= \left(\frac{1}{2}; \frac{7}{2}\right)$

$M_{QS} = \left(\frac{x_q + x_s}{2}; \frac{y_q + y_s}{2}\right)$

$= \left(\frac{3-2}{2}; \frac{6+1}{2}\right)$

$= \left(\frac{1}{2}; \frac{7}{2}\right)$

d) $m_{PR} = \frac{y_r - y_p}{x_r - x_p}$

$= \frac{1-6}{3+2}$

$= -1$

$m_{QS} = \frac{y_s - y_q}{x_s - x_q}$

$= \frac{1-6}{-2-3}$

$= 1$

e) $m_{PR} \times m_{QS} = -1$

Therefore, line PR is perpendicular to line QS

f) Diagonals bisect each other at 90°. → this means the quadrilateral is a rhombus
Diagonals are equal in length.
Therefore, the quadrilateral is a square.

g) $m_{RE} = \frac{y_e - y_r}{x_e - x_r}$

$= \frac{-1-1}{5-3}$

$= -1$

$\therefore m_{RE} = m_{PR}$ and points P, R and E are collinear.



Complete Exercise 1 on page 292.

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D. FINDING THE EQUATION OF A STRAIGHT LINE

You have already dealt with determining the equation of a straight line in Grade 10. In Grade 11 there are 2 ways of representing the equation of a straight line. A straight line can be represented as either $y = mx + c$ or $y - y_1 = m(x - x_1)$.

Equation of a straight line:

$y = mx + c$ or $y - y_1 = m(x - x_1)$

Where: $m \rightarrow$ **Gradient of the the line**

$c \rightarrow$ **The y-intercept**

$(x_1; y_1) \rightarrow$ **A point on the line**

Example 1: Determine the equation of a straight line which passes through the points $A(3;7)$ and $B(10;21)$.

You can determine the equation of the line using either of the above equations. In both methods you first have to determine the gradient of the line.

$$m = gradient = \frac{y_2 - y_1}{x_2 - x_1} = \frac{21-7}{10-3} = 2$$

Now substitute the gradient and a point on the graph into one of the above equations:

Method 1: using: $y = mx + c$

$y = 2x + c \rightarrow m = 2$

Sub $A(3;7) \rightarrow$ B could also be used

$\therefore 7 = 2(3) + c$

$\therefore c = 1$

$\therefore y = 2x + 1$

Method 2: using: $y - y_1 = m(x - x_1)$

$y - y_1 = 2(x - x_1) \rightarrow m = 2$

Sub $A(3;7) \rightarrow$ B could also be used

$\therefore y - 7 = 2(x - 3)$

$\therefore y - 7 = 2x - 6$

$\therefore y = 2x + 1$

Example 2: Determine the equation of a straight line which is:

a) Parallel to the line $y = -2x + 8$, passing through the point $A(3;7)$.

b) Perpendicular to the line $y = \frac{1}{3}x - 2$, with a y-intercept of 6.

c) Parallel to the x-axis and passes through the point $(5;-7)$

d) Parallel to the y-axis and passes through the point $(5;-7)$

a) $y = -2x + c \rightarrow$ line is // to $y = -2x + 8$

sub $A(3;7)$

$\therefore 7 = -2(3) + c$

$\therefore c = 13$

$\therefore y = -2x + 13$

b) $y = -3x + c \rightarrow$ line is $\perp$ to $y = \frac{1}{3}x - 2$

sub $c = 6 \rightarrow$ y-intercept is given as 6

$\therefore y = -3x + 6$

c) $y = -7$

A line parallel to the x-axis is a horizontal line.

d) $x = 5$

A line parallel to the y-axis is a vertical line.

Complete Exercise 2 on page 292.

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E. ANGLE OF INCLINATION OF A STRAIGHT LINE

The angle of inclination is the angle measured anti-clockwise from the x-axis (or a horizontal line) to a straight line.
The angle of inclination is always between 0° and 180°.

Angle of Inclination

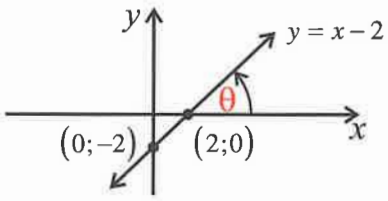
$$\tan\theta = m$$

Where: $m \rightarrow$ **Gradient of the line**

$\theta \rightarrow$ **The angle of inclination**

ANGLE OF INCLINATION WHEN THE GRADIENT OF A LINE IS POSITIVE ($m > 0$)

Example: Determine the angle of inclination of the line $y = x - 2$.



Recap:

if $\tan\theta = m$ then $\theta = \arctan(m)$

arctan is displayed on a calculator as $\tan^{-1}$

and is accessed by pressing shift tan.

The gradient of the line $y = 1x - 2$ is equal to **1**:

$\therefore \tan\theta = 1 \rightarrow$ in order to solve for θ use **shift** tan 1 on your calculator

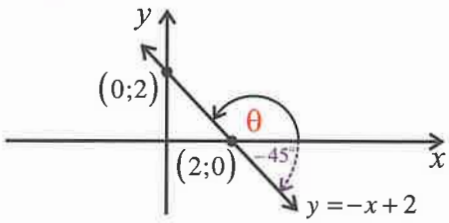
$\therefore \theta = 45°$

ANGLE OF INCLINATION WHEN THE GRADIENT OF A LINE IS NEGATIVE ($m < 0$)

It is important to note that a calculator will only provide answers for an angle between $\pm 90°$. A negative angle means that the angle is measured clockwise.

This means that if a line has a negative gradient you will have to add 180° to the answer on your calculator in order to obtain the angle which is measured anti-clockwise from the horizontal as demonstrated in the example below.

Example: Determine the angle of inclination of the line $y = -x + 2$.



The gradient of the line $y = -1x + 2$ is equal to **−1**:

$\therefore \tan\theta = -1 \rightarrow$ in order to solve for θ use **shift** tan −1 on your calculator

As the gradient is negative the calculator will give an answer of $-45°$. This is the angle measured clockwise from the horizontal as shown in the above diagram. In order to obtain θ you have to add 180°, as adjacent angles on a straight line add up to 180°.

$\therefore \theta = -45° + 180°$

$\therefore \theta = 135°$

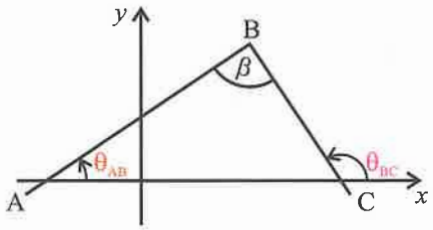
Complete Exercise 3 numbers 1 and 2 on page 293.

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USING THE ANGLE OF INCLINATION TO DETERMINE THE ANGLE BETWEEN TWO LINES

An important application of the angle of inclination is to determine the angle formed between two straight lines.

Example 1: Determine the value of β, the angle formed by lines AB and BC in the diagram below if the gradient of line AB is $\frac{1}{2}$ and the gradient of line BC is $-\frac{1}{3}$.



Method 1: Remember that the exterior angle of a triangle is equal to the sum of the opposite interior angles.

$\theta_{BC} = \beta + \theta_{AB}$ → exterior angle of a triangle

$\tan\theta_{AB} = \frac{1}{2}$

$\therefore \theta_{AB} = 26,57°$

$\theta_{BC} = \beta + \theta_{AB}$

$\therefore \beta = \theta_{BC} - \theta_{AB}$

$\therefore \beta = 161,57° - 26,57°$

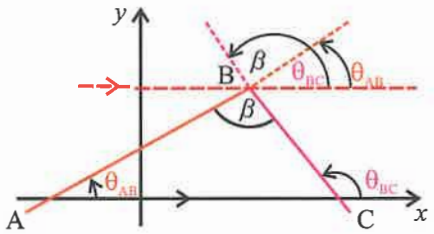
$\therefore \beta = 135°$

$\tan\theta_{BC} = -\frac{1}{3}$

$\therefore \theta_{BC} = -18,43° + 180°$ → gradient is negative

$\therefore \theta_{BC} = 161,57°$

Method 2: Extend lines AB and BC and draw in a horizontal line through B.



By extending the lines it can easily be seen that $\beta = \theta_{BC} - \theta_{AB}$

$\tan\theta_{AB} = \frac{1}{2}$

$\therefore \theta_{AB} = 26,57°$

$\beta = \theta_{BC} - \theta_{AB}$

$\therefore \beta = 161,57° - 26,57°$

$\therefore \beta = 135°$

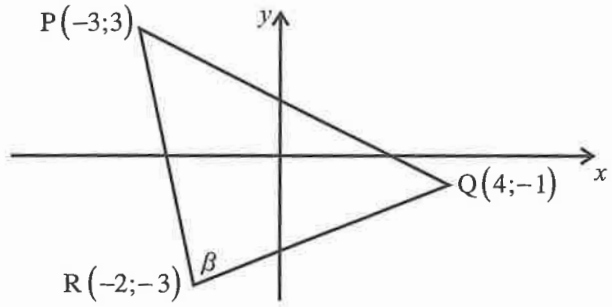
$\tan\theta_{BC} = -\frac{1}{3}$

$\therefore \theta_{BC} = -18,43° + 180°$ → gradient is negative

$\therefore \theta_{BC} = 161,57°$

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Example 2: Triangle PQR has coordinates $P(-3;3); Q(4;-1); R(-2;-3)$ as shown below. Determine:



a) **The gradient of PQ, QR and PR.**
b) **If triangle PQR is a right-angled triangle.**
c) **The size of β correct to one decimal place.**

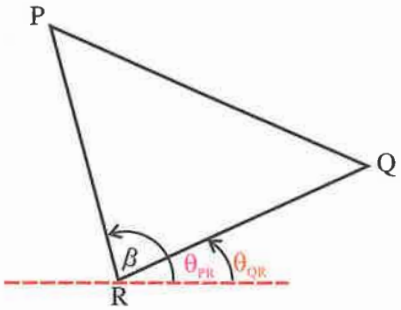
a) $m_{PQ} = \dfrac{y_q - y_p}{x_q - x_p}$ $= \dfrac{-1-3}{4+3}$ $= -\dfrac{4}{7}$

$m_{QR} = \dfrac{y_r - y_q}{x_r - x_q}$ $= \dfrac{-3+1}{-2-4}$ $= \dfrac{1}{3}$

$m_{PR} = \dfrac{y_r - y_p}{x_r - x_p}$ $= \dfrac{-3-3}{-2+3}$ $= -6$

b) As none of the products of the gradients are equal -1 ΔPQR is not a right-angled triangle.

c) In order to determine β draw in a horizontal line at R as shown below:



From the diagram it can be seen that $\beta = \theta_{PR} - \theta_{QR}$

$\tan \theta_{QR} = \dfrac{1}{3}$

$\therefore \theta_{QR} = 18,43^\circ$

$\tan \theta_{PR} = -6$

$\therefore \theta_{PR} = -80,54^\circ + 180^\circ \rightarrow$ gradient is negative

$\therefore \theta_{PR} = 99,46^\circ$

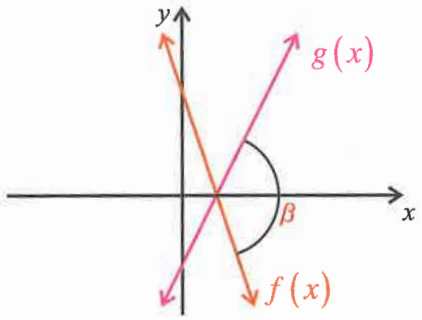
$\beta = \theta_{PR} - \theta_{QR}$

$\therefore \beta = 99,46^\circ - 18,43^\circ$

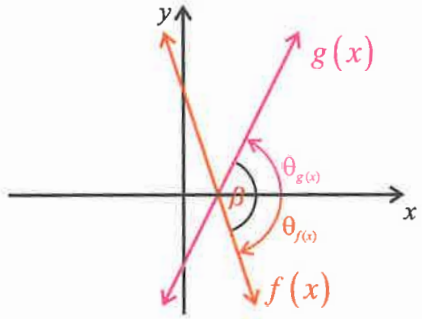
$\therefore \beta = 81,0^\circ \rightarrow$ correct to 1 decimal place

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Example 3: **The graphs of $f(x) = -3x + 6$ and $g(x) = 2x - 4$ are sketched below. Determine the size of angle β.**



From the diagram below it should be clear that $\beta = \theta_{g(x)} + \theta_{f(x)}$



$\tan \theta_{g(x)} = 2$

$\therefore \theta_{g(x)} = 63,43°$

$\tan \theta_{f(x)} = -3$

$\therefore \theta_{f(x)} = -71,57°$

It is important to understand that $\theta_{f(x)}$ is not the angle of inclination of $f(x)$ but the **size** of the angle measured clockwise from the x-axis. In this case it is not necessary to calculate the angle of inclination of $f(x)$ by adding $180°$, as $\beta = \theta_{g(x)} + \theta_{f(x)}$

$\therefore \beta = 63,43° + 71,57°$

$\therefore \beta = 135°$



Complete Exercise 3 number 3 on page 293 and the Mixed Exercise on pages 293-295.

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CHAPTER 5 – FUNCTIONS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
76	**A. Grade 10 Revision**	1	• Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Including linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions. • Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the y- axis. • Problem solving and graph work involving the prescribed functions. • Determine the average gradient between points and develop an intuitive understanding of gradient at a point.
76	• Axes of Symmetry		
76	• Asymptotes		
76	• Domain and Range		
76	• Function and Mapping Notation		
77	• Grade 10 Revision of The Parabola		
79	• Grade 10 Revision of the Hyperbola		
80	• Grade 10 Revision of the Exponential Function		
81	**B. The Parabola (Quadratic Function)**		
81	• The Parabola in the form of $y=a(x-p)^2+q$		
84	• The Parabola in the form of $y=ax^2+bx+c$		
87	• Determining the Equation of a Parabola		
87	• Given the turning point and one other point		
88	• Given the x-intercepts and one other point		
89	• Given the y-intercept and two other points		
90	**C. The Hyperbola**		
90	• Axes of Symmetry and Points of Intersection		
93	• Determining the Equation of a Hyperbola		
95	**D. The Exponential Function**		
96	• Steps to Sketching the Exponential Function		
99	• Determining the Equation of an Exponential Graph		
100	**E. Translations and Reflections of Graphs**		

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CHAPTER 5 – FUNCTIONS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
102	**F. Graph Interpretation**		
102	• lengths of vertical and horizontal lines		
103	• Interpretation and Deductions from Graphs		
107	• Real World Applications		
108	• Average Gradient (Average Rate of Change)		
109	**G. Trigonometric Functions**		
109	• Grade 10 Revision of the Sin, Cos and Tan Graphs		
110	• The effects of a (Change in Amplitude)		
110	• The effects of q (Vertical Shifts)		
111	• The effects of p (Horizontal Shifts)		
114	• The effects of k (Change in Period)		
116	**H. Interpretation of Trigonometric Functions**		

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A. GRADE 10 REVISION

AXES OF SYMMETRY

If a function has a line of symmetry, it means that the function is a mirror image of itself about the line of symmetry. In other words, if the function were folded along the line of symmetry it would duplicate the graph on the other side of the line.

ASYMPTOTES

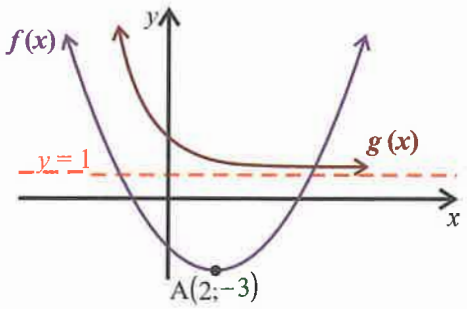
Asymptotes are imaginary lines that a graph approaches, but never touches or cuts.

DOMAIN AND RANGE

Domain: The domain refers to the possible ***x*-values** for which a graph is defined.

Range: The range refers to the possible ***y*-values** for which a graph is defined.

Example: **The graphs of $f(x)$ and $g(x)$ are sketched below. Using the graphs clearly state the domain and range of each graph. It is given that A is the turning point of the parabola $f(x)$.**



$f(x)$:

Domain: $x \in \mathbb{R}$

This can also be written as $x \in (-\infty;\infty)$.

Range: $y \in [-3;\infty)$

This can also be written as $y \geq -3$.

$g(x)$:

Domain: $x \in \mathbb{R}$

This can also be written as $x \in (-\infty;\infty)$.

Range: $y \in (1;\infty)$

This can also written as $y > 1$.

Notice that the turning point of a parabola and the asymptote of an exponential function determine the range of the functions.

FUNCTION AND MAPPING NOTATION

In Grade 10 you were introduced to different ways of representing functions. The different notations are summarised below:

$y = \ldots \rightarrow$ equation notation

$f(x) = \ldots \rightarrow$ function notation

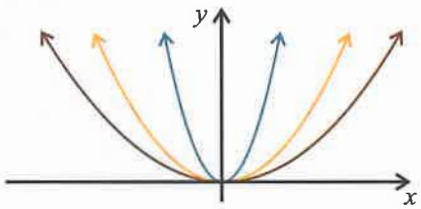
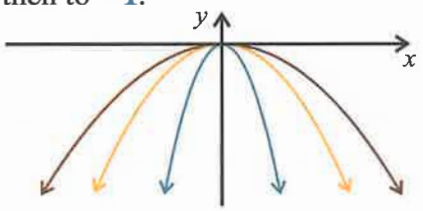
$f : x \rightarrow \ldots \rightarrow$ mapping notation

All of the above representations mean the same thing.

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GRADE 10 REVISION OF THE PARABOLA

In Grade 10 you were introduced to the parabola in the form of $y = ax^2 + q$. It was shown that a determines the shape of a parabola whilst q resulted in a vertical shift (up or down) and was the y-intercept. The properties of a parabola that you learnt in Grade 10 are summarised below:

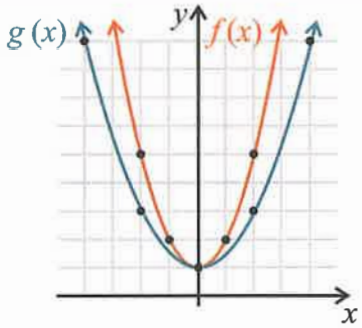
	$a > 0$	$a < 0$
Shape (a)	Concave up (happy face)	Concave down (sad face)
	As the value of a increases from 0 the graph is compressed and gets narrower. This is demonstrated in the graphs below where a is increased from **0,1** to 0,2 and then to **1**. 	As the value of a decreases from 0 the graph is compressed and gets narrower. This is demonstrated in the graphs below where a is decreased from **–0,1** to –0,2 and then to **–1**. 
Vertical Shift (q)	The value of q results in a vertical shift. If the value of q **is greater than 0** the graph will be **shifted up**. If q **is less than 0** the graph will be **shifted down**.	
Domain and Range	**The domain of a parabola is:** $x \in \mathbb{R} \rightarrow$ also written as $x \in (-\infty;\infty)$ **The range of a parabola is:** • $y \geq q$ if $a > 0 \rightarrow$ also written as $y \in [q;\infty)$ • $y \leq q$ if $a < 0 \rightarrow$ also written as $y \in (-\infty;q]$ **The domain and range of the parabola do not change from Grade 10 to Grade 11.**	

Example 1: Use a table to sketch the graphs of $f(x) = x^2 + 1$ and $g(x) = \frac{1}{2}x^2 + 1$.

$f(x)$:	x	–2	–1	0	1	2
	y	5	2	1	2	5

Choose values for x which will avoid fractions in order to make calculations easier.

$g(x)$:	x	–4	–2	0	2	4
	y	9	3	1	3	9



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Example 2: Using a table sketch the graphs of:

a) $g(x)=x^2$, $f(x)=x^2+3$ and $h(x)=x^2-4$.

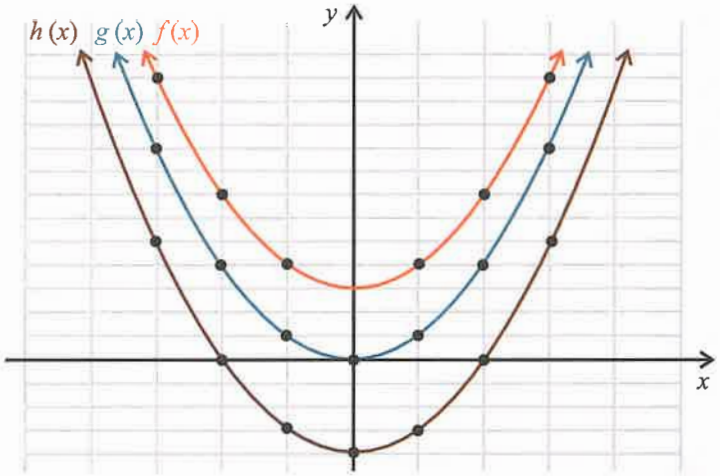
b) $g(x)=-x^2$, $f(x)=-x^2-3$ and $h(x)=-x^2+4$.

a)

$g(x)$:	x	–3	–2	–1	0	1	2	3
	y	9	4	1	0	1	4	9

$f(x)$:	x	–3	–2	–1	0	1	2	3
	y	12	7	4	3	4	7	12

$h(x)$:	x	–3	–2	–1	0	1	2	3
	y	5	0	–3	–4	–3	0	5

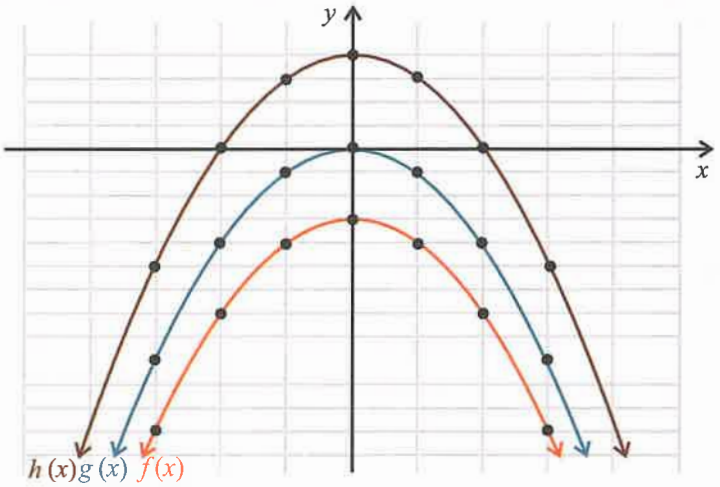


b)

$g(x)$:	x	–3	–2	–1	0	1	2	3
	y	–9	–4	–1	0	–1	–4	–9

$f(x)$:	x	–3	–2	–1	0	1	2	3
	y	–12	–7	–4	–3	–4	–7	–12

$h(x)$:	x	–3	–2	–1	0	1	2	3
	y	–5	0	3	4	3	0	–5



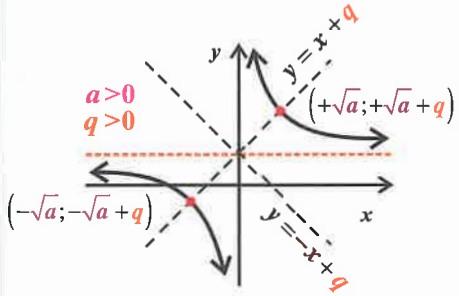
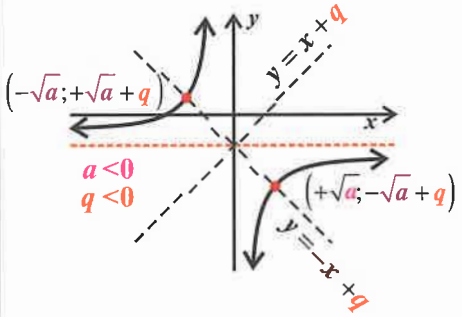
Notice that your graphs in b) are a reflection about the x-axis of the graphs in a). You can use your calculator in table mode to obtain the above tables.

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GRADE 10 REVISION OF THE HYPERBOLA

In Grade 10 you were introduced to the hyperbola in the form of $y=\frac{a}{x}+q$.

It was shown that a determines the shape of a hyperbola as well as the quadrants which the hyperbola lies in. The value of q resulted in a vertical shift (up or down) and the line $y=q$ is the horizontal asymptote. The properties of a hyperbola that you learnt in Grade 10 are summarised below:

<table>
<tr><th></th><th>$a>0$</th><th>$a<0$</th></tr>
<tr><td>Shape (a)</td><td>The graph lies in the 1st and 3rd quadrants formed by the asymptotes and the y-axis.
 
</td><td>The graph lies in the 2nd and 4th quadrants formed by the asymptotes and the y-axis.
 
</td></tr>
<tr><td>Axes of Symmetry</td><td colspan="2">In Grade 10 it was shown that the hyperbola was symmetrical about the lines $y=x+q$ and $y=-x+q$. In Grade 11 the hyperbola will be shifted left and right and the axes of symmetry will be shifted in the same manner.</td></tr>
<tr><td>Points of intersection with the Axes of Symmetry</td><td>If $a>0$ the axis of symmetry $y=x+q$ intersects the hyperbola at the points :
• $\left(+\sqrt{a}^{x};+\sqrt{a}^{y}+q\right)$
• $\left(-\sqrt{a}^{x};-\sqrt{a}^{y}+q\right)$</td><td>If $a<0$ the axis of symmetry $y=-x+q$ intersects the hyperbola at the points :
• $\left(-\sqrt{a}^{x};+\sqrt{a}^{y}+q\right)$
• $\left(+\sqrt{a}^{x};-\sqrt{a}^{y}+q\right)$</td></tr>
<tr><td></td><td colspan="2">As the function is shifted left and right in Grade 11, the points of intersection will also be shifted left and right.
The sign of a is not taken into account when calculating the points of intersection.</td></tr>
<tr><td>Domain and Range</td><td colspan="2">**The domain of a hyperbola in Grade 10 was:**
$x\in\mathbb{R},\ x\neq 0\rightarrow$ also written as $x\in(-\infty;\infty)\ x\neq 0$
This will change in Grade 11 as the graph is shifted left and right.
The range of a hyperbola is:
$y\in\mathbb{R},\ x\neq q\rightarrow$ also written as $y\in(-\infty;\infty),\ y\neq q$</td></tr>
</table>

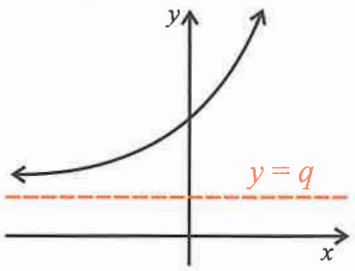
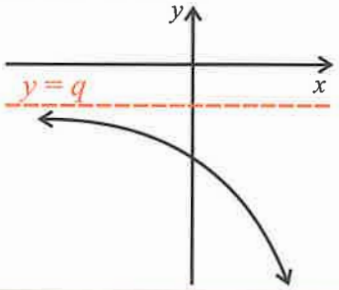
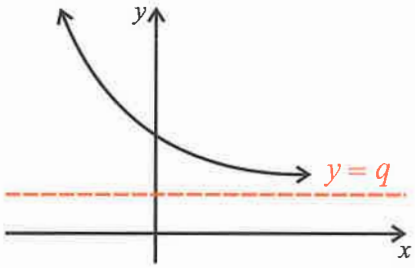
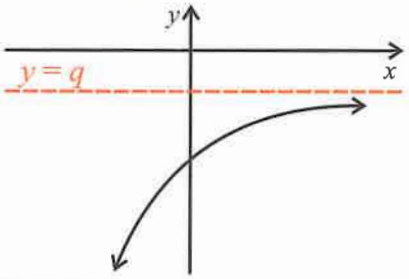
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GRADE 10 REVISION OF THE EXPONENTIAL FUNCTION

In Grade 10 you were introduced to the exponential function in the form of:

$$y = ab^x + q \text{ where } b > 0 \text{ and } b \neq 0$$

It was shown that a determines the shape of the graph as well as on which side of the asymptote the exponential function lies. Together with a the value of b determines whether the function is increasing or decreasing.
The value of q resulted in a vertical shift (up or down) and the line $y = q$ is the horizontal asymptote. The properties of the exponential function that you learnt in Grade 10 are summarised below:

<table>
<tr><td rowspan="4">Shape
(a and b)</td><td>$a > 0$ and $b > 1$</td><td>$a < 0$ and $b > 1$</td></tr>
<tr><td>The graph lies above the horizontal asymptote and is an increasing function.
 
</td><td>The graph lies below the horizontal asymptote and is a decreasing function.
 
</td></tr>
<tr><td>$a > 0$ and $0 < b < 1$</td><td>$a < 0$ and $0 < b < 1$</td></tr>
<tr><td>The graph lies above the horizontal asymptote and is a decreasing function.
 
</td><td>The graph lies below the horizontal asymptote and is an increasing function.
 
</td></tr>
<tr><td>Domain and Range</td><td colspan="2">The domain of an exponential function is:
$x \in \mathbb{R} \rightarrow$ also written as $x \in (-\infty;\infty)$
The range of an exponential function is:
• $y > q$ if $a > 0 \rightarrow$ also written as $y \in (q;\infty)$
• $y < q$ if $a < 0 \rightarrow$ also written as $y \in (-\infty;q)$
The domain and range of the exponential function do not change from Grade 10 to Grade 11.</td></tr>
</table>

Recap:

A function is increasing if the y-values increase as the x-values increase.
A function is decreasing if the y-values decrease as the x-values increase.

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B. THE PARABOLA (QUADRATIC FUNCTION)

In Grade 11 you will now work with shifting the parabola horizontally (left and right) as well as vertically. When sketching a parabola the equation of the parabola will usually be represented in 2 ways:

1. $y = a(x-p)^2 + q \rightarrow$ turning point form
2. $y = ax^2 + bx + c \rightarrow$ standard form

THE PARABOLA IN THE FORM OF $y = a(x-p)^2 + q$

When the parabola has the form : $y = a(x-p)^2 + q$

$a \rightarrow$ *shape*

$p \rightarrow x$ - value of the turning point (and horizontal shift)

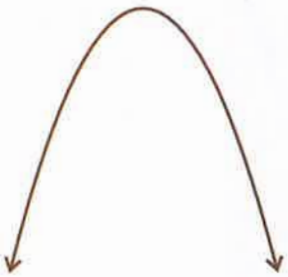
$q \rightarrow y$ - value of the turning point (and vertical shift)

STEPS TO SKETCHING THE PARABOLA:

1. **Test for shape:** if $a > 0 \rightarrow$ happy face parabola (concave up)
   if $a < 0 \rightarrow$ sad face parabola (concave down)
2. **Determine the x and y coordinates of the turning point (TP) from the equation.**
3. **Determine the y-intercept by letting x equal to 0.**
4. **Determine the x-intercept/s by letting y equal to 0.**



$a > 0$: Happy face parabola
(Concave up)

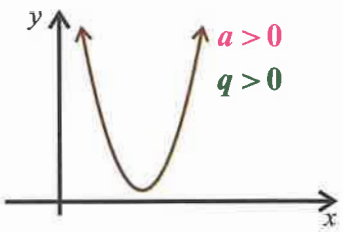


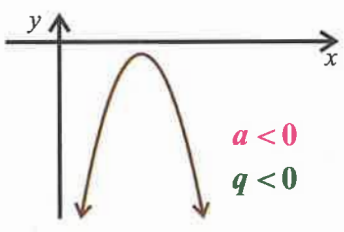
$a < 0$: Sad face parabola
(Concave down)

If $a > 0$ the y value of the TP will be the minimum value of the graph.
If $a < 0$ the y value of the TP will be the maximum value of the graph.

It is important to note that:

- **If the graph is 'happy' and the y-value of the turning point is positive, the graph will not have any x-intercepts.**
- **If the graph is 'sad' and the y-value of the turning point is negative, the graph will not have any x-intercepts.**





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Example 1: On the same set of axes, sketch the graphs of $f(x)=(x+2)^2-1$ and $g(x)=-(x-3)^2+4$.

$f(x)$:

1) Shape: $a>0 \rightarrow$ happy face

2) Turning point: $(-2;-1)$

3) y-int: let $x=0$

$y=(0+2)^2-1=3$

$\therefore y$-int: $(0;3)$

4) x-int: let $y=0$

$0=(x+2)^2-1$

$\therefore 0=x^2+4x+4-1$

$\therefore 0=x^2+4x+3$

$\therefore 0=(x+3)(x+1)$

$\therefore x=-3$ or $x=-1$

$\therefore x$-int's: $(-3;0)$ and $(-1;0)$

$g(x)$:

Shape: $a<0 \rightarrow$ sad face

Turning point: $(3;4)$

y-int: let $x=0$

$y=-(0-3)^2+4=-5$

$\therefore y$-int: $(0;-5)$

x-int: let $y=0$

$0=-(x-3)^2+4$

$\therefore 0=-\left(x^2-6x+9\right)+4$

$\therefore 0=-x^2+6x-9+4$

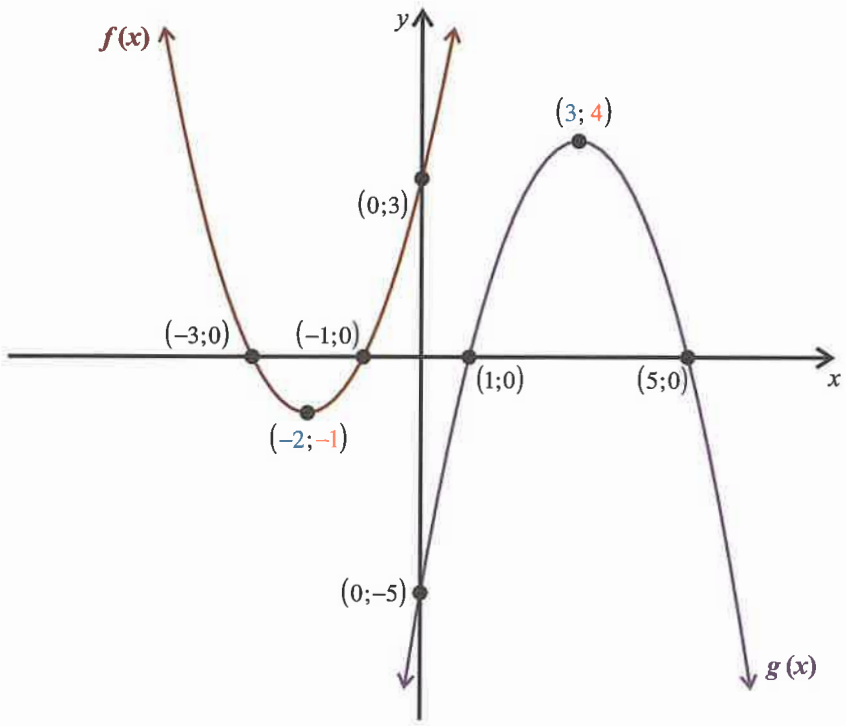
$\therefore 0=-x^2+6x-5$

$\therefore 0=x^2-6x+5$

$\therefore 0=(x-1)(x-5)$

$\therefore x=1$ or $x=5$

$\therefore x$-int's: $(1;0)$ and $(5;0)$



A sketch graph does not have to be accurate, but it does have to have the correct shape and you must show the intercepts with the axes as well as the turning point. Note that the sign of the *x*-coordinate of the turning point is usually opposite to the sign of *p* in the equation.

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Example 2: On the same set of axes, sketch the graphs of $f(x)=(x-3)^2-4$ and $h(x)=-2(x+3)^2+2$.

$f(x)$:

1) Shape: $a>0 \rightarrow$ happy face

2) Turning point: $(3;-4)$

3) y-int: let $x=0$

$y=(0-3)^2-4=5$

$\therefore$ y-int: $(0;5)$

4) x-int: let $y=0$

$0=(x-3)^2-4$

$\therefore 0=x^2-6x+9-4$

$\therefore 0=x^2-6x+5$

$\therefore 0=(x-1)(x-5)$

$\therefore x=1$ or $x=5$

$\therefore$ x-int's: $(1;0)$ and $(5;0)$

$h(x)$:

Shape: $a<0 \rightarrow$ sad face

Turning point: $(-3;2)$

y-int: let $x=0$

$y=-2(0+3)^2+2=-16$

$\therefore$ y-int: $(0;-16)$

x-int: let $y=0$

$0=-2(x+3)^2+2$

$\therefore 0=-2\left(x^2+6x+9\right)+2$

$\therefore 0=-2x^2-12x-18+2$

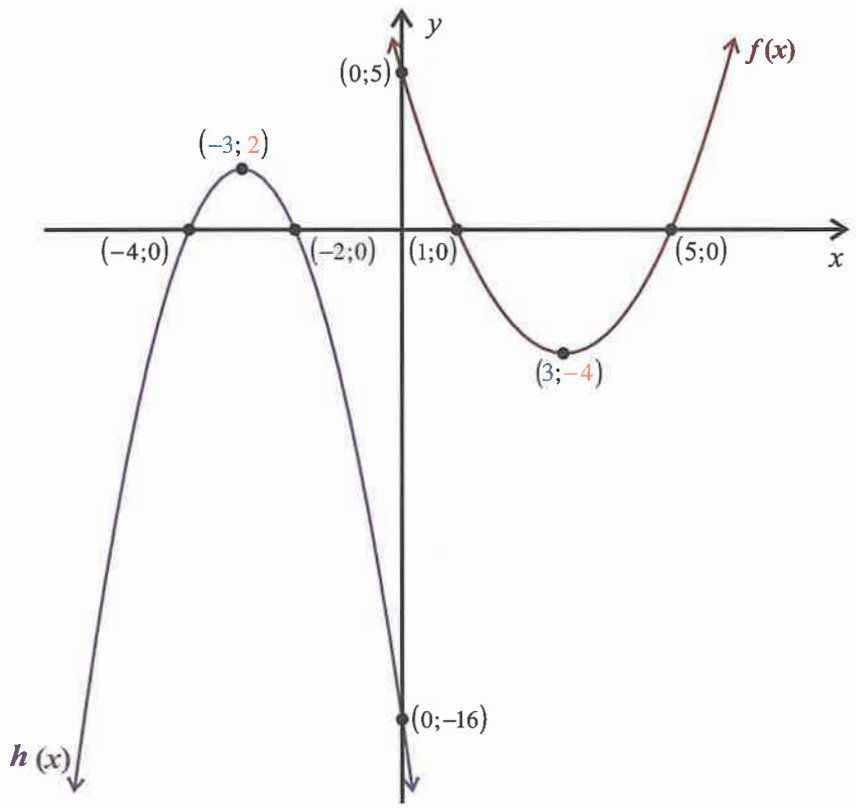
$\therefore 0=-2\left(x^2+6x+8\right) \rightarrow$ divide by -2

$\therefore 0=x^2+6x+8$

$\therefore 0=(x+2)(x+4)$

$\therefore x=-2$ or $x=-4$

$\therefore$ x-int's: $(-2;0)$ and $(-4;0)$





Complete Exercise 1 number 1 on page 303.

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THE PARABOLA IN THE FORM OF $y = ax^2 + bx + c$

When the parabola is in the form $y = ax^2 + bx + c$ the value of c is the y-intercept and as before a tells you about the shape of the parabola.

In order to determine the turning point of the parabola you can complete the square on the general equation as shown below.

$$y = ax^2 + bx + c$$

$$\therefore y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \rightarrow \text{add and subtract } \left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2$$

$$\therefore y = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$\therefore y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \rightarrow \text{LCD} = 4a^2$$

$$\therefore y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \rightarrow \text{multiply } a \text{ back in}$$

$$\therefore y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \rightarrow x\text{-coordinate of the TP and } y\text{-coordinate of the TP}$$

From this it can be seen that $x = -\frac{b}{2a}$ is the x-coordinate of the turning point.

This line is also the axis of symmetry. The parabola is always symmetrical about its turning point. This means that it is a mirror image of itself over a vertical line passing through the turning point.

STEPS TO SKETCHING THE PARABOLA:

The only difference in sketching the parabola in the form $y = ax^2 + bx + c$ is that you have to determine the turning point by using the axis of symmetry (AS) or by completing the square.

1. **Test for shape:** **if $a > 0 \rightarrow$ happy face parabola** (concave up) ☺
   if $a < 0 \rightarrow$ sad face parabola (concave down) ☹
2. **Determine the x and y coordinates of the turning point.**
   a) **Determine the Axis of Symmetry (AS) using the equation $x = \frac{-b}{2a}$.**
   b) **Substitute the AS into the equation of the parabola to find the y-coordinate of the turning point.**
3. **Determine the y-intercept by letting x equal to 0.**
4. **Determine the x-intercept/s by letting y equal to 0.**



The parabola only has 1 axis of symmetry (the line of $x = \frac{-b}{2a}$) and has no asymptotes.

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Example 1: **On the same set of axes, sketch the graphs of $g(x) = -2x^2 - 12x - 10$ and $f(x) = x^2 - 6x + 5$.**

$g(x)$:

1) Shape: $a < 0 \rightarrow$ sad face

2) AS: $x = \frac{-b}{2a} = \frac{-(-12)}{2(-2)} = -3$

$g(-3) = -2(-3)^2 - 12(-3) - 10 = 8$

$\therefore$ TP : $(-3;8)$

3) y-int : $(0;-10)$

4) x-int : let $y = 0$

$0 = -2x^2 - 12x - 10$

$\therefore 0 = -2\left(x^2 + 6x + 5\right) \rightarrow$ divide by -2

$\therefore 0 = x^2 + 6x + 5$

$\therefore 0 = (x+5)(x+1)$

$\therefore x = -5$ or $x = -1$

$\therefore x$-int 's: $(-5;0)$ and $(-1;0)$

$f(x)$:

Shape: $a > 0 \rightarrow$ happy face

AS: $x = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3$

$f(3) = (3)^2 - 6(3) + 5 = -4$

$\therefore$ TP: $(3;-4)$

y-int : $(0;5)$

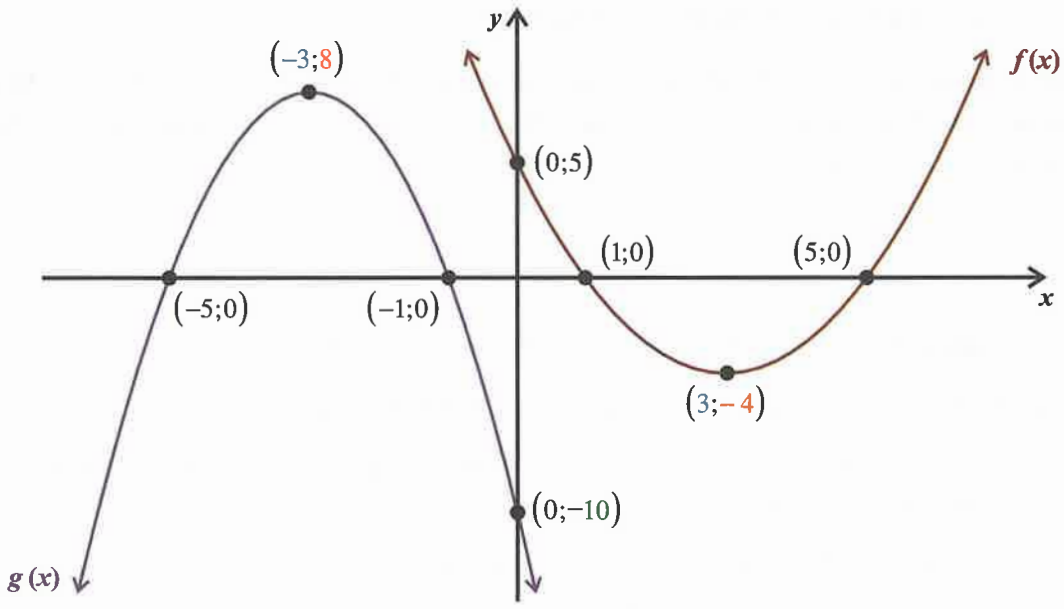
x-int : let $y = 0$

$0 = x^2 - 6x + 5$

$\therefore 0 = (x-1)(x-5)$

$\therefore x = 1$ or $x = 5$

$\therefore x$-int 's: $(1;0)$ and $(5;0)$



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Example 2: On the same set of axes, sketch the graphs of $f(x)=x^2-2x-3$ and $g(x)=-\frac{1}{4}x^2-3x-8$.

$f(x)$:

1) Shape: $a>0 \rightarrow$ happy face

2) AS: $x=\frac{-b}{2a}=\frac{-(-2)}{2(1)}=1$

$f(1)=(1)^2-2(1)-3=-4$

$\therefore$ TP: $(1;-4)$

3) y-int: $(0;-3)$

4) x-int: let $y=0$

$0=x^2-2x-3$

$\therefore 0=(x+1)(x-3)$

$x=-1$ or $x=3$

$\therefore x$-int's: $(-1;0)$ and $(3;0)$

$g(x)$:

Shape: $a<0 \rightarrow$ sad face

AS: $x=\frac{-b}{2a}=\frac{-(-3)}{2\left(-\frac{1}{4}\right)}=-6$

$g(-6)=-\frac{1}{4}(-6)^2-3(-6)-8=1$

$\therefore$ TP: $(-6;1)$

y-int: $(0;-8)$

x-int: let $y=0$

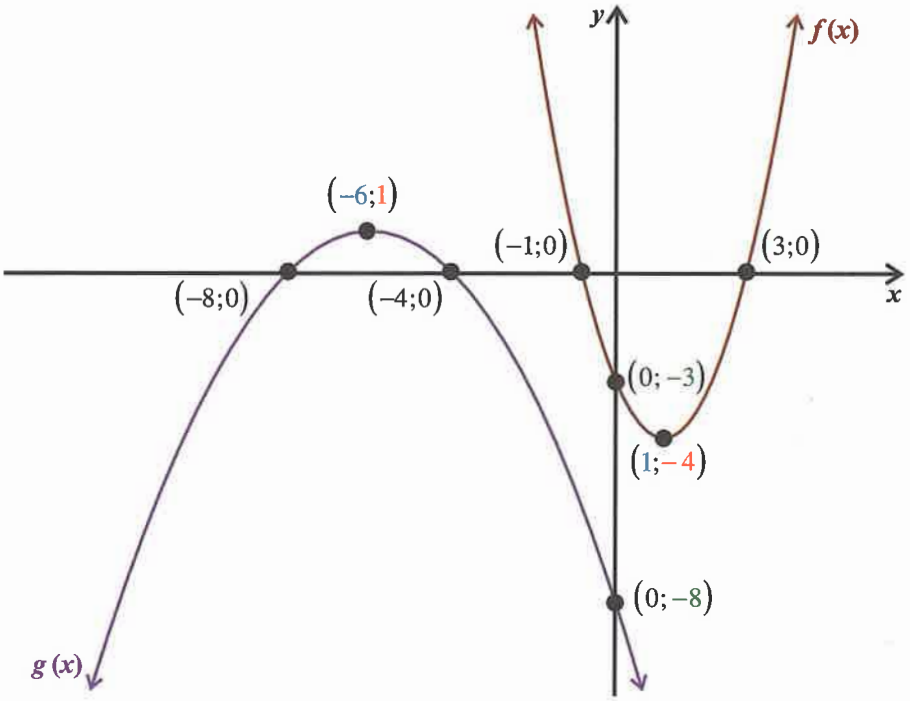
$0=-\frac{1}{4}x^2-3x-8 \rightarrow$ multiply by -4

$\therefore 0=x^2+12x+32$

$\therefore 0=(x+8)(x+4)$

$\therefore x=-8$ or $x=-4$

$\therefore x$-int's: $(-8;0)$ and $(-4;0)$





Complete Exercise 1 number 2 on page 303.

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DETERMINING THE EQUATION OF A PARABOLA

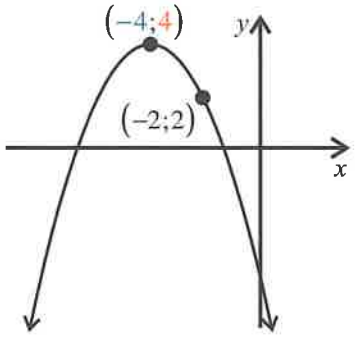
There are usually three ways in which you will be expected to find the equation of a parabola (quadratic function):

1. **You will be given the turning point and one other point on the curve.**
2. **You will be given the x-intercepts and one other point on the curve.**
3. **You will be given the y-intercept and two other point on the curve.**

1. GIVEN THE TURNING POINT AND ONE OTHER POINT

In this case, use the equation of the parabola in turning point form: $y = a(x-p)^2 + q$

Example: **Determine the equation of $f(x)$ shown below, if the point $(-4;4)$ is its turning point and $(-2;2)$ is a point on the graph. Write your answer in the form $f(x) = ax^2 + bx + c$.**



Turning Point (TP): $(-4;4)$

$\therefore y = a\left[x-(-4)\right]^2 + 4$

$\therefore y = a(x+4)^2 + 4$

Now substitute the other known point $(-2;2)$ into the above equation to solve for a.

$\therefore 2 = a(-2+4)^2 + 4$

$\therefore 2 = 4a + 4$

$\therefore -2 = 4a$

$\therefore a = -\frac{1}{2}$

$\therefore y = -\frac{1}{2}(x+4)^2 + 4$

$\therefore y = -\frac{1}{2}(x^2 + 8x + 16) + 4$ → don't forget the middle term

$\therefore y = -\frac{1}{2}x^2 - 4x - 8 + 4$

$\therefore f(x) = -\frac{1}{2}x^2 - 4x - 4$ → always write the equation in standard form



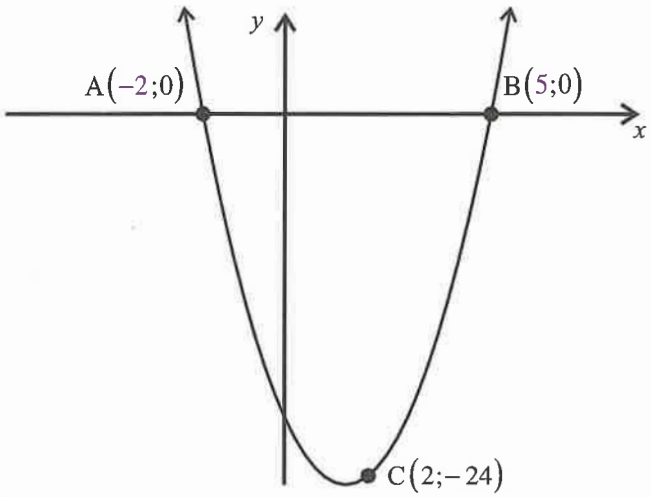
Complete Exercise 2 number 1 on page 303.

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2. GIVEN THE x-INTERCEPTS AND ONE OTHER POINT

Another form of the equation of the parabola is $y = a(x - R_1)(x - R_2)$.
In this form R_1 and R_2 are the roots (x-intercepts) of the equation.
When given the x-intercepts and another point, use the equation in this form in order to determine the equation of the parabola.

Example: **Determine the equation of the parabola shown below. Points $A(-2;0)$ and $B(5;0)$ are the x-intercepts and $C(2;-24)$ is a point on the graph.**



$\therefore y = a[x-(-2)][x-5]$

$\therefore y = a(x+2)(x-5)$

Now substitute the other known point $(2;-24)$ into the above equation to solve for a

$\therefore -24 = a(2+2)(2-5)$

$\therefore -24 = -12a$

$\therefore a = 2$

$\therefore y = 2(x+2)(x-5)$

$\therefore y = 2(x^2 - 3x - 10)$

$\therefore y = 2x^2 - 6x - 20$ → always write the equation in standard form



Complete Exercise 2 number 2 on page 303.

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3. GIVEN THE y-INTERCEPT AND TWO OTHER POINTS

In this case use the equation of the parabola in the form of $y=ax^2+bx+c$ and use simultaneous equations to solve for the values of a and b.

Example: **A parabola's y-intercept is at point $A(0;-2)$. If points $B(2;8)$ and $C(-6;-8)$ are points which lie on the graph, determine the equation of the parabola.**

$\therefore y=ax^2+bx+c \rightarrow c$ is the y-intercept (-2)

$y=ax^2+bx-2$

Substitute the point $B(2;8)$ into the equation:

$\therefore 8=a(2)^2+b(2)-2$

$\therefore 8=4a+2b-2$

$\therefore 10=4a+2b \rightarrow [1]$

Substitute the point $C(-6;-8)$ into the equation:

$\therefore -8=a(-6)^2+b(-6)-2$

$\therefore -8=36a-6b-2$

$\therefore 6b=36a+6$

$\therefore b=6a+1 \rightarrow [2]$

Substitute [2] into [1]:

$\therefore 10=4a+2(6a+1)$

$\therefore 10=4a+12a+2$

$\therefore 8=16a$

$\therefore a=\frac{1}{2} \rightarrow [3]$

Substitute [3] into [2]:

$\therefore b=6\left(\frac{1}{2}\right)+1$

$\therefore b=4$

$\therefore y=\frac{1}{2}x^2+4x-2$



Complete Exercise 2 number 3 on page 304.

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C. THE HYPERBOLA

In Grade 11 the hyperbola will be in the form of $y = \frac{a}{x-p} + q$.

The hyperbola will either lie in the 1st and 3rd quadrants formed by the asymptotes, or in the 2nd and 4th quadrants formed by the asymptotes. This is summarised below:

$a \rightarrow$ a is a constant which determines the quadrants that the hyperbola will lie in

If $a > 0$ the hyperbola will lie in the 1st and 3rd quadrants

If $a < 0$ the hyperbola will lie in the 2nd and 4th quadrants

$p \rightarrow$ p is the vertical asymptote (the line $x = p$)

$q \rightarrow$ q is the horizontal asymptote (the line $y = q$)

It is important to note that the value of p shifts the graph horizontally while q shifts the graph vertically.

AXES OF SYMMETRY AND POINTS OF INTERSECTION

The hyperbola has 2 axes of symmetry:

$$y = x - p + q \quad \text{and} \quad y = -(x-p) + q = -x + p + q$$

The axes of symmetry will cut the hyperbola at the following points:

If $a > 0$ the line $y = x - p + q$ will cut the hyperbola at:

$\left(+\sqrt{a}+p;+\sqrt{a}+q\right)$ and $\left(-\sqrt{a}+p;-\sqrt{a}+q\right)$

If $a < 0$ the line $y = -x + p + q$ will cut the hyperbola at:

$\left(-\sqrt{a}+p;+\sqrt{a}+q\right)$ and $\left(+\sqrt{a}+p;-\sqrt{a}+q\right)$

The sign of a is not taken into account when calculating the points of intersection.

Hyperbolas with p and q equal to 0 are shown below:

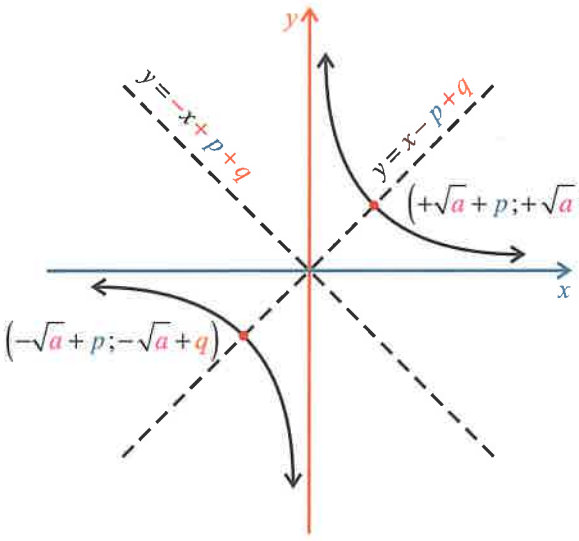


Figure 1: $a > 0$

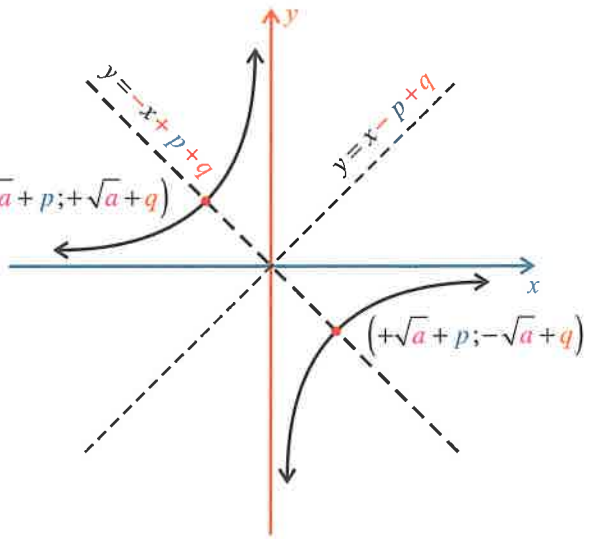


Figure 2: $a < 0$

The sign of p is always opposite to the sign of p in the denominator.

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STEPS TO SKETCHING THE HYPERBOLA:

1. **Determine the vertical and horizontal asymptotes and which quadrants the hyperbola lies in.**
2. **Determine the x-intercept by letting $y=0$ and the y-intercept by letting $x=0$.**
3. **If asked, determine the equations of the axes of symmetry.**
4. **If asked, determine the points of intersection of the axes of symmetry with the hyperbola.**

Example 1: Sketch the graph of $y=\frac{4}{x-2}+4$ showing both axes of symmetry and their point(s) of intersection with the graph.

1) Determine the asymptotes and quadrants:

$a=+4 \rightarrow a>0$

$\therefore$ The hyperbola lies in the 1st and 3rd quadrants

Asymptotes: $x=2 \rightarrow$ vertical

$y=4 \rightarrow$ horizontal

2) Determine the x and y-intercepts:

x-int: let $y=0$

$$0=\frac{4}{x-2}+4$$

$$\therefore -4=\frac{4}{x-2} \rightarrow \text{multiply both sides by } x-2$$

$$\therefore -4(x-2)=4$$

$$\therefore -4x+8=4$$

$$\therefore x=1$$

$\therefore x$-int: $(1;0)$

y-int: let $x=0$

$$y=\frac{4}{0-2}+4$$

$$\therefore y=2$$

$\therefore y$-int: $(0;2)$

3) Determine the equations of the axes of symmetry:

$y=x-p+q$

$\therefore y=x-2+4$

$\therefore y=x+2$

$y=-(x-p)+q$

$\therefore y=-(x-2)+4$

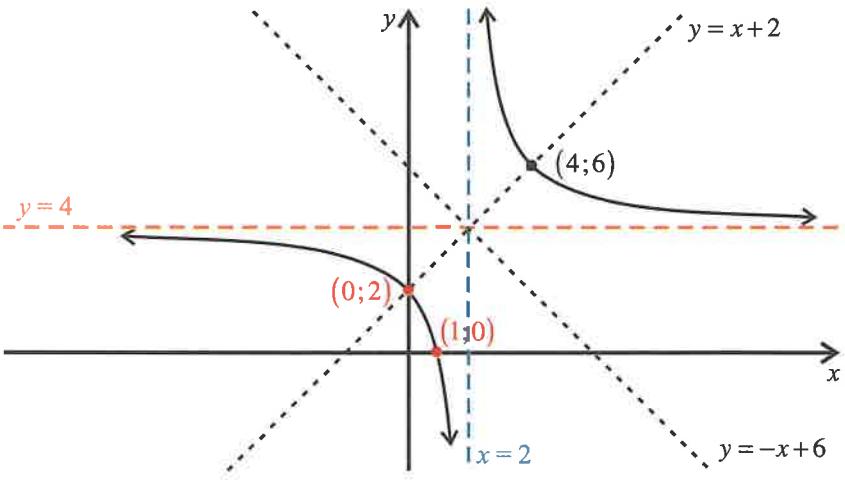
$\therefore y=-x+6$

4) Determine the points of intersection:

$\left(+\sqrt{a}+p;+\sqrt{a}+q\right)$ and $\left(-\sqrt{a}+p;-\sqrt{a}+q\right) \rightarrow a>0$

$=\left(\sqrt{4}+2;\sqrt{4}+4\right)$ and $\left(-\sqrt{4}+2;-\sqrt{4}+4\right)$

$=(4;6)$ and $(0;2)$



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Example 2: Sketch the graph of $y=\frac{-4}{x+2}+4$ showing both axes of symmetry and their point(s) of intersection with the graph.

1) Determine the asymptotes and quadrants:

$a=-4 \rightarrow a<0$

$\therefore$ The hyperbola lies in the 2^{nd} and 4^{th} quadrants

Asymptotes: $x=-2 \rightarrow$ vertical

$y=4 \rightarrow$ horizontal

2) Determine the x and y-intercepts:

x-int: let $y=0$

$$0=\frac{-4}{x+2}+4$$

$$\therefore -4=\frac{-4}{x+2} \rightarrow \text{multiply both sides by } x+2$$

$$\therefore -4(x+2)=-4$$

$$\therefore -4x-8=-4$$

$$\therefore x=-1$$

$\therefore x$-int: $(-1;0)$

y-int: let $x=0$

$$y=\frac{-4}{0+2}+4$$

$$\therefore y=2$$

$\therefore y$-int: $(0;2)$

3) Determine the equations of the axes of symmetry:

$y=x-p+q$

$$\therefore y=x-(-2)+4$$

$$\therefore y=x+6$$

$y=-(x-p)+q$

$$\therefore y=-[x-(-2)]+4$$

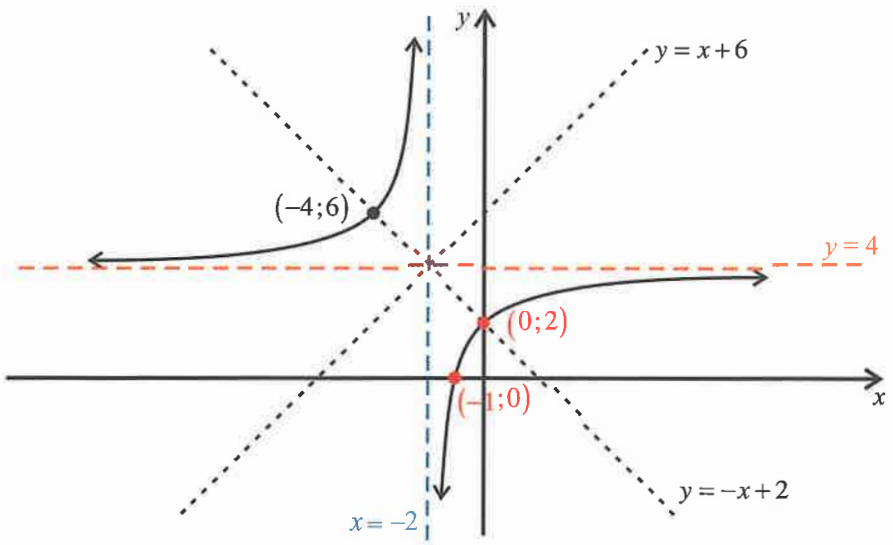
$$\therefore y=-x+2$$

4) Determine the points of intersection:

$\left(-\sqrt{a}+p;+\sqrt{a}+q\right)$ and $\left(\sqrt{a}+p;-\sqrt{a}+q\right) \rightarrow a<0$

$=\left(-\sqrt{4}-2;\sqrt{4}+4\right)$ and $\left(\sqrt{4}-2;-\sqrt{4}+4\right) \rightarrow$ the sign of a is not taken into account

$=(-4;6)$ and $(0;2)$





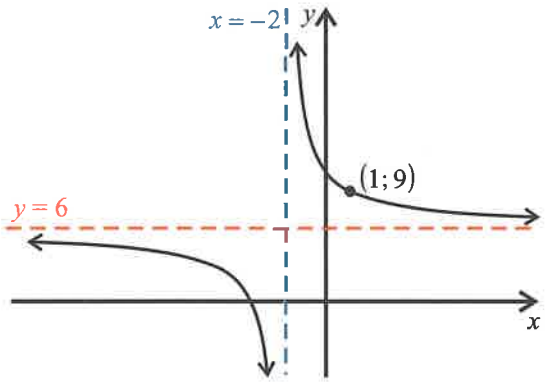
Complete Exercise 3 on page 304.

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DETERMINING THE EQUATION OF A HYPERBOLA

In most cases when finding the equation of a hyperbola you will be given the vertical and horizontal asymptotes or enough information to find them as well as a point on the graph. Once you have these values they can be substituted into p and q. The value of a can then be found by substituting the point on the graph into the equation $y = \frac{a}{x-p} + q$.

Example 1: Determine the equations of the hyperbolas shown below:



$y = \frac{a}{x-p} + q$

$\therefore y = \frac{a}{x+2} + 6 \rightarrow p = -2$ and $q = 6$

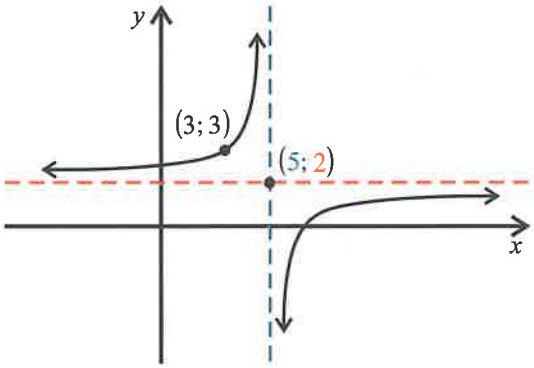
Sub the given point $(1;9)$ into the equation

$\therefore 9 = \frac{a}{1+2} + 6$

$\therefore 3 = \frac{a}{3} \rightarrow$ multiply both sides by 3

$\therefore a = 9$

$\therefore y = \frac{9}{x+2} + 6$



$y = \frac{a}{x-p} + q$

$\therefore y = \frac{a}{x-5} + 2 \rightarrow p = 5$ and $q = 2$

Sub the given point $(3;3)$ into the equation

$\therefore 3 = \frac{a}{3-5} + 2$

$\therefore 1 = \frac{a}{-2} \rightarrow$ multiply both sides by -2

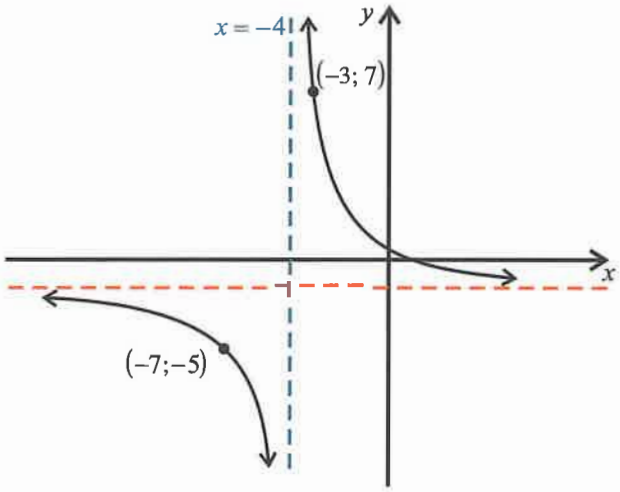
$\therefore a = -2$

$\therefore y = \frac{-2}{x-5} + 2$

The x-value anywhere on the vertical asymptote is the value of p and the y-value anywhere on the horizontal asymptote is the value of q.

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Example 2: **Determine the equation of the hyperbola shown below. All points shown are points that lie on the hyperbola and the line $x=-4$ is the equation of the vertical asymptote.**



In a case where only one asymptote is provided **simultaneous equations** will be required to determine the equation of the hyperbola.

$y=\frac{a}{x+4}+q \rightarrow$ it is given that $p=-4$

Sub point $(-3;7)$

$\therefore 7=\frac{a}{-3+4}+q$

$\therefore 7-q=a \rightarrow [1]$

Sub point $(-7;-5)$

$\therefore -5=\frac{a}{-7+4}+q$

$\therefore -5=\frac{a}{-3}+q \rightarrow [2]$

Sub [1] into [2]:

$\therefore -5=\frac{(7-q)}{-3}+q \rightarrow a=7-q$

$\therefore -5-q=\frac{7-q}{-3} \rightarrow$ multiply boths sides by -3

$\therefore 15+3q=7-q$

$\therefore 4q=-8$

$\therefore q=-2 \rightarrow [3]$

Sub [3] into [1]:

$a=7-(-2)=9$

$\therefore y=\frac{9}{x+4}-2$



Complete Exercise 4 on page 304.

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D. THE EXPONENTIAL FUNCTION

In Grade 11 the exponential function has the form:

$$y = ab^{x-p} + q \text{ where } b > 0 \text{ and } b \neq 1$$

The values of a, p and q shift the graph as follows:

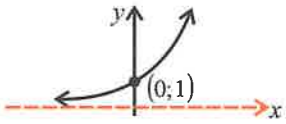
$a \rightarrow$ The sign of a determines which side of the asymptote the graph will lie (shape).
If $a > 0$ then the graph will lie above the asymptote.
If $a < 0$ then the graph will lie below the asymptote.

$p \rightarrow$ p shifts the graph left or right.

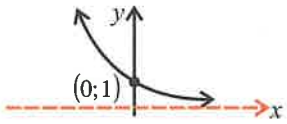
$q \rightarrow$ The line $y = q$ is the horizontal asymptote (the vertical shift).

In order to demonstrate the shape and y-intercept of the exponential function we will let $a = 1$, $p = 0$ and $q = 0$. This results in the standard graph of $y = b^x$ which is sketched below:

$a = 1$ and $b > 1$
Increasing function



$a = 1$ and $0 < b < 1$
Decreasing function



By letting $x = 0$ the y-intercept of the graph is:

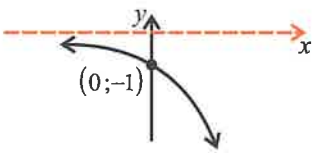
$y = b^0 \rightarrow$ any number raised to the power of 0 is equal to 1

$\therefore y = 1$

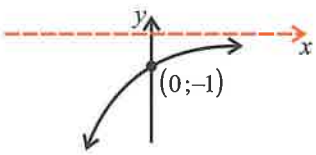
$\therefore$ The y-intercept is at the point $(0;1)$.

So, for the standard exponential graph where $a = 1$ the x-axis is the horizontal asymptote and the y-intercept will be at $(0;1)$. If $a = -1$ the graph is reflected about the x-axis and the y-intercept will be at $(0;-1)$ and the graph will lie below the horizontal asymptote, as sketched below.

$a = -1$ and $b > 1$
Decreasing function



$a = -1$ and $0 < b < 1$
Increasing function





The domain of an exponential function is:

$x \in \mathbb{R} \rightarrow$ also written as $x \in (-\infty;\infty)$

The range of an exponential function is:

- $y > q$ if $a > 0 \rightarrow$ also written as $y \in (q;\infty)$
- $y < q$ if $a < 0 \rightarrow$ also written as $y \in (-\infty;q)$

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STEPS TO SKETCHING THE EXPONENTIAL FUNCTION

1. **Check if the graph is above or below the asymptote:**
   if $a > 0 \rightarrow$ graph is above the asymptote
   if $a < 0 \rightarrow$ graph is below the asymptote

2. **Check if the function is increasing or decreasing:**
   if $b > 1$ *and* $a > 0 \rightarrow$ graph is increasing (as x increases so does the value of y)
   if $0 < b < 1$ *and* $a > 0 \rightarrow$ graph is decreasing (as x increases the value of y decreases)

   if $b > 1$ *and* $a < 0 \rightarrow$ graph is decreasing (as x increases the value of y decreases)
   if $0 < b < 1$ *and* $a < 0 \rightarrow$ graph is increasing (as x increases so does the value of y)

   Note : if b is raised to a negative 1 it is the same as writing $\frac{1}{b}$. $\rightarrow b^{-n} = \frac{1}{b^n}$

3. **Find the y intercept by setting x equal to 0.**

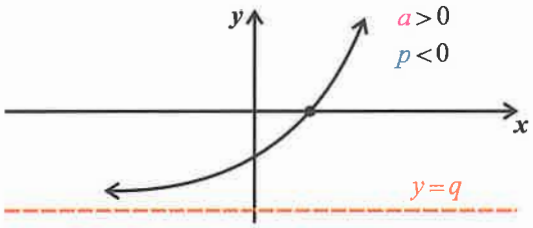
4. **Find the horizontal asymptote $(y = q)$.**

5. **Find the x-intercept by setting y equal to 0.**

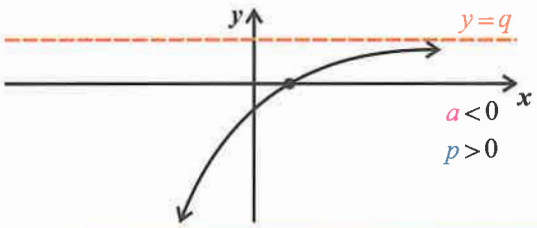
6. **Substitute a value for x in order to obtain another point on the graph and obtain the correct shape.**

The exponential function will only have an x-intercept if:

$a > 0$ and $q < 0 \rightarrow$ the horizontal asymptote lies below the x-axis



$a < 0$ and $q > 0 \rightarrow$ the horizontal asymptote lies above the x-axis



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Example 1: Sketch the graph of $f(x)=4(2)^{x-2}+1$.

1) $a>0 \rightarrow$ graph is above the horizontal asymptote

2) $b>1$ and $a>0$ $\therefore$ graph is increasing

3) Find the y-intercept:

$y-\text{int}: x=0$

$\therefore y=4(2)^{0-2}+1$

$\therefore y=4(2)^{-2}+1$

$\therefore y=4\left(\frac{1}{4}\right)+1$

$\therefore y=2$

$\therefore y\text{-int: } (0;2)$

4) $y=1$ is the horizontal asymptote

5) x-intercept: none $\rightarrow a>0$ and $q>0$

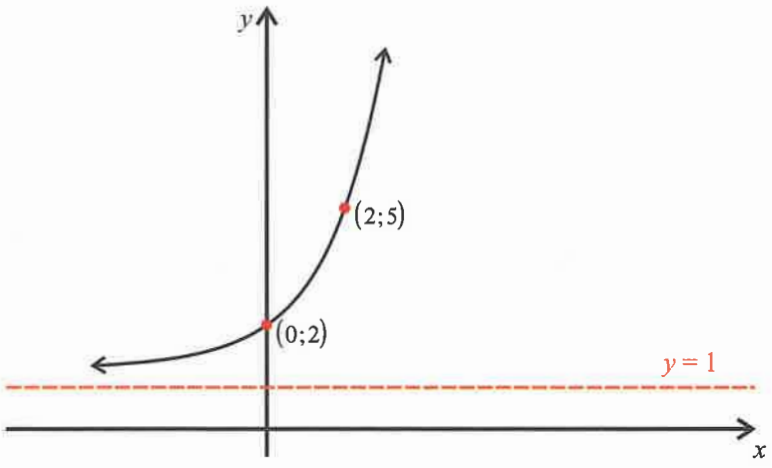
6) Substitute a value for x in order to obtain another point on the graph:
substitute $x=2 \rightarrow x=p$ is a good choice

$\therefore y=4(2)^{2-2}+1$

$\therefore y=4(1)+1$

$\therefore y=5$

$\therefore$ Additional point: $(2;5)$



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Example 2: Sketch the graph of $y = -\frac{1}{2}(4)^{-x+2} + 1$.

Note the minus sign in front of the x. Always rewrite the equation so that x is positive by using the rule that $b^{-x} = \left(\frac{1}{b}\right)^{x}$.

$\therefore y = -\frac{1}{2}(4)^{-(x-2)} + 1$

$\therefore y = -\frac{1}{2}\left(\frac{1}{4}\right)^{x-2} + 1$

1) $a < 0 \rightarrow$ graph is below the horizontal asymptote

2) $0 < b < 1$ and $a < 0$ $\therefore$ graph is increasing

3) Find the y-intercept:

$y-\text{int}: x = 0$

$\therefore y = -\frac{1}{2}\left(\frac{1}{4}\right)^{0-2} + 1 = -7$

$\therefore y\text{-int: } (0;-7)$

4) $y = 1$ is the horizontal asymptote

5) Find the x-intercept by setting $y = 0$:

$\therefore 0 = -\frac{1}{2}(4)^{-(x-2)} + 1$

$\therefore -1 = -\frac{1}{2}\left(\frac{1}{4}\right)^{x-2} \rightarrow$ multiply by -2

$\therefore 2 = \left(2^{-2}\right)^{x-2} \rightarrow \frac{1}{4} = 2^{-2}$

$\therefore 2^{1} = 2^{-2x+4}$

$\therefore 1 = -2x + 4$

$\therefore 2x = 3$

$\therefore x = \frac{3}{2}$

$\therefore x\text{-int: } \left(\frac{3}{2};0\right)$

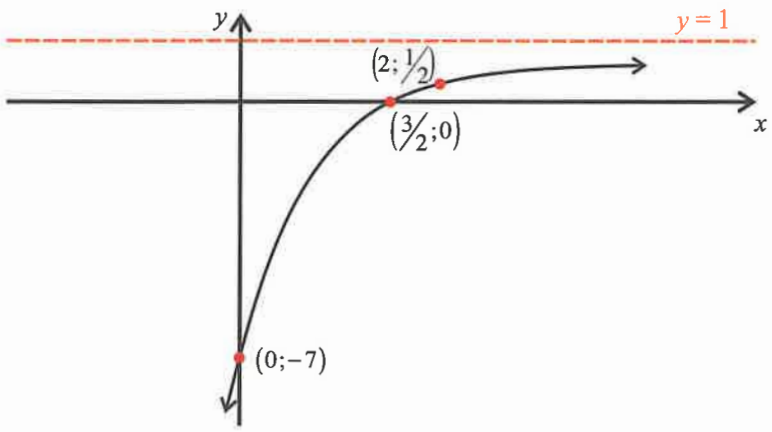
6) Substitute a value for x in order to obtain another point on the graph:
substitute $x = 2 \rightarrow p$ is a good choice

$\therefore y = -\frac{1}{2}(4)^{-2+2} + 1$

$\therefore y = -\frac{1}{2}(1) + 1$

$\therefore y = \frac{1}{2}$

$\therefore$ Additional point: $\left(2;\frac{1}{2}\right)$



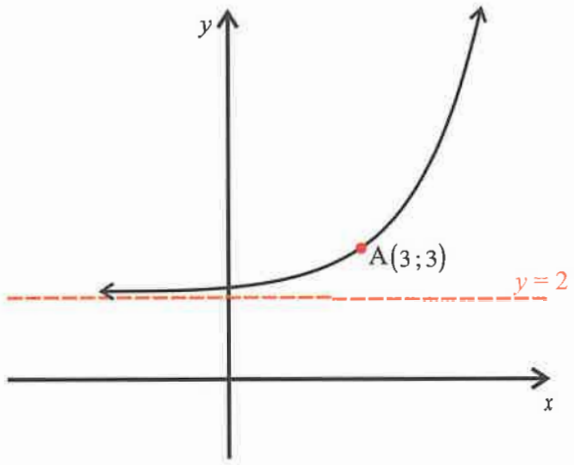
Complete Exercise 5 on page 304.

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DETERMINING THE EQUATION OF AN EXPONENTIAL GRAPH

In most cases, when finding the equation of an exponential graph you will be given the horizontal asymptote or enough information to find it. You will also be given a point or points which lie on the graph. Once you have the horizontal asymptote it can be substituted into q and the remaining unknowns can then be found by substituting the given point or points into the equation.

Example: **The exponential function below has the equation $f(x)=2^{x-p}+q$. The line of $y=2$ is the horizontal asymptote and $A(3;3)$ is a point on the graph. Determine the equation of the graph.**



$f(x)=2^{x-p}+q$

$\therefore f(x)=2^{x-p}+2 \rightarrow y=q$ is the horizontal asymptote

Sub $A(3;3)$

$\therefore 3=2^{3-p}+2$

$\therefore 1=2^{3-p} \rightarrow$ you need to get the same bases on both sides

$\therefore 2^0=2^{3-p} \rightarrow$ any number raised to the power of 0 is equal to 1 $\left(a^0=1\right)$

$\therefore 0=3-p$

$\therefore p=3$

$\therefore f(x)=2^{x-3}+2$



Complete Exercise 6 on page 305.

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E. TRANSLATIONS AND REFLECTIONS OF GRAPHS

As you have seen in the graphs that you have dealt with, graphs can be translated (shifted) horizontally and vertically.
It has been shown that the value of p is responsible for horizontal shifts and the value of q is responsible for vertical shifts.

A reflection in the x-axis causes all the y-values to change sign and leaves the x-values the same.

A reflection in the y-axis causes all the x-values to change sign and leaves the y-values the same.

Illustrative Example 1: Sketch the graph of $f(x)=x^2$. Then, on the same set of axes sketch the graphs of:

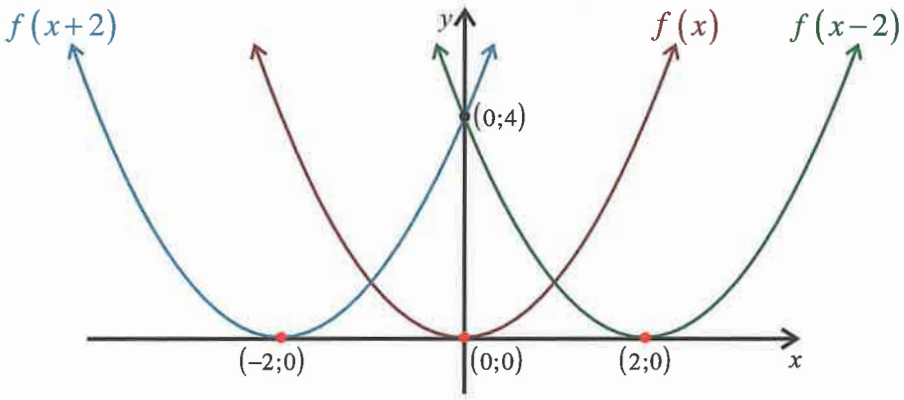
a) $f(x-2)$ **b)** $f(x+2)$

a) $f(x-2)=(x-2)^2$

This is a parabola with a turning point at $(2;0)$ and y-intercept of $(0;4)$. This is the graph of $f(x)$ shifted 2 units to the right.

b) $f(x+2)=(x+2)^2$

This is a parabola with a turning point at $(-2;0)$ and y-intercept of $(0;4)$ .This is the graph of $f(x)$ shifted 2 units to the left.



From the above graphs it should be clear that:

- The graph of $f(x-p)$ shifts the graph of $f(x)$ p units to the right.
- The graph of $f(x+p)$ shifts the graph of $f(x)$ p units to the left.

Illustrative Example 2: Given the function $f(x)=(x-3)^2+4$:

a) Determine the turning point of f. **b) Determine the turning point of $g(x)$ if $g(x)=f(x-1)$**

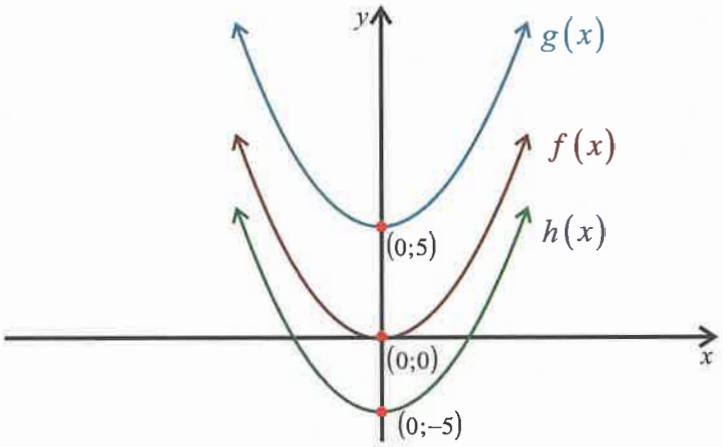
a) From the equation of $f(x)$ it can be seen that the turning point is $(3;4)$

b) $f(x-1)$ shifts the graph of $f(x)$ 1 unit to the right.

$\therefore$ TP of $g(x)$ is $(4;4)$

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Illustrative Example 3: **The graph of $f(x)=x^2$ is sketched below. The graphs of $g(x)=f(x)+5$ and $h(x)=f(x)-5$ are also sketched on the same set of axes.**



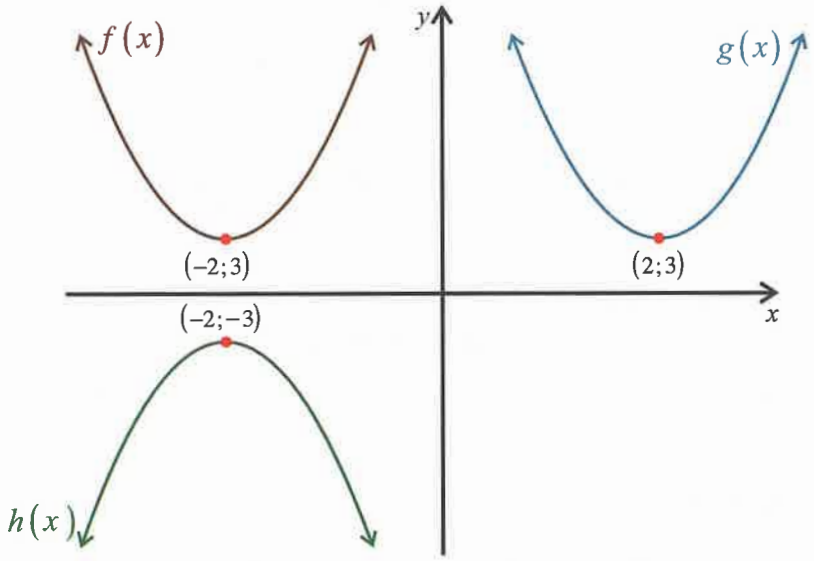
From the graphs it can be seen that $g(x)$ shifts $f(x)$ up 5 units.

The graph of $h(x)$ shifts $f(x)$ down 5 units.

In summary:

- The graph of $f(x)+q$ shifts the graph of $f(x)$ q units up.
- The graph of $f(x)-q$ shifts the graph of $f(x)$ q units down.

Illustrative Example 4: **The graph of $f(x)=(x+2)^2+3$ is sketched below. The graphs of $g(x)=f(-x)$ and $h(x)=-f(x)$ are also sketched on the same set of axes.**



From the graphs it can be seen that $g(x)$ is a reflection of $f(x)$ about the y-axis and the graph of $h(x)$ is a reflection of $f(x)$ about the x-axis.

In summary:

- The graph of $f(-x)$ is a reflection of the graph of $f(x)$ about the y-axis.
- The graph of $-f(x)$ is a reflection of the graph of $f(x)$ about the x-axis.



Complete Exercise 7 on page 305.

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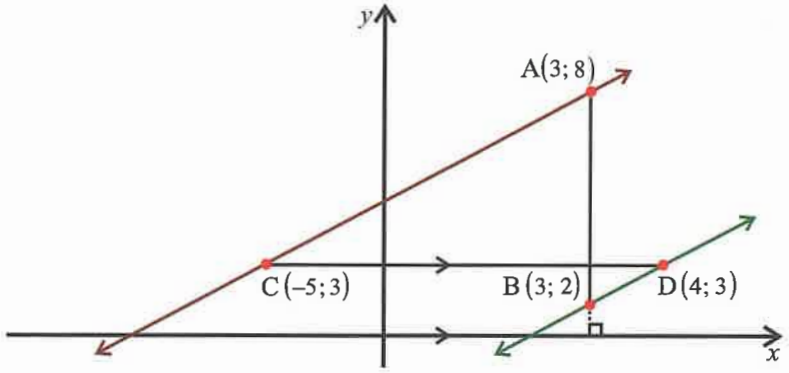
F. GRAPH INTERPRETATION

LENGTHS OF VERTICAL AND HORIZONTAL LINES

The length of any vertical line AB is: $AB = y_{top} - y_{bottom}$

The length of any horizontal line CD is: $CD = x_{right} - x_{left}$

Example 1: Determine the length of line AB and CD below:



$AB = y_{top} - y_{bot}$

$= y_a - y_b$

$= 8 - 2$

$= 6$ units

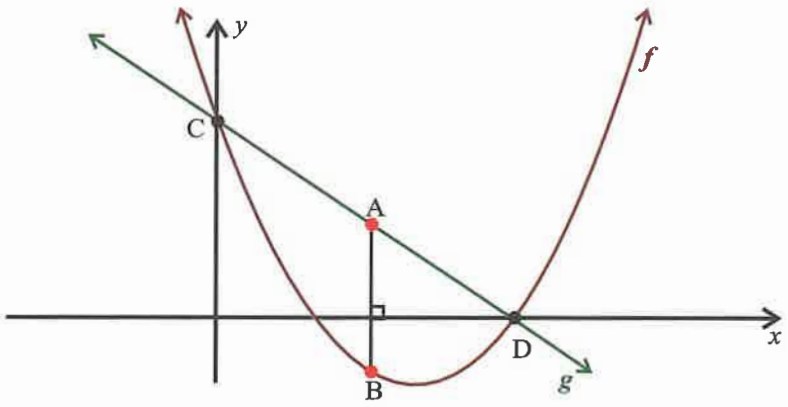
$CD = x_{right} - y_{left}$

$= x_d - x_c$

$= 4 - (-5)$

$= 9$ units

Example 2: Line AB below can move anywhere between points C and D. It is given that $f(x) = 2x^2 - 8x + 6$ and $g(x) = -2x + 6$. Write down an expression for the length of line AB in terms of x. Hence, determine the length of line AB when $x = 2$.



$AB = g(x) - f(x) \rightarrow y_{top} - y_{bot}$

$\therefore AB = -2x + 6 - (2x^2 - 8x + 6)$

$\therefore AB = -2x + 6 - 2x^2 + 8x - 6$

$\therefore AB = -2x^2 + 6x \rightarrow$ this is the length of line AB anywhere between points C and D

Now substitute $x = 2$ into the equation for AB above:

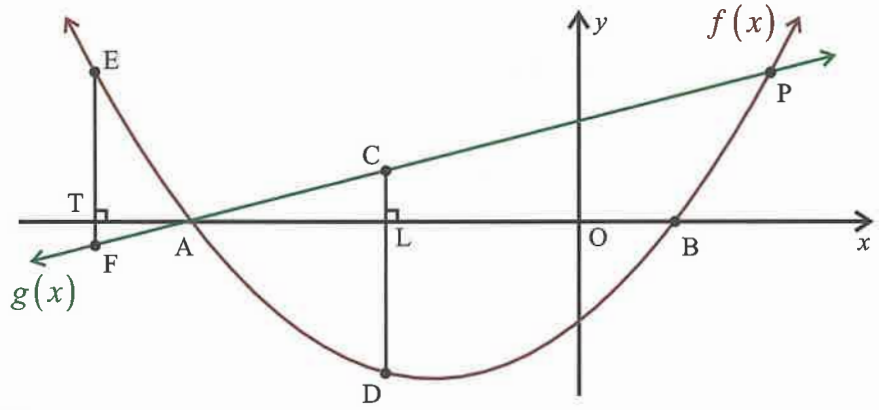
$\therefore AB = -2(2)^2 + 6(2)$

$= 4$ units at $x = 2$

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INTERPRETATION AND DEDUCTIONS FROM GRAPHS

Example 1: In the diagram below, $f(x)=x^2+3x-4$ and $g(x)=x+4$.



Determine:

a) The length of CD if OL = 2 units.

b) The length of OT if EF = 7 units.

c) The coordinates of A and B.

d) The coordinates of P.

e) The values of x for which $f(x)\geq 0$.

f) The values of x for which $f(x)\leq g(x)$.

g) The values of x for which $f(x).g(x)>0$.

a) $\text{CD}=g(x)-f(x)$

$=x+4-\left(x^2+3x-4\right)$

$=x+4-x^2-3x+4$

$=-x^2-2x+8$

Sub $x=-2\rightarrow$ OL is 2 units but L is on the negative x-axis $\therefore$ the x-value at L is -2

$\therefore \text{CD}=-(-2)^2-2(-2)+8$

$\therefore \text{CD}=8$ units

b) $\text{EF}=f(x)-g(x)$

$\therefore 7=x^2+3x-4-(x+4)\rightarrow$ EF = 7 units

$\therefore 7=x^2+2x-8$

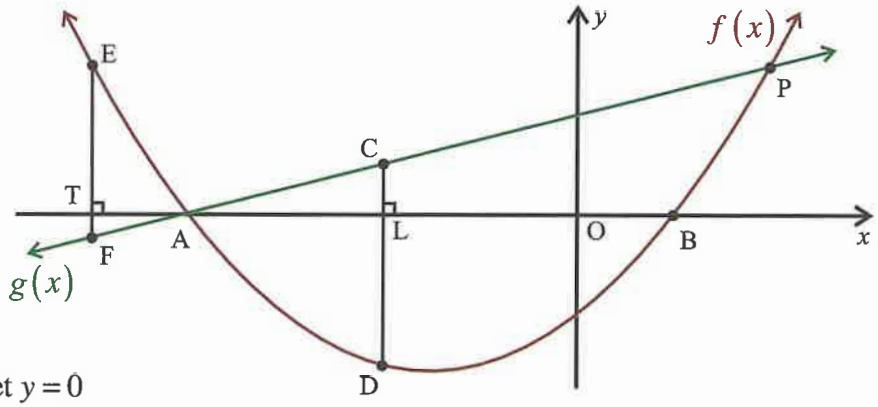
$\therefore x^2+2x-15=0$

$\therefore (x+5)(x-3)=0$

$\therefore x=-5$ or $x=3\rightarrow$ N/A as T is on the negative x-axis

$\therefore$ OT = 5 units $\rightarrow$ positive as a length cannot be negative

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c) x-int: let $y=0$

$\therefore x^2+3x-4=0$

$\therefore (x+4)(x-1)=0$

$\therefore x=-4$ or $x=1$

$\therefore$ A is the point $(-4;0)$ and B is the point $(1;0)$

d) P is a point of intersection of the graphs

$\therefore f(x)=g(x)$

$\therefore x^2+3x-4=x+4$

$\therefore x^2+2x-8=0$

$\therefore (x-2)(x+4)=0$

$\therefore x=2$ or $x=-4 \rightarrow$ this is the x-value at point A (the other point of intersection)

$\therefore$ The x-value at point P is 2

Sub $x=2$ into $g(x) \rightarrow f(x)$ could also be used (use the easier equation)

$\therefore g(2)=2+4=6$

$\therefore$ P is the point $(2;6)$

e) $f(x) \geq 0 \rightarrow$ this occurs where the graph of f is above the x-axis

$\therefore x \leq -4$ or $x \geq 1 \rightarrow$ also written as $x \in (-\infty;-4]$ or $[1;\infty)$

f) $f(x) \leq g(x) \rightarrow$ this occurs where the graph of f is below or intersects g

$\therefore f(x) \leq g(x)$ from points A to P

$\therefore -4 \leq x \leq 2 \rightarrow$ also written as $x \in [-4;2]$

g) $f(x).g(x)>0$

There are 2 possible ways this can occur:

1) $f(x)$ and $g(x)$ are both positive $\rightarrow$ '+' × '+' = '+'

This occurs from point B onwards.

2) $f(x)$ and $g(x)$ are both negative $\rightarrow$ '−' × '−' = '+'

This does not occur on this graph.

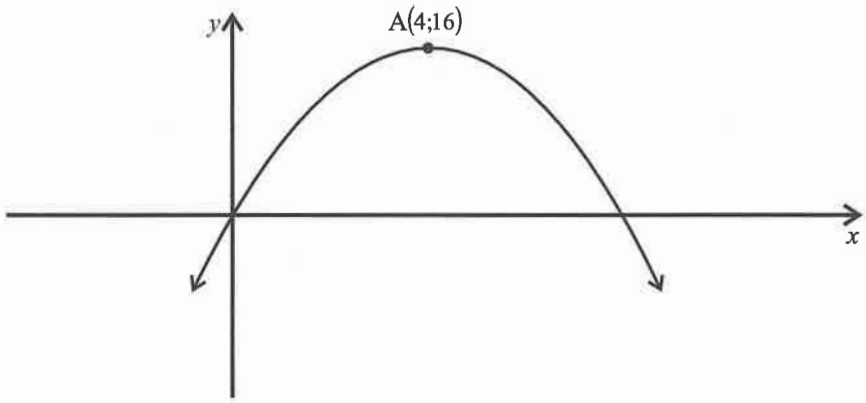
$\therefore x>1 \rightarrow$ also written as $x \in (1;\infty) \rightarrow x \neq 1$ as the question said greater than 0



A graph is positive where it lies above the x-axis and negative where it lies below the x-axis.

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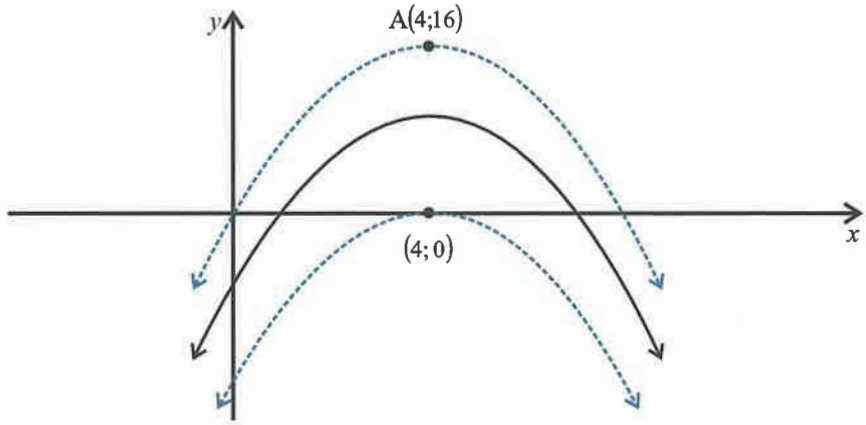
Example 2: The graph below represents $y = -x^2 + 8x$ with turning point $A(4;16)$.



For what values of t will the equation $-x^2 + 8x + t = 0$ have two roots which are positive and greater than 0?

In this question it is important to understand that t will shift the graph vertically up or down.

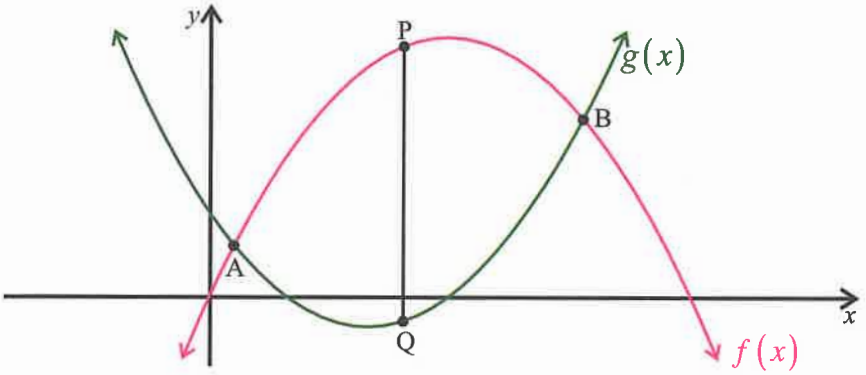
In order to have two positive roots the graph will have to lie between the two dotted graphs shown below. This means that the graph has to be shifted down, but by no more than 16 units.



$\therefore -16 < t < 0$ → also written as $t \in (-16;0)$

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Example 3: **It is given that $f(x)=-x^2+6x$ and $g(x)=x^2-4x+3$. The line PQ can lie anywhere between points A and B.**



a) Write down an expression for the length of line PQ.
b) Hence, determine the maximum length of PQ.

a) $PQ=f(x)-g(x)$

$\therefore PQ=-x^2+6x-\left(x^2-4x+3\right)$

$\therefore PQ=-x^2+6x-x^2+4x-3$

$\therefore PQ=-2x^2+10x-3$ → this expression represents the length of line PQ

The above equation is a parabola, where the *y*-value represents the length of line PQ for a particular *x*-value between A and B. It was shown in graph sketching that the *y*-coordinate of the turning point is the maximum or minimum value of the graph. Therefore, the *y*-coordinate of the turning point of PQ will be the maximum length of PQ.

b) $PQ=-2x^2+10x-3$

AS: $x=\dfrac{-b}{2a}$

$\therefore x=\dfrac{-10}{2(-2)}=\dfrac{5}{2}$ → this is the *x*-value where the length of PQ will be a maximum

Sub $x=\dfrac{5}{2}$ into the equation of PQ to obtain the maximum value:

$\therefore PQ_{max}=-2\left(\frac{5}{2}\right)^2+10\left(\frac{5}{2}\right)-3$

$=\dfrac{19}{2}$ units

$\therefore$ The maximum length of PQ is 9,5 units.

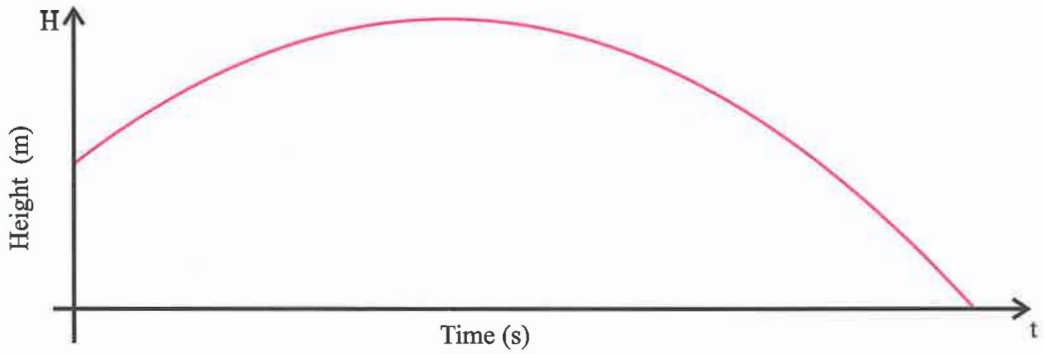


Complete Exercise 8 on pages 305 and 306.

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REAL WORLD APPLICATIONS

Example: **A Grade 11 mathematics student builds a water rocket and launches it from the roof of his school. He determines that the height above the ground from the moment it is launched can be modelled by the function $H(t) = -5t^2 + 20t + 20$, which is sketched below:**



a) Determine the height of the school roof.
(Hint: This is the height the rocket was launched from)

b) Determine the rocket's maximum height above the ground.

c) Determine the time the rocket was in the air for correct to the nearest second.

a) The y-intercept of the graph represents its initial starting height:

$\therefore H(0) = -5(0)^2 + 20(0) + 20$

$= 20$ metres

b) The y-coordinate of the turning point represents the rocket's maximum height :

$H(t) = -5t^2 + 20t + 20$

AS: $x = \dfrac{-b}{2a}$

$x = \dfrac{-20}{2(-5)} = 2 \rightarrow$ this is the time when the rocket will reach its maximum height

$\therefore H(2) = -5(2)^2 + 20(2) + 20$

$= 40$ metres

$\therefore$ The maximum height the rocket reaches is 40 metres.

c) The x-intercepts of the graph represent when the rocket is on the ground:

$0 = -5t^2 + 20t + 20 \rightarrow$ divide both sides by -5

$\therefore 0 = t^2 - 4t - 4 \rightarrow$ use the quadratic formula

$\therefore t = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)} \rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore t = \dfrac{4 \pm 4\sqrt{2}}{2}$

$\therefore t = 4{,}83$ or $t = -0{,}83$

but $t \neq -0{,}83 \rightarrow$ time cannot be negative

$\therefore t = 5$ seconds $\rightarrow$ the question said the answer must be correct to the nearest second

$\therefore$ The rocket was in the air for 5 seconds.

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AVERAGE GRADIENT (AVERAGE RATE OF CHANGE)

The average gradient (m) is the average rate of change between two points.
The average gradient between the points $(x_1; y_1)$ and $(x_2; y_2)$ is given by:

$$\text{average gradient} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

The average gradient between two points on a curve is the gradient of a straight line which joins the two points together.

Illustrative Example:

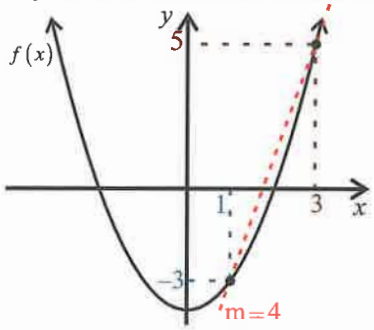
Given the function $f(x) = x^2 - 4$, determine the average gradient of the function between $x = 1$ and $x = 3$.

In order to determine the average gradient you need the y values where $x = 1$ and $x = 3$.

$f(1) = (1)^2 - 4 = -3$

$f(3) = (3)^2 - 4 = 5$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(3) - f(1)}{x_2 - x_1} = \frac{5 - (-3)}{3 - 1} = \frac{8}{2} = 4$$



Now let us examine the gradient between the following points on the function:

Point A	Point B	Gradient
$(1;-3)$	$(3;5)$	$m = \frac{5-(-3)}{3-1} = 4$
$(1;-3)$	$(2;0)$	$m = \frac{0-(-3)}{2-1} = 3$
$(1;-3)$	$(1,5;-1,75)$	$m = \frac{-1,75-(-3)}{1,5-1} = 2,5$
$(1;-3)$	$(1,2;-2,56)$	$m = \frac{-2,56-(-3)}{1,2-1} = 2,2$
$(1;-3)$	$(1,1;-2,79)$	$m = \frac{-2,79-(-3)}{1,1-1} = 2,1$
$(1;-3)$	$(1,05;-2,8975)$	$m = \frac{-2,8975-(-3)}{1,05-1} = 2,05$
$(1;-3)$	$(1,005;-2,989975)$	$m = \frac{-2,989975-(-3)}{1,005-1} = 2,005$

From the table it can be seen that as you calculate the average gradient using points closer to A, the gradient starts to approach **2**.

As the points get closer together, the gradient of line starts to approximate the gradient of the graph at a point. From this it can be seen that the gradient at point A is equal to 2.

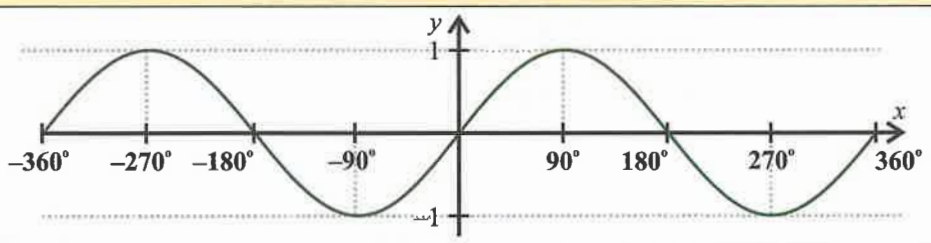
Complete Exercise 9 on pages 306 and 307.

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G. TRIGONOMETRIC FUNCTIONS

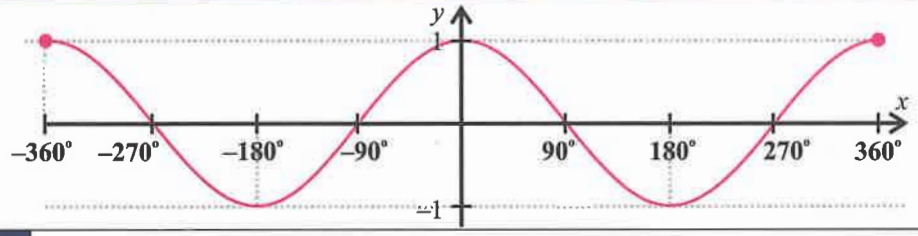
GRADE 10 REVISION OF THE SIN, COS AND TAN GRAPHS

$y = \sin x$ where $x \in [-360^\circ; 360^\circ]$



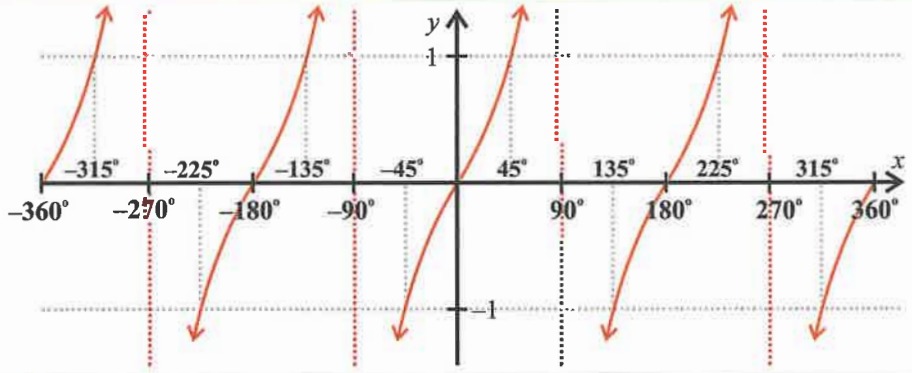
Period	The period of the standard sin graph is 360°.
Domain	The domain of the standard unrestricted sin graph is $x \in \mathbb{R}$. In the above graph the domain has been restricted to $x \in [-360^\circ; 360^\circ]$
Range	The range of a standard sin graph is: $y \in [-1; 1]$

$y = \cos x$ where $x \in [-360^\circ; 360^\circ]$



Period	The period of the standard cos graph is 360°.
Domain	The domain of the standard unrestricted cos graph is $x \in \mathbb{R}$. In the above graph the domain has been restricted to $x \in [-360^\circ; 360^\circ]$.
Range	The range of a standard cos graph is: $y \in [-1; 1]$

$y = \tan x$ where $x \in [-360^\circ; 360^\circ]$



Period	The period of the standard tan graph is 180°.
Domain	The domain of the standard unrestricted tan graph is: $x \in \mathbb{R}$, $x \neq \pm 90^\circ + 180^\circ k$ where $k \in \mathbb{Z}$ In the above graph the domain has been restricted to $x \in [-360^\circ; 360^\circ]$ but $x \neq \pm 90^\circ$ and $x \neq \pm 270^\circ$
Range	The range of a standard tan graph is: $y \in \mathbb{R}$. → also written as $y \in (-\infty; \infty)$

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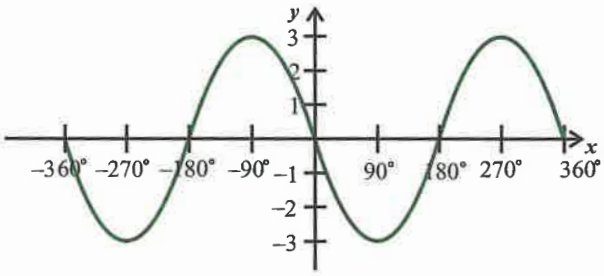
THE EFFECTS OF a (CHANGE IN AMPLITUDE)

If there is a change in amplitude the y-values of the standard graphs will be multiplied by a. This causes a change in the amplitude by a factor of a. A change in amplitude stretches or compresses the graphs vertically.
The graphs will be in the forms of:

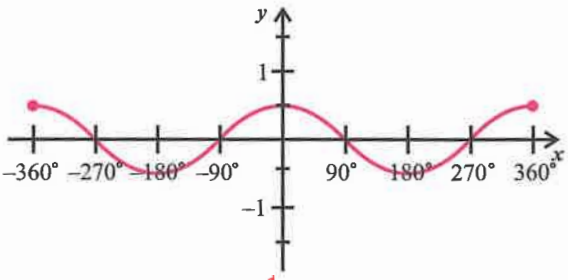
$$y = a\sin(x) \qquad y = a\cos(x) \qquad y = a\tan(x)$$

Example: On separate axes sketch the graphs of:

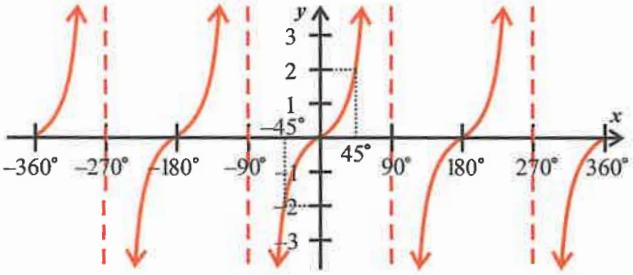
$$y = -3\sin x,\ y = \frac{1}{2}\cos x \text{ and } y = 2\tan x \text{ where } x \in [-360°; 360°]$$



$y = -3\sin x$



$y = \frac{1}{2}\cos x$



$y = 2\tan x$



The standard tan graph (where a is equal to 1) has a y-value of 1 at 45 degrees. In this case, as a is equal to 2, the y-value at 45 degrees is equal to 2.

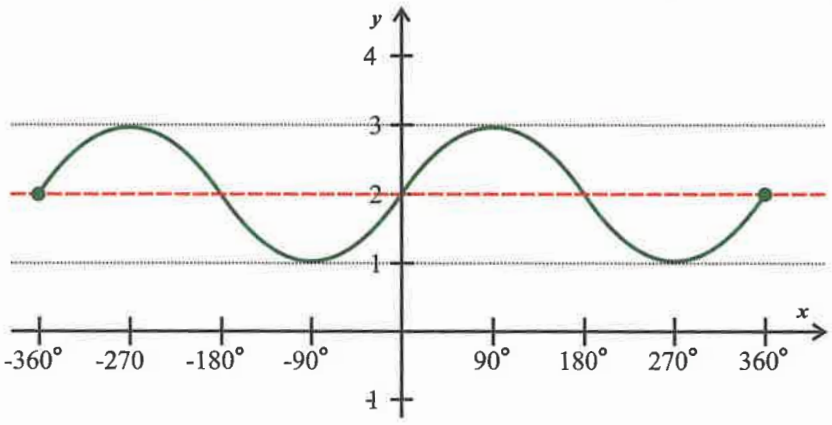
The domain is restricted in the question. Therefore, the graph is drawn from –360 to 360 degrees.

THE EFFECTS OF q (VERTICAL SHIFTS)

If q is **added** to the **graph** it **will shift up** by q units.
If q is **subtracted** from the **graph** it **will shift down** by q units.
The graphs will be in the forms of:

$$y = \sin(x) \pm q \qquad y = \cos(x) \pm q \qquad y = \tan(x) \pm q$$

Example 1: Sketch the graph of $y = \sin x + 2$ where $x \in [-360°; 360°]$



$y = \sin x + 2$

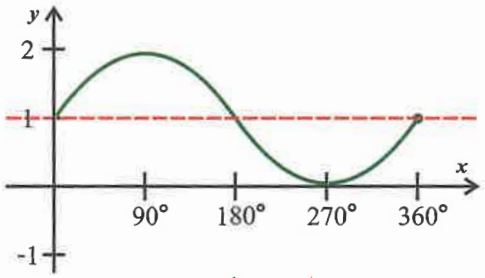


You can think of the line $y = q$ as the new x-axis.

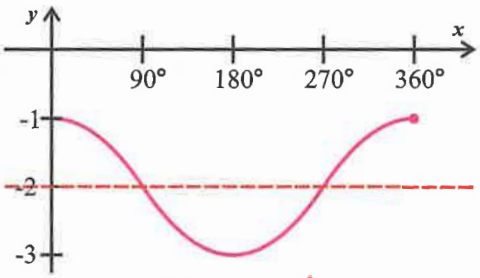
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Example 2: On separate axes sketch the graphs of:

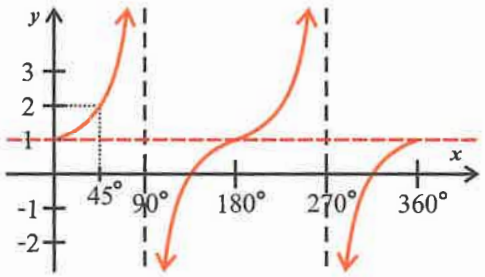
$$y = \sin x + 1, \quad y = \cos x - 2 \quad \text{and} \quad y = \tan x + 1 \quad \text{where } x \in [0°;360°]$$



$y = \sin x + 1$



$y = \cos x - 2$



$y = \tan x + 1$

The standard tan graph (where q is equal to 0) has a y-value of 1 at 45 degrees. In this case, as q is equal to 1, the y-value at 45 degrees is equal to 2. This is because one is added to all of the y-values.

THE EFFECTS OF p (HORIZONTAL SHIFTS)

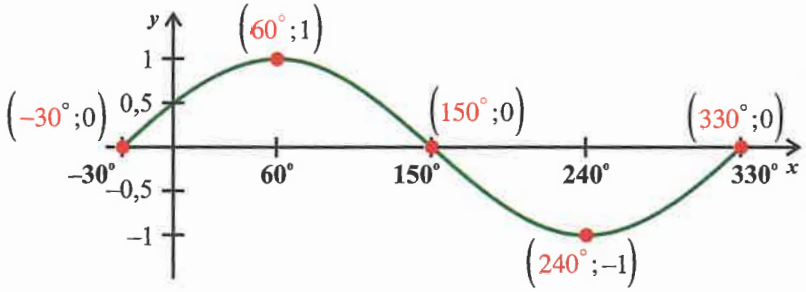
In the case of a left shift, $p°$ will be added to the angle.
The graphs will be in the forms of:

$$y = \sin(x + p°) \qquad y = \cos(x + p°) \qquad y = \tan(x + p°)$$

Example 1: Sketch the graph of $y = \sin(x + 30°)$ where $x \in [-30°;330°]$

This is a shift of $30°$ left. To sketch the graph, take every key value (intercepts and turning points) and subtract $30°$ from the x-coordinates.

Original Point	Shift	New point
$(0°;0)$	$(0° - 30°;0)$	$(-30°;0)$
$(90°;1)$	$(90° - 30°;1)$	$(60°;1)$
$(180°;0)$	$(180° - 30°;0)$	$(150°;0)$
$(270°;-1)$	$(270° - 30°;-1)$	$(240°;-1)$
$(360°;0)$	$(360° - 30°;0)$	$(330°;0)$



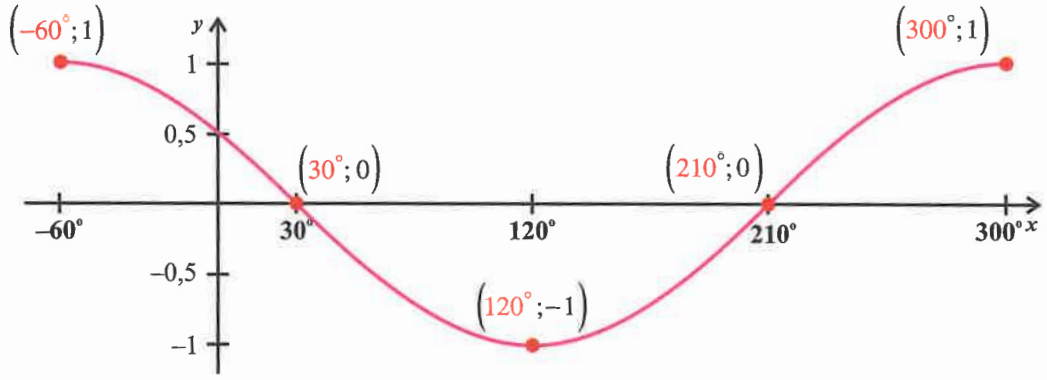
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Example 2: On separate axes sketch the graphs of:

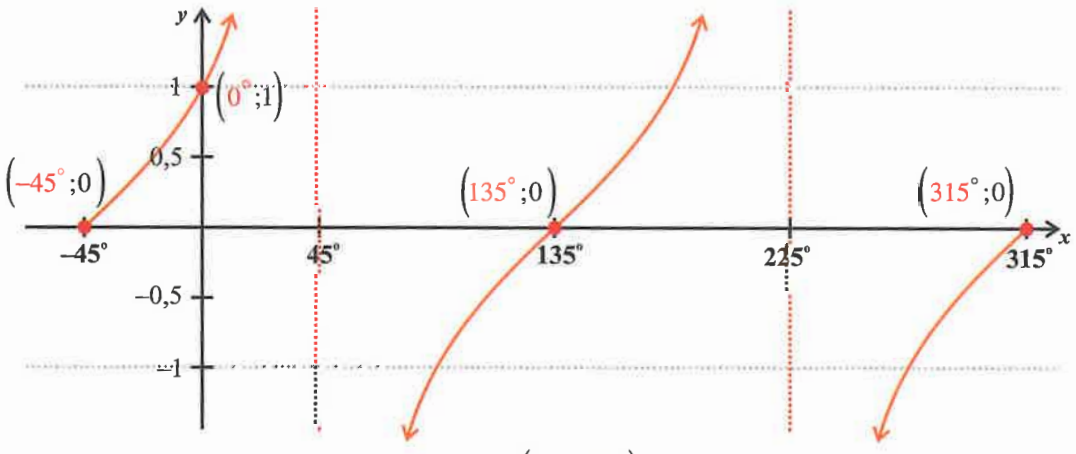
$y = \cos(x + 60^\circ)$ **where** $x \in [-60^\circ; 300^\circ]$ **and** $y = \tan(x + 45^\circ)$ **where** $x \in [-45^\circ; 315^\circ]$

As you did in the previous example draw up a table and subtract 60° and 45° from the x-coordinates of the key values of the cos and tan graph respectively.

$y = \cos(x + 60^\circ)$			$y = \tan(x + 45^\circ)$		
Original Point	**Shift**	**New point**	**Original Point**	**Shift**	**New point**
$(0^\circ;1)$	$(0^\circ - 60^\circ;1)$	$(-60^\circ;1)$	$(0^\circ;0)$	$(0^\circ - 45^\circ;0)$	$(-45^\circ;0)$
$(90^\circ;0)$	$(90^\circ - 60^\circ;0)$	$(30^\circ;0)$	$(45^\circ;1)$	$(45^\circ - 45^\circ;1)$	$(0^\circ;1)$
$(180^\circ;-1)$	$(180^\circ - 60^\circ;-1)$	$(120^\circ;-1)$	asymptote $x = 90^\circ$	asymptote $x = 90^\circ - 45^\circ$	asymptote $x = 45^\circ$
$(270^\circ;0)$	$(270^\circ - 60^\circ;0)$	$(210^\circ;0)$	$(180^\circ;0)$	$(180^\circ - 45^\circ;0)$	$(135^\circ;0)$
$(360^\circ;1)$	$(360^\circ - 60^\circ;1)$	$(300^\circ;1)$	asymptote $x = 270^\circ$	asymptote $x = 270^\circ - 45^\circ$	asymptote $x = 225^\circ$
			$(360^\circ;0)$	$(360^\circ - 45^\circ;0)$	$(315^\circ;0)$



$y = \cos(x + 60^\circ)$



$y = \tan(x + 45^\circ)$

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In the case of a right shift, p° will be subtracted from the angle.
The graphs will be in the forms of:

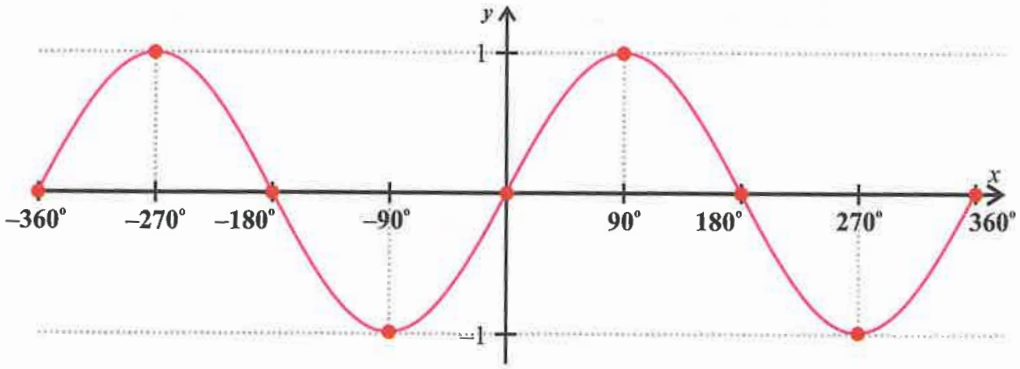
$$y=\sin(x-p^\circ) \qquad y=\cos(x-p^\circ) \qquad y=\tan(x-p^\circ)$$

Example: Sketch the graph of $y=\cos(x-90^\circ)$ where $x\in[-360^\circ;360^\circ]$

This is a shift of 90° right. To sketch the graph, take every key value (intercepts and turning points) and add 90° to the x-coordinates.

$y=\cos(x-90^\circ)$					
Original Point	**Shift**	**New point**	**Original Point**	**Shift**	**New point**
$(-450^\circ;0)$	$(-450^\circ+90^\circ;0)$	$(-360^\circ;0)$	$(0^\circ;1)$	$(0^\circ+90^\circ;1)$	$(90^\circ;1)$
$(-360^\circ;1)$	$(-360^\circ+90^\circ;1)$	$(-270^\circ;1)$	$(90^\circ;0)$	$(90^\circ+90^\circ;0)$	$(180^\circ;0)$
$(-270^\circ;0)$	$(-270^\circ+90^\circ;0)$	$(-180^\circ;0)$	$(180^\circ;-1)$	$(180^\circ+90^\circ;-1)$	$(270^\circ;-1)$
$(-180^\circ;-1)$	$(-180^\circ+90^\circ;-1)$	$(-90^\circ;-1)$	$(270^\circ;0)$	$(270^\circ+90^\circ;0)$	$(360^\circ;0)$
$(-90^\circ;0)$	$(-90^\circ+90^\circ;0)$	$(0^\circ;0)$	$(360^\circ;1)$	$(360^\circ+90^\circ;1)$	$(450^\circ;1)$

You do not sketch the point $(450^\circ;1)$ as it is not within the domain that was given in the question.



Notice that a graph of $\cos\theta$ shifted 90° to the right is a graph of $\sin\theta$.

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THE EFFECTS OF k (CHANGE IN PERIOD)

A change in period stretches or compresses the graph horizontally.
In the case of a change in period the angle will be multiplied by a factor of k.

$$y=\sin(kx) \qquad y=\cos(kx) \qquad y=\tan(kx)$$

If $0<k<1$ the graph is stretched and the period increases.
If $k>1$ the graph is compressed and the period decreases.

Period of a graph:
In order to calculate the period of a graph use the formula:

$$\text{new period}=\frac{\text{original period}}{k}$$

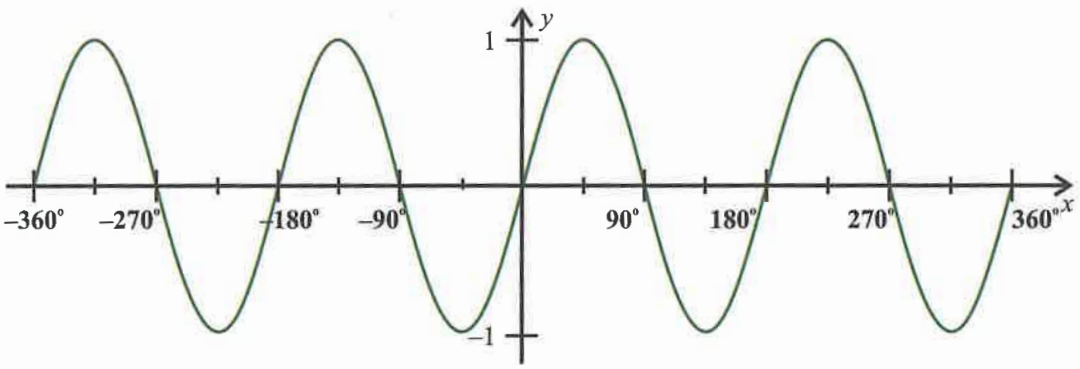
Example 1: Sketch the graph of $y=\sin(2x)$ where $x\in[-360^\circ;360^\circ]$

$$\text{new period}=\frac{360^\circ}{2}=180^\circ \rightarrow \text{one cycle is completed in } 180^\circ \text{ instead of } 360^\circ$$

In order to sketch the graph, take every key value (intercepts and turning points) and divide the x-coordinates by 2.

Original Point	New point	Original Point	New point
$(\pm 720^\circ;0)$	$(\pm 360^\circ;0)$	$(\pm 360^\circ;0)$	$(\pm 180^\circ;0)$
$(\pm 630^\circ;\mp 1)$	$(\pm 315^\circ;\mp 1)$	$(\pm 270^\circ;\mp 1)$	$(\pm 135^\circ;\mp 1)$
$(\pm 540^\circ;0)$	$(\pm 270^\circ;0)$	$(\pm 180^\circ;0)$	$(\pm 90^\circ;0)$
$(\pm 450^\circ;\pm 1)$	$(\pm 225^\circ;\pm 1)$	$(\pm 90^\circ;\pm 1)$	$(\pm 45^\circ;\pm 1)$
Starting point stays at: $(0^\circ;0)$			

The starting point of a graph does not change with a change in period.



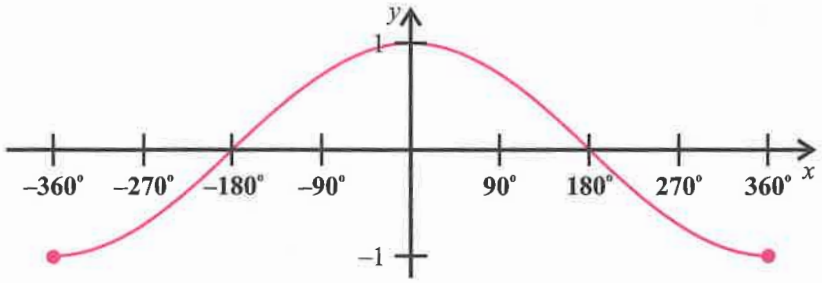
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Example 2: Sketch the graph of $y = \cos\left(\frac{1}{2}x\right)$ where $x \in [-360°; 360°]$

new period $= \frac{360°}{\frac{1}{2}} = 720° \rightarrow$ one cycle is completed in $720°$ instead of $360°$

In order to sketch the graph, take every key value (intercepts and turning points) and divide the x-coordinates by a $\frac{1}{2}$ (this is the same as multiplying by 2).

Original Point	New point	Original Point	New point
$(-180°;-1)$	$(-360°;-1)$	$(90°;0)$	$(180°;0)$
$(-90°;0)$	$(-180°;0)$	$(180°;-1)$	$(360°;-1)$
Starting point stays: $(0°;1)$			

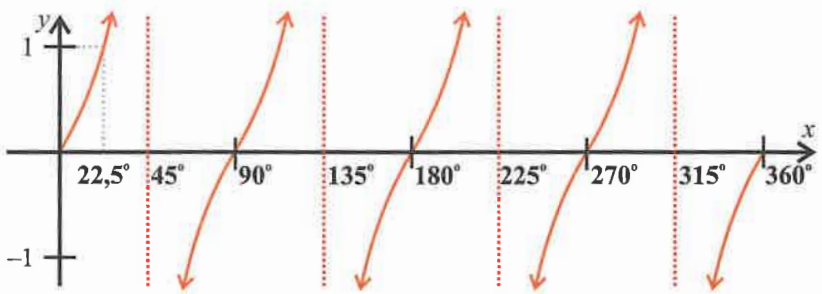


Example 3: Sketch the graph of $y = \tan(2x)$ where $x \in [0°; 360°]$

new period $= \frac{180°}{2} = 90° \rightarrow$ one cycle is completed in $90°$ instead of $180°$

In order to sketch the graph, take every key value (intercepts and turning points) and divide the x-coordinates by 2.

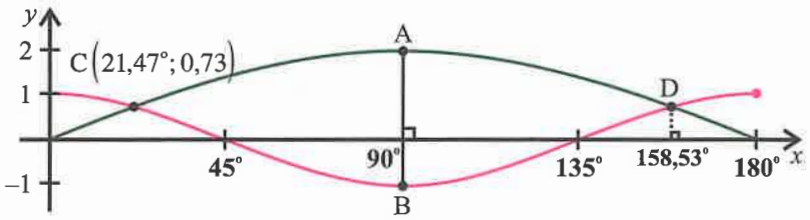
Original Point	New point	Original Point	New point
$(0°;0)$	$(0°;0)$	$(360°;0)$	$(180°;0)$
$(45°;1)$	$(22,5°;1)$	asymptote $x = 450°$	asymptote $x = 225°$
asymptote $x = 90°$	asymptote $x = 45°$	$(540°;0)$	$(270°;0)$
$(180°;0)$	$(90°;0)$	asymptote $x = 630°$	asymptote $x = 315°$
asymptote $x = 270°$	asymptote $x = 135°$	$(720°;0)$	$(360°;0)$



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H. INTERPRETATION OF TRIGONOMETRIC FUNCTIONS

Example 1: **The graphs of $f(x)=a\sin(x)$ and $g(x)=\cos(kx)$ are sketched below for $x\in[0°;180°]$:**



a) Determine the values of a and k.

b) Determine the length of line AB.

c) Determine the coordinates of point D correct to 2 decimal places.

d) For which values of x is $f(x)\geq g(x)$.

e) For which values of x is $\dfrac{f(x)}{g(x)}\leq 0$.

a) The standard sin graph starts at $(0;0)$ and has an amplitude of 1. The above graph starts at $(0;0)$ and has an amplitude of 2.

$\therefore a=2$

The standard cos graph has a period of $360°$. The above graph has a period of $180°$.

$\therefore \dfrac{360°}{k}=180°$

$\therefore k=2$

b) $AB=f(x)-g(x)$

$\therefore AB=2-(-1)$

$\therefore AB=3$ units

c) In order to solve for the y-value of D substitute the given x-value into the equation of either $f(x)$ or $g(x)$.

$f(158,53°)=2\sin(158,53°)\rightarrow g(x)$ could also be used

$=2\sin(158,53°)$

$=0,73$

$\therefore$ D is the point $(158,53°;0,73)$

d) f lies above or on g from points C to D.

$\therefore 21,47°\leq x\leq 158,53°$

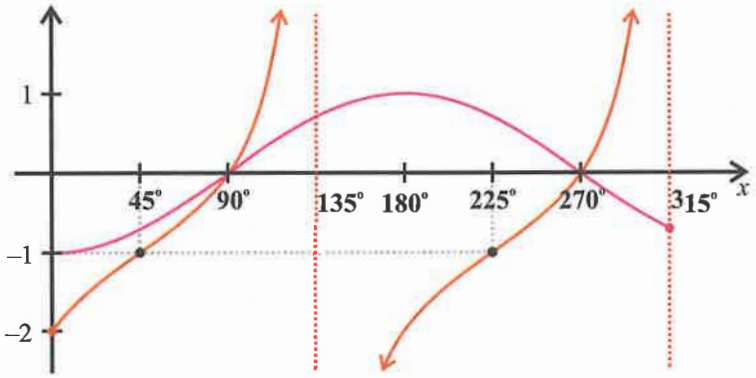
e) $\dfrac{f(x)}{g(x)}\leq 0$ when one graph is postive and the other is negative or $f(x)=0$.

$\therefore x=0°$ or $x=180°$ or $45°<x<135°$

Note that $x\neq 45°$ and $x\neq 135°$ as $g(x)=0$ at these points and you cannot divide by 0.

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Example 2: **The graphs of $f(x)=\tan(x-p)+q$ and $g(x)=a\cos x$ are sketched below for $x\in[0°;315°]$:**



a) Determine the equation of $g(x)$.

b) Determine the equation of $f(x)$.

c) For which values of x is $f(x)=g(x)$?

d) For which values of x is $g(x)\leq f(x)$?

e) Write down the equation of $h(x)$ if $h(x)$ is the graph of $g(x)$ shifted 30° to the left and 2 units up.

a) The standard cos graph starts at $(0;1)$. It can be seen on the graph that this is a standard cos graph which has been reflected about the x-axis.

$\therefore a=-1$

$\therefore g(x)=-\cos x$

b) The standard tan graph has an asymptote at $90°$. In the above graph the asymptote is at $135°$. This means the graph has been shifted right by $45°$.

$\therefore p=45°$

It can also be seen that the graph has been shifted one unit down.

$\therefore q=-1$

$\therefore f(x)=\tan(x-45°)-1$

c) This is where the graphs intersect each other.

$\therefore x=90°$ or $x=270°$

d) This occurs where the graph of g is below or equal to the graph of f.

$\therefore 90°\leq x<135°$ or $270°\leq x<315°$

This can also be written as: $x\in[90°;135°)$ or $x\in[270°;315°)$

e) $h(x)=g(x+30°)+2$

$=-\cos(x+30°)+2$



Complete Exercise 10 on pages 307 and 308 and the Mixed Exercise on pages 308 and 309.

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CHAPTER 6 – TRIGONOMETRY

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
119	**A. Grade 10 Revision**	8	• Derive and use the identities: $\sin^2\theta+\cos^2\theta=1$ and $\tan\theta=\frac{\sin\theta}{\cos\theta}$
119	• Trigonometric Ratios		
119	• Special Angles		
120	• Trigonometric Ratios in the Cartesian Plane		• Derive the reduction formula.
123	**B. Identities**		• Determine the general solution and/or the specific solutions of trigonometric equations.
123	• Proofs		
124	• Application of Identities		
127	**C. Reduction Formulas**		
128	• Negative Angles		
129	• Co-Functions		
130	• Using Reduction Formulas, Co-functions and Negative Angles with Numbers		
133	**D. Trigonometric Equations**		
133	• General Solutions		
135	• Types of General Solutions		
139	• Equations in a Specified Domain		
140	**E. Restrictions in Trigonometric Equations**		

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A. GRADE 10 REVISION

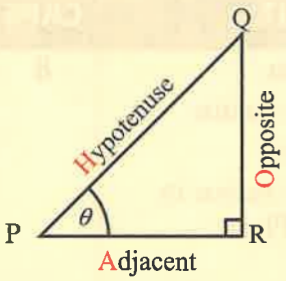
TRIGONOMETRIC RATIOS

In Grade 10 you learnt the following trigonometric ratios for a right-angled triangle:

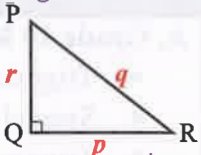
$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \rightarrow \frac{\text{Oh}}{\text{Hell}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \rightarrow \frac{\text{Another}}{\text{Hour}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} \rightarrow \frac{\text{Of}}{\text{Algebra}}$$



Recap:
Labelling sides of Triangles:



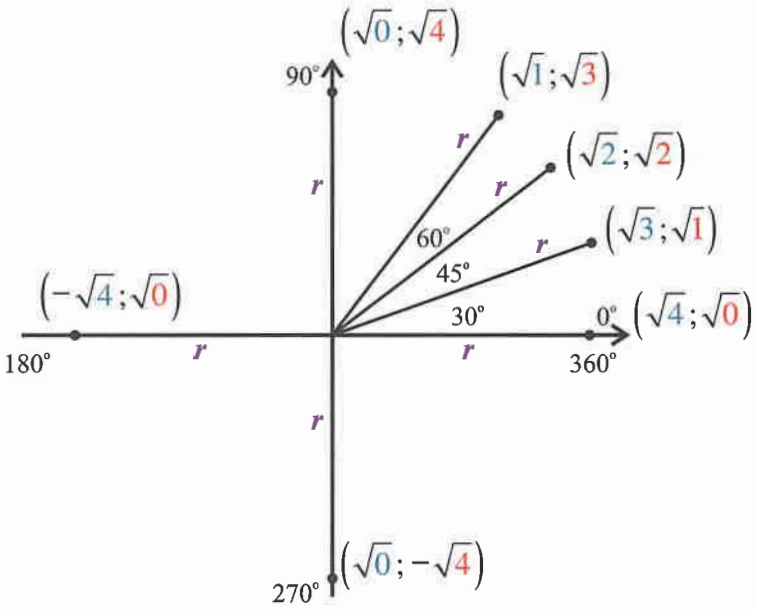
Side PR opposite $\hat{Q}$ is labelled q
Side QR opposite $\hat{P}$ is labelled p
Side PQ opposite $\hat{R}$ is labelled r

It was also shown that the functions could also be defined as:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

SPECIAL ANGLES

The value of $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be solved without the use of a calculator if θ is one of the special angles shown in the fan diagram below



From the above diagram it can be seen that at:

$\theta = 0^\circ \rightarrow (x;y) = (2;0)$

$\theta = 30^\circ \rightarrow (x;y) = (\sqrt{3};1)$

$\theta = 45^\circ \rightarrow (x;y) = (\sqrt{2};\sqrt{2})$

$\theta = 60^\circ \rightarrow (x;y) = (1;\sqrt{3})$

$\theta = 90^\circ \rightarrow (x;y) = (0;2)$

$\theta = 180^\circ \rightarrow (x;y) = (-2;0)$

$\theta = 270^\circ \rightarrow (x;y) = (0;-2)$

$\theta = 360^\circ \rightarrow (x;y) = (2;0)$

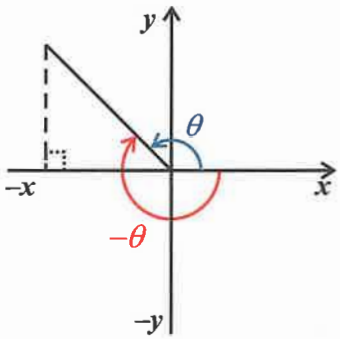
When dealing with special angles r is always equal to 2.

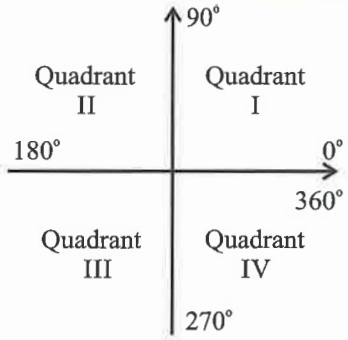
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TRIGONOMETRIC RATIOS IN THE CARTESIAN PLANE

In Grade 10 you learnt to deal with trigonometric functions in the different quadrants of the Cartesian plane.

The Cartesian planes below show how angles are measured in the Cartesian plane and where the relative quadrants lie. If an angle is measured **anti-clockwise** it is **positive** and if measured **clockwise** it is **negative**.





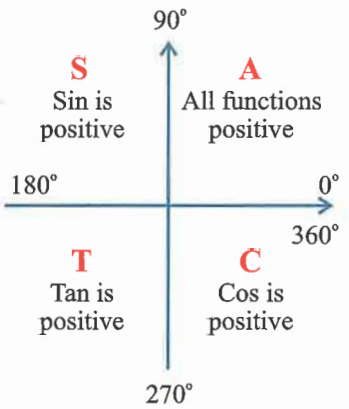
From the above diagrams it can be seen which trigonometric functions are positive or negative in the relative quadrants.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$	$\sin\theta = \frac{y}{r} = \frac{+}{+} = +$	$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$	$\sin\theta = \frac{y}{r} = \frac{-}{+} = -$
$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$	$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$	$\cos\theta = \frac{x}{r} = \frac{-}{+} = -$	$\cos\theta = \frac{x}{r} = \frac{+}{+} = +$
$\tan\theta = \frac{y}{x} = \frac{+}{+} = +$	$\tan\theta = \frac{y}{x} = \frac{+}{-} = -$	$\tan\theta = \frac{y}{x} = \frac{-}{-} = +$	$\tan\theta = \frac{y}{x} = \frac{-}{+} = -$

Remember that r is a length and is always positive. The line drawn from the origin to a point is known as the terminal arm. In the diagram above the terminal arm is drawn in the second quadrant.

There are two mnemonics that may be used to remember which functions are positive in the different quadrants. These are also shown in the diagram below.

1. Moving **anti-clockwise** from the **first quadrant:** **A**ll **S**tudents **T**ake **C**are
2. Moving **anti-clockwise** from the **fourth quadrant** the word: **CAST**



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Example 1: If $\sin\theta = \frac{2}{3}$ and $\theta \in [90°; 270°]$ determine the value of $\sin^2\theta + \cos^2\theta$, without the use of a calculator.

With this type of question you have to draw a diagram in order to solve the problem.

Step 1: Determine which quadrant θ is in and draw a diagram:

You are given that $\sin\theta = \frac{2}{3}$.

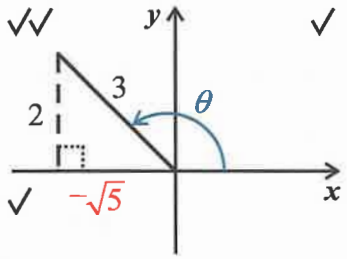
$\frac{2}{3}$ is positive, therefore θ lies in quadrants I or II as $\sin\theta$ is positive in these quadrants.

Therefore place a tick in the 1st and 2nd quadrant. You now have to use the restriction in the question. You are told that θ is between 90° and 270°. This means that θ is in the 2nd or 3rd quadrant. Therefore place a tick in the 2nd and 3rd quadrant. The quadrant with two ticks is where both constraints are satisfied and this is the quadrant that the terminal arm lies in.

$\sin\theta = \frac{2}{3} \rightarrow \frac{y}{r}$ therefore the y-value is 2 and r is 3.

You can now draw the diagram as shown below:

The value of $-\sqrt{5}$ in red is calculated in step 2 and then added to the diagram.



Step 2: Calculate the value of the unknown side using the Theorem of Pythagoras:

$r^2 = x^2 + y^2$

$\therefore 3^2 = x^2 + 2^2$

$\therefore x^2 = 5$

$\therefore x = -\sqrt{5}$ or $x = \sqrt{5} \rightarrow$ N/A as the terminal arm lies in the second quadrant

$\therefore x = -\sqrt{5}$

Step 3: Now solve the problem using your diagram.

$$\sin^2\theta + \cos^2\theta = \left(\frac{2}{3}\right)^2 + \left(\frac{-\sqrt{5}}{3}\right)^2 \rightarrow \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

$$= \frac{4}{9} + \frac{5}{9} = 1$$

A common mistake is to write the above answer as:

$$\sin^2\left(\frac{2}{3}\right) + \cos^2\left(\frac{-\sqrt{5}}{3}\right) = 1$$

This is incorrect! The sides of your triangle can't be written as the angle.

In this question the terminal arm has a length of 3 units and lies in the 2nd quadrant. Remember that the length of a line can never be negative. No matter which quadrant the terminal arm lies in, r is always positive.

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Example 2: If $\tan\theta = \frac{4}{3}$ and $\cos\theta < 0$ determine, without a calculator the value of:

a) $\frac{\sin\theta}{\cos\theta}$ b) $\cos\theta\tan\theta$

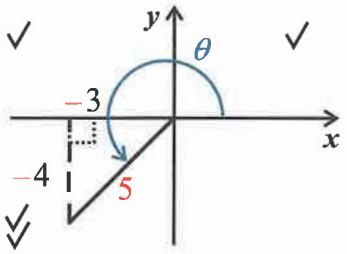
Step 1: Determine which quadrant θ is in and draw a diagram:

$\tan\theta$ is positive and $\cos\theta$ is negative. Therefore the terminal arm lies in quadrant III. Remember as θ is in the **3rd quadrant** the values of x **and** y **are negative.**

$\tan\theta = \frac{4}{3} \rightarrow \frac{y}{x}$ $\therefore y$ is -4 and x is -3

You can now draw the diagram as shown below:

The value of 5 in red is calculated in step 2 and then added to the diagram.



Step 2: Calculate the value of the unknown side using the Theorem of Pythagoras:

$r^2 = x^2 + y^2$

$\therefore r^2 = (-3)^2 + (-4)^2$

$\therefore r^2 = 25$

$\therefore r = 5 \rightarrow x \neq -5$, r is always positive as it is a length

Step 3: Now solve the problem using your diagram.

a) $\frac{\sin\theta}{\cos\theta}$

$= \frac{-4}{5} \div \frac{-3}{5}$

$= \frac{-4}{5} \times \frac{5}{-3} \rightarrow$ 'tip and times'

$= \frac{4}{3}$

b) $\cos\theta\tan\theta$

$= \frac{-3}{5}\left(\frac{-4}{-3}\right) \rightarrow x$ and y are negative in quadrant III

$= \frac{-4}{5}$

Complete Exercise 1 on page 323.

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B. IDENTITIES

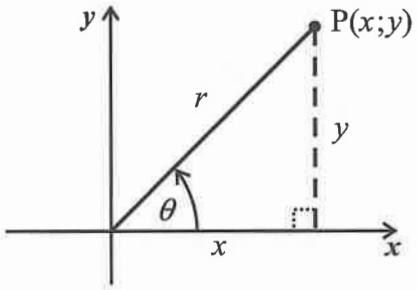
Identities are used in order to simplify trigonometric expressions.
In grade 11 there are two fundamental identities you are required to know:

$$1)\ \sin^2\theta+\cos^2\theta=1 \qquad 2)\ \tan\theta=\frac{\sin\theta}{\cos\theta}$$

Take note that $\tan\theta$ is undefined when $\cos\theta=0$. This is the reason for the asymptotes on the tan graph which you dealt with in Chapter 5 (pg. 109). This will be covered in more detail later in the chapter when dealing with trigonometric equations.

PROOFS

To prove the identity $\sin^2\theta+\cos^2\theta=1$ and $\tan\theta=\frac{\sin\theta}{\cos\theta}$, draw the diagram with point $P(x;y)$ as shown below:



Proof that $\sin^2\theta+\cos^2\theta=1$:

Reading from the diagram:

$$\sin^2\theta+\cos^2\theta$$
$$=\left(\frac{y}{r}\right)^2+\left(\frac{x}{r}\right)^2$$
$$=\frac{y^2}{r^2}+\frac{x^2}{r^2}$$
$$=\frac{x^2+y^2}{r^2}\rightarrow\boxed{1}$$

Using Pythagoras and the diagram:

$$x^2+y^2=r^2\rightarrow\boxed{2}$$

Sub $\boxed{2}$ into $\boxed{1}$:

$$\therefore\frac{x^2+y^2}{r^2}=\frac{r^2}{r^2}=1\rightarrow x^2+y^2=r^2$$

Proof that $\tan\theta=\frac{\sin\theta}{\cos\theta}$:

Reading from the diagram:

$$\frac{\sin\theta}{\cos\theta}$$
$$=\frac{y}{r}\div\frac{x}{r}\rightarrow\text{tip and times}$$
$$=\frac{y}{\not r}\times\frac{\not r}{x}$$
$$=\frac{y}{x}$$
$$=\tan\theta$$



The identity $\sin^2\theta+\cos^2\theta=1$ can also be written as:

$$\sin^2\theta=1-\cos^2\theta \quad\text{or}\quad \cos^2\theta=1-\sin^2\theta$$

This identity is known as the square identity

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APPLICATION OF IDENTITIES

In most cases you will have to use identities to simplify expressions or to prove that the left hand side of an identity is equal to the right hand side of the identity.

Tips when solving problems using identities:

- Write expressions in terms of $\sin\theta$ and $\cos\theta$ only, by using $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
- Note that: $\frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \rightarrow 1 \div \tan\theta = \frac{1}{1} \div \frac{\sin\theta}{\cos\theta} = \frac{1}{1} \times \frac{\cos\theta}{\sin\theta} = \frac{\cos\theta}{\sin\theta}$.
- When dealing with fractions find a LCD and add.
- Factorise expressions where possible. (Usually do this after finding a LCD)

Example 1: Express $\tan\theta + \frac{\cos\theta}{1+\sin\theta}$ as a single trigonometric ratio.

$$\tan\theta + \frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta(1+\sin\theta) + \cos\theta(\cos\theta)}{\cos\theta(1+\sin\theta)} \rightarrow \text{LCD} = \cos\theta(1+\sin\theta)$$

$$= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1+\sin\theta)} \rightarrow \sin^2\theta + \cos^2\theta = 1$$

$$= \frac{\sin\theta + 1}{\cos\theta(1+\sin\theta)}$$

$$= \frac{1}{\cos\theta}$$

Example 2: Express $\frac{\sin\theta}{\tan\theta}$ as a single trigonometric ratio.

$$\frac{\sin\theta}{\tan\theta}$$

$$= \sin\theta \div \frac{\sin\theta}{\cos\theta} \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \sin\theta \times \frac{\cos\theta}{\sin\theta} \rightarrow \text{'tip and times'}$$

$$= \cos\theta$$

Example 3: Determine the value of $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$.

$$(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$$

$$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 2\sin\theta\cos\theta \rightarrow \text{don't forget the middle term}$$

$$= \sin^2\theta + \cos^2\theta \rightarrow \text{square identity}$$

$$= 1$$



Complete Exercise 2 number 1 on page 323.

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Example 4: Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

When proving identities, only work with one side at time. You then show that the left hand side (LHS) is equal to the right hand side (RHS).

$$\text{LHS} = \tan^2\theta - \sin^2\theta$$

$$= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \rightarrow \text{write the equation in terms of } \sin\theta \text{ and } \cos\theta \text{ and find a LCD}$$

$$= \frac{\sin^2\theta - \sin^2\theta.\cos^2\theta}{\cos^2\theta} \rightarrow \text{take out a common factor (factorise)}$$

$$= \frac{\sin^2\theta(1-\cos^2\theta)}{\cos^2\theta}$$

$$= \frac{\sin^2\theta(\sin^2\theta)}{\cos^2\theta} \rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} \therefore \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \tan^2\theta\sin^2\theta = \text{RHS}$$

Example 5: Prove that $\dfrac{2\cos^2\theta + \cos\theta - 1}{2\sin\theta\cos\theta - \sin\theta} = \dfrac{1+\cos\theta}{\sin\theta}$

$$\text{LHS} = \frac{2\cos^2\theta + \cos\theta - 1}{2\sin\theta\cos\theta - \sin\theta} \rightarrow \text{factorise the numerator and denominator}$$

$$= \frac{(2\cos\theta - 1)(\cos\theta + 1)}{\sin\theta(2\cos\theta - 1)}$$

$$= \frac{\cos\theta + 1}{\sin\theta} = \text{RHS}$$

Example 6: Prove that $\dfrac{1 - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$

$$\text{LHS} = \frac{1 - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta} \rightarrow 1 = \sin^2\theta + \cos^2\theta, \text{ difference of squares}$$

$$= \frac{\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta}{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)} \rightarrow \text{rewrite in the order } \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$$

$$= \frac{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta}{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)} \rightarrow \text{factorise, } \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = (\sin\theta - \cos\theta)^2$$

$$= \frac{(\sin\theta - \cos\theta)^2}{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}$$

$$= \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \text{RHS}$$

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Example 7: Prove that $\dfrac{1-\cos\theta}{\sin\theta}=\dfrac{\sin\theta}{1+\cos\theta}$

In this example it is not possible to simplify either side. We are therefore going to create a difference of squares by multiplying by 1 in the form of $\dfrac{1+\cos\theta}{1+\cos\theta}$.

$$\text{LHS}=\frac{1-\cos\theta}{\sin\theta}\times\frac{1+\cos\theta}{1+\cos\theta}\rightarrow \text{multiply by 1}$$

$$=\frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}\rightarrow \text{difference of squares}$$

$$=\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}\rightarrow 1-\cos^2\theta=\sin^2\theta$$

$$=\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\frac{\sin\theta}{1+\cos\theta}=\text{RHS}$$



$1+\cos\theta$ is known as the conjugate of $1-\cos\theta$. You can always obtain a difference of squares by multiplying by the conjugate.

Example 8: Prove that $\dfrac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}=\dfrac{1+\tan\theta}{1-\tan\theta}$

$$\text{RHS}=(1+\tan\theta)\div(1-\tan\theta)\rightarrow \text{split apart a } \frac{\text{fraction}}{\text{fraction}}$$

$$=\left[1+\left(\frac{\sin\theta}{\cos\theta}\right)\right]\div\left[1-\left(\frac{\sin\theta}{\cos\theta}\right)\right]\rightarrow \text{find a LCD of } \cos\theta \text{ for each bracket}$$

$$=\left[\frac{\cos\theta+\sin\theta}{\cos\theta}\right]\div\left[\frac{\cos\theta-\sin\theta}{\cos\theta}\right]\rightarrow \text{'tip and times'}$$

$$=\frac{\cos\theta+\sin\theta}{\cos\theta}\times\frac{\cos\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\rightarrow \text{this is as simple as the RHS can be, now work with the LHS}$$

$$\text{LHS}=\frac{1+2\sin\theta\cos\theta}{1-2\sin^2\theta}\rightarrow 1=\sin^2\theta+\cos^2\theta$$

$$=\frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta-2\sin^2\theta}\rightarrow \text{factorise, } \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=(\sin\theta+\cos\theta)^2$$

$$=\frac{(\sin\theta+\cos\theta)^2}{\cos^2\theta-\sin^2\theta}\rightarrow \text{factorise, difference of squares}$$

$$=\frac{(\sin\theta+\cos\theta)^2}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}=\frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}=\text{RHS}$$

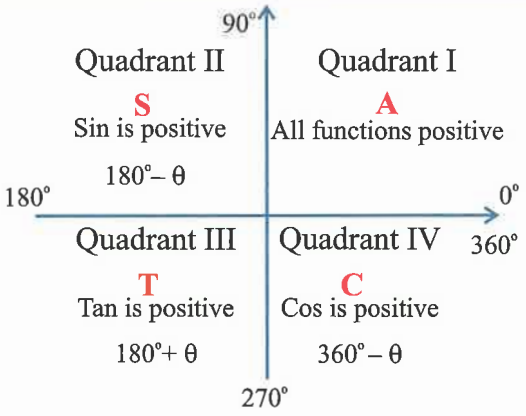


Complete Exercise 2 number 2 on pages 323 and 324.

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C. REDUCTION FORMULAS

Reduction formulas enable us to write a trigonometric ratio of any angle in terms of an acute angle (an angle less than 90°). The quadrants are labelled according to the CAST diagram. This diagram can be used to determine the sign of the trigonometric ratio in the various quadrants as you have previously done.



QUADRANT II	QUADRANT I
$\sin(180^\circ-\theta)=+\sin\theta$ $\cos(180^\circ-\theta)=-\cos\theta$ $\tan(180^\circ-\theta)=-\tan\theta$	You do not use reduction formulas on angles less than 90°.
QUADRANT III	**QUADRANT IV**
$\tan(180^\circ+\theta)=+\tan\theta$ $\sin(180^\circ+\theta)=-\sin\theta$ $\cos(180^\circ+\theta)=-\cos\theta$	$\cos(360^\circ-\theta)=+\cos\theta$ $\tan(360^\circ-\theta)=-\tan\theta$ $\sin(360^\circ-\theta)=-\sin\theta$

Example 1: Simplify $\dfrac{\cos(180^\circ+\theta)\sin(360^\circ-\theta)}{\cos(360^\circ-\theta)\sin(180^\circ-\theta)}$

$$\frac{\cos(180^\circ+\theta)\sin(360^\circ-\theta)}{\cos(360^\circ-\theta)\sin(180^\circ-\theta)}$$

$$=\frac{(-\cos\theta)(-\sin\theta)}{(+\cos\theta)(+\sin\theta)}$$

$$=1$$

Example 2: Simplify $\sin^2(360^\circ-\theta)-\cos(360^\circ-\theta)\cos(180^\circ+\theta)$

$$\left[\sin(360^\circ-\theta)\right]^2-\cos(360^\circ-\theta)\cos(180^\circ+\theta)$$

$$=(-\sin\theta)^2-(\cos\theta)(-\cos\theta)$$

$$=\sin^2\theta+\cos^2\theta$$

$$=1$$

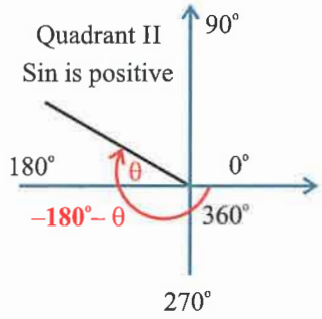
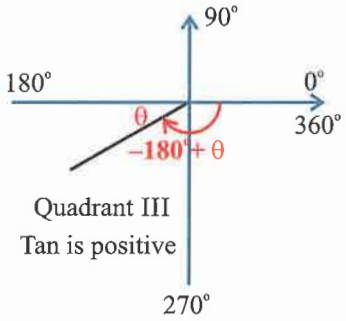
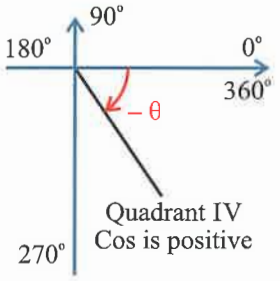


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NEGATIVE ANGLES

Earlier in this chapter (pg. 120) it was shown that angles measured clockwise are negative. The same CAST diagram that was used for reduction formula can be used for negative angles. For instance, if given the ratio $\sin(-180^\circ-\theta)$ this is equivalent to moving 180° clockwise and then moving another θ° clockwise. This would take you into the second quadrant where sin is positive and cos and tan are negative. The negative angles which you are required to know and understand are shown in the table below.

$-180^\circ-\theta$	$-180^\circ+\theta$	$-\theta$
		
$\sin(-180^\circ-\theta)=+\sin\theta$ $\cos(-180^\circ-\theta)=-\cos\theta$ $\tan(-180^\circ-\theta)=-\tan\theta$	$\tan(-180^\circ+\theta)=+\tan\theta$ $\sin(-180^\circ+\theta)=-\sin\theta$ $\cos(-180^\circ+\theta)=-\cos\theta$	$\cos(-\theta)=+\cos\theta$ $\sin(-\theta)=-\sin\theta$ $\tan(-\theta)=-\tan\theta$

In all of the above expressions, θ has to be an acute angle.

It is important to note that adding or subtracting 360° does not change the value of a trigonometric expression.

$\sin(\theta\pm360^\circ)=+\sin\theta$ $\quad\cos(\theta\pm360^\circ)=+\cos\theta$ $\quad\tan(\theta\pm360^\circ)=+\tan\theta$

Example: Simplify $\sin(-180^\circ-\theta)\cos(-180^\circ+\theta)\tan(-\theta)-1+\cos^2(\theta-360^\circ)$.

$\sin(-180^\circ-\theta)\cos(-180^\circ+\theta)\tan(-\theta)-1+\left[\cos(\theta-360^\circ)\right]^2$ → use negative angles

$=(\sin\theta)(-\cos\theta)(-\tan\theta)-1+(\cos\theta)^2$

$=(\sin\theta)(-\cancel{\cos\theta})\left(-\frac{\sin\theta}{\cancel{\cos\theta}}\right)-1+\cos^2\theta$

$=\sin^2\theta-1+\cos^2\theta \rightarrow \sin^2\theta+\cos^2\theta=1$

$=1-1$

$=0$

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CO-FUNCTIONS

Co-functions allow us to write a ratio of sin in terms of cos and vice versa. The co-functions you are required to know are shown below:

$$\sin(90^\circ-\theta)=\cos\theta \qquad \sin(90^\circ+\theta)=\cos\theta$$

$$\cos(90^\circ-\theta)=\sin\theta \qquad \cos(90^\circ+\theta)=-\sin\theta$$

If θ is an acute angle, $90^\circ-\theta$ lies in the 1st quadrant and $90^\circ+\theta$ lies in the 2nd quadrant. This is why all co-ratios are positive except for $\cos(90^\circ+\theta)$.

Example 1: Simplify $\sin(180^\circ-\theta)\cos(90^\circ+\theta)-\sin(90^\circ-\theta)\cos(360^\circ-\theta)$.

$\sin(180^\circ-\theta)\cos(90^\circ+\theta)-\sin(90^\circ-\theta)\cos(360^\circ-\theta)$

$=(\sin\theta)(-\sin\theta)-(\cos\theta)(\cos\theta)$

$=-\sin^2\theta-\cos^2\theta$ → take out a HCF of -1

$=-(\sin^2\theta+\cos^2\theta)$ → $\sin^2\theta+\cos^2\theta=1$

$=-1$

Example 2: Simplify $\dfrac{\sin(\theta-180^\circ)\sin(360^\circ-\theta)-\cos(180^\circ-\theta)\cos(360^\circ-\theta)}{\sin(90^\circ-\theta)}$.

$\dfrac{\sin(\theta-180^\circ)\sin(360^\circ-\theta)-\cos(180^\circ-\theta)\cos(360^\circ-\theta)}{\sin(90^\circ-\theta)}$ → $\theta-180^\circ=-180^\circ+\theta$

$=\dfrac{(-\sin\theta)(-\sin\theta)-(-\cos\theta)(\cos\theta)}{\cos\theta}$ → do not cancel over a $\pm$ sign

$=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta}$ → $\sin^2\theta+\cos^2\theta=1$

$=\dfrac{1}{\cos\theta}$

Example 3: Simplify $\cos^2(360^\circ+\theta)+\cos^2(90^\circ+\theta)$

$\left[\cos(360^\circ+\theta)\right]^2+\left[\cos(90^\circ+\theta)\right]^2$ → adding 360° does not change the ratio

$=(\cos\theta)^2+(-\sin\theta)^2$

$=\cos^2\theta+\sin^2\theta$

$=1$



Complete Exercise 3 number 2 on page 324.

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USING REDUCTION FORMULAS, CO-FUNCTIONS AND NEGATIVE ANGLES WITH NUMBERS

Up until now we have only dealt with the above principles using variables. We will now work with actual angles. When dealing with angles always use reduction formulas to write a ratio in terms of an acute angle. In most cases this angle will be a special angle. If there are angles which are not special angles, left in the numerator and denominator check if they sum to 90°. If they do, use co-functions.

Example 1: Simplify $\sin 330^\circ \cos 120^\circ + \tan 225^\circ$ without a calculator.

$$\sin\underbrace{330^\circ}_{\text{quadrant 4}}\cos\underbrace{120^\circ}_{\text{quadrant 2}}+\tan\underbrace{225^\circ}_{\text{quadrant 3}}$$

$$=\left[\sin\left(360^\circ-30^\circ\right)\right]\left[\cos\left(180^\circ-60^\circ\right)\right]+\tan\left(180^\circ+45^\circ\right) \rightarrow \text{you don't have to show this step}$$

$$=\left(-\sin 30^\circ\right)\left(-\cos 60^\circ\right)+\tan 45^\circ \rightarrow \text{use special angles}$$

$$=\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)+\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\tfrac{1}{4}+1$$

$$=\tfrac{5}{4}$$

Example 2: Evaluate $\dfrac{\cos 150^\circ \tan 225^\circ \sin 270^\circ}{\sin\left(-60^\circ\right)\cos 480^\circ}$ without a calculator.

$$\frac{\cos 150^\circ \tan 225^\circ \sin 270^\circ}{\sin\left(-60^\circ\right)\cos 480^\circ} \rightarrow \text{if the angle is greater than } 360^\circ\text{, subtract } 360^\circ$$

$$=\frac{\cos\left(180^\circ-30^\circ\right)\tan\left(180^\circ+45^\circ\right)(-1)}{-\sin 60^\circ \cos 120^\circ} \rightarrow \sin 270^\circ = -1,\ 270^\circ \text{ is a special angle}$$

$$=\frac{\left(-\cos 30^\circ\right)\tan\left(45^\circ\right)(-1)}{-\sin 60^\circ \cos\left(180^\circ-60^\circ\right)} \rightarrow \cos\left(180^\circ-60^\circ\right)=-\cos 60^\circ=-\frac{1}{2}$$

$$=\frac{\left(-\tfrac{\sqrt{3}}{2}\right)(1)(-1)}{\left(-\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{1}{2}\right)}$$

$$=\frac{1}{\tfrac{1}{2}}$$

$$=2$$

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Example 3: Determine the value of $\dfrac{3\cos^2 150^\circ \cos 180^\circ}{-\tan 315^\circ - \cos^2 240^\circ}$ without a calculator.

$$\frac{3\left(\cos 150^\circ\right)^2 \cos 180^\circ}{-\tan 315^\circ - \left(\cos 240^\circ\right)^2}$$

$$= \frac{3\left[\cos\left(180^\circ - 30^\circ\right)\right]^2 [-1]}{-\tan\left(360^\circ - 45^\circ\right) - \left[\cos\left(180^\circ + 60^\circ\right)\right]^2} \rightarrow \cos 180^\circ = -1,\ 180^\circ \text{ is a special angle}$$

$$= \frac{-3\left(-\cos 30^\circ\right)^2}{-\left(-\tan 45^\circ\right) - \left(-\cos 60^\circ\right)^2} \rightarrow \text{special angles}$$

$$= \frac{-3\left(-\sqrt{3}/2\right)^2}{-(-1) - \left(-\frac{1}{2}\right)^2} \rightarrow \text{split apart a } \frac{\text{fraction}}{\text{fraction}},\ \frac{-3\left(-\sqrt{3}/2\right)^2}{1 - \left(-\frac{1}{2}\right)^2} = \left[-3\left(-\sqrt{3}/2\right)^2\right] \div \left[1 - \left(-\frac{1}{2}\right)^2\right]$$

$$= \frac{-9}{4} \div \frac{3}{4}$$

$$= \frac{-9}{4} \times \frac{4}{3} \rightarrow \text{'tip and times'}$$

$$= -3$$

Example 4: Determine the value of $\dfrac{\sin 33^\circ \tan 115^\circ}{\cos 123^\circ \tan 295^\circ}$ without a calculator.

$$\frac{\sin 33^\circ . \tan 115^\circ}{\cos 123^\circ . \tan 295^\circ}$$

$$= \frac{\sin 33\left[\tan\left(180^\circ - 65^\circ\right)\right]}{\cos\left(180^\circ - 57^\circ\right)\tan\left(360^\circ - 65^\circ\right)}$$

$$= \frac{\sin 33^\circ\left(-\tan 65^\circ\right)}{-\cos 57^\circ\left(-\tan 65^\circ\right)}$$

$$= \frac{\sin 33^\circ}{-\cos 57^\circ} \rightarrow \text{the angles sum to } 90^\circ, \text{ use co-functions}$$

$$= \frac{\sin(90^\circ - 57^\circ)}{-\cos 57^\circ}$$

$$= \frac{\cos 57^\circ}{-\cos 57^\circ} \rightarrow \sin(90^\circ - 57^\circ) = \cos 57^\circ$$

$$= -1$$

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Example 5: Determine the value of $\dfrac{\tan 210^\circ \sin 240^\circ \sin 20^\circ}{\cos 225^\circ \sin 135^\circ \cos 70^\circ}$ without a calculator.

$$\frac{\tan 210^\circ \sin 240^\circ \sin 20^\circ}{\cos 225^\circ \sin 135^\circ \cos 70^\circ}$$

$$= \frac{\tan(180^\circ + 30^\circ)\sin(180^\circ + 60^\circ)\sin 20^\circ}{\cos(180^\circ + 45^\circ)\sin(180^\circ - 45^\circ)\cos 70^\circ} \rightarrow \text{not special angles, don't use a calculator}$$

$$= \frac{(\tan 30^\circ)(-\sin 60^\circ)(\sin 20^\circ)}{(-\cos 45^\circ)(\sin 45^\circ)(\cos 70^\circ)}$$

$$= \frac{\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{\sqrt{3}}{2}\right)(\sin 20^\circ)}{\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)(\cos 70^\circ)} \rightarrow \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$= \frac{\left(-\frac{1}{2}\right)(\sin 20^\circ)}{\left(-\frac{1}{2}\right)(\cos 70^\circ)}$$

$$= \frac{\sin 20^\circ}{\cos 70^\circ} \rightarrow \text{the angles sum to } 90^\circ, \text{ use co-functions}$$

$$= \frac{\cos 70^\circ}{\cos 70^\circ} \rightarrow \sin(90^\circ - 70^\circ) = \cos 70^\circ$$

$$= 1$$

Example 6: Given that $\cos 72^\circ = k$, express the following in terms of k:

a) $\sin 252^\circ$ **b) $\sin 18^\circ$** **c) $\tan 72^\circ$**

Always draw a diagram for this type of question. Then use reduction formulas, negative angles and co-functions to write all angles in terms of 72°, then use your diagram.

$\cos 72^\circ = \frac{k}{1} \rightarrow \frac{x}{r}$

It can be seen that the hypotenuse (r) is equal to 1 and x is equal to k. Pythagoras can now be used to calculate the value of y in terms of k, as shown in the diagram alongside.

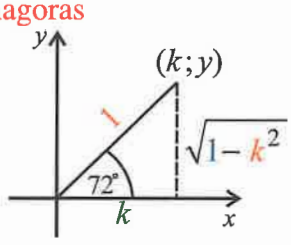
$x^2 + y^2 = r^2 \rightarrow$ Pythagoras

$\therefore k^2 + y^2 = 1^2$

$\therefore y^2 = 1 - k^2$

$\therefore y = \pm\sqrt{1-k^2}$

$\therefore y = +\sqrt{1-k^2} \rightarrow$ 1st quad $\therefore y$ is positive



a) $\sin 252^\circ$

$= \sin(180^\circ + 72^\circ)$

$= -\sin 72^\circ$

$= -\sqrt{1-k^2}$

b) $\sin 18^\circ$

$= \sin(90^\circ - 72^\circ)$

$= \cos 72^\circ \rightarrow \cos\theta = \frac{x}{r}$

$= k$

c) $\tan 72^\circ \rightarrow \tan\theta = \frac{y}{x}$

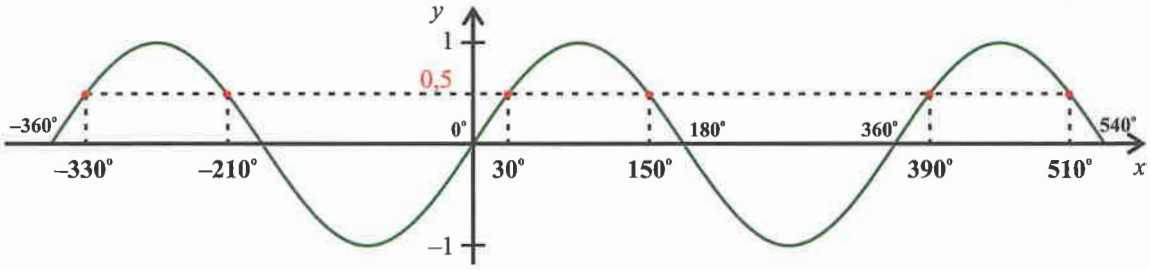
$= \frac{\sqrt{1-k^2}}{k}$

Complete Exercise 4 on pages 324 and 325.

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D. TRIGONOMETRIC EQUATIONS

Illustrative Example 1: Using the diagram below, solve for x if $\sin x = 0{,}5$ and $x \in [-360^\circ ; 540^\circ]$.



If we examine the above graph it can be seen that there are 6 solutions to the above equation: $x = -330^\circ$ or $x = -210^\circ$ or $x = 30^\circ$ or $x = 150^\circ$ or $x = 390^\circ$ or $x = 510^\circ$.

If a calculator was used to solve the problem, only 1 solution would be obtained ($x = 30^\circ$). This is known as the **reference angle (RA)**. By adding and subtracting 360° from the **reference angle** the solutions of $x = -330^\circ$ or $x = 390^\circ$ can be obtained. In order to obtain the other solutions you take $180^\circ - \text{RA}$ to obtain 150°. By adding and subtracting 360° from 150° the solutions of $x = -210^\circ$ or $x = 510^\circ$ can be obtained. This is because the Sine graph has a period of 360° and therefore repeats itself every 360°.

GENERAL SOLUTIONS

By applying the above principles the general solutions shown below can be obtained:

if $\sin\theta = \text{m}$ and $-1 \le \text{m} \le 1$ then:

$\theta = \text{RA} + 360^\circ\, k$ or $\theta = 180^\circ - \text{RA} + 360^\circ\, k$

$k \in \mathbb{Z}$

if $\cos\theta = \text{m}$ and $-1 \le \text{m} \le 1$ then:

$\theta = \pm\text{RA} + 360^\circ\, k$ $k \in \mathbb{Z}$

if $\tan\theta = \text{m}$ and $\text{m} \in \mathbb{R}$ then:

$\theta = \text{RA} + 180^\circ\, k$ $k \in \mathbb{Z}$

In order to find the reference angle on the calculator:

$\sin\theta/\tan\theta/\cos\theta = \text{m}$

The RA = Shift $\sin\theta/\tan\theta/\cos\theta$ m

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Example 1: Determine the general solution of $\sin\theta = \frac{\sqrt{3}}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$ → use shift sin $\frac{\sqrt{3}}{2}$ to obtain the RA

$\therefore$ RA $= 60^\circ$

$\therefore \theta = 60^\circ + 360^\circ k$ or $\theta = 180^\circ - 60^\circ + 360^\circ k$; $k \in \mathbb{Z}$ → always write $k \in \mathbb{Z}$

$= 120^\circ + 360^\circ k$

Example 2: Determine the general solution of $\cos\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$ → use shift cos $\frac{1}{2}$ to obtain the RA

$\therefore$ RA $= 60^\circ$

$\therefore \theta = \pm 60^\circ + 360^\circ k$; $k \in \mathbb{Z}$ → always write $k \in \mathbb{Z}$

Example 3: Determine the general solution of $\tan\theta = 1$

$\tan\theta = 1$ → use shift tan 1 to obtain the RA

$\therefore$ RA $= 45^\circ$

$\therefore \theta = 45^\circ + 180^\circ k$; $k \in \mathbb{Z}$ → always write $k \in \mathbb{Z}$

Example 4: Determine the general solution of $\cos(\theta + 10^\circ) = 0$

$\cos(\theta + 10^\circ) = 0$ → use shift cos 0 to obtain the RA

$\therefore$ RA $= 90^\circ$

$\therefore \theta + 10^\circ = 90^\circ + 360^\circ k$ or $\theta + 10^\circ = -90^\circ + 360^\circ k$; $k \in \mathbb{Z}$ → always write $k \in \mathbb{Z}$

$\therefore \theta = 80^\circ + 360^\circ k$ or $\theta = -100^\circ + 360^\circ k$

Example 5: Determine the general solution of $\cos\theta = \sin 20^\circ$

$\cos\theta = \sin 20^\circ$ → use shift cos(sin 20°) to obtain the RA

$\therefore$ RA $= 70^\circ$

$\therefore \theta = \pm 70^\circ + 360^\circ k$; $k \in \mathbb{Z}$ → always write $k \in \mathbb{Z}$

Example 6: Determine the general solution of $\sin(2x - 15^\circ) = 0{,}8$

$\sin(2x - 15^\circ) = 0{,}8$ → use shift sin 0,8 to obtain the RA

$\therefore$ RA $= 53{,}130...^\circ$ → only round off at the end

$\therefore 2x - 15^\circ = 53{,}130...^\circ + 360^\circ k$ or $2x - 15^\circ = 180^\circ - 53{,}130...^\circ + 360^\circ k$; $k \in \mathbb{Z}$

$\therefore 2x = 68{,}13...^\circ + 360^\circ k$ or $2x = 141{,}86...^\circ + 360^\circ k$ → divide both sides by 2

$\therefore x = 34{,}07^\circ + 180^\circ k$ or $x = 70{,}93^\circ + 180^\circ k$



Complete Exercise 5 number 1 on page 325.

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TYPES OF GENERAL SOLUTIONS

In most cases there will be 5 different types of trigonometric equations you will have to solve in Grade 11. These involve equations:

1) **Where $\sin\theta = \cos\theta$**
2) **Using co-functions**
3) **Requiring factorisation**
4) **Using the square identity**
5) **Using reduction formulas and negative angles**

1) Equations where $\sin\theta = \cos\theta$:

Example 1: Solve for θ if $\sin\theta = \cos\theta$.

$\sin\theta = \cos\theta \rightarrow$ divide both sides by $\cos\theta$

$\therefore \tan\theta = 1$

$\therefore RA = 45^\circ$

$\therefore \theta = 45^\circ + 180^\circ k\ ;\ k \in \mathbb{Z}$

Example 2: Solve for x, correct to 1 decimal place, if $2\cos x = 3\sin x$.

$2\cos x = 3\sin x \rightarrow$ divide both sides by $\cos x$

$\therefore 2 = 3\tan x \rightarrow$ divide both sides by 3

$\therefore \frac{2}{3} = \tan x$

$\therefore RA = 33,690...^\circ \rightarrow$ only round off at the end

$\therefore x = 33,7^\circ + 180^\circ k\ ;\ k \in \mathbb{Z}$

2) Equations using co-functions:

Example 1: Solve for θ if $\sin 2\theta = \cos\theta$.

$\sin 2\theta = \cos\theta \rightarrow$ you can't divide by $\cos\theta$ as the angles are different, use co-functions

$\therefore \sin 2\theta = \sin\left(90^\circ - \theta\right) \rightarrow \cos\theta = \sin\left(90^\circ - \theta\right)$

$\therefore RA = 90^\circ - \theta \rightarrow 2\theta$ could also be used as the reference angle

$\therefore 2\theta = 90^\circ - \theta + 360^\circ k$ or $2\theta = 180^\circ - \left(90^\circ - \theta\right) + 360^\circ k\ ;\ k \in \mathbb{Z}$

$\therefore 3\theta = 90^\circ + 360^\circ k$ or $2\theta = 180^\circ - 90^\circ + \theta + 360^\circ k$

$\therefore \theta = 30^\circ + 120^\circ k$ or $\theta = 90^\circ + 360^\circ k$

Example 2: Solve for θ if $\cos\left(\theta + 10^\circ\right) = -\sin\frac{\theta}{2}$ correct to 1 decimal place.

$\cos\left(\theta + 10^\circ\right) = -\sin\frac{\theta}{2} \rightarrow$ angles are different, use co-functions

$\therefore \cos\left(\theta + 10^\circ\right) = \cos\left(90^\circ + \frac{\theta}{2}\right) \rightarrow \cos\left(90^\circ + \frac{\theta}{2}\right) = -\sin\left(\frac{\theta}{2}\right)$

$\therefore RA = 90^\circ + \frac{\theta}{2} \rightarrow \theta + 10^\circ$ could also be used as the reference angle

$\therefore \theta + 10^\circ = 90^\circ + \frac{\theta}{2} + 360^\circ k$ or $\theta + 10^\circ = -\left(90^\circ + \frac{\theta}{2}\right) + 360^\circ k\ ;\ k \in \mathbb{Z}$

$\therefore \frac{\theta}{2} = 80^\circ + 360^\circ k$ or $\theta = -10^\circ - 90^\circ - \frac{\theta}{2} + 360^\circ k$

$\therefore \theta = 160^\circ + 720^\circ k$ or $\frac{3\theta}{2} = -100^\circ + 360^\circ k$

$\therefore \theta = 160^\circ + 720^\circ k$ or $\theta = -66,7^\circ + 240^\circ k$

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3) Equations requiring factorisation

Example 1: Determine the general solution of $2\cos^2\theta+\cos\theta-1=0$.

$2\cos^2\theta+\cos\theta-1=0 \rightarrow$ this is in the form $2x^2+x-1$ and can be factorised

$\therefore (2\cos\theta-1)(\cos\theta+1)=0$

$\therefore \cos\theta=\frac{1}{2}$ or $\cos\theta=-1$

$\therefore RA=60^\circ$ or $RA=180^\circ$

$\therefore \theta=\pm 60^\circ+360^\circ k$ or $\theta=\pm 180^\circ+360^\circ k;\ k\in\mathbb{Z}$

Example 2: Solve for x if $2\sin x\cos x=\sin x$.

$2\sin x\cos x=\sin x \rightarrow$ never divide both sides by $\sin x$ as it could be equal to 0

$\therefore 2\sin x\cos x-\sin x=0 \rightarrow$ factorise, take out a HCF of $\sin x$

$\therefore \sin x(2\cos x-1)=0$

$\therefore \sin x=0$ or $\cos x=\frac{1}{2}$

$\therefore RA=0^\circ$ or $RA=60^\circ$

$\therefore x=0^\circ+360^\circ k$ or $x=180^\circ-0^\circ+360^\circ k$ or $x=\pm 60^\circ+360^\circ k;\ k\in\mathbb{Z}$



You can only divide by $\sin x$ or $\cos x$ if you obtain $\tan x$ by doing so. Otherwise division by 0 could occur and solutions will be lost.

Example 3: Solve for x if $2\sin^2 x-\sin x\cos x-6\sin x=-3\cos x$

$2\sin^2 x-\sin x\cos x-6\sin x=-3\cos x$

$\therefore 2\sin^2 x-\sin x\cos x-6\sin x+3\cos x=0 \rightarrow$ use grouping if there are more than 3 terms

$\therefore 2\sin x(\sin x-3)-\cos x(\sin x-3)=0$

$\therefore (2\sin x-\cos x)(\sin x-3)=0$

$\therefore 2\sin x-\cos x=0$ *or* $\sin x-3=0$

$\therefore 2\sin x=\cos x$ $\therefore \sin x=3$

$\therefore \frac{2\sin x}{\cos x}=\frac{\cos x}{\cos x}$ $\therefore$ No solution as $-1\le\sin x\le 1$

$\therefore 2\tan x=1$

$\therefore \tan x=\frac{1}{2}$

$RA=26{,}565...^\circ \rightarrow$ only round off at the end

$\therefore x=26{,}67^\circ+180^\circ k\ ;k\in\mathbb{Z}$

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4) Equations using the square identity

Example 1: Determine the general solution of $3\cos^2\theta - 5\sin\theta - 1 = 0$.

$3\cos^2\theta - 5\sin\theta - 1 = 0$

This is in the form of $3x^2 - 5y - 1 = 0$ and cannot be factorised. When this occurs you change the square ratio to match the middle term using the square identity. In this case you express $\cos^2\theta$ in terms of $\sin^2\theta$ using $\cos^2\theta = 1 - \sin^2\theta$.

$\therefore 3\left(1 - \sin^2\theta\right) - 5\sin\theta - 1 = 0$

$\therefore 3 - 3\sin^2\theta - 5\sin\theta - 1 = 0$

$\therefore -3\sin^2\theta - 5\sin\theta + 2 = 0 \rightarrow$ multiply both sides by -1

$\therefore 3\sin^2\theta + 5\sin\theta - 2 = 0 \rightarrow$ this is in the form $3x^2 + 5x + 2 = 0$ and can be factorised

$\therefore (3\sin\theta - 1)(\sin\theta + 2) = 0$

$\therefore \sin\theta = \frac{1}{3}$ or $\sin\theta = -2 \rightarrow$ no solution, $-1 \le \sin\theta \le 1$

$\therefore RA = 19,471...^\circ$

$\therefore \theta = 19,471...^\circ + 360^\circ k$ or $\theta = 180^\circ - \left(19,471...^\circ\right) + 360^\circ k$; $k \in \mathbb{Z}$

$\therefore \theta = 19,47^\circ + 360^\circ k$ or $\theta = 160,53^\circ + 360^\circ k$

You only have to write $k \in \mathbb{Z}$ once.

Example 2: Determine the general solution of $2\sin^2\theta - \cos\theta + 1 = 0$.

$2\sin^2\theta - \cos\theta + 1 = 0 \rightarrow \sin^2\theta = 1 - \cos^2\theta$

$\therefore 2\left(1 - \cos^2\theta\right) - \cos\theta + 1 = 0$

$\therefore 2 - 2\cos^2\theta - \cos\theta + 1 = 0$

$\therefore -2\cos^2\theta - \cos\theta + 3 = 0$

$\therefore 2\cos^2\theta + \cos\theta - 3 = 0 \rightarrow$ this is in the form $2x^2 + x - 3 = 0$ and can be factorised

$\therefore (\cos\theta - 1)(2\cos\theta + 3) = 0$

$\therefore \cos\theta = 1$ or $\cos\theta = -3/2 \rightarrow$ no solution, $-1 \le \cos\theta \le 1$

$\therefore RA = 0^\circ$

$\therefore \theta = \pm 0^\circ + 360^\circ k$; $k \in \mathbb{Z}$

$\therefore \theta = 360^\circ k$

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Example 3: Determine the general solution of $6\sin^2\theta - 7\sin\theta\cos\theta - 2 = 0$.

$6\sin^2\theta - 7\sin\theta\cos\theta - 2 = 0$

This equation is in the form of $6x^2 - 7xy - 2 = 0$. If the middle term contains a $\sin\theta$ and $\cos\theta$ express the constant in terms of $\sin^2\theta + \cos^2\theta$.

$\therefore 6\sin^2\theta - 7\sin\theta\cos\theta - 2(1) = 0 \rightarrow$ you can always multiply the constant by 1

$\therefore 6\sin^2\theta - 7\sin\theta\cos\theta - 2(\sin^2\theta + \cos^2\theta) = 0 \rightarrow 1 = \sin^2\theta + \cos^2\theta$

$\therefore 6\sin^2\theta - 7\sin\theta\cos\theta - 2\sin^2\theta - 2\cos^2\theta = 0$

$\therefore 4\sin^2\theta - 7\sin\theta\cos\theta - 2\cos^2\theta = 0 \rightarrow$ this is in the form $4x^2 - 7xy - 2y^2 = 0$

$\therefore (\sin\theta - 2\cos\theta)(4\sin\theta + \cos\theta) = 0$

$\therefore \sin\theta = 2\cos\theta \quad$ or $\quad 4\sin\theta = -\cos\theta$

$\therefore \tan\theta = 2 \quad$ or $\quad \tan\theta = -\frac{1}{4}$

$\therefore RA = 63,434...^\circ \quad$ or $\quad RA = -14,036...^\circ$

$\therefore \theta = 63,43^\circ + 180^\circ k \quad$ or $\quad \theta = -14,04^\circ + 180^\circ k;\ k \in \mathbb{Z}$

5) Equations using the negative angles and reduction formulas

Example 1: Determine the general solution of $\sin\theta = -\sin 40^\circ$.

$\sin\theta = -\sin 40^\circ$

$\therefore \sin\theta = \sin(-40^\circ) \rightarrow$ same trigonometric function on both sides, $\therefore$ the RA is -40°

$\therefore RA = -40^\circ$

$\therefore \theta = -40^\circ + 360^\circ k \quad$ or $\quad \theta = 180^\circ - (-40^\circ) + 360^\circ k, \quad k \in \mathbb{Z}$

$= 220^\circ + 360^\circ k$

Example 2: Solve for θ if $\cos(\theta + 5^\circ) = -\cos 2\theta$ correct to 1 decimal place.

$\cos(\theta + 5^\circ) = -\cos 2\theta$

$\therefore \cos(\theta + 5^\circ) = \cos(180^\circ - 2\theta) \rightarrow \cos(180^\circ - 2\theta) = -\cos 2\theta$

$\therefore RA = 180^\circ - 2\theta$

$\therefore \theta + 5^\circ = 180^\circ - 2\theta + 360k \quad$ or $\quad \theta + 5^\circ = -(180^\circ - 2\theta) + 360^\circ k$

$\therefore 3\theta = 175^\circ + 360^\circ k \qquad \therefore -\theta = -185^\circ + 360^\circ k$

$\therefore \theta = 58,3^\circ + 120^\circ k \qquad \therefore \theta = 185^\circ - 360^\circ k, \quad k \in \mathbb{Z}$



Complete Exercise 5 number 2 on page 325.

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EQUATIONS IN A SPECIFIED DOMAIN

The general solution to a trigonometric equation consists of all the possible solutions for a general domain. You will now work with solutions in a specified domain.

In order to determine solutions in a specified domain you have to substitute values for k into the general solution and determine all of the solutions which lie within the specified domain.

Example 1: Solve for θ if $\sin\theta = \sin 40^\circ$ and $\theta \in [0^\circ; 360^\circ]$

$\sin\theta = \sin 40^\circ$

$RA = 40^\circ$

$\therefore \theta = 40^\circ + 360^\circ k$ or $\theta = 180^\circ - 40^\circ + 360^\circ k\ ;\ k \in \mathbb{Z}$

$= 140^\circ + 360^\circ k$

You can now draw up a table of the values of θ for different values of k :

k	$\theta = 40^\circ + 360^\circ k$		$\theta = 140^\circ + 360^\circ k$	
–1	$\theta = -320^\circ$	×	$\theta = -220^\circ$	×
0	$\theta = 40^\circ$	✓	$\theta = 140^\circ$	✓
1	$\theta = 400^\circ$	×	$\theta = 500^\circ$	×

From the above table it can be seen that there are two solutions which lie in the given interval:

$\therefore \theta = 40^\circ$ or $\theta = 140^\circ$ → this is also written as $\theta \in \{40^\circ; 140^\circ\}$



The table method on your calculator can also be used to draw the above table. No marks are awarded for the table, only for your answers. Page 233 on calculator tips demonstrates this procedure.

Example 2: Solve for θ if $\tan\theta = 1/\sqrt{3}$ and $\theta \in [-360^\circ; 360^\circ]$

$\tan\theta = 1/\sqrt{3}$

$RA = 30^\circ$

$\therefore \theta = 30^\circ + 180^\circ k\ ;\ k \in \mathbb{Z}$

k	θ		k	θ	
–3	$\theta = -510^\circ$	×	0	$\theta = 30^\circ$	✓
–2	$\theta = -330^\circ$	✓	1	$\theta = 210^\circ$	✓
–1	$\theta = -150^\circ$	✓	2	$\theta = 390^\circ$	×

From the above table it can be seen that there are 4 solutions which lie in the given interval:

$\therefore \theta = -330^\circ$ or $\theta = -150^\circ$ or $\theta = 30^\circ$ or $\theta = 210^\circ$

This is also written as $\theta \in \{-330^\circ; -150^\circ; 30^\circ; 210^\circ\}$



Complete Exercise 5 number 3 on page 325.

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E. RESTRICTIONS IN TRIGONOMETRIC EQUATIONS

An expression or equation will be undefined if division by 0 occurs. For this reason it is important to note the following properties of the trig ratio tan:

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\tan\theta$ is undefined when $\cos\theta = 0$ $\therefore$ when $\theta = \pm 90° + 360°k \quad ; \quad k \in \mathbb{Z}$



$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$$

$\therefore \tan 2\theta$ is undefined when $\cos 2\theta = 0$

$\therefore 2\theta = \pm 90° + 360°k \; ; k \in \mathbb{Z}$

$\therefore \theta = \pm 45° + 180°k$

Example 1: Solve for θ if $\tan\theta\sin\theta = \tan\theta$

$\tan\theta\sin\theta = \tan\theta$

$\therefore \tan\theta\sin\theta - \tan\theta = 0$

$\therefore \tan\theta(\sin\theta - 1) = 0 \rightarrow$ never divide by $\tan\theta$ as you lose an answer $(\tan\theta = 0)$

$\therefore \tan\theta = 0$ or $\sin\theta = 1$

$\therefore \theta = 180°k$ or $\therefore \theta = 90° + 360°k \rightarrow$ N/A

The equation is undefined when $\cos\theta = 0$ $\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta}\right)$,

$\therefore \cos\theta = 0$ when $\theta = 90° + 360°k \; ; \; k \in \mathbb{Z}$

Example 2: For which values of x is the equation $\dfrac{\sin^2 x}{\sin 2x} = \dfrac{\tan x}{2}$ undefined?

For the equation to be undefined $\sin 2x = 0$ or $\cos x = 0 \rightarrow$ the equation contains $\tan x$

$\sin 2x = 0$ or $\cos x = 0$

$\therefore RA = 0°$ or $RA = 90°$

$\therefore 2x = 0° + 360°k$ or $2x = 180° - 0° + 360°k$ or $x = \pm 90° + 360°k$

$\therefore x = 180°k$ or $x = 90° + 180°k$ or $x = \pm 90° + 360°k$



Complete Exercise 6 on page 325 and the Mixed Exercise on pages 325-327.

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CHAPTER 7 – MEASUREMENT

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
142	**A. Surface Area and Volume Formulas**	7	• Use and apply all of the formulas for surface area and volume which you learnt in Grade 10.
142	• Right Prisms		
142	• Scaling Factors of Prisms		
143	• Cylinders		
143	• Pyramids		
144	• Spheres		
144	• Hemispheres		
145	• Cones		
146	• Mixed Examples		

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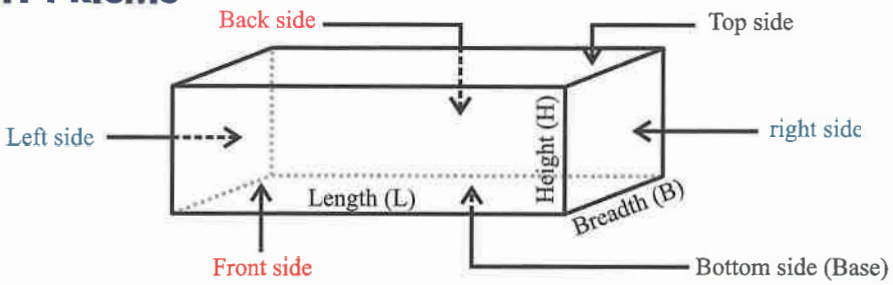
A. SURFACE AREA AND VOLUME FORMULAS

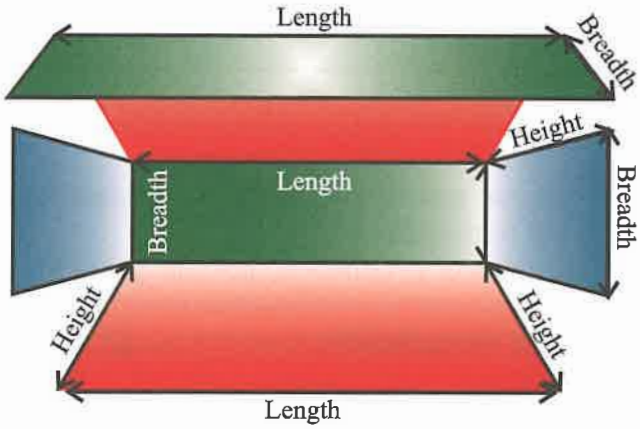
In general, the surface area is the sum of all the areas of all the shapes that cover the exterior surface of the object.

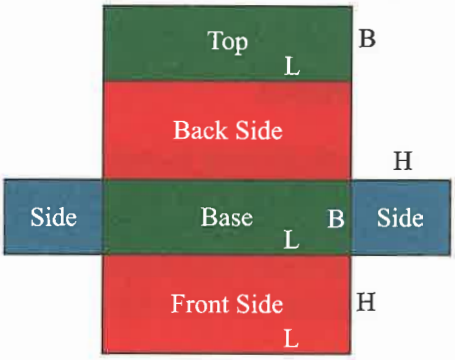
The volume of a shape represents the amount of space that a shape can occupy. In general, the volume of a shape is the area of the base shape multiplied by the perpendicular height it is extended by.

Area and surface area are always expressed in units^2.
Volume is always expressed in units^3.

RIGHT PRISMS







Right Prisms

Surface Area: $SA = 2BH + 2LH + 2LB$
Volume: $V = L \times B \times H$

SCALING FACTORS OF PRISMS

Change	Effect on Surface Area
All 3 sides multiplied by a factor of k.	**Surface area increases by a factor of k^2**

Change	Effect on Volume
1 side multiplied by a factor of k.	**Volume is increased by a factor of k**
Any 2 sides multiplied by a factor of k.	**Volume is increased by a factor of k^2**
All 3 sides multiplied by a factor of k.	**Volume is increased by a factor of k^3**

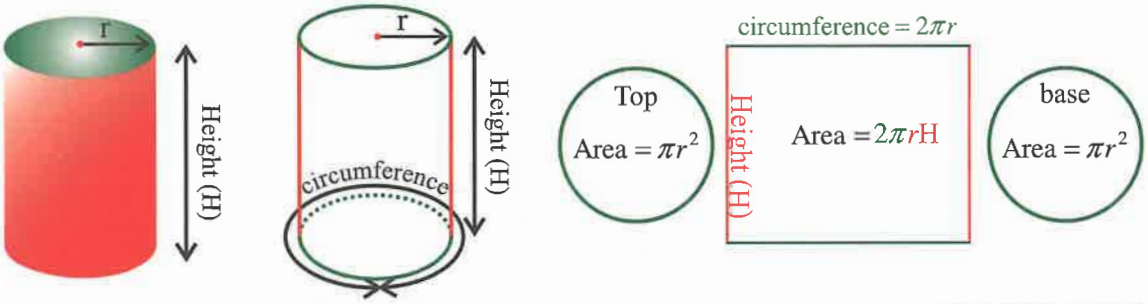
There is no general rule for the change in surface area if one or two sides are multiplied by a factor of k. In these cases the new area has to be worked out using all the formulas that you know.

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CYLINDERS

A cylinder is made up of two circles and a rectangle. The area of the rectangle is the circumference of the circle multiplied by the height of the cylinder as shown below.

The formula for the surface area of a cylinder has to be adjusted depending on whether it is closed or open on either end.



Right Cylinders

Surface Area:

$SA = 2\pi rH + 2\pi r^2 \rightarrow$ closed cylinder

$SA = 2\pi rH + \pi r^2 \rightarrow$ cylinder open on one end

$SA = 2\pi rH \rightarrow$ cylinder open on both ends

Volume:

$V = \pi r^2 \times H$

PYRAMIDS

In Grade 10 you learnt to calculate the volume and surface area of 2 triangular based pyramids as well as rectangular based pyramids.

The methods and formulas for calculating the areas of pyramids do not change for the different types of pyramids.

Pythagoras will often have to be used to calculate the pyramid or slant height.

Pyramids:

Surface Area:

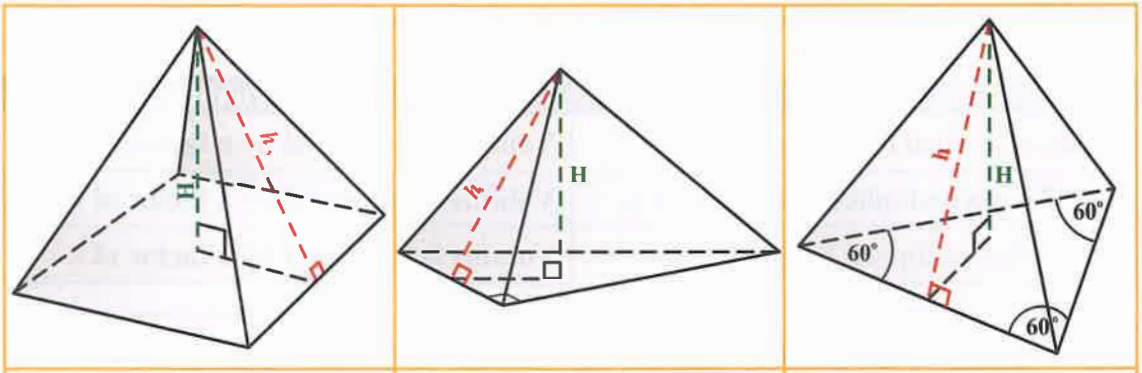
SA = area base + area of triangular faces

Area of triangular face $= \frac{1}{2} base \times \perp h_s$ *(slant)*

Volume:

$$V = \frac{\text{area of base} \times \text{Height}}{3}$$

$$= \frac{1}{3}(\text{area of base} \times \text{Height})$$

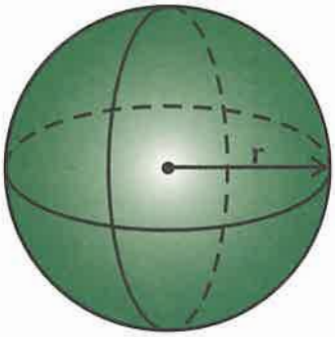


Rectangular Pyramid	Triangular Pyramid	Equilateral based Pyramid

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SPHERES

A sphere is a perfectly symmetrical ball with all points on the surface of the sphere being equidistant from its centre.
The formulas for surface area and volume of a sphere are shown below:

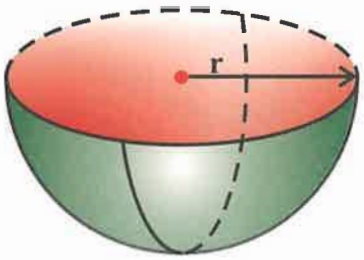


Spheres:

Surface Area: $SA = 4\pi r^2$	Volume: $V = \frac{4}{3}\pi r^3$

HEMISPHERES

A hemisphere is half a sphere. Therefore, the volume of a hemisphere is simply half that of a sphere. However, if the hemisphere is closed (has a solid surface for a base) or solid you have to take into account the area of the base of the hemisphere. If the hemisphere is closed the area of the base (πr^2) has to be added.



Closed Hemispheres:

Surface Area:	Volume:
$SA = \frac{1}{2}(4\pi r^2) + \pi r^2 = 3\pi r^2$	$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$

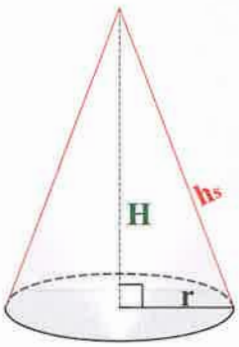
Open Hemispheres:

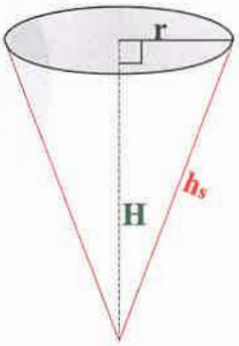
Surface Area:	Volume:
$SA = \frac{1}{2}(4\pi r^2) = 2\pi r^2$	$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$

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CONES

The formulas for determining the volume and surface area for cones are shown below. As with hemispheres it is important to take note of whether the cone is closed (solid) or open (hollow). If the cone is closed, the area of the base has to be taken into account when calculating the surface area of cones.
Pythagoras will often have to be used to calculate the cone height or slant height.





Closed Cones:

$$\text{Volume} = \frac{1}{3}\pi r^2 H$$

$$\text{Surface Area} = \frac{1}{2}\text{circumference of the base} \times \text{slant height} + \text{area of the base}$$

$$= \frac{1}{2}(2\pi r)(h_s) + \pi r^2$$

$$= \pi r h_s + \pi r^2$$

Open Cones:

$$\text{Volume} = \frac{1}{3}\pi r^2 H$$

$$\text{Surface Area} = \frac{1}{2}\text{circumference of the base} \times \text{slant height}$$

$$= \frac{1}{2}(2\pi r)(h_s)$$

$$= \pi r h_s$$

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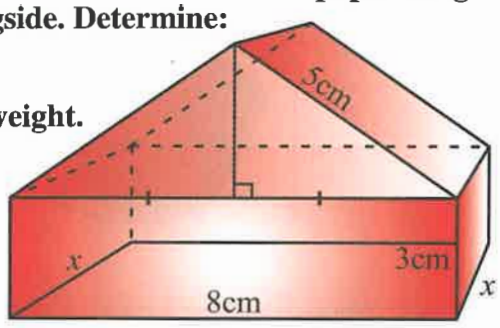
MIXED EXAMPLES

Example 1: **A glass paperweight is constructed from a triangular and rectangular prism and is then filled with 144 cm^3 of red water. The paper weight with its dimensions is shown alongside. Determine:**

a) The value of x.

b) The surface area of the paperweight.

c) If all the dimensions are multiplied by 3, what volume of water will be required to fill the paperweight?



a) You are given that the volume of the paperweight is 144 cm^3. The first step in calculating the volume is to determine the height of the triangular prism using the Theorem of Pythagoras:

$H^2 = 5^2 - 4^2$ → the base of each right-angled triangle face is $4\,\text{cm}\left(\frac{1}{2}\times 8\right)$

$\therefore H = 3\,\text{cm}$ → this is the perpendicular height of the face of the triangular prism

$V_{total} = V_{rectangular\ prism} + V_{triangular\ prism}$

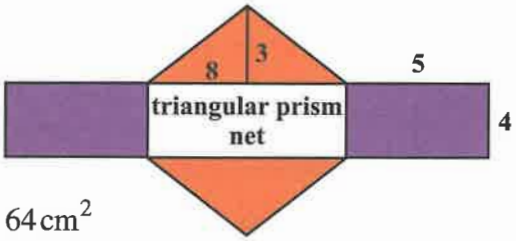
$V_{rectangular\ prism} = 8\times 3\times x = 24x$

$V_{triangular\ prism} = \frac{1}{2}(8)(3)(x) = 12x$

$\therefore 144 = 24x + 12x$ → the volume of the paper weight is $144\,\text{cm}^3$

$\therefore 144 = 36x$

$\therefore x = 4\,\text{cm}$



b) $SA_{total} = SA_{triangular\ prism} + SA_{rectangular\ prism}$

$SA_{triangular\ prism} = 2\left[\frac{1}{2}(8)(3)\right] + 2\left[5(4)\right] = 64\,\text{cm}^2$

$SA_{rectangular\ prism} = 2\left[(8)(3)\right] + 2\left[3(4)\right] + 8(4) = 104\,\text{cm}^2$

$\therefore SA_{total} = 64 + 104$

$= 168\,\text{cm}^2$



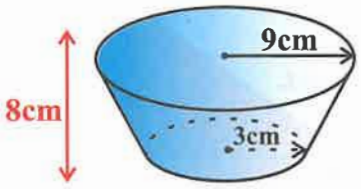
c) For any shape, if all dimensions are multiplied by a factor of k, the volume is increased by a factor of k^3.

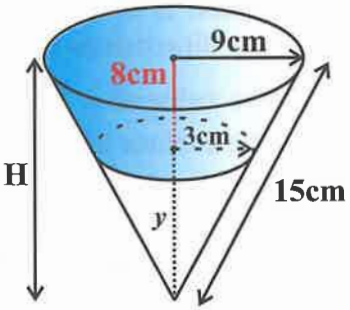
$\therefore V_{new} = 144\times 3^3$

$\therefore V_{new} = 3888\,\text{cm}^3$

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Example 2: **An art student makes a clay bowl which is 8cm high and has a radius of 9 cm at the top and 3 cm at the base. In order to make the bowl he follows the instructions below, given to him by a Grade 11 Mathematics learner:**





1) Construct a hollow cone with a slant height of 15cm and radius of 9cm.	2) At height y, cut the cone and construct a solid base for the bowl with a radius of 3cm.

a) Determine y, the height at which the cone has to be cut.

b) Hence, determine the volume of the bowl.

c) The art student wishes to coat the outside surface of the bowl with enamel. He can purchase enamel in tins of 100 ml. If one tin covers an area of 90 cm^2, how many tins will he need to purchase?

a) Using the Theorem of Pythagoras: $15^2 = (y+8)^2 + 9^2$

$\therefore 15^2 = y^2 + 16y + 64 + 9^2$

$\therefore y^2 + 16y - 80 = 0$

$\therefore (y-4)(y+20) = 0$

$\therefore y = 4$ or $y = -20 \rightarrow$ N/A as a height cannot be negative

$\therefore y = 4\,\text{cm}$

b) $V_{bowl} = V_{cone} - V_{cut\ off}$

$V_{cone} = \frac{1}{3}\pi r^2 H \rightarrow H = y + 8$

$= \frac{1}{3}\pi(9)^2(8+4) \rightarrow y$

$= 324\pi\,\text{cm}^3$

$V_{cut\ off} = \frac{1}{3}\pi r^2 y$

$= \frac{1}{3}\pi(3)^2(4) \rightarrow y$

$= 12\pi\,\text{cm}^3$

$\therefore V_{bowl} = 324\pi - 12\pi$

$= 312\pi$

$= 980{,}18\,\text{cm}^3$

Note that volume is measured in cm^3.

c) $SA_{Total} = SA_{cone} - SA_{cut\ off} + SA_{base}$

$SA_{cone} = \pi r h_s$

$= \pi(9)(15) = 135\pi\,\text{cm}^2$

You need to calculate h_s of the cut off:

$h_s^2 = 3^2 + 4^2 \rightarrow h_s^2 = 3^2 + y$

$\therefore h_s = 5\,\text{cm}$

$SA_{cut\ off} = \pi r h_s$

$= \pi(3)(5) = 15\pi\,\text{cm}^2$

$SA_{base} = \pi r^2$

$= \pi(3)^2 = 9\pi\,\text{cm}^2$

$\therefore SA_{Total} = 135\pi - 15\pi + 9\pi$

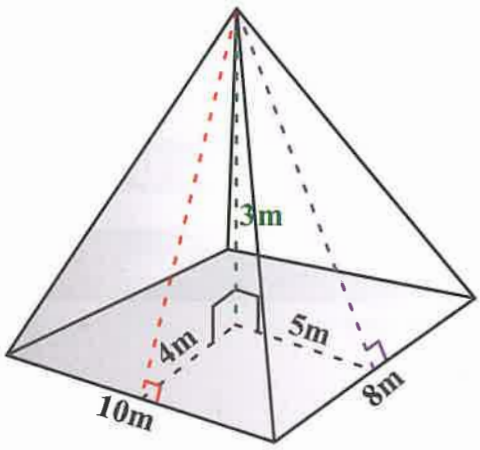
$= 405{,}27\,\text{cm}^2$

$\therefore$ Number of tins $= \frac{405{,}27}{90} = 4{,}5$

$\therefore$ 5 tins of enamel are needed.

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Example 3: Determine the volume and surface area of the pyramid shown below:





It is important to understand that if the pyramid has a square base, the surface area of its triangular faces will all be the same. However, in this case the base is a rectangle and therefore only the opposite faces have the same area.

The first step in calculating the surface area is to determine the slant height of the triangular face using the Theorem of Pythagoras. You can then calculate the surface area of the base and then the total surface area.

△ face with 10m base	△ face with 8m base	Base of pyramid
$h_s^2 = 4^2 + 3^2$	$h_s^2 = 5^2 + 3^2$	$SA = l \times b$
$= 25$	$= 34$	$= 8 \times 10$
$\therefore h_s = 5\,m$	$\therefore h_s = \sqrt{34}\,m$	$= 80\,m^2$
$\therefore SA = \frac{1}{2}(base)(h_s)$	$\therefore SA = \frac{1}{2}(base)(h_s)$	
$\therefore SA = \frac{1}{2}(10)(5)$	$\therefore SA = \frac{1}{2}(8)(\sqrt{34})$	
$\therefore SA = 25\,m^2$	$\therefore SA = 4\sqrt{34}\,m^2$	

$SA_{total} = 2(25) + 2(4\sqrt{34}) + 80$ → there are 2 of each triangular face

$= 176,65\,m^2$

Volume of the pyramid:

$$V = \frac{\text{area of base} \times \text{Height}}{3}$$

$$= \frac{80 \times 3}{3}$$

$$\therefore V = 80\,m^3$$



Complete Mixed Exercise on pages 340 and 341.

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CHAPTER 8 – EUCLIDEAN GEOMETRY

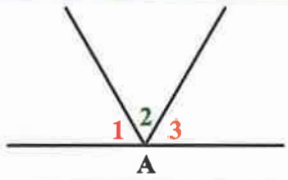
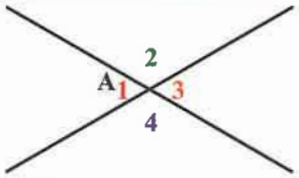
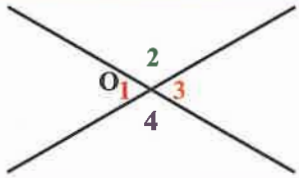
WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
150	**A. Revision of Lines and Triangles**	7	• Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles. • Solve circle geometry problems, providing reasons for statements when required. • Prove riders.
150	• Straight Lines		
150	• Parallel Lines		
150	• Triangles		
151	• Congruency		
152	**B. Circle Terminology**		
153	**C. Circle Theorems**		
153	• Theorem 1:Line segment through centre and midpoint		
153	• Theorem 1: (Converse)		
154	• Theorem 2: Perpendicular bisector of a chord		
156	• Theorem 3: Angle at centre is twice the angle at the circumference		
158	• Theorem 4: Angles in same segment are equal		
159	• Theorem 4: (Converse)		
159	• Theorem 3 and 4: (Corollaries)		
161	• Theorem 5: Opposite angles of a cyclic quadrilateral		
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164	• Theorem6: Tangents from the same point are equal		
166	• Theorem7: Tan-Chord Theorem		
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167	• Determining which is the angle in the alternate segment		
169	**D. Mixed Examples**		
173	• Tips for Solving Riders		
173	• Summary of Theorems, Converses and Corollaries		

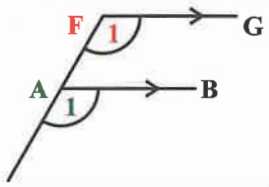
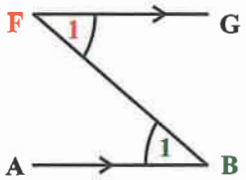
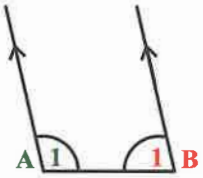
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A. REVISION OF LINES AND TRIANGLES

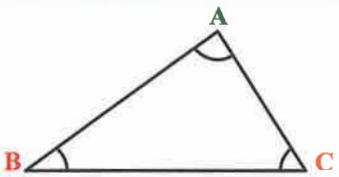
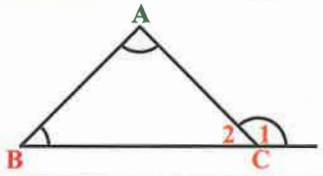
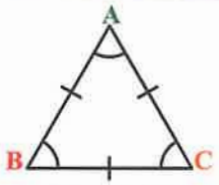
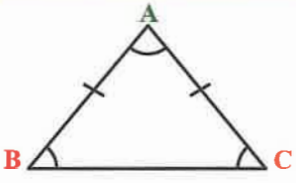
STRAIGHT LINES

Adjacent angles on a straight line are supplementary.	Angles around a point sum to 360°.	Vertically opposite angles are equal.
		
$\hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 180°$	$\hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_4 = 360°$	$\hat{O}_1 = \hat{O}_3$ and $\hat{O}_2 = \hat{O}_4$

PARALLEL LINES

Corresponding angles are equal (F shape).	Alternate angles are equal (Z or N shape).	Co-interior angles are supplementary (U shape).
		
$\hat{F}_1 = \hat{A}_1$	$\hat{F}_1 = \hat{B}_1$	$\hat{A}_1 + \hat{B}_1 = 180°$

TRIANGLES

The interior angles of a triangle are supplementary.	The exterior angle of a triangle is equal to the sum of the interior opposite angles.
	
$\hat{A} + \hat{B} + \hat{C} = 180°$	$\hat{C}_1 = \hat{A} + \hat{B}$
In an equilateral triangle all sides are equal and all angles are equal to 60°.	**Angles opposite equal sides are equal. Sides opposite equal angles are equal.**
	
$\hat{A} = \hat{B} = \hat{C} = 60°$ and $AB = AC = BC$	$\hat{B} = \hat{C}$ and $AB = AC$

To prove that 2 lines are parallel you have to show 1 of the following:

- **Alternate angles are equal**
- **Corresponding angles are equal**
- **Co-interior angles are supplementary**

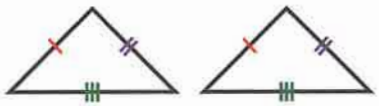
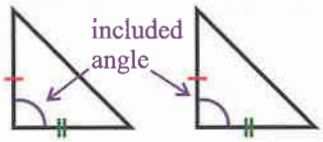
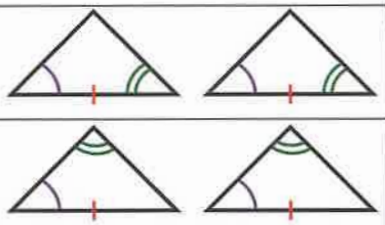
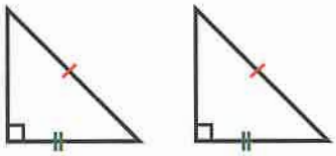
You should also take note of the following terminology:

- **Supplementary angles add up to 180°**
- **Complementary angles add up to 90°**

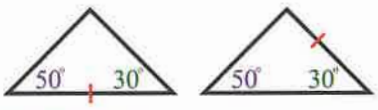
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CONGRUENCY

If triangles are congruent it means that they are exactly the same, i.e. all corresponding angles and sides in each triangle are equal.
There are 4 conditions for triangles to be congruent.
If any of the conditions listed below are true then the triangles are congruent.
The symbol ≡ is used for congruency.

Condition / Reason	Example	Explanation
Side Side Side (SSS)		**If triangles have three corresponding sides that are equal, the triangles are congruent.**
Side Angle Side (SAS)		**If triangles have two corresponding sides and an included angle that are equal, the triangles are congruent.**
Angle Angle Side Side Angle Angle Angle Side Angle (AAS) / (SAA) / (ASA)		**If triangles have two corresponding angles and a single corresponding side that are equal, the triangles are congruent.**
Right angle Hypotenuse Side (RHS)		**If the hypotenuse and one other corresponding side are equal in right-angled triangles, the triangles are congruent.**

Using AAS, the sides that are equal have to be the corresponding sides. The triangles below are not congruent as the equal sides are not the same relative to the equal angles.

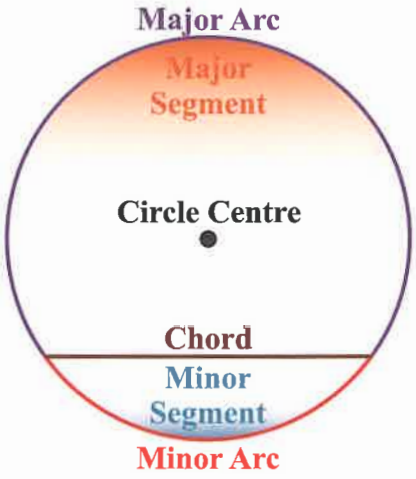


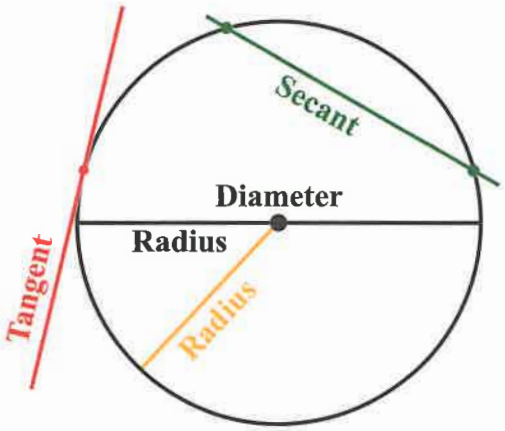
It is important for you to know the properties of quadrilaterals on pg. 65.

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B. CIRCLE TERMINOLOGY

When dealing with circles and circle theorems it is important that you understand the following terminology often used when dealing with circle problems. The diagrams below show the various parts of a circle which you are required to be familiar with.





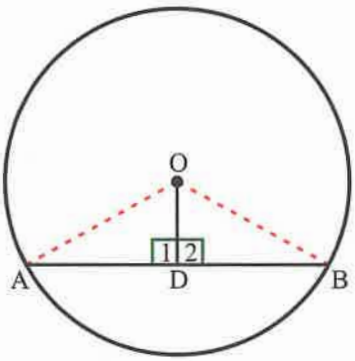
Term	Explanation
Radius	A line drawn from the **centre of the circle** to a point on the **circumference**.
Diameter	A line joining two points on the circumference of the circle **passing through the circle centre**.
Tangent	A line drawn from a point outside the circle which **touches the circle at only one point**, called the point of contact. The **tangent** is always **perpendicular to the radius** at the point of contact.
Chord	A line joining two points on the circumference of the circle which **does not** pass through the circle centre.
Secant	A line which cuts through two points on the circumference of a circle.
Theorem	A theorem is a proof of a statement.
Converse	A converse uses the conclusion of a theorem to prove what is given in a theorem.
Corollary	A statement that follows a theorem which is true as a result of the theorem.
Axiom	A general statement which is accepted to be true without a proof.
Rider	The name given to a problem in geometry.

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C. CIRCLE THEOREMS

THEOREM 1: LINE SEGMENT THROUGH CENTRE AND MIDPOINT

The line drawn from the centre of a circle perpendicular to a chord bisects the chord.



Given: O is the centre of the circle and $\hat{D}_1 = \hat{D}_2 = 90°$. D lies on chord AB.

Required to prove (RTP): $AD = DB$.

Construction: Join OA and OB.

Proof:

Statement	Reason
In $\triangle OAD$ and $\triangle OBD$:	
$OA = OB$	Radii
$OD = OD$	Common side
$\hat{D}_1 = \hat{D}_2 = 90°$	Given
$\therefore \triangle OAD \equiv \triangle OBD$:	RHS
$\therefore AD = DB$	Congruent triangles, $\triangle OAD \equiv \triangle OBD$:

When using this theorem to solve riders the reason used is:
Line from centre ⊥ to chord

THEOREM 1: (CONVERSE)

The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Given: O is the centre of the circle and $AD = DB$. D lies on chord AB.

Statement	Reason
$\hat{D}_1 = \hat{D}_2 = 90°$	Line from centre to midpoint of chord

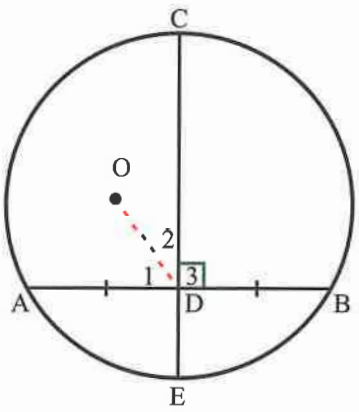


You will have to be able to apply the converse theorem in questions when solving riders. However, you will not have to prove the converse theorem.

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THEOREM 2: PERPENDICULAR BISECTOR OF A CHORD

The perpendicular bisector of a chord passes through the centre of the circle.



Given: A circle has a chord AB and CDE is a perpendicular bisector of AB.

Required to prove (RTP): The centre of the circle lies on CDE.

Proof: This theorem will be proved by contradiction. A proof by contradiction is a proof where we will make an assumption and then prove it to be incorrect. Assume that the circle has centre O and its centre does not lie on CDE.

Construction: Draw OD from the centre to the midpoint of the chord.

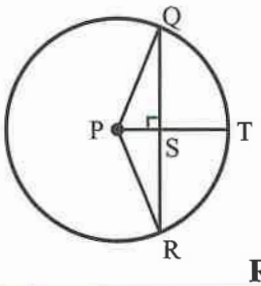
Statement	Reason
$\hat{D}_3 = 90°$	Given $(CD \perp AB)$
$\hat{D}_1 = \hat{D}_2 = 90°$	Line from centre $\perp$ to chord
$\therefore \hat{D}_1 + \hat{D}_2 + \hat{D}_3 = 3(90°) = 270°$	
But $\hat{D}_1 + \hat{D}_2 + \hat{D}_3 = 180°$	Adj. $\angle$'s on a str. line are supplementary
$\therefore$ The assumption that the circle centre O does not lie on CDE is incorrect. $\therefore$ By contradiction, the perpendicular bisector of a chord passes through the centre of the circle.	



Do not use proof by contradiction to solve geometry riders.

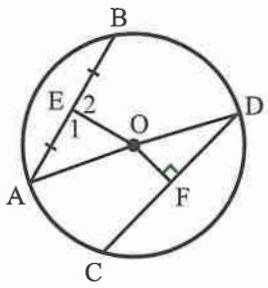
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Example 1: **In the diagram below, QR = 24 cm and PS = 5 cm. P is the centre of circle. Determine the length of ST.**



Statement	Reason
$QS = RS = 12\,cm$	Line from centre $\perp$ to chord
$QP^2 = PS^2 + QS^2$ $\therefore QP^2 = 5^2 + 12^2 = 169$ $\therefore QP = 13$	Pythag.
$QP = PT = 13\,cm$	Radii
$ST = PT - PS$ $= 13 - 5$ $= 8\,cm$	

Example 2: **In the diagram below O is the centre of the circle. AE = BE, OE = 3 cm and AB = 8 cm. If OF = 4 cm, determine the length of chord CD.**



Statement	Reason
$AE = EB = 4\,cm$	Given $AE = EB$ and $AB = 8\,cm$
$\hat{E}_1 = \hat{E}_2 = 90^\circ$	Line from centre to midpoint of chord
$\therefore OA^2 = OE^2 + AE^2$ $\therefore OA^2 = 3^2 + 4^2$ $\therefore OA^2 = 25$ $\therefore OA = 5\,cm$	Pythag. Given $OE = 3\,cm$
$OA = OD = 5\,cm$	Radii
$FD^2 = OD^2 - OF^2$ $\therefore FD^2 = 5^2 - 4^2 = 9$ $\therefore FD = 3\,cm$	Given $OF = 4\,cm$
$CF = FD$	Line from centre $\perp$ to chord
$\therefore CD = 6\,cm$	



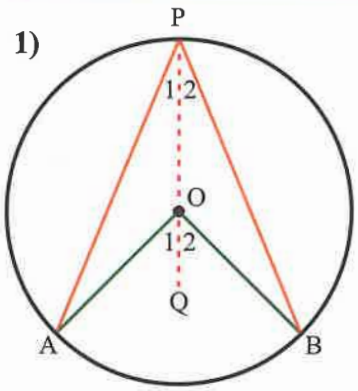
Complete Exercise 1 on page 344.

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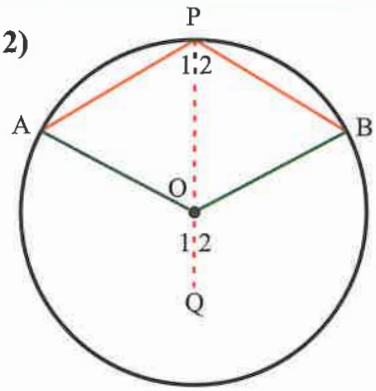
THEOREM 3: ANGLE AT CENTRE IS TWICE THE ANGLE AT THE CIRCUMFERENCE

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc on the circumference of a circle.

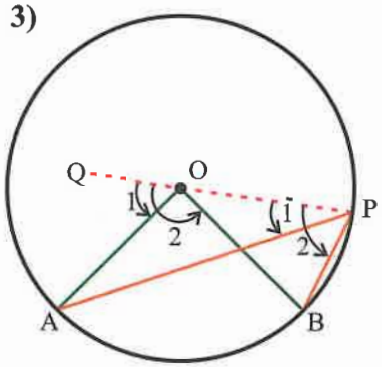
1)



2)



3)



Given: O is the centre of the circle, with arc AB subtending $A\hat{O}B$ at the centre and $A\hat{P}B$ at the circumference.

Required to prove (RTP): $A\hat{O}B = 2A\hat{P}B$.

Construction: Draw PO and produce (extend) to Q

Proof:

Statement	Reason
$\hat{O}_1 = \hat{P}_1 + \hat{A}$ and $\hat{O}_2 = \hat{P}_2 + \hat{B}$	Ext. $\angle$ of Δ
$OP = OA = OB$	Radii
$\hat{P}_1 = \hat{A}$ and $\hat{P}_2 = \hat{B}$	Angles opp. = sides
$\therefore \hat{O}_1 = 2\hat{P}_1$ and $\hat{O}_2 = 2\hat{P}_2$	

In diagrams 1 and 2:

$\hat{O}_1 + \hat{O}_2 = 2\hat{P}_1 + 2\hat{P}_2$ $= 2(\hat{P}_1 + \hat{P}_2)$ $\therefore A\hat{O}B = 2A\hat{P}B$	

In diagram 3:

$\hat{O}_2 - \hat{O}_1 = 2\hat{P}_2 - 2\hat{P}_1$ $= 2(\hat{P}_2 - \hat{P}_1)$ $\therefore A\hat{O}B = 2A\hat{P}B$	

When using this theorem to solve riders the reason used is:
$\angle$ at centre $= 2\angle$ at circumference

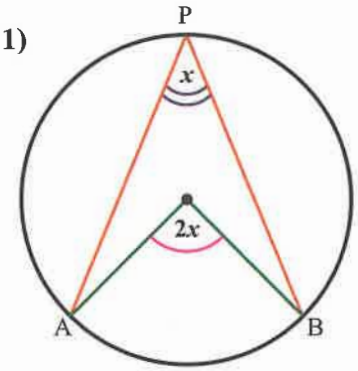


The notation $A\hat{O}B$ refers to the angle formed between lines AO and OB. The word subtending means creates. Note that for diagram 2, the angle at the centre is always larger than 180^o (a reflex angle).

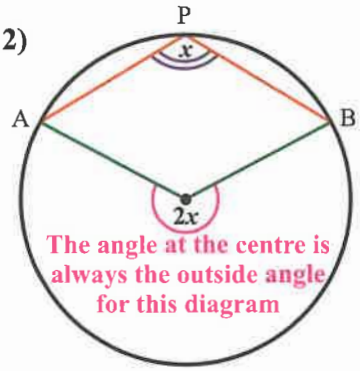
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Determining which angle is the angle at the circumference and the angle at the centre:

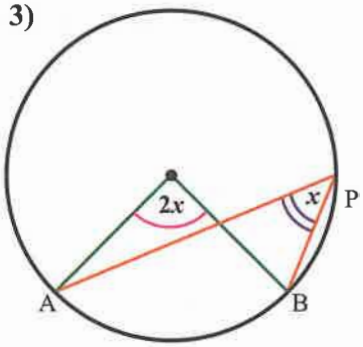
1)



2)



3)

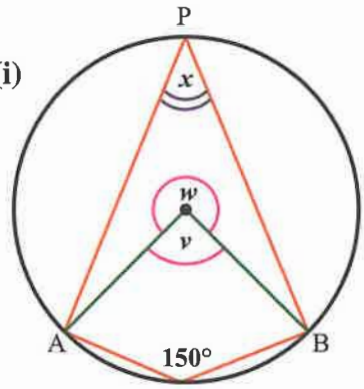


For the three cases shown above, to determine which is the angle at the centre follow the radii from the centre to the two points where they touch the circumference on the circle (points A and B). The angle between these lines is the angle at the centre. Then follow the lines to the point where they meet each other at the circumference (point P). This will form the angle at the circumference.

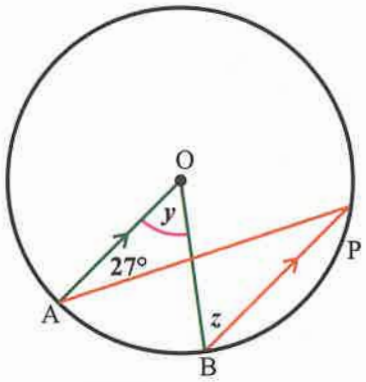
Example: **In the diagrams below, determine the values of v, w, x, y, and z.**

Remember that diagrams are not drawn to scale.

(i)



(ii)



Statement	Reason
Diagram (i):	
$w = 2(150^\circ) = 300^\circ$	$\angle$ at centre $= 2\angle$ at circumference
$v = 360^\circ - 300^\circ = 60^\circ$	$\angle$'s around a point
$x = 60^\circ \div 2 = 30^\circ$	$\angle$ at centre $= 2\angle$ at circumference
Diagram (ii):	
$\hat{P} = 27^\circ$	Alt. $\angle$'s $(OA \,//\, PB)$
$y = 2\hat{P} = 2(27^\circ) = 54^\circ$	$\angle$ at centre $= 2\angle$ at circumference
$z = y = 54^\circ$	Alt. $\angle$'s $(OA \,//\, PB)$

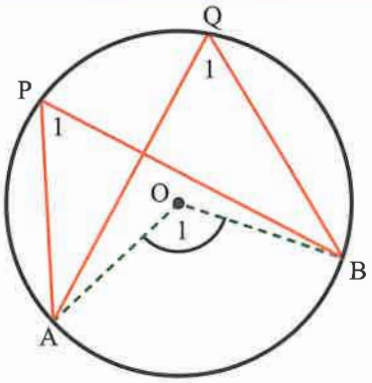
The converse to Theorem 3 is false. If an angle at the circumference of a circle is half the size of an angle subtended by the same arc at a point inside the circle, it does not mean that this is the circle centre.

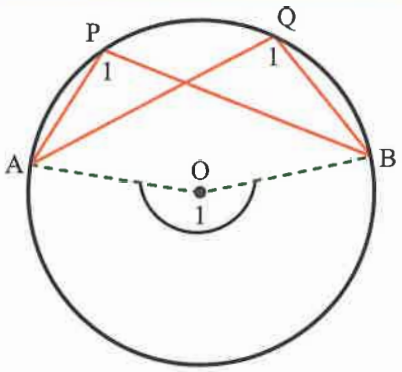
Complete Exercise 2 on page 344.

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THEOREM 4: ANGLES IN SAME SEGMENT ARE EQUAL

The angles subtended by a chord of the circle, on the same side of the chord, are equal.





Given: A, B, P and Q are points on the circumference of the circle with centre O.

Required to prove (RTP): $\hat{P}_1 = \hat{Q}_1$.

Construction: Join AO and OB.

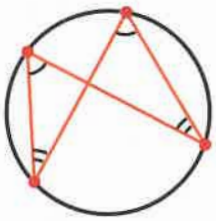
Proof:

Statement	Reason
$\hat{O}_1 = 2\hat{P}_1$	$\angle$ at centre $= 2\angle$ at circumference
$\hat{O}_1 = 2\hat{Q}_1$	$\angle$ at centre $= 2\angle$ at circumference
$\therefore 2\hat{P}_1 = 2\hat{Q}_1$	
$\therefore \hat{P}_1 = \hat{Q}_1$	

When using this theorem to solve riders the reason used is:
$\angle$'s in same segment



When solving riders this theorem can be seen by the characteristic bowtie in the circle. It is important to note that every corner of the bowtie must lie on the circumference of the circle. Do not confuse this with theorem 3. In theorem 3 a corner of the bowtie will lie at the centre of the circle.



Theorem 4-
angles in same segment



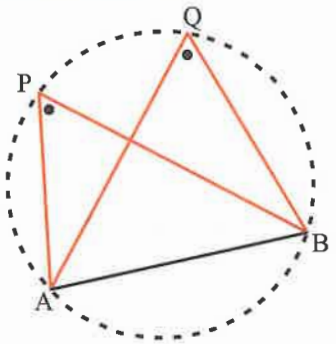
Theorem 3-
$\angle$ at centre $= 2\angle$ at circumference

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THEOREM 4: (CONVERSE)

If a line segment joining two points subtends two equal angles at two other points on the same side of the line segment, then these four points are concyclic.

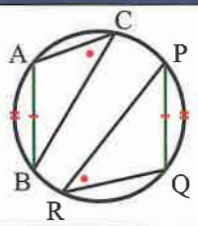
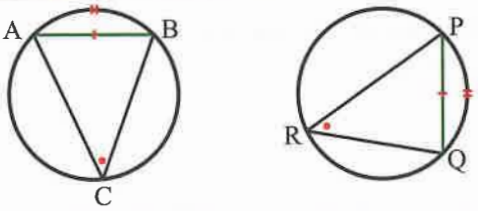
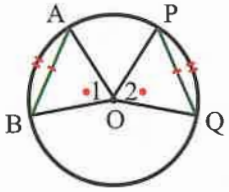
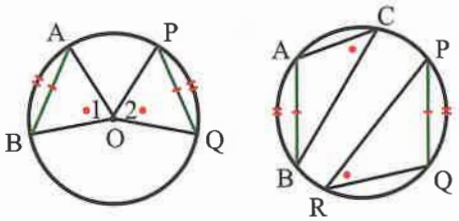
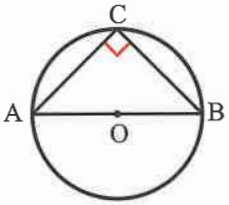
If 4 points are concyclic it means that the points lie on the circumference of a circle. A cyclic quadrilateral is a quadrilateral whose vertices lie on the circumference of a circle, or a quadrilateral which can have a circle drawn around it which passes through the vertices of the quadrilateral.



Given: $\hat{P} = \hat{Q}$ **and both angles are subtended by chord AB.**

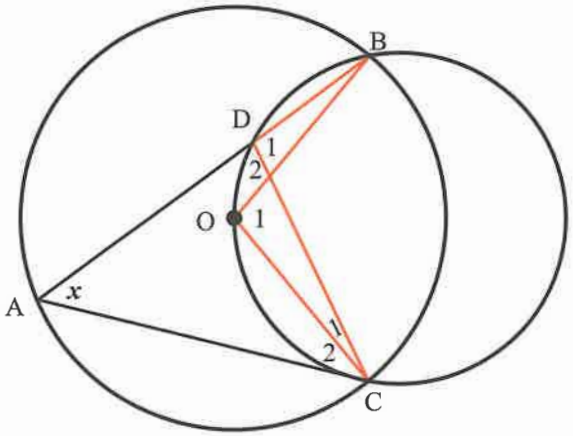
Statement	Reason
ABQP is a cyclic quadrilateral	Line seg. subt. = ∠'s on the same side

THEOREM 3 AND 4: (COROLLARIES)

Equal chords/arcs subtend equal angles on the circumference of a circle.	Angles subtended by chords/arcs of equal length on 2 different circles with equal radii are equal.
	
Given: AB = PQ ∴ $\hat{C} = \hat{R}$	Given: AB = PQ ∴ $\hat{C} = \hat{R}$
Equal chords/arcs subtend equal angles at the centre of a circle.	Chords/arcs are equal if they subtend equal angles at the circumference or circle centre.
	
Given: AB = PQ ∴ $\hat{O}_1 = \hat{O}_2$	Given: $\hat{O}_1 = \hat{O}_2$ or $\hat{C} = \hat{R}$ ∴ AB = PQ
The diameter of a circle subtends a right angle on the circumference of the circle.	
Given: AB is a diameter ∴ $\hat{C} = 90°$ Reason: ∠ in a semi-circle	

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Example 1: **In the diagram below, A, B and C are points on circle centre O. B, C and D are points on the smaller circle passing through O. Prove that AD = DC.**



Statement	Reason
$\hat{O}_1 = 2x$	$\angle$ at centre $= 2\angle$ at circumference
$\hat{O}_1 = \hat{D}_1 = 2x$	$\angle$'s in same segment
$\hat{D}_1 = \hat{A} + (\hat{C}_1 + \hat{C}_2)$ $\therefore 2x = x + (\hat{C}_1 + \hat{C}_2)$ $\therefore (\hat{C}_1 + \hat{C}_2) = x$ $\therefore D\hat{A}C = D\hat{C}A = x$	Ext. $\angle$ of a Δ
$\therefore DA = DC$	$\angle$'s opp. = sides

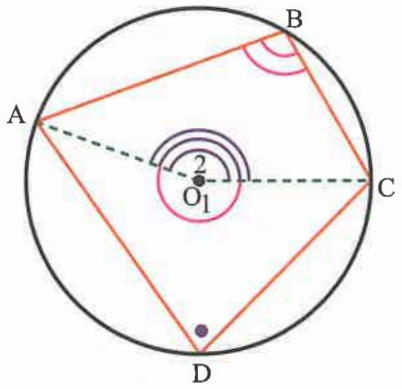
As you progress, you will have to apply previous theorems when solving riders using the new theorems that you learn.

Complete Exercise 3 on page 345.

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THEOREM 5: OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL

The opposite angles of a cyclic quadrilateral are supplementary.



Given: ABCD is a cyclic quadrilateral

Required to prove (RTP): $\hat{B}+\hat{D}=180^\circ$ and $\hat{A}+\hat{C}=180^\circ$.

Construction: Join AO and OC.

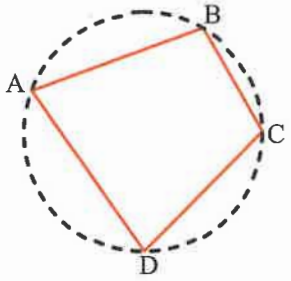
Proof:

Statement	Reason
$\hat{O}_1=2\hat{B}$	$\angle$ at centre $=2\angle$ at circumference
$\hat{O}_2=2\hat{D}$	$\angle$ at centre $=2\angle$ at circumference
$\hat{O}_1+\hat{O}_2=360^\circ$ $\therefore 2\hat{B}+2\hat{D}=360^\circ$ $\therefore \hat{B}+\hat{D}=180^\circ$	$\angle$'s around a point
$\therefore \hat{A}+\hat{C}=180^\circ$	Int. $\angle$'s of quad. sum to 360°

When using this theorem to solve riders the reason used is:
Opp. $\angle$'s of a cyclic quad.

THEOREM 5: (CONVERSE)

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.



Given: $\hat{B}+\hat{D}=180^\circ$ or $\hat{A}+\hat{C}=180^\circ$.

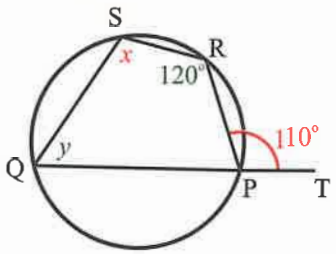
Statement	Reason
ABCD is a cyclic quadrilateral	**Opp. $\angle$'s supplementary**

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THEOREM 5: (COROLLARY)

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

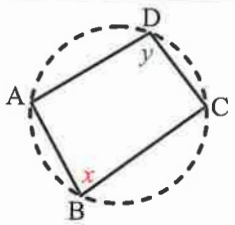
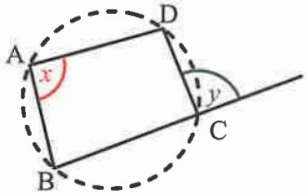
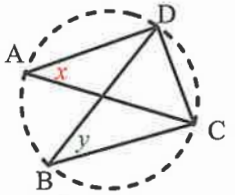
Illustrative Example: In the diagram below, QPRS is a cyclic quadrilateral. QP is produced to point T. Determine the values of x and y.



Statement	Reason
$x = 110°$	Ext. ∠ of a cyclic quad.
$y + 120° = 180°$	Opp. ∠'s of a cyclic quad.
$\therefore y = 60°$	

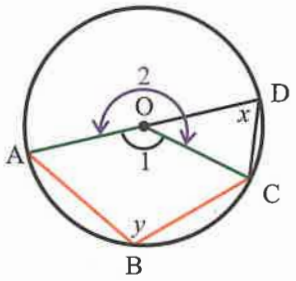
PROVING THAT A QUADRILATERAL IS CYCLIC

When proving that a quadrilateral is cyclic, there are 3 properties to look for. If you prove any 1 of the properties below, a quadrilateral is cyclic.

	Property	Diagram
1.	If one pair of opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic. In other words, in the diagram alongside, if: $x + y = 180°$ then ABCD is cyclic.	
2.	If the exterior angle of a quadrilateral is equal to the interior opposite angle, the quadrilateral is cyclic. In other words, in the diagram alongside, if: $x = y$ then ABCD is cyclic.	
3.	If a line subtends equal angles at 2 different points on the same side of the line, the four points are concyclic (angles in same segment). In other words, in the diagram alongside, if: $x = y$ then ABCD is cyclic.	

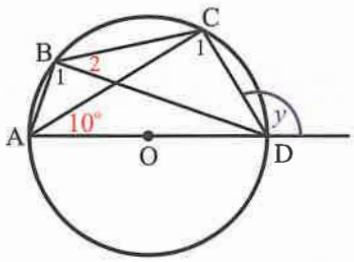
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Example 1: **In the diagram below ABCD is a cyclic quadrilateral and AOD is a diameter of the circle. Determine the values of x and y if $\hat{O}_1$ is 120°.**



Statement	Reason
$O_2 = 360° - 120° = 240°$	∠'s around a point
$\therefore y = \frac{1}{2}(240°)$ $\therefore y = 120°$	∠ at centre = 2∠ at circumference
$x = 180° - y$ $\therefore x = 180° - 120° = 60°$	Opp. ∠'s of a cyclic quad.

Example 2: **Prove that ABCD is a cyclic quadrilateral and then determine the value of y in the diagram below.**



Statement	Reason
$\hat{B}_1 = 90°$	∠ in a semi circle
$\hat{C}_1 = 90°$	∠ in a semi circle
$\therefore \hat{B}_1 = \hat{C}_1$ $\therefore$ ABCD is a cyclic quad.	AD subtends = ∠'s
$\hat{B}_2 = C\hat{A}D = 10°$	∠'s in same segment
$y = \hat{B}_1 + \hat{B}_2$	Ext. ∠ of a cyclic quad.
$\therefore y = 90° + 10°$ $\therefore y = 100°$	



Complete Exercise 4 on pages 345 and 346 .

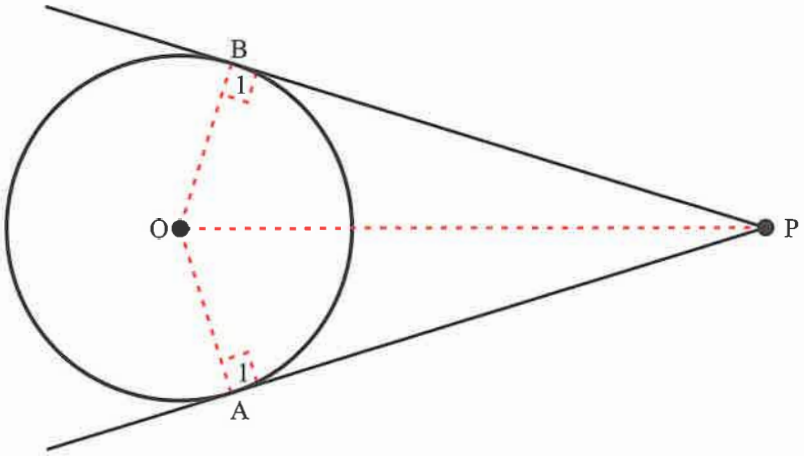
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THEOREM 6: TANGENTS FROM THE SAME POINT ARE EQUAL

If two tangents are drawn from the same point outside a circle they are equal in length.

Recap:

A tangent is a straight line which touches a circle at one point. A tangent is always perpendicular to the radius at the point of contact.



Given: **Circle with centre O and tangents PA and PB are drawn from point P and touch the circle at points A and B.**

Required to prove (RTP): PA = PB.

Construction: Join OA, OB and OP.

Proof:

Statement	Reason
In △PAO and △PBO:	
$\hat{A}_1 = \hat{B}_1 = 90^\circ$	Tan ⊥ radius
PO = PO	Common side
OA = OB	Radii
∴△PAO ≡△PBO	RHS
∴ PA = PB	Congruent triangles, △PAO ≡△PBO

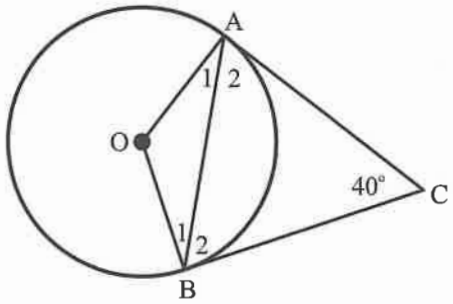
When using this theorem to solve riders the reason used is:
Tangents from same pt.

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Example 1: **In the diagram below O is the centre of the circle and points A and B are points on the circle. Tangents AC and BC are drawn from point C.**

a) Prove that AOBC is a cyclic quadrilateral.

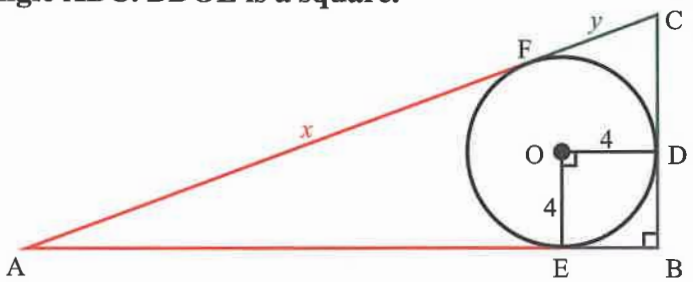
b) Determine the size of angle $\hat{B}_2$.



a) Statement	Reason
$O\hat{B}C = O\hat{A}C = 90°$	Tan $\perp$ radius
$\therefore$ AOBC is a cyclic quad.	Opp. $\angle$'s supplementary

b) Statement	Reason
$AC = BC$	Tangents from same pt.
$\hat{A}_2 + \hat{B}_2 = 180° - 40° = 140°$	$\angle$'s in a Δ
$\therefore \hat{A}_2 = \hat{B}_2 = 70°$	$\angle$'s opp. = sides

Example 2: **In the diagram below O is the centre of the circle with radius 4 cm. AC, CB and AB are tangents to the circle at points F, D and E respectively. If AC is 12cm in length determine the perimeter of triangle ABC. BDOE is a square.**

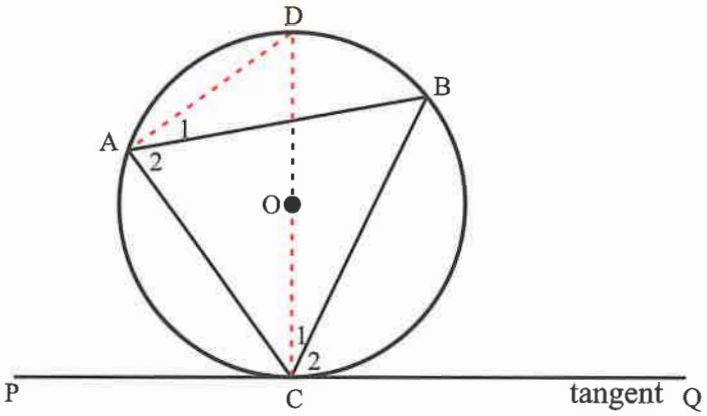


Statement	Reason
$EB = DB = 4\,cm$	Opp. sides of a square are equal
$CF = CD = y$	Tangents from same pt.
$AF = AE = x$	Tangents from same pt.
$AC = x + y = 12$	Given
Perimeter $= AF + FC + AE + EB + BD + DC$ $\therefore P = x + y + x + 4 + 4 + y$ $= 2x + 2y + 8$ $= 2(x + y) + 8$ $= 2(12) + 8$ $= 32\,cm$	

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THEOREM 7: TAN-CHORD THEOREM

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment.



Given: PCQ is a tangent, CB is a chord and A is a point on the major arc.

Required to prove (RTP): $B\hat{C}Q = B\hat{A}C$.

Construction: Draw diameter CD and join AD.

Proof:

Statement	Reason
$\hat{C}_1 + \hat{C}_2 = 90°$	Tan $\perp$ radius
$\hat{A}_1 + \hat{A}_2 = 90°$	$\angle$ in a semi-circle
$\therefore \hat{C}_1 + \hat{C}_2 = \hat{A}_1 + \hat{A}_2$	
$\hat{A}_1 = \hat{C}_1$	$\angle$'s in same segment
$\therefore \hat{A}_2 = \hat{C}_2$	
$\therefore B\hat{A}C = B\hat{C}Q$	

When using this theorem to solve riders the reason used is:
$\angle$ between tan. and chord

THEOREM 7: (CONVERSE)

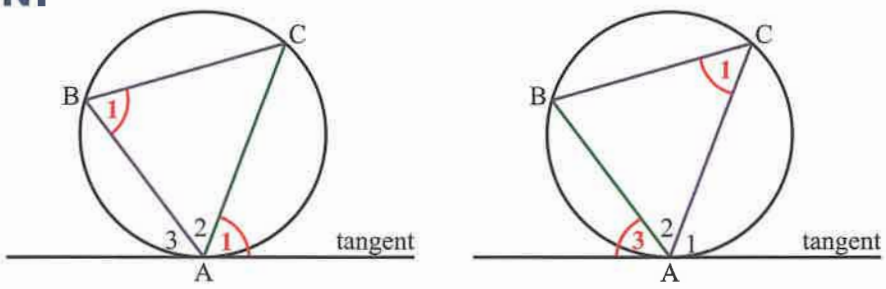
If the angle between a line and chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

Given: $B\hat{C}Q = B\hat{A}C$.

Statement	Reason
PQ is a tangent to the circle	$\angle$ between line and chord

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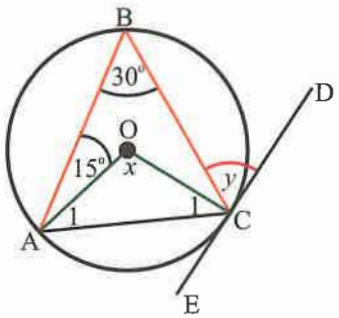
DETERMINING WHICH IS THE ANGLE IN THE ALTERNATE SEGMENT



Place your fingers on either end of the **chord.** Follow the **lines** from either end of the chord to the point where they meet on the circumference of the circle. This creates the **angle** in the **alternate segment**. The **angle** from the tangent to the **chord** is equal to the angle in the **alternate segment**. So in the diagrams above:

$$\hat{A}_1 = \hat{B}_1 \text{ and } \hat{A}_3 = \hat{C}_1$$

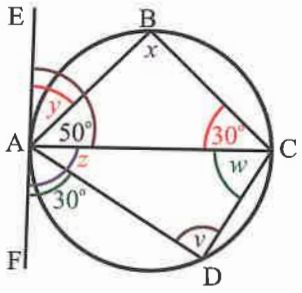
Example 1: Determine the values of x and y in the diagram below. O is the circle centre and DE is a tangent to the circle at C.



Statement	Reason
$x = 60°$	$\angle$ at centre $= 2\angle$ at circumference
$OA = OC$	Radii
$\therefore \hat{A}_1 = \hat{C}_1 = 60°$	$\angle$'s opp. = sides and $\angle$'s in a Δ
$y = 15° + \hat{A}_1$ $\therefore y = 15° + 60° = 75°$	$\angle$ between tan. and chord

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Example 2: **Determine the values of v, w, x, y and z in the diagram below. EF is a tangent to the circle at A.**

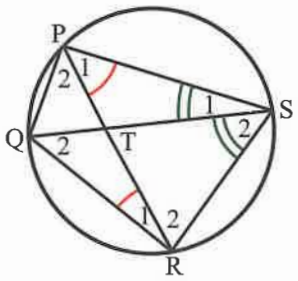


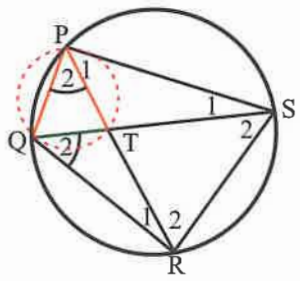
Statement	Reason
$y = 30^\circ$	∠ between tan. and chord
$w = 30^\circ$	∠ between tan. and chord
$x = 180^\circ - 50^\circ - 30^\circ = 100^\circ$	∠'s in a △
$z + 30^\circ = x = 100^\circ$ $\therefore z = 100^\circ - 30^\circ = 70^\circ$	∠ between tan. and chord
$v = 50^\circ + y = 50^\circ + 30^\circ = 80^\circ$	∠ between tan. and chord

Note that opposite angles of a cyclic quad. could also be used to determine angle v.

Example 3: **In the diagram below PQRS is a cyclic quad. $\hat{P}_1 = \hat{R}_1$ and $\hat{S}_1 = \hat{S}_2$. Prove that QR is a tangent to circle PQT.**

It is often useful to draw the circle onto your diagram in order to see where the tangent and chords lie. In order to prove that QR is a tangent you have to prove that $\hat{Q}_2 = \hat{P}_2$.





Statement	Reason
$\hat{Q}_2 = \hat{P}_1$	∠'s in same segment
$\hat{P}_1 = \hat{R}_1$	Given
$\hat{R}_1 = \hat{S}_1$	∠'s in same segment
$\hat{S}_1 = \hat{S}_2$	Given
$\hat{S}_2 = \hat{P}_2$	∠'s in same segment
But $\hat{S}_2 = \hat{S}_1 = \hat{R}_1 = \hat{P}_1 = \hat{Q}_2$	
$\therefore \hat{Q}_2 = \hat{P}_2$ $\therefore$ QR is a tangent to circle PQT	∠ between line and chord

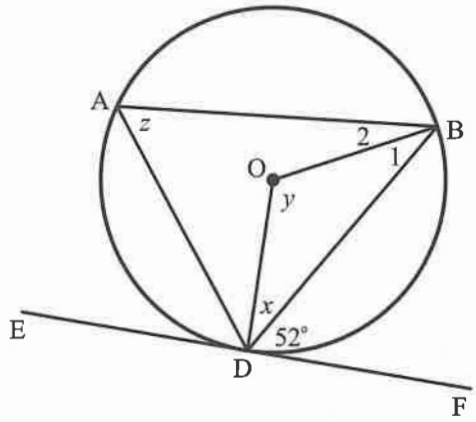


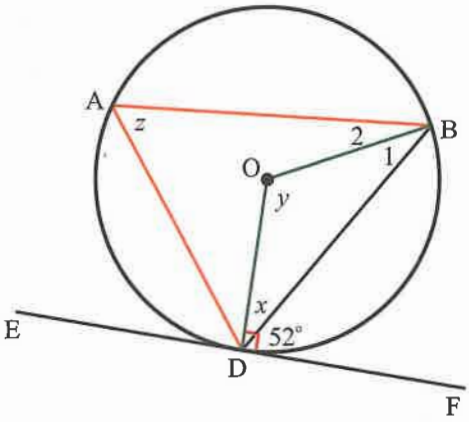
Complete Exercise 5 on page 346.

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D. MIXED EXAMPLES

Example 1: In the diagram below, O is the centre of the circle and EDF is a tangent to the circle at D. Determine the values of x, y and z.





Statement	Reason
$z = 52°$	$\angle$ between tan. and chord
$y = 104°$	$\angle$ at centre $= 2\angle$ at circumference
$x + 52° = 90°$ $\therefore x = 38°$	Tan $\perp$ radius

It is important to realise that there is often more than 1 method of solving geometry riders. The solution below is an alternative method of solving the above rider. From now, only 1 method will be shown. However, this does not mean that it is the only way of solving a geometry rider.

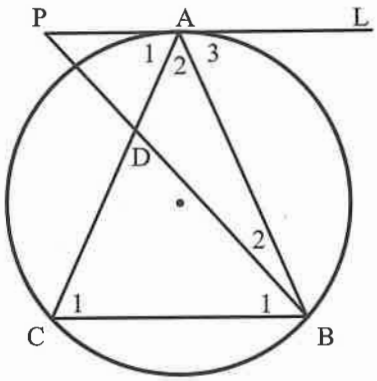
Statement	Reason
$x = 90° - 52° = 38°$	Tan $\perp$ radius
$OD = OB$	Radii
$\therefore x = \hat{B}_1 = 38°$	$\angle$'s opp. = sides
$\therefore y = 180° - 2(38°) = 104°$	$\angle$'s in a Δ
$z = 52°$	$\angle$ at centre $= 2\angle$ at circumference

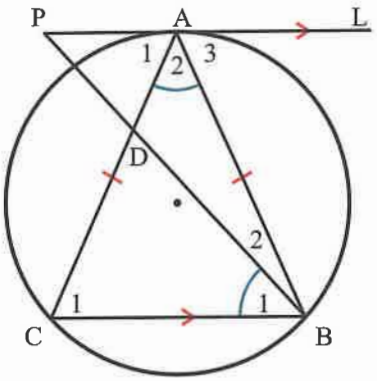
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Example 2: **In the diagram below, $AB = AC$, $AP//BC$ and $\hat{A}_2 = \hat{B}_1$. Prove that:**

a) PAL is a tangent to the circle at A.

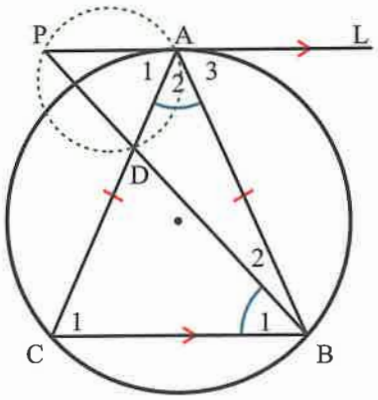
b) AB is a tangent to the circle ADP.





a) Statement	Reason
$\hat{A}_1 = \hat{C}_1$	Alt. ∠'s (PA//CB)
$AC = AB$	Given
$\therefore \hat{C}_1 = \hat{B}_1 + \hat{B}_2$	∠'s opp. = sides
$\therefore \hat{A}_1 = \hat{B}_1 + \hat{B}_2$	
$\therefore$ PAL is a tangent at A	∠ between line and chord

In order to answer part b it is helpful to draw circle ADP:

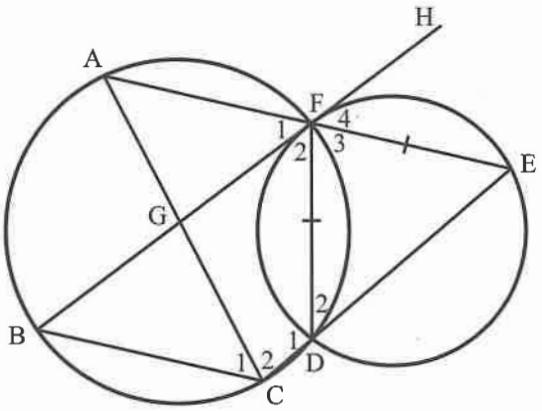


b) Statement	Reason
$\hat{P} = \hat{B}_1$	Alt. ∠'s (PA//CB)
$\hat{A}_2 = \hat{B}_1$	Given
$\therefore \hat{P} = \hat{A}_2$	
$\therefore$ AB is a tangent to circle ADP	∠ between line and chord

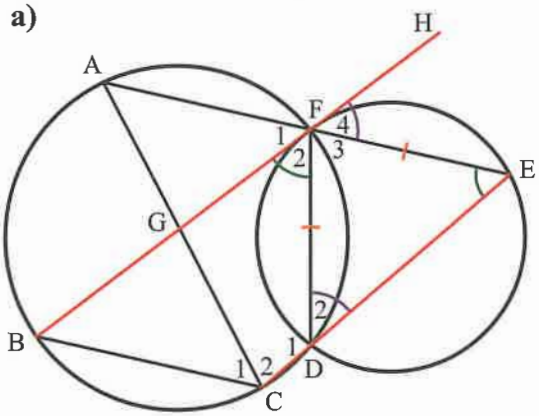
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Example 3: **In the diagram below, the circles intersect at F and D. BFH is a tangent to the smaller circle at F. AFE is a straight line drawn such that FD = FE. CDE is a straight line and chords AC and BF intersect at G. Prove that:**

a) BH//CE. **b) BCEF is a parallelogram.** **c) AC = BF**

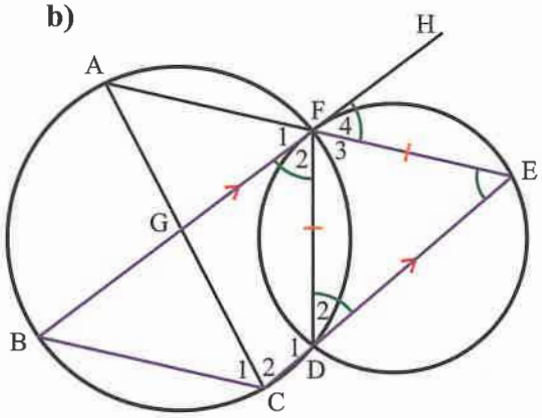


a)

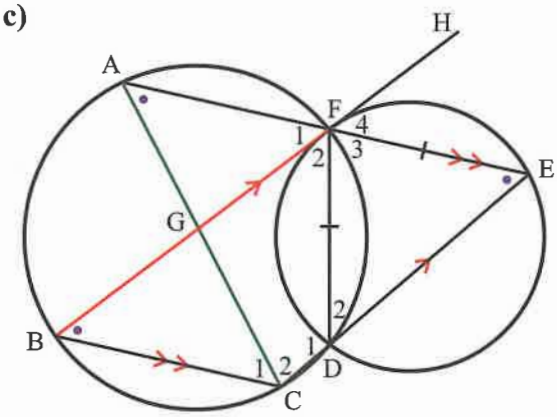


a) Statement	Reason
$\hat{F}_2 = \hat{E}$	∠ between tan. and chord
$\hat{D}_2 = \hat{E} = \hat{F}_2$	∠'s opp. = sides
∴ BH // CE	Alt. ∠'s =

b)



c)



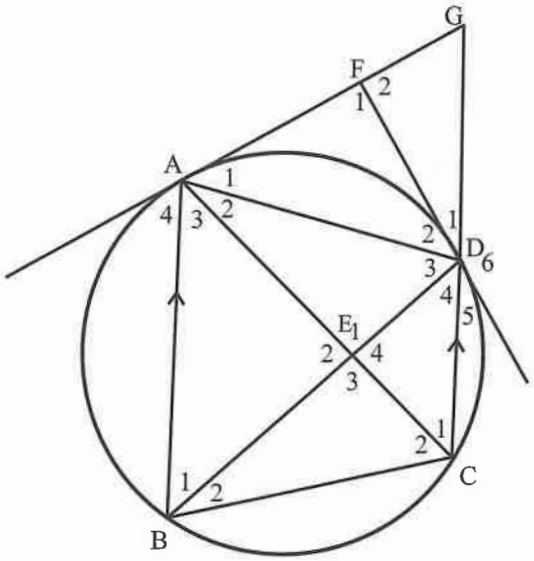
b) Statement	Reason
$\hat{F}_4 = \hat{D}_2$	∠ between tan. and chord
$\hat{D}_2 = \hat{B}$	Ext. ∠ of a cyclic quad. BFDC
$\therefore \hat{F}_4 = \hat{B}$	
∴ FE//BC	Corr. ∠'s are =
∴ BCEF is a parm.	Both pairs of opp. sides //

c) Statement	Reason
$\hat{B} = \hat{E}$	Opp. ∠'s of a parm are =
$\hat{B} = \hat{A}$	∠'s in same segment
$\therefore \hat{A} = \hat{E}$	
∴ AC = CE	sides opp. = ∠'s
CE = BF	Opp. sides of a parm are =
∴ AC = BF	

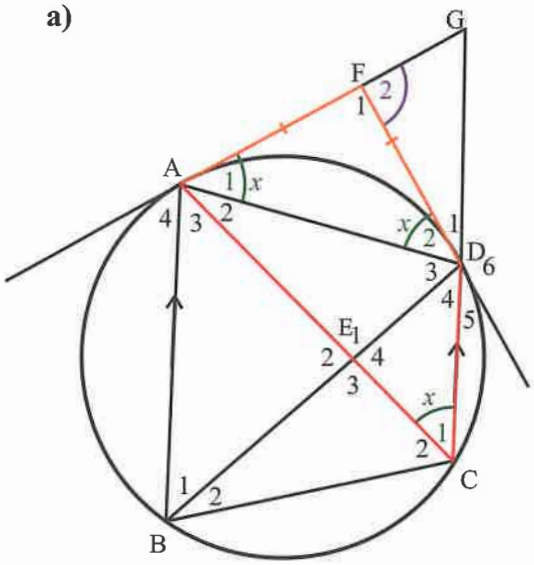
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Example 4: **In the diagram below AB and CD are chords of the circle such that AB//DC. AG is a tangent to the circle at A and FD is a tangent to the circle at D. CD is produced to G and AC and BD intersect at circle centre E. If $\hat{A}_1 = x$ and $\hat{C}_2 = y$, prove that:**

a) $\hat{F}_2 = 2\hat{C}_1$ **b) AEDF is a cyclic quad.** **c)** $\hat{B}_1 + \hat{G} = \hat{C}_1 + \hat{C}_2$

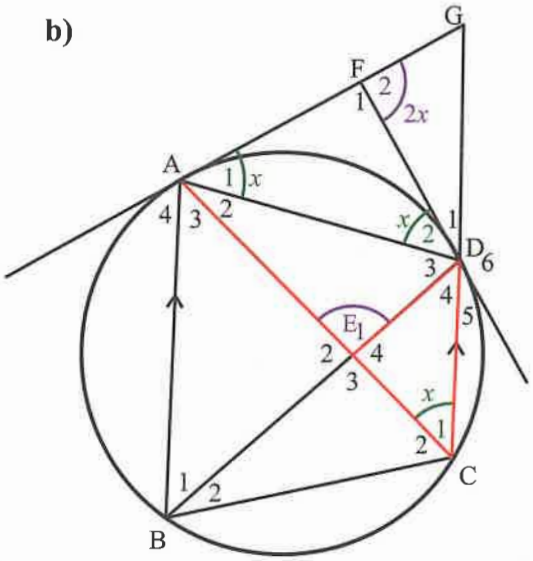


a)



a) Statement	Reason
AF = FD	Tangents from same pt.
$\hat{A}_1 = \hat{D}_2 = x$	∠'s opp. = sides
$\hat{D}_2 = \hat{C}_1 = x$	∠ between tan. and chord
$\hat{F}_2 = \hat{A}_1 + \hat{D}_2 = x + x = 2x$	Ext. ∠ of a △
$\therefore \hat{F}_2 = 2\hat{C}_1$	

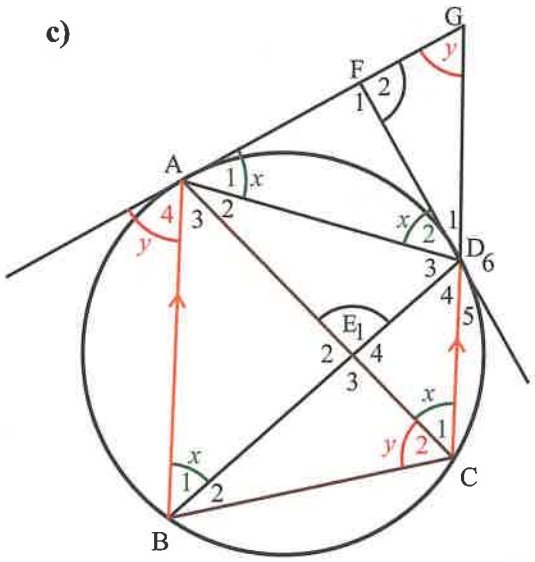
b)



b) Statement	Reason
$\hat{E}_1 = 2\hat{C}_1 = 2x$	∠ at centre = 2∠ at circumference
$\therefore \hat{E}_1 = \hat{F}_2 = 2x$	$\hat{F}_2 = 2x$ (proved above)
∴ AEDF is a cyclic quad.	Ext. ∠ = opp. int. ∠

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c)



c) Statement	Reason
$\hat{C}_1 + \hat{C}_2 = x + y$	$\hat{C}_1 = x$ (proved above)
$\hat{A}_4 = \hat{C}_2 = y$	$\angle$ between tan. and chord
$\hat{G} = \hat{A}_4 = y$	Corr. $\angle$'s (AB // GC)
$\hat{B}_1 = \hat{C}_1 = x$	$\angle$'s in same segment
$\therefore \hat{B}_1 + \hat{G} = x + y$	
$\therefore \hat{B}_1 + \hat{G} = \hat{C}_1 + \hat{C}_2$	

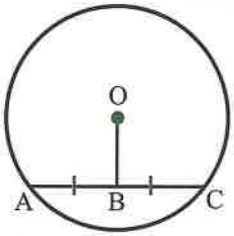
TIPS FOR SOLVING RIDERS

1) **Fill in all the given information on the diagram.**
2) **Colour is a useful tool. Highlight what the question has asked you to find and use the same colours to mark equal angles and sides.**
3) **Take note of what conditions you need to prove.**
4) **Always provide a reason for your statement unless the question tells you that it is not required.**



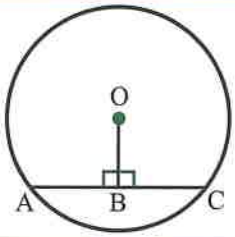
Complete the Mixed Exercise on pages 347-350.

SUMMARY OF THEOREMS, CONVERSES AND COROLLARIES



Given:	Conclusion:
AC ⊥ OB	AB = BC

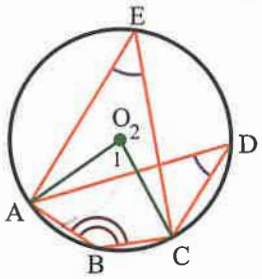
Reason:
Line from centre ⊥ to chord



Given:	Conclusion:
AB = BC	AC ⊥ OB

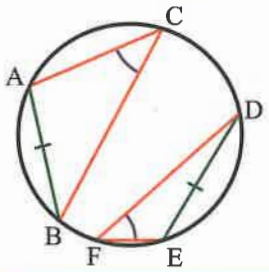
Reason:
Line from centre to midpoint of chord

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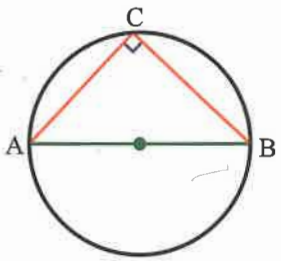
Given:	Conclusion:
A, B, C, D and E are points on the circle with centre O.	$\hat{O}_1 = 2\hat{E} = 2\hat{D}$ $\hat{O}_2 = 2\hat{B}$

Reason:
∠ at centre = 2∠ at circumference



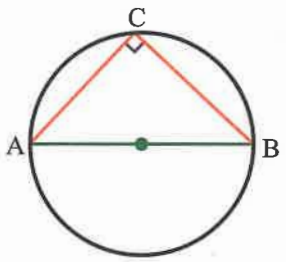
Given:	Conclusion:
AB = DE	$\hat{C} = \hat{F}$

Reason:
= chords subtend = ∠'s



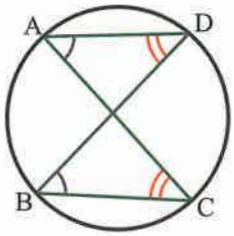
Given:	Conclusion:
AB is a diameter	$\hat{C} = 90°$

Reason:
∠ in a semi - circle



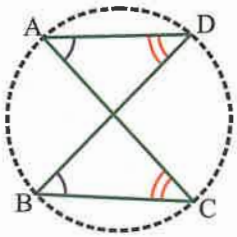
Given:	Conclusion:
$\hat{C} = 90°$	AB is a diameter

Reason:
Chord subtends a 90° ∠



Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A} = \hat{B}$ and $\hat{D} = \hat{C}$

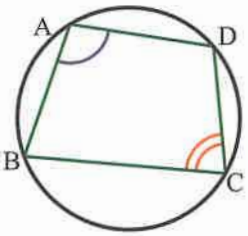
Reason:
∠'s in same segment



Given:	Conclusion:
$\hat{A} = \hat{B}$ or $\hat{D} = \hat{C}$	ABCD is a cyclic quadrilateral

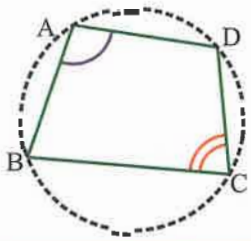
Reason:
Line seg. subt. = ∠'s on the same side

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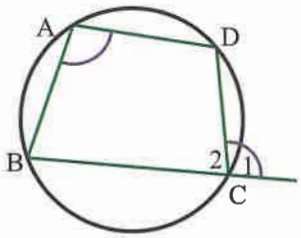
Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A}+\hat{C}=180^\circ$

Reason:
Opp. ∠'s of a cyclic quad.



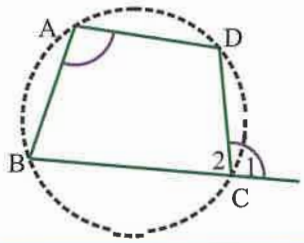
Given:	Conclusion:
$\hat{A}+\hat{C}=180^\circ$	ABCD is a cyclic quadrilateral

Reason:
Opp. ∠'s supplementary



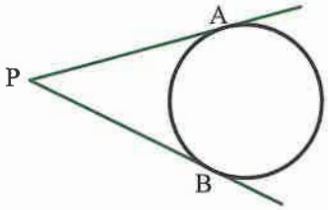
Given:	Conclusion:
ABCD is a cyclic quadrilateral	$\hat{A}=\hat{C}_1$

Reason:
Ext. ∠ of a cyclic quad.



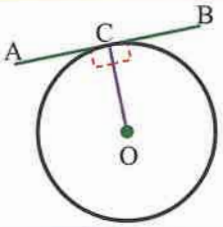
Given:	Conclusion:
$\hat{A}=\hat{C}_1$	ABCD is a cyclic quadrilateral

Reason:
Ext. ∠ = opp. Interior ∠



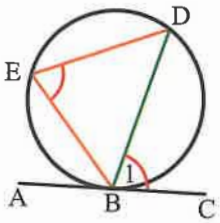
Given:	Conclusion:
AP and BP are tangents	AP = BP

Reason:
Tangents from same pt.



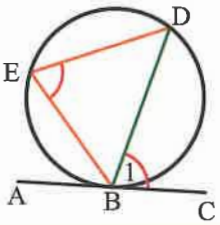
Given:	Conclusion:
AB is a tangent at C	OC ⊥ AB
OC ⊥ AB	AB is a tangent at C

Reason:
Tangent ⊥ to radius
Line to ⊥ radius



Given:	Conclusion:
ABC is a tangent	$\hat{B}_1=\hat{E}$

Reason:
∠ between tan. and chord



Given:	Conclusion:
$\hat{B}_1=\hat{E}$	ABC is a tangent

Reason:
∠ between line and chord

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CHAPTER 9 – SINE, COSINE AND AREA RULES

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
177	**A. Revision**	8	• Establish and prove the area, sine and cosine rule.
177	• Sides, angles and areas of triangles		• Apply the area, sine and cosine rule in order to solve problems in 2 dimensions.
177	• Angles of elevation and depression		
178	**B. The Area Rule**		
178	• Proof		
179	**C. The Sine Rule**		
180	• The Sine Rule Proof		
181	• The Ambiguous Case		
182	**D. The Cosine Rule**		
182	• The Cosine Rule Proof		
183	• Directions on a bearing		
184	• Mixed Examples		
187	**E. Summary**		
187	• The Area Rule		
187	• The Sine Rule		
187	• The Cosine Rule		

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A. REVISION

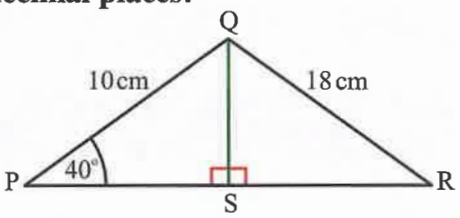
SIDES, ANGLES AND AREAS OF TRIANGLES

Example: In the diagram alongside, $\hat{P} = 40°$, $QS \perp PR$, $PQ = 10\,cm$ and $QR = 18\,cm$. Determine, correct to 2 decimal places:

a) The length of QS.

b) The size of $\hat{R}$.

c) The area of $\triangle PQR$.



a) $\sin 40° = \frac{QS}{10} \rightarrow \frac{\text{opposite}}{\text{hypotenuse}}$

$\therefore QS = 10 \sin 40° = 6,43\,cm$

b) $\sin \hat{R} = \frac{6,43}{18} \rightarrow \frac{\text{opposite}}{\text{hypotenuse}}$

$\therefore \hat{R} = 20,93°$

c) $\cos 40° = \frac{PS}{10} \rightarrow \frac{\text{adjacent}}{\text{hypotenuse}}$

$\therefore PS = 10 \cos 40° = 7,66\,cm$

$\cos 20,93° = \frac{RS}{18} \rightarrow \frac{\text{adjacent}}{\text{hypotenuse}}$

$\therefore RS = 18 \cos 20,93° = 16,81\,cm$

$\therefore PR = 7,66 + 16,81 = 24,47$

Area of $\triangle PQR = \frac{1}{2}$ base $\times \perp$ height

$\therefore$ Area of $\triangle PQR = \frac{1}{2}(24,47)(6,43) = 78,67\,cm^2$

Recap (pg. 119):

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

ANGLES OF ELEVATION AND DEPRESSION

The **angle of elevation** is the angle between the horizontal and the line of sight **up** to an object. This is shown in figure 1 alongside.

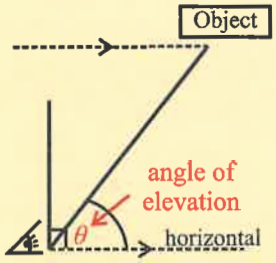


Figure 1: Angle of elevation

The **angle of depression** is the angle between the horizontal and the line of sight **down** to an object. This is shown in figure 2 alongside.

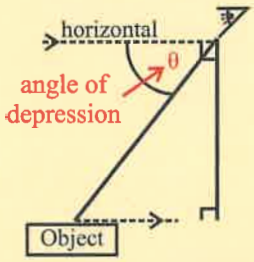


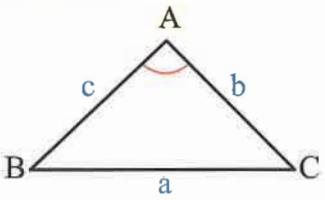
Figure 2: Angle of depression

Note that the angle of elevation is always equal to the angle of depression due to alternate angles between parallel lines.

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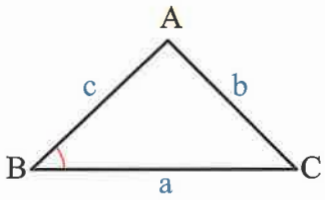
B. THE AREA RULE

The area rule is used to determine the area of triangles which are not right angled triangles or triangles where the perpendicular height is not known. The area rule is summarised in the diagrams below:



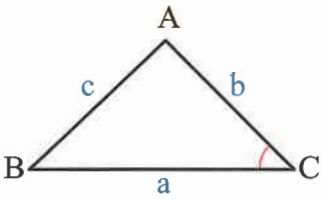
Angle A known

$$A_{\triangle ABC} = \frac{1}{2}bc\sin\hat{A}$$



Angle B known

$$A_{\triangle ABC} = \frac{1}{2}ac\sin\hat{B}$$



Angle C known

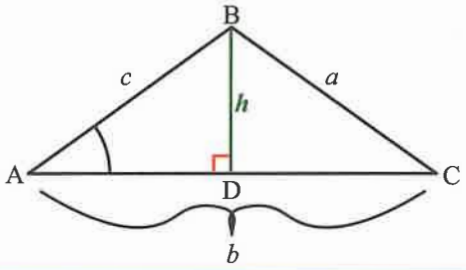
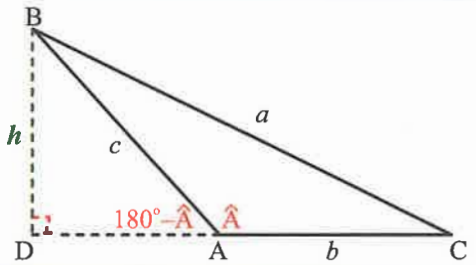
$$A_{\triangle ABC} = \frac{1}{2}ab\sin\hat{C}$$



Use the Area Rule when given two sides and an included angle.
An included angle is the angle between the two given sides.

PROOF

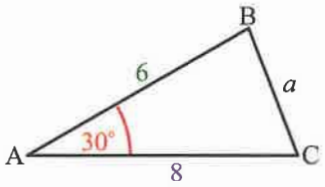
The proof of the area rule is shown below for an acute triangle and an obtuse triangle.

Acute Angled Triangle	Obtuse Angled Triangle
	
Area of $\triangle ABC = \frac{1}{2}b\times h \rightarrow [1]$	Area of $\triangle ABC = \frac{1}{2}b\times h \rightarrow [1]$
In $\triangle ADB$:	In $\triangle ADB$: In
$\sin\hat{A} = \frac{h}{c}$	$\sin(180-\hat{A}) = \frac{h}{c} \rightarrow \sin(180-\hat{A}) = \sin\hat{A}$
$\therefore h = c\sin\hat{A} \rightarrow [2]$	$\therefore h = c\sin\hat{A} \rightarrow [2]$
Sub 2 into [1]:	Sub 2 into [1]:
$\therefore$ Area of $\triangle ABC = \frac{1}{2}bc\sin\hat{A}$	$\therefore$ Area of $\triangle ABC = \frac{1}{2}bc\sin\hat{A}$
Similarly it can be proved that:	**Similarly it can be proved that:**
Area of $\triangle ABC = \frac{1}{2}ac\sin\hat{B}$	Area of $\triangle ABC = \frac{1}{2}ac\sin\hat{B}$
Area of $\triangle ABC = \frac{1}{2}ab\sin\hat{C}$	Area of $\triangle ABC = \frac{1}{2}ab\sin\hat{C}$

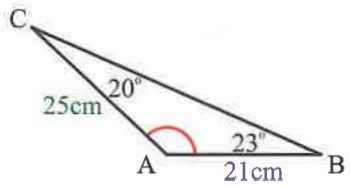
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Example 1: **Determine the area of the triangles shown below correct to 1 decimal place where necessary:**

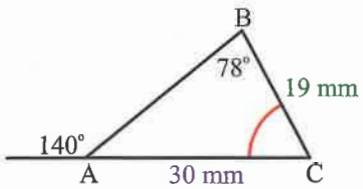
1)



2)



3)



1)

$$A = \frac{1}{2}(8)(6)\sin 30°$$

$$= 12 \text{ units}^2$$

2)

$$A = \frac{1}{2}(21)(25)\sin \hat{A}$$

$$\hat{A} = 180° - (20° + 23°) = 137°$$

$$\therefore A = \frac{1}{2}(21)(25)\sin 137°$$

$$\therefore A = 179{,}0 \text{ cm}^2$$

3)

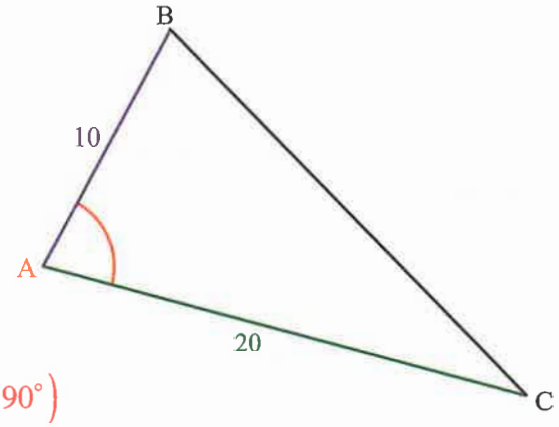
$$A = \frac{1}{2}(30)(19)\sin \hat{C}$$

$$\hat{C} = 140° - 78° = 62°$$

$$\therefore A = \frac{1}{2}(30)(19)\sin 62°$$

$$\therefore A = 251{,}6 \text{ mm}^2$$

Example 2: **Determine the size/s of $\hat{A}$ if ΔABC alongside has an area of 50m^2.**



$$\text{Area} = \frac{1}{2}(AB)(AC)\sin \hat{A}$$

$$\therefore 50 = \frac{1}{2}(10)(20)\sin \hat{A}$$

$$\therefore \sin \hat{A} = \frac{1}{2}$$

$$\therefore \hat{A} = 30° \rightarrow \text{this is an acute angle } (\text{less than } 90°)$$

It is important to note that it is possible for $\hat{A}$ to be an obtuse angle. You have to give both possible solutions to $\hat{A}$.

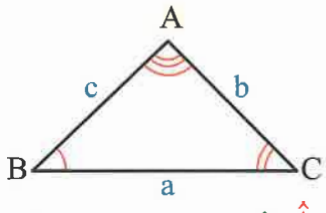
$$\therefore \hat{A} = 30° \text{ or } \hat{A} = 180° - 30° = 150° \rightarrow \text{obtuse angle}$$



Complete Exercise 1 on page 358.

C. THE SINE RULE

The Sine Rule is used to determine an unknown length or angle of a triangle which is not right angled.



$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} \quad \text{or} \quad \frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

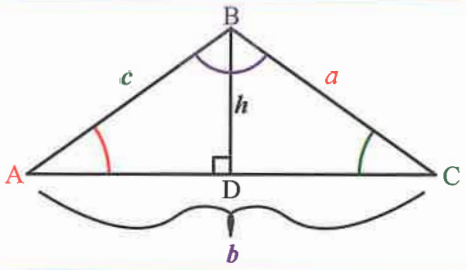
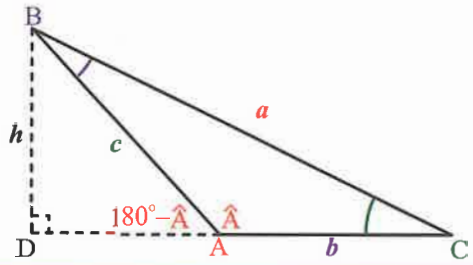


Use the Sine Rule when given two sides and an angle that is not between the given sides or if two angles and a side are given.

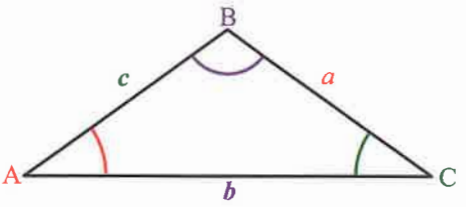
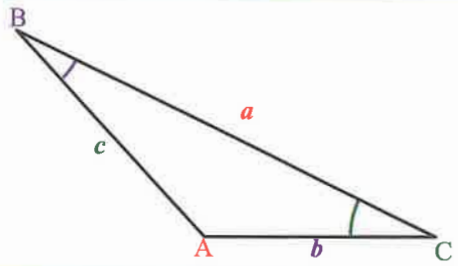
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THE SINE RULE PROOF

The proof of the Sine rule is shown below for an acute triangle and an obtuse triangle:

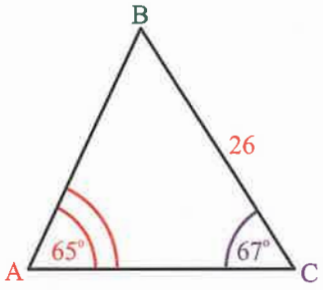
Acute Angled Triangle	Obtuse Angled Triangle
	
In $\triangle ADB$:	In $\triangle ADB$:
$\sin \hat{A} = \frac{h}{c}$	$\sin(180° - \hat{A}) = \frac{h}{c} \rightarrow \sin(180° - \hat{A}) = \sin \hat{A}$
$\therefore h = c \sin \hat{A} \rightarrow \boxed{1}$	$\therefore h = c \sin \hat{A} \rightarrow \boxed{1}$
In $\triangle CDB$:	In $\triangle CDB$:
$\sin \hat{C} = \frac{h}{a}$	$\sin \hat{C} = \frac{h}{a}$
$\therefore h = a \sin \hat{C} \rightarrow \boxed{2}$	$\therefore h = a \sin \hat{C} \rightarrow \boxed{2}$
$\therefore \boxed{1} = \boxed{2}$	$\therefore \boxed{1} = \boxed{2}$
$\therefore c \sin \hat{A} = a \sin \hat{C}$	$\therefore c \sin \hat{A} = a \sin \hat{C}$
$\therefore \frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c}$	$\therefore \frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c}$
Similarly it can be proved that:	**Similarly it can be proved that:**
$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c} = \frac{\sin \hat{B}}{b}$	$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c} = \frac{\sin \hat{B}}{b}$

The Sine Rule can also be proved using the Area Rule as shown below:

Acute Angled Triangle	Obtuse Angled Triangle
	
From the area rule: $\frac{1}{2} b c \sin \hat{A} = \frac{1}{2} a b \sin \hat{C} = \frac{1}{2} a c \sin \hat{B}$	
Now divide all sides by $\frac{1}{2} abc$ **:** $\therefore \frac{\sin \hat{A}}{a} = \frac{\sin \hat{C}}{c} = \frac{\sin \hat{B}}{b}$	

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Example: Solve $\triangle ABC$ in the diagram alongside:



When a question asks you to solve a triangle you have to solve for all unknown angles and sides.

$$\frac{AB}{\sin \hat{C}} = \frac{BC}{\sin \hat{A}}$$

$$\therefore AB = \frac{26 \sin 67^\circ}{\sin 65^\circ} = 26,41 \text{ units}$$

$$\frac{AC}{\sin \hat{B}} = \frac{BC}{\sin \hat{A}} \rightarrow \text{use the side that is given rather than a side you have calculated}$$

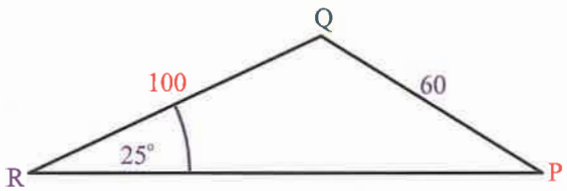
$$\therefore AC = \frac{26 \sin 48^\circ}{\sin 65^\circ} \rightarrow \hat{B} = 180^\circ - 65^\circ - 67^\circ \rightarrow \text{angles in a triangle}$$

$$\therefore AC = 21,32 \text{ units}$$

THE AMBIGUOUS CASE

When solving for an angle in a triangle it is possible for there to be 2 possible solutions for an angle.
This occurs when the angle you are solving for is opposite the longest given side.

Illustrative Example: Solve for $\hat{P}$ in the $\triangle PQR$ below:



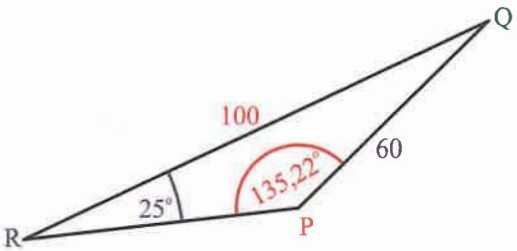
$$\frac{\sin \hat{P}}{RQ} = \frac{\sin \hat{R}}{60}$$

$$\therefore \frac{\sin \hat{P}}{100} = \frac{\sin 25^\circ}{60}$$

$$\therefore \sin \hat{P} = \frac{100 \sin 25^\circ}{60} \rightarrow \text{use shift } \sin^{-1}\left(\frac{100 \sin 25^\circ}{60}\right) \text{ to solve for the angle}$$

$$\therefore \hat{P} = 44,78^\circ$$

However, we know from trigonometric equations that it is possible to have another solution to $\hat{P}$. It is also possible for $\hat{P} = 180^\circ - 44,78^\circ = 135,22^\circ$. This is illustrated in the triangle below.



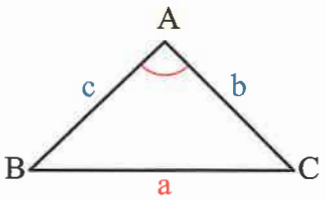
Always give both possible answers when dealing with the ambiguous case.

Complete Exercise 2 on page 358.

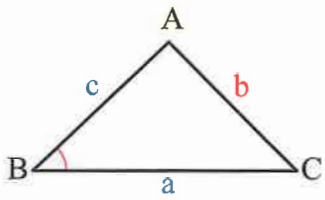
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D. THE COSINE RULE

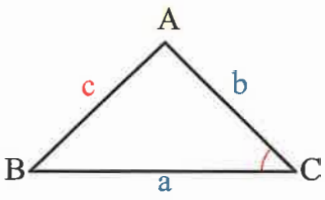
The Cosine Rule is also used to determine the length of an unknown side or angle in a triangle that is not a right angled triangle.



Angle A known



Angle B known



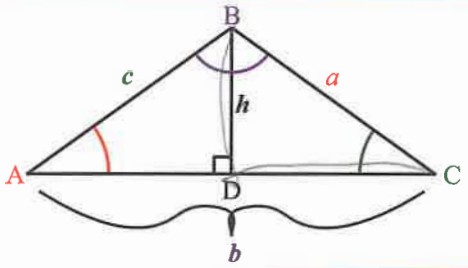
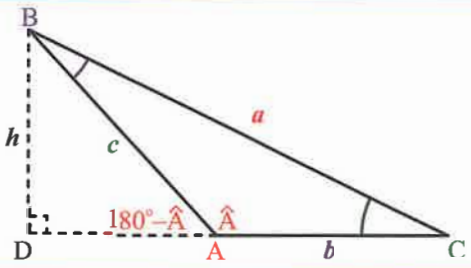
Angle C known

$a^2 = b^2 + c^2 - 2(b)(c)\cos\hat{A}$ $b^2 = a^2 + c^2 - 2(a)(c)\cos\hat{B}$ $c^2 = a^2 + b^2 - 2(a)(b)\cos\hat{C}$



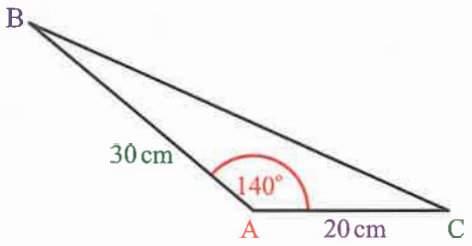
Only use the Cosine Rule if you have a triangle where you are given all three sides or two sides and an included angle. If a side of a triangle has a square root in it, the Cosine Rule is the most likely rule to use.

THE COSINE RULE PROOF

Acute Angled Triangle	Obtuse Angled Triangle
	
In $\triangle BDC$:	In $\triangle BDC$:
$a^2 = BD^2 + CD^2 \rightarrow$ pythag	$a^2 = BD^2 + CD^2 \rightarrow$ pythag
$= BD^2 + (b - AD)^2 \rightarrow CD = b - AD$	$= BD^2 + (b + AD)^2 \rightarrow CD = b + AD$
$= BD^2 + b^2 - 2bAD + AD^2 \rightarrow \boxed{1}$	$= BD^2 + b^2 + 2bAD + AD^2 \rightarrow \boxed{1}$
But using pythag. in $\triangle ABD$:	But using pythag. in $\triangle ABD$:
$c^2 = AD^2 + BD^2 \rightarrow \boxed{2}$	$c^2 = AD^2 + BD^2 \rightarrow \boxed{2}$
Sub $\boxed{2}$ into $\boxed{1}$	Sub $\boxed{2}$ into $\boxed{1}$
$\therefore a^2 = b^2 + c^2 - 2bAD \rightarrow \boxed{3}$	$\therefore a^2 = b^2 + c^2 + 2bAD \rightarrow \boxed{3}$
Also in $\triangle ABD$: $\cos\hat{A} = \frac{AD}{c}$	Also in $\triangle ABD$: $\cos(180° - \hat{A}) = \frac{AD}{c}$
$\therefore AD = c\cos\hat{A} \rightarrow \boxed{4}$	$\therefore -\cos\hat{A} = \frac{AD}{c} \rightarrow \cos(180° - \hat{A}) = -\cos\hat{A}$
Sub $\boxed{4}$ into $\boxed{3}$	$AD = -c\cos\hat{A} \rightarrow \boxed{4}$
$\therefore a^2 = b^2 + c^2 - 2bc\cos\hat{A}$	Sub $\boxed{4}$ into $\boxed{3}$
	$\therefore a^2 = b^2 + c^2 - 2bc\cos\hat{A}$

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Example 1: Determine the length of BC in triangle ABC alongside:



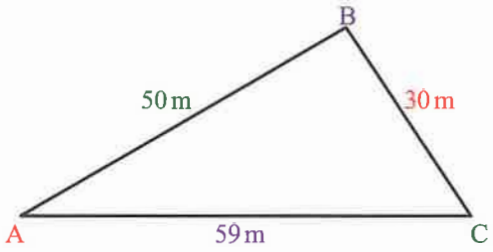
$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos\hat{A}$

$\therefore BC^2 = 30^2 + 20^2 - 2(30)(20)\cos 140°$

$\therefore BC^2 = 2219,25...$ → don't forget to square root this answer

$\therefore BC = 47,11\text{cm}$

Example 2: Determine the size of $\hat{B}$ in $\triangle ABC$ alongside:



$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\hat{B}$

$\therefore 59^2 = 50^2 + 30^2 - 2(50)(30)\cos\hat{B}$

$\therefore 81 = -2(50)(30)\cos\hat{B}$

$\therefore \cos\hat{B} = \dfrac{81}{-2(50)(30)}$

$\therefore \hat{B} = 91,55°$



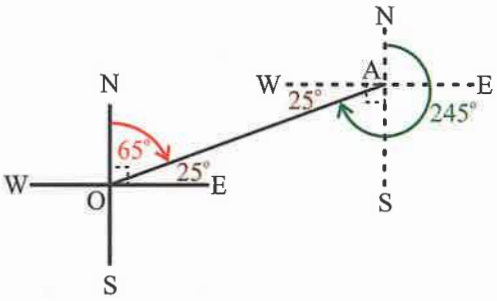
Complete Exercise 3 on pages 358 and 359.

DIRECTIONS ON A BEARING

Bearing is a way of measuring the direction of one object from another. Bearing is usually measured clockwise from north. Bearing can also be expressed in terms of North, East, South and West.

Illustrative Example: In the diagram below, write down the bearing of:

a) A from O. **b) O from A.**



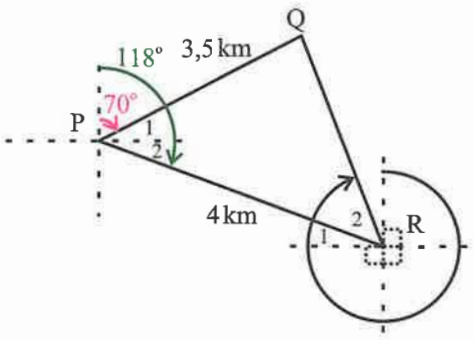
a) The bearing of A from O is 65°. This can also be written as N 65°E. This means move 65° east from north.

b) The bearing of O from A is 245°. This can also be written as W 25°S. This means move 25° south from west.

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MIXED EXAMPLES

Example 1: Phindile and Michael are taking part in a scavenger hunt. They both leave from point P. Phindile walks 3,5km on a bearing of 70° to point Q and Michael walks 4km on a bearing of 118° to point R.



a) How far are Phindile and Michael from each other at points Q and R?
b) At what bearing must Michael walk to reach Phindile?

a)

$$QR^2 = PQ^2 + PR^2 - 2(PQ)(PR)\cos Q\hat{P}R$$

$$P_1 = 90° - 70° = 20°$$

$$P_2 = 118° - 90° = 28°$$

$$Q\hat{P}R = 28° + 20° = 48°$$

$$\therefore QR^2 = (3,5)^2 + (4)^2 - 2(3,5)(4)\cos 48°$$

$$= 9,51...$$

$$\therefore QR = 3,08\,km$$

b)

$$\hat{P}_2 = \hat{R}_1 = 28° \rightarrow \text{alternate angles}$$

$$\frac{\sin \hat{R}_2}{3,5} = \frac{\sin(\hat{P}_1 + \hat{P}_2)}{3,08}$$

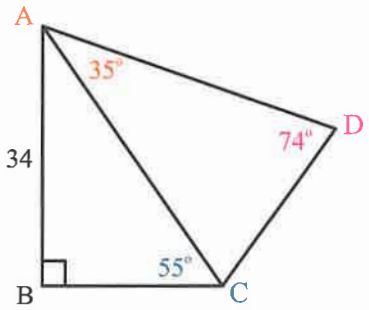
$$\therefore \sin \hat{R}_2 = \frac{3,5\sin(48°)}{3,08}$$

$$\therefore \hat{R}_2 = 57,62°$$

He must walk on a bearing of

$$270° + 28° + 57,62° = 355,62°$$

Example 2: In the diagram below, determine the length of CD.



$$\frac{CD}{\sin 35°} = \frac{AC}{\sin 74°}$$

$$\therefore CD = \frac{AC\sin 35°}{\sin 74°} \rightarrow \text{now determine the value of AC in the right angled triangle}$$

$$\sin 55° = \frac{AB}{AC} = \frac{34}{AC}$$

$$\therefore AC = \frac{34}{\sin 55°} \rightarrow \text{do not round off}$$

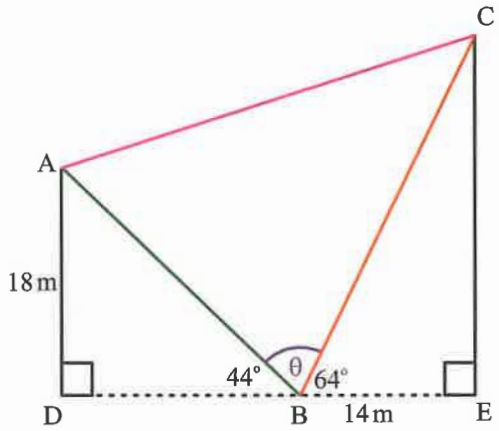
$$\therefore CD = \frac{34\sin 35°}{\sin 55°\sin 74°} \rightarrow AC = \frac{34}{\sin 55°}$$

$$\therefore CD = 24,77 \text{ units}$$

Do not round off during calculations. However, you may round off in your final answer.

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Example 3: **A triangular advertising banner is erected between 2 poles and secured to a point on the floor as shown in the diagram below. For stability a steel wire is run through the perimeter of the banner.**



a) If one side of the banner is to be painted and one can of paint covers a surface area of $35m^2$, how many cans are required?

b) Determine, to the nearest metre, the length of steel wire required.

a) $\text{Area} = \frac{1}{2}AB.BC\sin\theta$

$\sin 44° = \frac{18}{AB}$

$\therefore AB = \frac{18}{\sin 44°} = 25,912...$

$\cos 64° = \frac{14}{BC}$

$\therefore BC = \frac{14}{\cos 64°} = 31,936...$

$\therefore \text{Area} = \frac{1}{2}(25,912...).(31,936...)\sin(180° - 44° - 64°)$ → adj. ∠'s on a straight line

$= \frac{1}{2}(25,912...).(31,936...)\sin(72°)$

$= 393,52\,m^2$

$\text{Number of cans} = \frac{\text{Total area}}{35} = \frac{393,52}{35} = 11,24$

$\therefore$ 12 cans are required. → don't round down, 11 cans will not cover the entire banner

b) $\text{Perimeter} = AB + BC + AC$

$\therefore AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta$

$= (25,91)^2 + (31,94)^2 - 2(25,91)(31,94)\cos 72°$

$= 1180,028...$

$\therefore AC = 34,35$

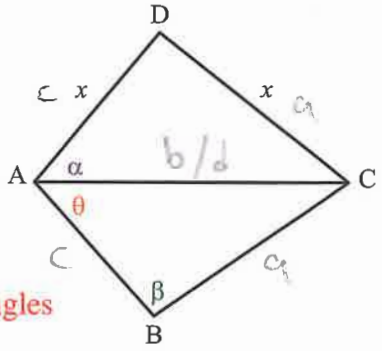
$\therefore \text{Perimeter} = 25,91 + 31,94 + 34,35 = 92,2$

$\therefore$ 93 m of steel wire is required → 92 m will not cover the entire perimeter

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Example 4: Show that the length of BC in the diagram alongside is:

$$BC = \frac{x\sin 2\alpha \sin\theta}{\sin\alpha \sin\beta}$$



Start solving in the triangle which has BC in it:

$\frac{BC}{\sin\theta} = \frac{AC}{\sin\beta}$ → use the side that is common to both triangles

$\therefore BC = \frac{AC\sin\theta}{\sin\beta}$

Now solve for AC in triangle ADC:

$\frac{AC}{\sin\hat{D}} = \frac{DC}{\sin\alpha}$ → $\hat{D} = 180° - \hat{A} - \hat{C} = 180° - \alpha - \alpha = 180° - 2\alpha$

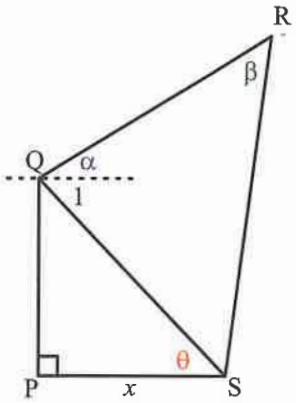
$\therefore AC = \frac{x\sin(180° - 2\alpha)}{\sin\alpha}$ → $DC = x$

$\therefore AC = \frac{x\sin 2\alpha}{\sin\alpha}$ → $\sin(180° - 2\alpha) = \sin 2\alpha$

$\therefore BC = \frac{x\sin 2\alpha \sin\theta}{\sin\alpha \sin\beta}$

Example 5: In the diagram below, the angle of elevation of R from Q is α and the angle of elevation of Q from S is θ. If $PS = x$ and $\hat{R} = \beta$, prove that

$$SR = \frac{x\sin(\alpha+\theta)}{\cos\theta\sin\beta}.$$



Start solving in the triangle which has SR in it:

$\hat{Q}_1 = \hat{S} = \theta$ → alternate angles

$\frac{SR}{\sin\hat{Q}} = \frac{QS}{\sin\beta}$ → use the side that is common to both triangles

$\therefore \frac{SR}{\sin(\theta+\alpha)} = \frac{QS}{\sin\beta}$

$\therefore SR = \frac{QS\sin(\theta+\alpha)}{\sin\beta}$

Now solve for QS in triangle PQS:

$\cos\theta = \frac{x}{QS}$ → $\frac{\text{adjacent}}{\text{hypotenuse}}$

$\therefore QS = \frac{x}{\cos\theta}$

$\therefore SR = \frac{x\sin(\theta+\alpha)}{\cos\theta\sin\beta}$ → $SR = \frac{QS\sin(\theta+\alpha)}{\sin\beta}$ and $QS = \frac{x}{\cos\theta}$



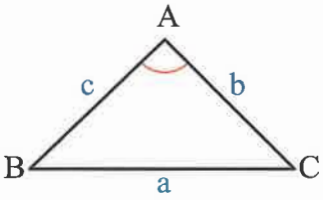
Complete the Mixed Exercise on pages 359 and 360.

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E. SUMMARY

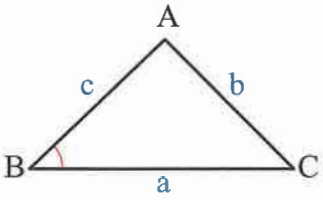
The rules that you have learnt are summarised below. Remember to only use the rules below for triangles which are not right angled triangles.

THE AREA RULE



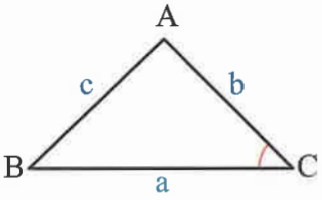
Angle A known

$$A_{\triangle ABC} = \frac{1}{2}bc\sin\hat{A}$$



Angle B known

$$A_{\triangle ABC} = \frac{1}{2}ac\sin\hat{B}$$

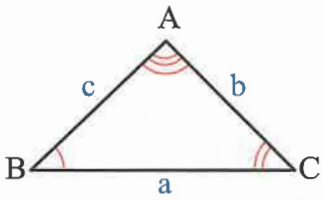


Angle C known

$$A_{\triangle ABC} = \frac{1}{2}ab\sin\hat{C}$$

Use the Area Rule when given two sides and an included angle.
An included angle is the angle between the two given sides.

THE SINE RULE

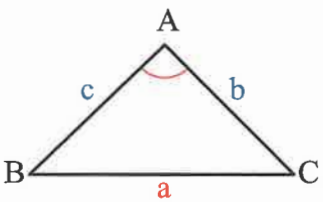


$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}} \quad \text{or} \quad \frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$$

Use the Sine Rule when given two sides and an angle that is not between the given sides or if two angles and a side are given.

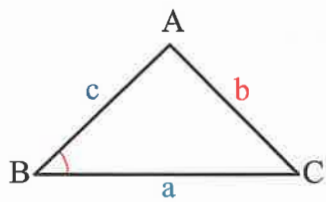
Always look for the ambiguous case when you are solving for angles. The ambiguous case occurs when solving for the angle opposite the longest given side.

THE COSINE RULE



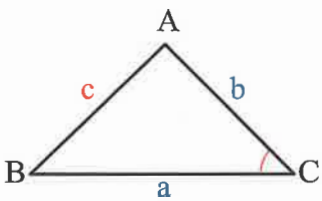
Angle A known

$$a^2 = b^2 + c^2 - 2(b)(c)\cos\hat{A}$$



Angle B known

$$b^2 = a^2 + c^2 - 2(a)(c)\cos\hat{B}$$



Angle C known

$$c^2 = a^2 + b^2 - 2(a)(b)\cos\hat{C}$$

Only use the Cosine Rule if you have a triangle where you are given all three sides or two sides and an included angle. If a side of a triangle has a square root in it, the Cosine Rule is the most likely rule to use.

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CHAPTER 10 – FINANCIAL MATHEMATICS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
189	**A. Simple and Compound Growth and Decay**	3	• Use simple and compound decay formulae to solve problems (including straight line depreciation and depreciation on a reducing balance).
189	• Simple Growth and Decay		
190	• Compound Growth and Decay		
192	**B. The Effects of Different Compounding Periods**		• Understand the effect of different periods of compounding growth and decay (including effective and nominal interest rates).
195	**C. Time Lines**		
195	• Interest Rate Changes		
196	• Additional Withdrawals or Deposits		
197	• Mixed Examples		• Convert a nominal interest rate to an effective interest rate and vice versa.
198	• Nominal and Effective Interest Rates		

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A. SIMPLE AND COMPOUND GROWTH AND DECAY

In Grade 10 you dealt with the concepts of simple and compound interest. You will now deal with simple and compound decay. After most items are purchased they depreciate (lose value over time). The formulas for decay allow us to determine the value of an item after a certain amount of time.

SIMPLE GROWTH AND DECAY

In simple interest or decay, the amount that an item or investment increases or decreases by is calculated on the original amount and is the same in each period. Another name for simple decay is straight line deprecation.

Simple Interest	Simple Decay
$A = P(1+in)$	$A = P(1-in)$

Where: $A \rightarrow$ Final Amount

$P \rightarrow$ Principal amount / starting value

$i \rightarrow$ Interest rate (written as a decimal, divide the given interest rate by 100)

$n \rightarrow$ Number of time periods

Example 1: **Lerato purchases furniture for her apartment to the value of R20 000. The furniture depreciates at 9% per annum on the straight line method. What will her furniture be worth after 5 years?**

$A = P(1-in) \rightarrow$ simple decay

$= 20\,000\left[1-(0{,}09)(5)\right] \rightarrow 9\% = \frac{9}{100}$

$= \text{R}11\,000$

Example 2: **Janet invested R18 000 into a high risk, short term unit trust account for 3 years. After 3 years she finds her investment has decreased and is only worth R13 680.**

a) Using the straight line method, determine the rate per annum at which her investment decreased.

b) Janet then invests the remainder of her money into an account that guarantees an interest rate of 3,5% per annum, simple interest. How much will she have in her account after 10 years?

a) $A = P(1-in) \rightarrow$ simple decay

$\therefore 13\,680 = 18\,000\left[1-(i)(3)\right] \rightarrow$ divide both sides by 18 000

$\therefore 0{,}76 = 1-3i \rightarrow$ never round off during calculations in financial maths

$\therefore 3i = 1-0{,}76 \rightarrow$ divide both sides by 3

$\therefore i = \frac{0{,}24}{3} = 0{,}08 \rightarrow 1-0{,}76 = 0{,}24$

$\therefore i = 8\%$

b) $A = P(1+in) \rightarrow$ simple interest

$= 13\,680\left[1+0{,}035(10)\right]$

$= \text{R}18\,468$

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COMPOUND GROWTH AND DECAY

When dealing with compound interest and decay, interest or depreciation is calculated on the value of an investment or item at the end of each period. Another name for compound decay is depreciation on a reducing balance.

Compound Interest	Compound Decay
$A = P(1+i)^n$	$A = P(1-i)^n$

Where: $A \rightarrow$ Final Amount

$P \rightarrow$ Principal amount / starting value

$i \rightarrow$ Interest rate (written as a decimal, divide the given interest rate by 100)

$n \rightarrow$ Number of time periods

Example 1: Tebogo purchases a car for R80 000 that depreciates at a compound rate of 11% per annum.

a) Calculate the book value of the car after 5 years.

b) If the price of a new car increases at 8% per annum compounded annually, determine the price of a new car after 5 years.

The book value of an item is the value of the item after it has depreciated.

a) $A = P(1-i)^n \rightarrow$ compound decay

$= 80\,000(1-0{,}11)^5$

$= R44\,672{,}48$

b) $A = P(1+i)^n \rightarrow$ compound interest

$= 80\,000(1+0{,}08)^5$

$= R117\,546{,}25$

Example 2: A car dealer, in error, uses the straight line method to calculate a car's value. The car was originally purchased for R120 000. The dealer calculates the car to be worth R93 442,56 as it is 3 years old. What rate of depreciation would have to be used to give the same result using the reducing balance method?

$A = P(1-i)^n \rightarrow$ compound decay

$\therefore 93\,442{,}56 = 120\,000[1-i]^3$

$\therefore \frac{93\,442{,}56}{120\,000} = [1-i]^3 \rightarrow$ cube root both sides

$\therefore \sqrt[3]{0{,}778\ldots} = 1-i \rightarrow$ do not round off during financial maths calculations

$\therefore i = 1-0{,}92$

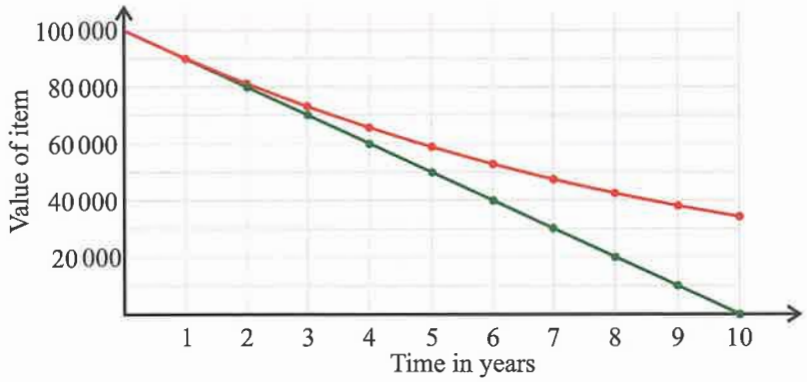
$\therefore i = 0{,}08$

$\therefore i = 8\%$

In financial maths, you may only round off answers in the final step. Inflation is always compound interest, unless otherwise stated.

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Example 3: **The graphs below represent an item which depreciates at 10% per annum. One graph represents straight line depreciation and the other reducing balance depreciation.**



a) **Which graph represents straight line depreciation? Give a reason for your answer.**
b) **Explain the reason for the differences between the two graphs.**
c) **Determine the initial cost of the item.**
d) **Hence, determine the equation of each graph in the form A = ...**

a) The green graph. The amount the item decreases by each year is the same. This represents straight line depreciation.

b) In straight line depreciation the item depreciates by the same amount based on the original cost price. In reducing balance (compound) depreciation, the depreciation amount is based on the value of the item in the previous period (in this case the previous year). After one year both items depreciate by the same amount, after which the amount that the item depreciates on the reducing balance becomes increasingly less, while the amount that the item depreciates by using the straight line method remains the same.

c) The y-intercept of the graph represents the initial cost of the item. Therefore, the initial cost of the item was R100 000.

d) Straight line: $A = P(1-in)$

$\therefore A = 100\,000(1-0{,}1n)$

Reducing balance: $A = P(1-i)^n$

$\therefore A = 100\,000(1-0{,}1)^n$

Example 4: **The volume of water in a steam iron decreases by 18,5% per minute that it is in use. If the iron has been used for 6 minutes and there is 29,31 ml of water remaining in the iron, determine, correct to the nearest millilitre, how many millilitres of water the iron initially had?**

$A = P(1-i)^n \rightarrow$ compound decay

$29{,}31 = P(1-0{,}185)^6 \rightarrow 29{,}31$ is the volume left after 6 minutes

$$\therefore P = \frac{29{,}31}{(1-0{,}185)^6}$$

$\therefore P = 100\,\text{ml}$



Complete Exercise 1 on page 365.

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B. THE EFFECTS OF DIFFERENT COMPOUNDING PERIODS

Up to this point you have only dealt with compound interest and decay that is compounded annually. In most real world applications interest is compounded more often than once a year. When this is the case, you have to adjust the formulas and use an interest rate and value for n which matches the compounding period. In other words, if compounding is performed monthly, the interest rate must be divided by 12 to obtain a monthly interest rate and n must be in months.

Illustrative Example: Determine the value R100 will grow to in 5 years, if interest is calculated at 12% per annum compounded:

a) Monthly. b) Quarterly . c) Semi-annually. d) Daily.

a) As interest is compounded monthly, you have to divide the given interest rate by 12 to obtain the monthly interest rate. It is important to note that as interest is compounded monthly, the number of periods (n) in the formula must also be in months.

$$A = P(1+i)^n$$

$$= 100\left(1+\frac{0,12}{12}\right)^{12\times 5} \rightarrow \text{compounded monthly, } n \text{ is the number of months}$$

$$= R181,67$$

b) As interest is compounded quarterly, divide the given interest rate by 4. The number of periods (n) must be in quarters. Remember that there are 4 quarters in a year and 1 quarter is 3 months.

$$A = P(1+i)^n$$

$$= 100\left(1+\frac{0,12}{4}\right)^{4\times 5} \rightarrow \text{compounded quarterly, } n \text{ is the number of quarters}$$

$$= R180,61$$

c) As interest is compounded semi-annually, divide the given interest rate by 2. The number of periods (n) must be in half years. Semi-annually means half yearly.

$$A = P(1+i)^n$$

$$= 100\left(1+\frac{0,12}{2}\right)^{2\times 5} \rightarrow \text{compounded semi-annually, } n \text{ is the number of half years}$$

$$= R179,08$$

d) As interest is compounded daily, divide the given interest rate by 365. The number of periods (n) must be in days.

$$A = P(1+i)^n$$

$$= 100\left(1+\frac{0,12}{365}\right)^{365\times 5} \rightarrow \text{compounded daily, } n \text{ is the number of days}$$

$$= R182,19$$



The more often interest is compounded, the greater the value of an investment at the end of a period.

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Example 1: **Rose invested R7 000 into a savings account for 4 years. If the savings account offers an interest rate of 7% per annum compounded monthly, determine how much she will have after 4 years.**

$A = ?\quad P = 7000\quad i = \frac{0,07}{12}\quad n = 4 \times 12$

$A = P(1+i)^n \rightarrow$ compound interest

$= 7000\left(1+\frac{0,07}{12}\right)^{4\times12} \rightarrow$ compounded monthly, n is the number of months

$= R9254,38 \rightarrow$ only round off at the end

Example 2: **Lindi takes out a loan of R5 000 from a micro-lender at an interest rate of 27% p.a compounded monthly. If the loan has to be paid back after 9 months how much will Lindi have to pay back to the micro-lender?**

$A = ?\quad P = 5000\quad i = \frac{0,27}{12}\quad n = 9 \rightarrow$ do not multiply by 12, n is the number of months

$A = P(1+i)^n \rightarrow$ compound interest

$= 5000\left(1+\frac{0,27}{12}\right)^{9} \rightarrow$ compounded monthly, n is the number of months

$= R6108,57 \rightarrow$ only round off at the end

Example 3: **Ravaash invests R2 500 into a savings account that offers an interest rate of 6,25% p.a. compounded quarterly. How much will Ravaash have after 8,5 years?**

$A = ?\quad P = 2500\quad i = \frac{0,0625}{4}\quad n = 8,5 \times 4 \rightarrow$ there are 4 quarters in a year

$A = P(1+i)^n \rightarrow$ compound interest

$= 2500\left(1+\frac{0,0625}{4}\right)^{4\times8,5} \rightarrow$ compounded quarterly, n is the number of quarters

$= R4235,21 \rightarrow$ only round off at the end

Example 4: **George invests R6 000 into an account which offers an interest rate of 11% p.a. compounded semi-annually. How much will he have after 3,5 years?**

$A = ?\quad P = 6000\quad i = \frac{0,11}{2}\quad n = 3,5 \times 2 \rightarrow$ there are two 6 month periods in a year

$A = P(1+i)^n \rightarrow$ compound interest

$= 6000\left(1+\frac{0,11}{2}\right)^{2\times3,5} \rightarrow$ compounded semi-annually, n is the number of half years

$= R8728,07 \rightarrow$ only round off at the end

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Example 5: **Thandi's car which was bought 5 years ago is now worth R80 434,88. What was the original purchase price of the car if it depreciated at 8% p.a. compounded daily?**

$A = 80\,434{,}88 \quad P = ? \quad i = \frac{0{,}08}{365} \quad n = 5 \times 365$

$A = P(1-i)^n \rightarrow$ compound decay

$\therefore 80\,434{,}88 = P\left(1-\frac{0{,}08}{365}\right)^{5\times 365}$ → compounded daily, n is the number of days

$\therefore P = \frac{80\,434{,}88}{\left(1-\frac{0{,}08}{365}\right)^{5\times 365}}$

$\therefore P = R120\,000$

Example 6: **An investment grows from R6 700 to R8 510,59 in 3 years. Determine the interest rate received if interest was compounded monthly.**

$A = 8510{,}59 \quad P = 6700 \quad i = \frac{?}{12} \quad n = 3 \times 12$

$A = P(1+i)^n \rightarrow$ compound interest

$\therefore 8510{,}59 = 6700\left(1+\frac{i}{12}\right)^{3\times 12}$ → compounded monthly, n is the number of months

$\therefore \frac{8510{,}59}{6700} = \left(1+\frac{i}{12}\right)^{36}$ → take the $\sqrt[36]{\ }$ of both sides

$\therefore 1+\frac{i}{12} = \sqrt[36]{\frac{8510{,}59}{6700}}$

$\therefore \frac{i}{12} = \sqrt[36]{\frac{8510{,}59}{6700}} - 1$ → do not round off

$i = 12\left(\sqrt[36]{\frac{8510{,}59}{6700}} - 1\right)$

$\therefore i = 0{,}08$

$\therefore i = 8\%$ → per annum compounded monthly



Remember that in financial maths you must never round off until you reach your final answer. However, you may round off between questions.

It is important to remember that inflation is always compound interest. If the compounding period is not given assume it to be annually.



Complete Exercise 2 on page 365.

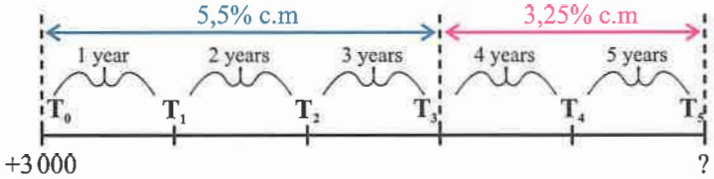
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C. TIME LINES

It is useful to draw a timeline for problems where there is a change to the interest rate as well as problems where there are additional withdrawals and/or deposits.

INTEREST RATE CHANGES

Example 1: **Nawaaz invests R3 000 into a savings account at an interest rate of 5,5% p.a compounded monthly. After 3 years the interest rate drops to 3,25% p.a. compounded monthly. How much will he have after 5 years?**

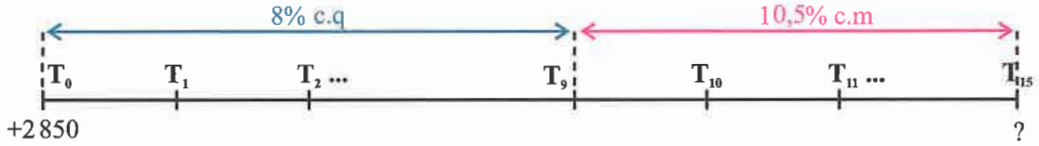


The investment earns interest at 5,5% for the first 3 years and then at 3,25% for the last 2 years.

$$\therefore A = 3000\left(1+\frac{0,055}{12}\right)^{3\times12}\left(1+\frac{0,0325}{12}\right)^{2\times12}$$

$$= R3774,05$$

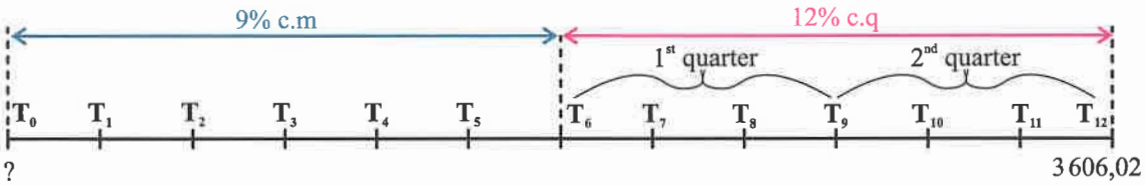
Example 2: **R2 850 is invested at 8% p.a. compounded quarterly for 15 years. After 9 years the interest rate changes to 10,5% p.a. compounded monthly. How much is the investment worth after 15 years?**



$$\therefore A = 2850\left(1+\frac{0,08}{4}\right)^{9\times4}\left(1+\frac{0,105}{12}\right)^{6\times12}$$ → 10,5% c.m is earned for 6 years

$$= R10885,95$$

Example 3: **In 1 year an investment grows to an amount of R3 606,02. For the first 6 months the interest rate is 9% p.a. compounded monthly. After this the interest rate changes to 12% p.a. compounded quarterly. How much was initially invested?**



$$\therefore 3606,02 = P\left(1+\frac{0,09}{12}\right)^{6}\left(1+\frac{0,12}{4}\right)^{2}$$ → 6 months is 2 quarters of a year $= \frac{1}{2}\times4$

$$\therefore P = \frac{3606,02}{\left(1+\frac{0,09}{12}\right)^{6}\left(1+\frac{0,12}{4}\right)^{2}} = R3250$$

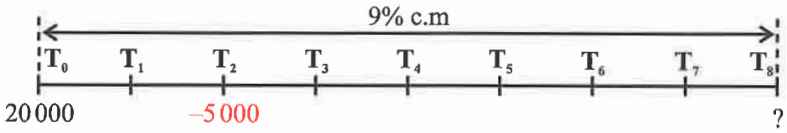


With changes in interest rates you multiply the brackets together.

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ADDITIONAL WITHDRAWALS OR DEPOSITS

Illustrative Example: **R20 000 is deposited into an account which offers an interest rate of 9% p.a. compounded monthly. 2 years later R5 000 is withdrawn from the account. How much will be in the account after 8 years?**



First determine the amount that would be in the account after 8 years if the additional deposit did not occur:

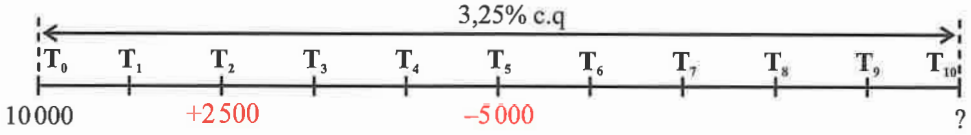
$$A = 20000\left(1+\frac{0,09}{12}\right)^{8\times12}$$

This amount includes the R5000 withdrawal as well as the interest that it earned for the 6 years that is wasn't in the account for. Therefore, you have to remove the R5000 with its interest for 6 years.

$$A = 20000\left(1+\frac{0,09}{12}\right)^{8\times12} - 5000\left(1+\frac{0,09}{12}\right)^{6\times12}$$

$$=R32\,415,66$$

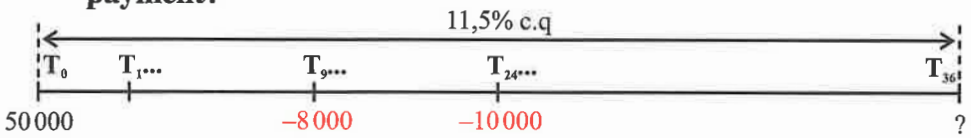
Example 1: **Ted invests R10 000 into an account that offers an interest rate of 3,25% p.a. compounded quarterly. After 2 years he deposits an additional R2 500 into the account and 3 years later withdraws R5 000. How much will he have in his account after 10 years?**



$$A = 10000\left(1+\frac{0,0325}{4}\right)^{10\times4} + 2500\left(1+\frac{0,0325}{4}\right)^{8\times4} - 5000\left(1+\frac{0,0325}{4}\right)^{5\times4}$$

$$=R11\,182,68$$

Example 2: **Romy takes out a loan of R50 000 at an interest rate of 11,5% compounded quarterly. 9 months after taking out the loan she repays R8 000 and 15 months later repays a further R10 000. 3 years after taking out the loan she pays off the loan. What is the value of the final payment?**



$$A = 50000\left(1+\frac{0,115}{4}\right)^{\frac{36}{3}} - 8000\left(1+\frac{0,115}{4}\right)^{\frac{27}{3}} - 10000\left(1+\frac{0,115}{4}\right)^{\frac{12}{3}}$$

$$=R48\,731,49$$

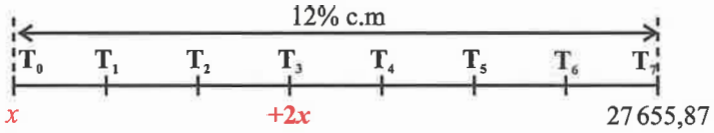


There are 3 months in a quarter of a year. To convert months to quarters you have to divide by 3. When dealing with deposits or withdrawals, *n* is always the amount of time left in the account from the time that the deposit or withdrawal is made, not the time it has been in the account.

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MIXED EXAMPLES

Example 1: **x Rand is invested into an account offering an interest rate of 12% p.a. compounded monthly. 3 years later $2x$ Rand is deposited into the account. After 7 years there is R27 655,87 in the account. Determine the value of x.**

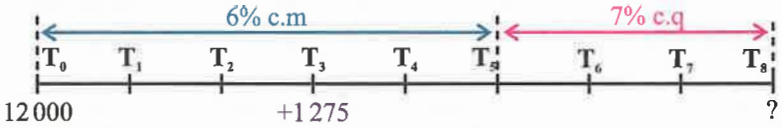


$$27\,655{,}87 = x\left(1+\frac{0{,}12}{12}\right)^{7\times12} + 2x\left(1+\frac{0{,}12}{12}\right)^{4\times12} \quad \rightarrow \text{take out a HCF of } x$$

$$\therefore 27\,655{,}87 = x\left[\left(1+\frac{0{,}12}{12}\right)^{7\times12} + 2\left(1+\frac{0{,}12}{12}\right)^{4\times12}\right]$$

$$\therefore x = \frac{27\,655{,}87}{\left[\left(1+\frac{0{,}12}{12}\right)^{7\times12} + 2\left(1+\frac{0{,}12}{12}\right)^{12\times4}\right]} = \text{R}5\,000$$

Example 2: **R12 000 is invested at 6% p.a. compounded monthly. After 5 years the interest rate changes to 7% p.a. compounded quarterly. After 3 years an additional R1 275 is deposited. What will the value of the investment be after 8 years?**

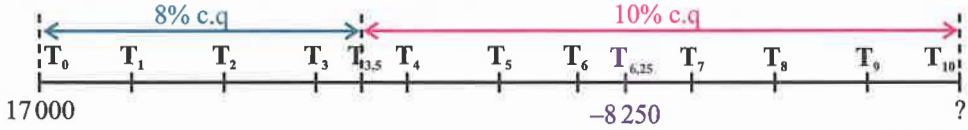


$$A = 12\,000\left(1+\frac{0{,}06}{12}\right)^{5\times12}\left(1+\frac{0{,}07}{4}\right)^{3\times4} + 1275\left(1+\frac{0{,}06}{12}\right)^{2\times12}\left(1+\frac{0{,}07}{4}\right)^{3\times4}$$

$$\therefore A = \text{R}21\,702{,}06$$

Example 3: **R17 000 is invested at 8% p.a. compounded quarterly for the first 3 and a half years and 10% p.a. compounded quarterly for the remaining period. After 75 months R8 250 is withdrawn. What is the total amount in the account after 10 years?**

You have to choose whether to work in months or years on your timeline. It is generally easiest to work in years if there are interest rate changes and additional deposits or withdrawals. To convert months to years, divide by 12.



$$A = 17\,000\left(1+\frac{0{,}08}{4}\right)^{3{,}5\times4}\left(1+\frac{0{,}10}{4}\right)^{6{,}5\times4} - 8\,250\left(1+\frac{0{,}10}{4}\right)^{3{,}75\times4}$$

$$\therefore A = \text{R}30\,677{,}27$$



Complete Exercise 3 on pages 365 and 366.

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NOMINAL AND EFFECTIVE INTEREST RATES

The nominal interest rate is the rate that is quoted in the question. The effective interest rate is the equivalent annual interest rate that you would receive once compounding has been taken into account. In order to compare interest rates with different compounding periods, it is often useful to convert nominal rates into effective rates.

Illustrative Example: Suppose x Rand is invested for a period of 1 year at an interest rate of 12% p.a. compounded monthly. What interest rate compounded yearly (the effective interest rate) would yield the same result .

First determine the final amount that would be obtained after one year:

$$A = P(1+i)^n$$

$$\therefore A = x\left(1+\frac{0,12}{12}\right)^{12} \rightarrow \boxed{1}$$

Now determine the final amount that would be obtained using the effective interest rate:

$$A = P(1+i)^n$$

$$\therefore A = x\left(1+i_{eff}\right)^1 \rightarrow \boxed{2}$$

$\boxed{1}$ must be equal to $\boxed{2}$

$$\therefore x\left(1+i_{eff}\right)^1 = x\left(1+\frac{0,12}{12}\right)^{12} \rightarrow \text{divide both sides by } x$$

$$\therefore \left(1+i_{eff}\right) = \left(1+\frac{0,12}{12}\right)^{12}$$

$$\therefore i_{eff} = \left(1+\frac{0,12}{12}\right)^{12} - 1$$

$$\therefore i_{eff} = 0,1268250301$$

$$\therefore i_{eff} = 12,68\%$$

This means that if you compound annually at a rate of 12,68% p.a. you will obtain the same result if you compounded at a rate of 12% p.a. monthly.
From the above example it should be clear that the formula for converting a nominal rate to an effective rate is:

Nominal to Effective:

$$\left(1+i_{eff}\right)^1 = \left(1+\frac{i_{nom}}{m}\right)^m$$

Where: i_{eff} → Effective interest rate

i_{nom} → Nominal interest rate

m → The number of times a year the nominal interest rate is compounded



The effective interest rate is always higher than the nominal interest rate.

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Example 1: Convert an interest rate of 15% p.a. compounded monthly to an effective interest rate.

$$(1+i_{eff})^1 = \left(1+\frac{i_{nom}}{m}\right)^m$$

$$\therefore 1+i_{eff} = \left(1+\frac{0,15}{12}\right)^{12}$$

$$\therefore i_{eff} = \left(1+\frac{0,15}{12}\right)^{12} - 1 = 0,1608$$

$$\therefore i_{eff} = 16,08\%$$

Example 2: Convert an interest rate of 8,25% p.a. compounded quarterly to an effective interest rate.

$$(1+i_{eff})^1 = \left(1+\frac{i_{nom}}{m}\right)^m$$

$$\therefore 1+i_{eff} = \left(1+\frac{0,0825}{4}\right)^4$$

$$\therefore i_{eff} = \left(1+\frac{0,0825}{4}\right)^4 - 1 = 0,08508761943$$

$$\therefore i_{eff} = 8,51\%$$

Example 3: Convert an effective interest rate of 17% p.a. to a nominal rate compounded daily.

$$(1+i_{eff})^1 = \left(1+\frac{i_{nom}}{m}\right)^m$$

$$\therefore 1+0,17 = \left(1+\frac{i_{nom}}{365}\right)^{365} \rightarrow \text{take the } \sqrt[365]{\ } \text{ of both sides}$$

$$\therefore \sqrt[365]{1+0,17} = 1+\frac{i_{nom}}{365}$$

$$\therefore \frac{i_{nom}}{365} = \sqrt[365]{1+0,17} - 1$$

$$\therefore i_{nom} = 0,157037521$$

$$\therefore i_{nom} = 15,70\%$$



When using the effective interest rate to find the value of an investment, you cannot use a rounded off value. Always use the exact rate.



Complete Exercise 4 on page 366 and the Mixed Exercise on pages 366 and 367.

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CHAPTER 11 – PROBABILITY

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
201	**A. Grade 10 Revision**	6	• Understand and work with dependent and independent events, in addition to all the probability which you learnt in Grade 10. • Use Venn diagrams or contingency tables, as well as tree diagrams as aids to solving probability problems.
201	• Probability of an Event Occurring and Sample Space		
201	• Theoretical and Experimental Probability		
201	• Notation		
201	• Mutually Exclusive Events		
202	• The Addition Rule		
202	• Complementary Events		
204	**B. Dependent and Independent Events**		
204	• The Product Rule		
206	**C. Venn Diagrams**		
210	**D. Tree Diagrams**		
212	**E. Contingency Tables**		

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A. GRADE 10 REVISION

PROBABILITY OF AN EVENT OCCURRING AND SAMPLE SPACE

In Grade 10 you learnt that a sample space consists of all the possible outcomes for an event to occur. For example, if a fair dice is thrown the possible outcomes are obtaining a 1 through to a 6. The sample space would be: $S = \{1;2;3;4;5;6\}$.

Event E is a possible outcome from the sample space. For example, the event E could be obtaining a 1. Event E would be denoted as: $E = \{1\}$

The probability of event E occurring is: $P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$

In general, E is the total number of ways a specific event can occur. S is the total number of possible outcomes for the event.

THEORETICAL AND EXPERIMENTAL PROBABILITY

In most cases, it takes a large number of trials before experimental and theoretical probabilities approach the same value.

Consider the situation of tossing a fair coin. The theoretical probability of obtaining a head is 50% and the theoretical probability of obtaining a tail is 50%. However, this does not mean that if a coin is tossed 10 times you will obtain 5 heads and 5 tails (this is experimental probability). There would have to be a large number of tosses before half the outcomes are heads and the other half are tails.

NOTATION

The probability of an event occurring will always lie between 0 and 1, where 0 is an impossible event and 1 is a certain event. It is essential that you are familiar with the notations below which are used in most probability questions:

- $P(A) \rightarrow$ The probability of event A occurring
- $P(A') \rightarrow$ The probability of event A **not** occurring (this is also called the **complement** of A)
- $P(A \text{ or } B) = P(A \cup B) \rightarrow$ The probability of A **or** B occurring. $\cup$ is the symbol for **or**, it is also known as **union**.
- $P(A \text{ and } B) = P(A \cap B) \rightarrow$ The probability of A and B occurring. $\cap$ is the symbol for **and**, it is also known as **intersection**.
- $P(A \mid B) \rightarrow$ The probability of A occurring **given** that B has occurred.

MUTUALLY EXCLUSIVE EVENTS

If two events are mutually exclusive it means that they cannot occur in the same trial. For example, if a trial is the tossing of a dice and event A is obtaining a 6 while event B is obtaining a 2, it is impossible to obtain events A and B together.

If a trial is picking a card and event A is selecting a queen and event B is selecting a heart, it is possible to obtain events A and B together. This is an example of events which are not mutually exclusive.

For mutually exclusive events : $P(A \text{ and } B) = 0$.

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THE ADDITION RULE

The addition rule states: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

If events A and B are mutually exclusive $P(A \text{ and } B) = 0$ and the addition rule simplifies to: $P(A \text{ or } B) = P(A) + P(B)$.



The addition rule can also be written as:
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

COMPLEMENTARY EVENTS

If two events are complementary:
$$P(\text{not } A) = P(A') = 1 - P(A)$$

There are 2 conditions that have to be true for events to be complementary:

1) **The events have to be mutually exclusive**
2) $P(A) + P(B) = 1$

Example 1: If a fair dice is rolled what is the probability of getting a multiple of 3?

The only multiples of 3 in the sample space are 3 and 6.

$$P(3 \text{ or } 6) = P(3) + P(6) - P(3 \text{ and } 6)$$
$$= \frac{1}{6} + \frac{1}{6} - 0 \rightarrow \text{Mutually exclusive, } \therefore P(3 \text{ and } 6) = 0$$
$$= \frac{2}{6} = \frac{1}{3}$$

Example 2: A card is selected at random from a fair pack of playing cards:

a) **Determine the probability of selecting a jack.**
b) **Determine the probability of selecting a heart.**
c) **Determine the probability of selecting a jack or a heart.**

a) $P(\text{jack}) = \frac{4}{52}$

b) $P(\text{heart}) = \frac{13}{52}$

c) $P(\text{jack or heart}) = P(\text{jack}) + P(\text{heart}) - P(\text{jack and heart})$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$
$$= \frac{16}{52}$$
$$= \frac{4}{13}$$
$$= 0,3077$$
$$= 30,77\%$$



In a fair pack of playing cards there are 52 cards and no jokers. Unless otherwise stated always assume that a deck of cards is fair.

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Example 3: **Given sample space $s=\{6;7;8;9;10\}$ determine whether or not the following events are complementary:**

a) Selecting an even number and a multiple of 5.
b) Selecting a multiple of 3 and a prime number.
c) Selecting an odd number and a multiple of 2.

a) The set of even numbers can be given by $A=\{6;8;10\}$ and the set of multiples of 5 can be given by $B=\{10\}$. As the events share a common element (10) they are not mutually exclusive. Therefore they are not complementary events.

b) The set of multiples of 3 can be given by $A=\{6;9\}$ and the set of prime numbers can be given by $B=\{7\}$. Therefore the events are mutually exclusive.

$P(A)=\frac{2}{5}$ and $P(B)=\frac{1}{5}$

$\therefore P(A)+P(B)=\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$

$\therefore P(A)+P(B)\neq 1$

$\therefore$ the events are not complementary.

c) The set of odd numbers can be given by $A=\{7;9\}$ and the set of multiples of 2 can be given by $B=\{6;8;10\}$. Therefore the events are mutually exclusive.

$P(A)=\frac{2}{5}$ and $P(B)=\frac{3}{5}$

$\therefore P(A)+P(B)=\frac{2}{5}+\frac{3}{5}=\frac{5}{5}$

$\therefore P(A)+P(B)=1$

$\therefore$ the events are complementary.

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B. DEPENDENT AND INDEPENDENT EVENTS

Two events are said to be independent if the outcome of one event does not affect the probability of another event occurring. If the outcome of one event changes the probability that another event occurs, the events are said to be dependent.

For example, if a card is selected from a deck of cards and then placed back in the deck and another card is drawn, the drawing of the first card will not affect the outcome of drawing the second card. These would be independent events.

If the first card was not placed back in the deck, drawing the first card would affect the outcome of drawing the second card. These would be dependent events.

In most cases, events that take place with replacement are independent events and those without replacement are dependent events.

THE PRODUCT RULE

1) The product rule for independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

2) The product rule for dependent events:

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

Remember that $P(A \text{ and } B)$ can also be written as $P(A \cap B)$.

$P(B \mid A)$ means the probability of B occurring given that A has occurred.

Example 1: In a game, a player has to roll a dice and toss a coin. In order to win, the player has to get a 6 and a head. If a player gets a 6 and a tail he is allowed another turn.

a) What is the probability of winning?

b) What is the probability of winning or getting another turn?

a) As tossing a coin has no effect on the outcome of rolling a dice the events are independent.

Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

$P(6) = \frac{1}{6}$ and $P(\text{head}) = \frac{1}{2}$ and $P(\text{tail}) = \frac{1}{2}$

$\therefore P(6 \text{ and head}) = P(6) \times P(\text{head})$

$$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} = 0,0833 = 8,33\%$$

b) In this case the events are still independent. However, there are 2 options that you are looking for:

$P(6 \text{ and head}) \text{ or } P(6 \text{ and tail}) = P(6) \times P(\text{head}) + P(6) \times P(\text{tail})$

$$= \left(\frac{1}{6} \times \frac{1}{2}\right) + \left(\frac{1}{6} \times \frac{1}{2}\right)$$

$$= \frac{2}{12} = 0,1667 = 16,67\%$$

In order to prove that events are independent you have to show that:

$$P(A \text{ and } B) = P(A) \times P(B)$$

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Example 2: If 3 blue balls and 7 red balls are placed in a bag and 2 balls are randomly chosen without replacement, determine the probability of choosing:

a) A blue ball and then a red ball.
b) A red ball and then a blue ball.
c) A red ball and a blue ball in an any order.
d) 2 blue balls.

As the balls are chosen without replacement, the probability of the second event is dependent on the colour of the first ball. Therefore, the events are dependent.

a) Dependent events: $P(A \text{ and } B) = P(A) \times P(B|A)$

$P(\text{blue}) = \frac{3}{10} \rightarrow$ probability of a blue first

$P(\text{red}|\text{blue}) = \frac{7}{9} \rightarrow$ probability of a red given that a blue has been chosen

$\therefore P(\text{blue and red}) = P(\text{blue}) \times P(\text{red}|\text{blue})$

$$= \frac{3}{10} \times \frac{7}{9} = \frac{7}{30} = 0,2333 = 23,33\%$$

It is important to remember that once a ball has been chosen there is 1 less ball in the bag when the second ball is chosen.

b) Dependent events: $P(A \text{ and } B) = P(A) \times P(B|A)$

$P(\text{red}) = \frac{7}{10} \rightarrow$ probability of a red first

$P(\text{blue}|\text{red}) = \frac{3}{9} \rightarrow$ probability of a blue given that a red has been chosen

$\therefore P(\text{red and blue}) = P(\text{red}) \times P(\text{blue}|\text{red})$

$$= \frac{7}{10} \times \frac{3}{9} = \frac{7}{30} = 0,2333 = 23,33\%$$

c) Dependent events: $P(A \text{ and } B)$ or $P(B \text{ and } A) = P(A) \times P(B|A)$ or $P(B) \times P(A|B)$

$P(\text{blue and red})$ or $P(\text{red and blue}) = \frac{7}{30} + \frac{7}{30} = \frac{14}{30} = 0,4666 = 46,67\%$

d) Dependent events: $P(A \text{ and } B) = P(A) \times P(B|A)$

$P(\text{blue}) = \frac{3}{10} \rightarrow$ probability of a blue first

$P(\text{blue}|\text{blue}) = \frac{2}{9} \rightarrow$ probability of a blue given that a blue has been chosen

$\therefore P(\text{blue and blue}) = P(\text{blue}) \times P(\text{blue}|\text{blue})$

$$= \frac{3}{10} \times \frac{2}{9} = \frac{1}{15} = 0,0666 = 6,67\%$$

In probability the word 'and' usually means multiply and the word 'or' means add.

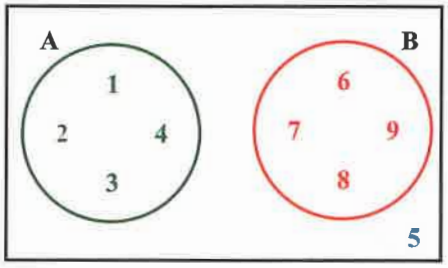
Complete Exercise 1 on page 373.

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C. VENN DIAGRAMS

Venn diagrams are a graphical way of representing a sample space and its events. Venn diagrams were covered in Grade 10 and we will now expand on them. The illustrative examples below revise Venn diagrams from Grade 10.

Illustrative Example 1: In the Venn diagram below, the sample space represents the first 9 natural numbers. Set A contains the natural numbers less than 5 and set B contains the natural numbers greater than 5.



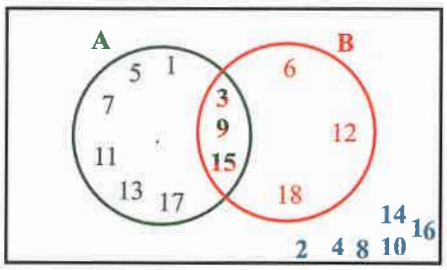
From the above diagram it can be seen that events A and B **are mutually exclusive** (no overlap) but **not complementary** as there is an event which lies neither in A or B (the number 5). The entire sample space is represented in the diagram as shown above. Probability can easily be determined from the diagram by counting the number of outcomes in each event as shown below:

$$P(A)=\frac{4}{9} \text{ and } P(B)=\frac{4}{9} \text{ and } P(A \text{ or } B)=P(A\cup B)=P(A)+P(B)=\frac{4}{9}+\frac{4}{9}=\frac{8}{9}$$

$$P(\text{not } A)=P(A')=\frac{4}{9}+\frac{1}{9}=\frac{5}{9}$$

For mutually exclusive events: $P(A \text{ or } B)=P(A\cup B)=P(A)+P(B)$.

Illustrative Example 2: In the Venn diagram below, the sample space represents the first 18 natural numbers. Set A contains the odd numbers and set B contains the multiples of 3.



Set A and B are **not** mutually exclusive as there are events which are common to both sets. This is the **intersection of the sets** and is written as: A **and** $B=A\cap B=\{3;9;15\}$.

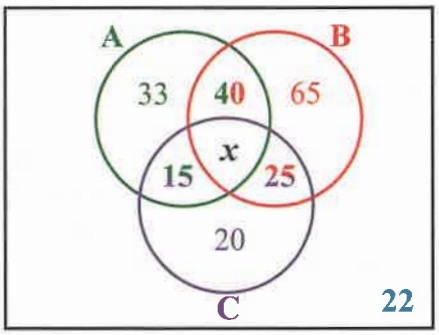
The $P(A \text{ or } B)$ can be read by counting directly from the diagram: $P(A \text{ or } B)=\frac{12}{18}$ or by using the formula: $P(A \overset{or}{\cup} B)=P(A)+P(B)-P(A \overset{and}{\cap} B)=\frac{9}{18}+\frac{6}{18}-\frac{3}{18}=\frac{12}{18}$

For events that are not mutually exclusive

$$P(A \overset{or}{\cup} B)=P(A)+P(B)-P(A \overset{and}{\cap} B).$$

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Example 1: **Thabo polled 250 people to determine if they liked shampoo A, B or C. His results are shown in the Venn diagram below:**



a) How many people like all of the shampoos?

b) Determine the probability of randomly picking a person that likes shampoos A or B.

c) Determine the probability of randomly picking a person that likes shampoo C and doesn't like shampoo B.

d) Determine the probability of randomly picking a person that only likes shampoos A and B.

e) Determine the probability of randomly picking a person that doesn't like shampoos C or B.

f) What is the probability of randomly picking a person who doesn't like any of the shampoos?

a) $x = 250 - (65+40+33+15+25+20+22)$ → don't forget the 22 who don't like any

$\therefore x = 30$

b) $P(A \overset{or}{\cup} B) = \dfrac{33+65+40+15+25+30}{250} = \dfrac{208}{250} = 0,832 = 83,2\%$

c) $P(C \overset{and}{\cap} B') = \dfrac{20}{250} + \dfrac{15}{250} = \dfrac{35}{250} = 0,14 = 14\%$

d) $P(A \overset{and}{\cap} B \text{ only}) = \dfrac{40}{250} = \dfrac{4}{25} = 0,16 = 16\%$

e) $P(C' \overset{or}{\cup} B') = \dfrac{33}{250} + \dfrac{22}{250} = \dfrac{55}{250} = \dfrac{11}{50} = 0,22 = 22\%$

f) $P(A' \overset{or}{\cup} B' \overset{or}{\cup} C') = \dfrac{22}{250} = \dfrac{11}{125} = 0,088 = 8,8\%$

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Example 2: **In a study 160 Grade 11 learners were asked if they take Biology, Accounting or Science. The results of the study are shown below:**

- **80 people said they take Biology**
- **70 people said they take Accounting**
- **80 people said they take Science**
- **42 people said they take Biology and Science**
- **36 people said they take Accounting and Science**
- **136 people said they take at least one of the subjects**
- **14 said they take all 3 subjects.**

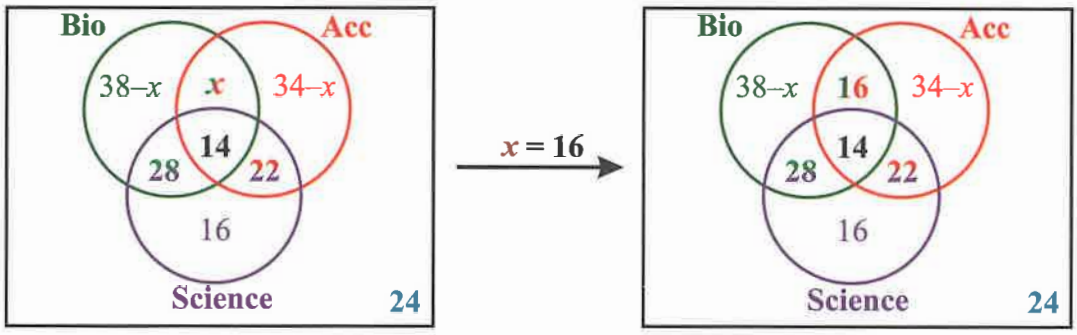
a) **Record the above information in a Venn diagram.**

b) **How many people take none of the subjects?**

c) **How many people take Biology and Accounting but not Science?**

d) **What is the probability that a person chosen at random takes at least two of the subjects?**

a)



Method:

Step 1: Determine the number of people that take none of the subjects. There are 160 people in the sample space and 136 people who take at least one of the subjects.

$\therefore$ the number of people who take none of the subjects $= 160 - 136 = 24$

Step 2: It is given that **14** people take all of the subjects. Write this down in the intersection of all the subjects.

Step 3: It is given that 42 people take Biology and Science. However, this includes the **14** people that take all of the subjects. The 14 has to be deducted from the 42 people who take Biology and Science. The same must be applied to Accounting and Science.

$\therefore$ Biology and Science $= 42 - 14 = \mathbf{28}$ Accounting and Science $= 36 - 14 = \mathbf{22}$

Step 4: You can now work out the number of people who take only Science.

Number of people who take only Science $= 80 - 28 - 14 - 22 = \mathbf{16}$

Step 5: As you don't know the number of people who take Biology and Accounting let that be x.

You can now work out the number of people that take only Biology and only Accounting.

$\therefore$ Biology only $= 80 - 28 - 14 - x$ Accounting $= 70 - 14 - 22 - x$

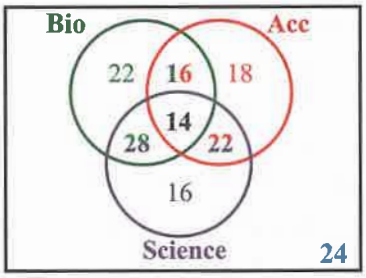
$= \mathbf{38 - x}$ $= \mathbf{34 - x}$

Step 6: You can now calculate the value of x. There were a total of 160 people:

$\therefore 38 - x + x + 34 - x + 28 + 14 + 22 + 16 + 24 = 160$

$\therefore x = 16$

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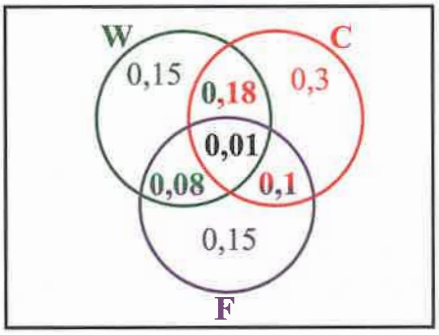
b) 24

c) 16

d) At least 2 subjects means 2 or more subjects (all intersections):
(Biology and Accounting) **or** (Biology and Science) **or** (Accounting and Science) **or** (Biology and Accounting and Science)

$$\therefore P(\text{at least two subjects}) = \frac{16}{160} + \frac{28}{160} + \frac{22}{160} + \frac{14}{160}$$
$$= \frac{80}{160} = \frac{1}{2} = 0.5 = 50\%$$

Example 3: **50 Grade 11 learners were asked if their school lunch consisted of a sweet (W), chocolate (C) or fruit (F). The probabilities of each are shown in the Venn diagram below.**



Determine the probability that a learner selected at random will have:

a) **A sweet and a chocolate and a fruit.**

b) **None of the above foods.**

c) **A sweet and a chocolate but not a fruit.**

d) **Only fruit.**

e) **A fruit or a sweet.**

a) $P(W \cap C \cap F) = 0,01 = 1\%$ → the intersection of all 3 sets

b) $P(W' \cup C' \cup F') = 1 - (0,15 + 0,18 + 0,3 + 0,01 + 0,08 + 0,1 + 0,15)$
$= 0,03 = 3\%$

c) $P(W \cap C \cap F') = 0,18 = 18\%$ → the intersection of sweets and chocolate

d) $P(\text{F only}) = 0,15 = 15\%$ → only fruit

e) $P(F \cup W) = 1 - P(\text{C only or none of the foods})$
$= 1 - (0,3 + 0,03)$
$= 0,67 = 67\%$

This question could also have been solved by adding all the elements in the sets excluding the choice of chocolate only.

Complete Exercise 2 on pages 373 and 374.

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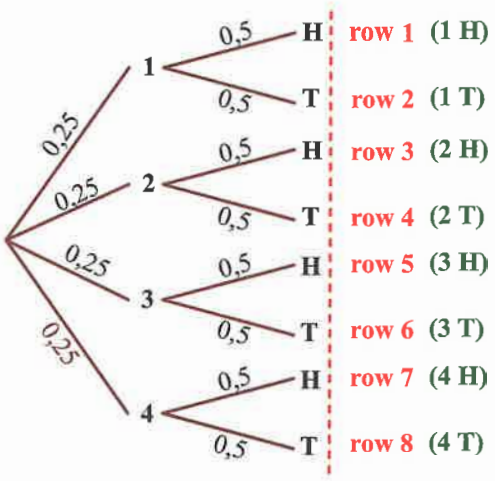
D. TREE DIAGRAMS

When there are 2 or more consecutive events taking place, it is often useful to represent the possible solutions on a tree diagram. Tree diagrams are constructed by showing all possible events. They can be used for dependent or independent events. When dealing with tree diagrams always multiply along branches and add probabilities moving down branches at the end. This is shown in the examples below.

Example 1: In a game, a four sided die with the numbers 1 to 4 is rolled and a fair coin is tossed.

a) Draw a tree diagram to represent the above situation.
b) Determine the probability of rolling a 3 and getting a head.
c) Determine the probability of rolling an even number.
d) Determine the probability of rolling a 1 or a head.

a)



b) $P(3 \text{ and a head}) = 0,25 \times 0,5 = 0,125 = 12,5\% \rightarrow$ row 5

c) $P(\text{even}) = 0,25 \times 0,5 + 0,25 \times 0,5 + 0,25 \times 0,5 + 0,25 \times 0,5 \rightarrow$ rows 3,4,7 and 8

$= 0,5$

$= 50\%$

d) $P(1 \text{ or a head}) = 5(0,25 \times 0,5) = 0,625 = 62,5\% \rightarrow$ rows 1,2,3,5 and 7



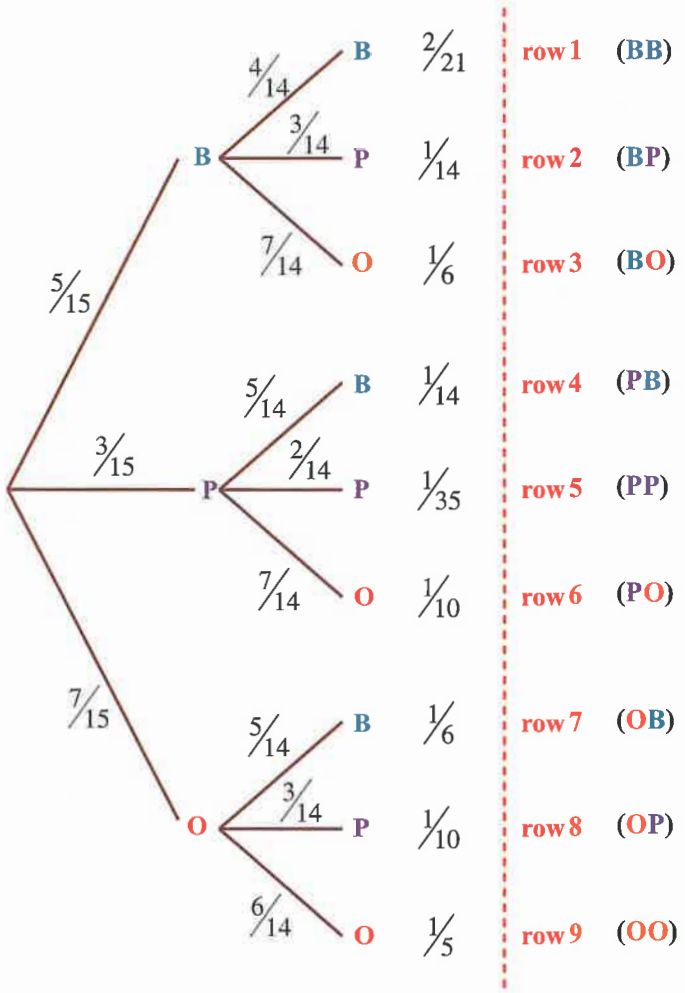
When dealing with tree diagrams multiply across branches and add when moving down. Write the probability of an event occurring at the top of the branches and the event at the end of the branch.

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Example 2: **5 blue balls, 3 purple balls and 7 orange balls are placed in a bag. If 2 balls are then selected without replacement:**

a) Draw a tree diagram to represent the above scenario.
b) Determine the probability of selecting a purple or blue ball.
c) Determine the probability of the 2nd ball being purple.
d) Determine the probability of selecting 2 orange balls.

a)



b) $P(\text{B or P}) = \frac{2}{21} + \frac{1}{14} + \frac{1}{6} + \frac{1}{14} + \frac{1}{35} + \frac{1}{10} + \frac{1}{6} + \frac{1}{10}$ → all rows other than 9

$= \frac{4}{5}$

$= 0,8$

$= 80\%$

c) $P(\text{2nd ball being P}) = \frac{1}{14} + \frac{1}{35} + \frac{1}{10}$ → rows 2,5 and 8

$= \frac{1}{5}$

$= 0,2$

$= 20\%$

d) $P(\text{O and O}) = \frac{1}{5}$ → row 9

$= 0,2$

$= 20\%$



Complete Exercise 3 on pages 374 and 375 .

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E. CONTINGENCY TABLES

Contingency tables are statistical tables that show the relationships between two or more variables. They are often used to determine whether or not events are independent.

Example 1: A group of 540 people with green or blue eyes were randomly selected in order to determine whether or not green or blue eyes are dependent on gender. The results are tabulated below:

	Male	Female	Totals
Green eyes	183	147	**330**
Blue Eyes	117	93	**210**
Totals	**300**	**240**	**540**

a) If a person is selected at random determine the probability that she is a female with green eyes.

b) If a person is selected at random and they have green eyes, determine the probability that she is female.

c) After analysing the results, a Grade 11 learner concludes that the probability of having green eyes is independent of gender. Is he correct? Substantiate your answer with relevant calculations. Give all answers correct to 2 decimal places.

a) $P(\text{female} \cap \text{green eyes}) = \frac{147}{540}$ → 147 females with green eyes out of 540 people

$$= \frac{49}{180}$$
$$= 27,22\%$$

b) $P(\text{female} \mid \text{green eyes}) = \frac{147}{330}$ → there are 330 people with green eyes

$$= \frac{49}{110}$$
$$= 44,55\%$$

c) For events to be independent: $P(\text{female} \cap \text{green eyes}) = P(\text{green eyes}) \times P(\text{female})$

$$P(\text{green eyes}) \times P(\text{female}) = \frac{330}{540} \times \frac{240}{540} = \frac{22}{81} = 0,27$$

$$P(\text{female} \cap \text{green eyes}) = \frac{147}{540} = \frac{49}{180} = 0,27$$

$\therefore P(\text{female} \cap \text{green eyes}) = P(\text{green eyes}) \times P(\text{female})$

$\therefore$ the events are independent and the learner is correct.

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Example 2: **A study was conducted between two airports in order to determine whether or not planes departing late was independent of the airports. Based on the results the contingency table below was drawn.**

	Airport A	Airport B	Totals
# of planes departed late	450		**550**
# of planes departed on time		300	
Totals	**600**	**400**	

a) Fill in the missing values in the table.

b) Determine the probability of a plane departing late.

c) Determine the probability of a plane departing late from Airport A.

d) Determine the probability of a plane departing late from Airport B.

e) A manager at Airport A claims that the probability of a plane departing late is independent of the airport that it departs from. Is the manager correct?

f) Based on your findings, if any, which airport should a traveller choose to depart from?

a)

	Airport A	Airport B	Totals
# of planes departed late	450	**100**	**550**
# of planes departed on time	**150**	300	**450**
Totals	**600**	**400**	**1000**

b) $P(\text{late}) = \frac{550}{1000} \rightarrow 550$ late out of 1000 planes

$= \frac{11}{20} = 55\%$

c) $P(\text{late} \mid \text{Airport A}) = \frac{450}{600} \rightarrow 450$ late out of 600 planes at Airport A

$= \frac{3}{4} = 75\%$

d) $P(\text{late} \mid \text{Airport B}) = \frac{100}{400} \rightarrow 100$ late out of 400 planes at Airport B

$= \frac{1}{4} = 25\%$

e) For events to be independent: $P(\text{Airport A} \cap \text{late}) = P(\text{Airport A}) \times P(\text{late})$

$P(\text{Airport A}) \times P(\text{late}) = \frac{600}{1000} \times \frac{550}{1000} = \frac{33}{100} = 0,33$

$P(\text{Airport A} \cap \text{late}) = \frac{450}{1000} = \frac{9}{20} = 0,45$

$\therefore P(\text{Airport A} \cap \text{late}) \neq P(\text{Airport A}) \times P(\text{late})$

$\therefore$ The manager is incorrect.

f) Travellers should choose to depart from airport B as they are 3 times more likely to depart late from airport A than from airport B.



Complete Exercise 4 on page 375 and the Mixed Exercise on pages 376 and 377.

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CHAPTER 12 – STATISTICS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
215	**A. Revision of Terminology**	10	• Represent measures of central tendency and dispersion in univariate numerical data by: using ogives and calculating the variance and standard deviation of sets of data manually (for small sets of data) and using calculators (for larger sets of data) and representing results graphically. • Represent skewed data in box and whisker diagrams, and frequency polygons. • Identify Outliers.
215	• Measures of Central Tendency		
215	• Five Number Summary		
215	• Measures of Dispersion		
215	• Calculation of the Mean		
215	• Calculation of quartiles and percentiles		
216	**B. Histograms and Frequency Polygons**		
217	• Frequency Polygons		
220	**C. Ogives**		
224	**D. Variance and Standard Deviation**		
226	• Standard Deviation with Grouped Data		
227	**E. Symmetrical and Skewed Data**		
228	• Identification of Outliers		

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A. REVISION OF TERMINOLOGY

MEASURES OF CENTRAL TENDENCY

Mode: The most common value in a data set.
Median (Q_2): The median is the middle value in an ordered data set.
Mean ($\bar{x}$): The average of a data set.

FIVE NUMBER SUMMARY

Minimum Value in the set: The lowest value in a data set.
Lower Quartile (Q_1): The median of the lower half of an ordered data set.
Median (Q_2): The median is the middle value in an ordered data set.
Upper Quartile (Q_3): The median of the upper half of an ordered data set.
Maximum Value in the set: The highest value in a data set.

The Five Number Summary is represented visually on a box and whisker diagram.

MEASURES OF DISPERSION

Range: The difference between the largest value and smallest value in a data set.

Range = largest value – smallest value.

Inter-quartile range (IQR): The difference between the upper and lower quartiles.

$IQR = Q_3 - Q_1$.

CALCULATION OF THE MEAN

The mean is the most commonly used measure of central tendency. It is commonly known as the average. The mean is the sum of the data values in the data set, divided by the number of data points in the data set.

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\text{sum of the values in the data set}}{\text{number of elements in the data set}}$$

CALCULATION OF QUARTILES AND PERCENTILES

Median (Q_2)

The median is the middle value of an ordered data set. When determining the median of a data set the **data must always be ordered.**
Use the formula below to determine the position of the median.

$$\text{Position of median} = \frac{n+1}{2}$$

where n is the number of elements in the data set.

Lower (Q_1) and Upper Quartiles (Q_3)

The lower and upper quartiles are the median of the lower and upper half of the ordered data set respectively. You can therefore calculate the upper and lower quartile by determining the median of the lower and upper halves.

Percentiles

A percentile is the value below which a specific percentage of data elements lie.

$$\text{Position of percentile} = \frac{\text{percentile}}{100} \times \text{number of elements in the data set}$$

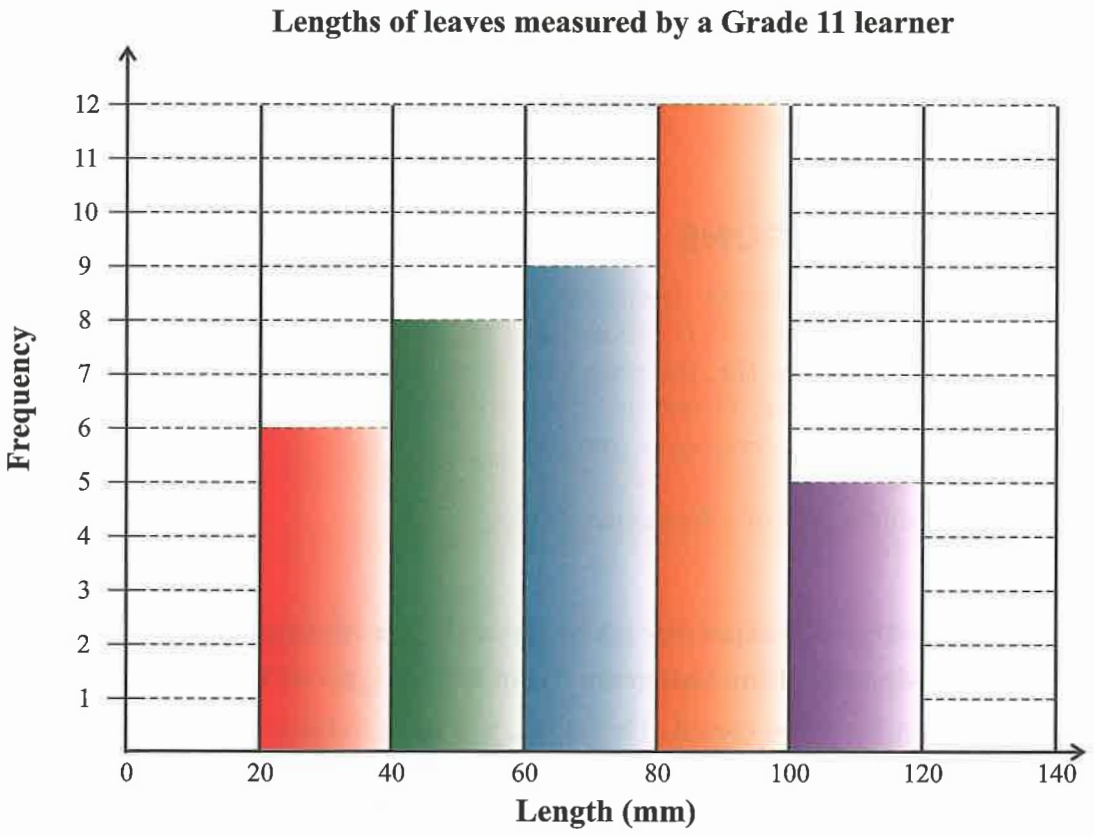
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B. HISTOGRAMS AND FREQUENCY POLYGONS

Histograms are a graphical representation of grouped data. They are similar to bar graphs but are usually used to represent continuous data that has been sorted into groups (frequencies). There are no gaps between the bars and each bar is usually the same width. The width of the bar represents the group intervals.

Example 1: In a Biology experiment, a Grade 11 pupil was required to measure the lengths of 40 different leaves. The frequency table below shows the results of the experiment. Draw a histogram for the data below.

Length (L) in millimetres	Frequency
$20 < L \leq 40$	**6**
$40 < L \leq 60$	**8**
$60 < L \leq 80$	**9**
$80 < L \leq 100$	**12**
$100 < L \leq 120$	**5**

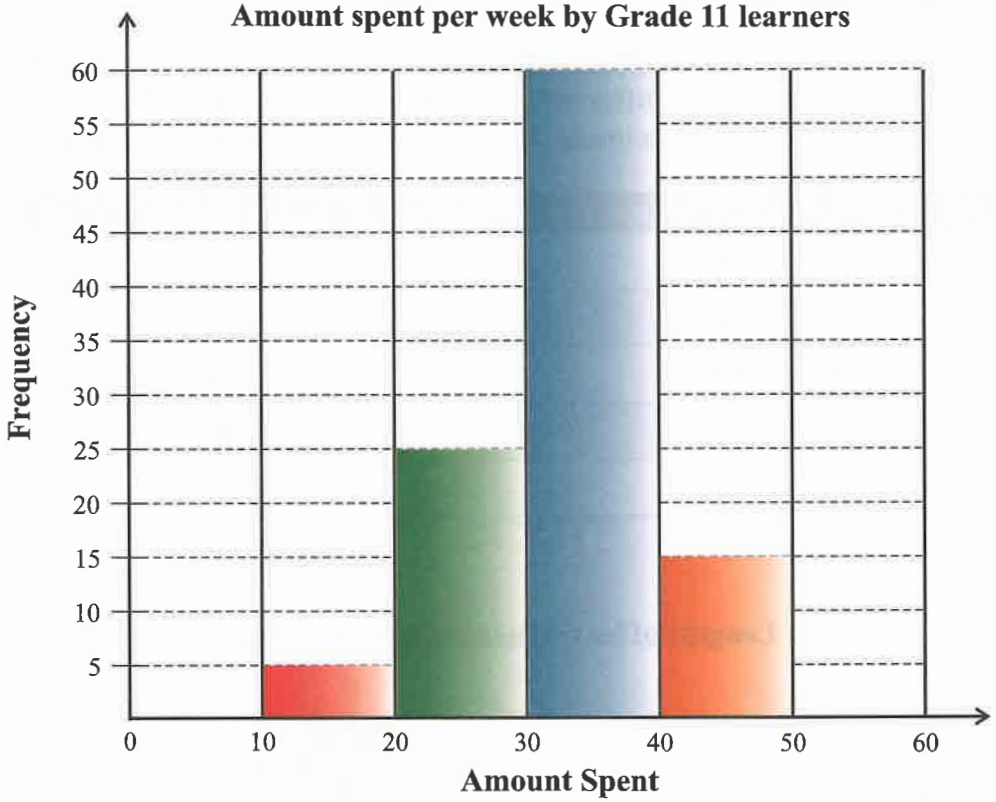


In a histogram the height of the bars represent the frequency in each category. This is only true if the bars are equal in width. In Grade 11 you will only have to work with histograms that have bars of equal length.

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Example 2: The frequency table below represents the amount of money (in Rands) that a group of Grade 11 learners spend on their cellular phones in 1 week. Draw a histogram for the data below.

Amount Spent	$10 \leq R < 20$	$20 \leq R < 30$	$30 \leq R < 40$	$40 \leq R < 50$
Number of learners	5	25	60	15



FREQUENCY POLYGONS

Just as a histogram shows frequency distribution using bars, a frequency polygon shows frequency distribution using a line graph. A frequency polygon is constructed by joining the midpoints of a histogram with straight lines. A polygon is a closed shape. Therefore, it has to start and end on an axis. There are two ways to construct a frequency polygon:

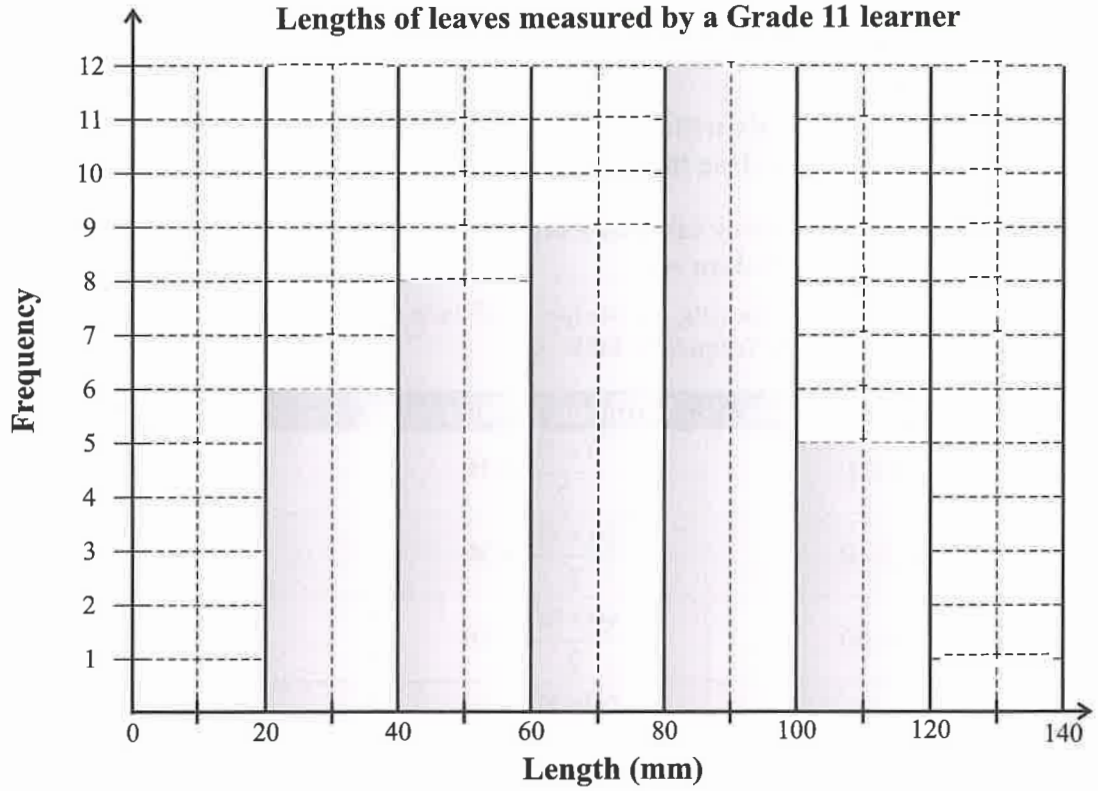
1. Using a histogram.
2. Using the midpoints of a frequency table.

Steps to drawing a frequency polygon using a histogram:

1. Join the midpoints of the histogram (from left to right) with straight lines.
2. The polygon has to be closed. Therefore, you have to have an interval on either side of the histogram. These intervals will have a frequency of 0. Join the midpoints of the 0 frequency intervals to the line graph.

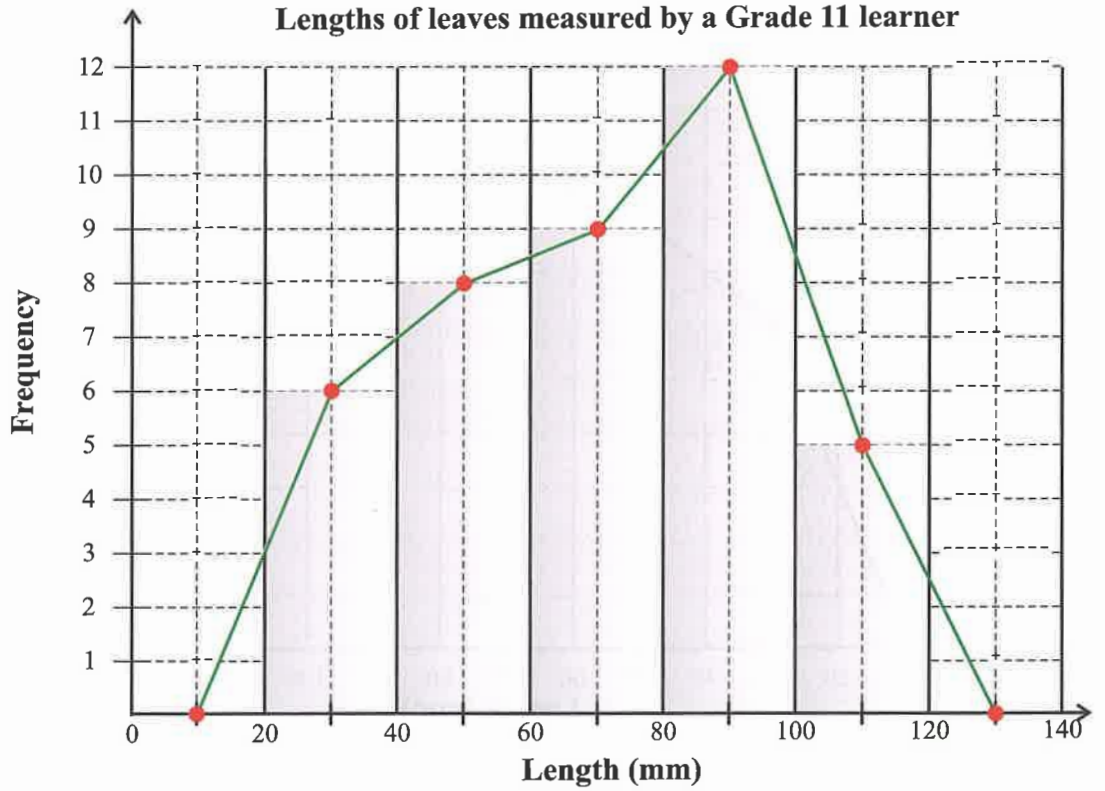
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Example: Use the histogram in example 1 on page 216 to draw a frequency polygon for the data set. The histogram is shown below.





To construct the frequency polygon below, join the midpoints of each interval. The midpoints of the 0 frequency intervals are joined in order to close the polygon. If the 0 frequency interval falls on the negative x-axis only draw the line up to the y-axis. Note that the area of a frequency polygon is always equal to the total area of the histogram.



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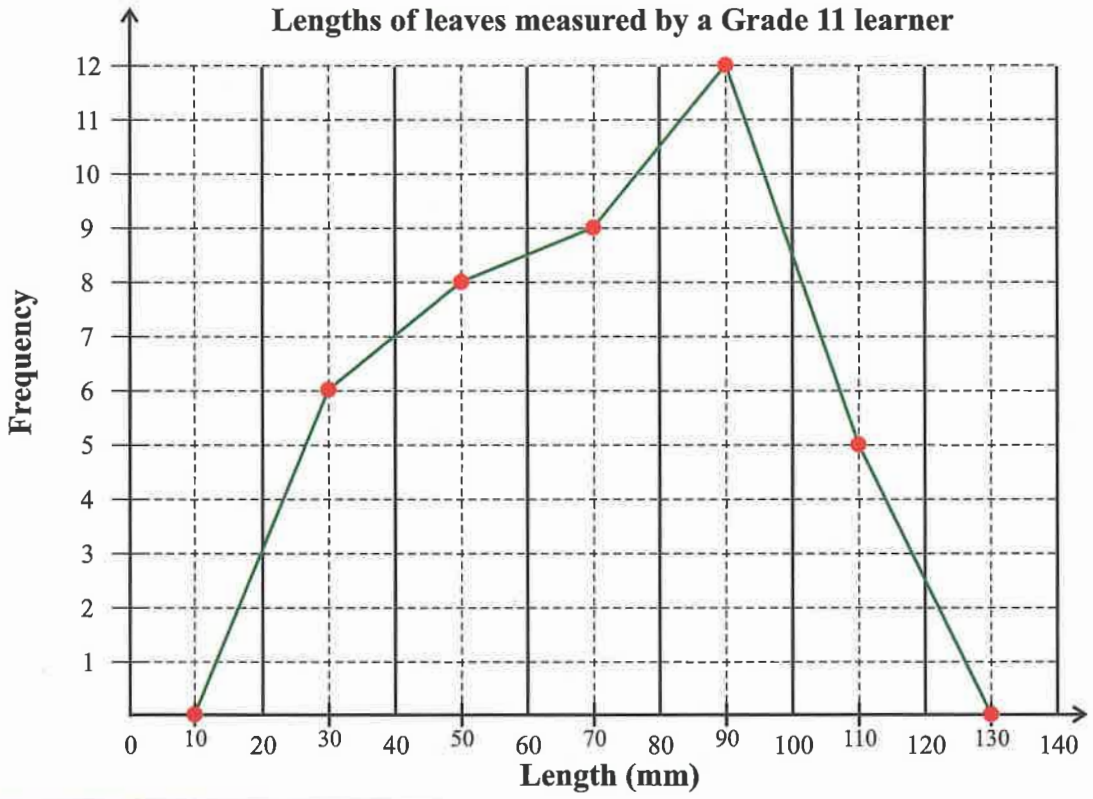
Steps to drawing a frequency polygon using the midpoints of a frequency table:

1. Determine the midpoint of each interval. This is also known as the class mark.
2. Plot the midpoints against the frequency of each interval and join the points with straight lines.
3. Once again the intervals with zero frequencies have to be added on either side of the bars in order to close the polygon.

Example: Use the Frequency table in example 1 on page 216 to draw a frequency polygon for the data set.

You have to add a column showing the midpoint of each class. It is also useful to show the zero frequency classes in the frequency table.

Length (L) in millimetres	Midpoint of interval	Frequency
$0 < L \leq 20$	$\frac{0+20}{2} = 10$	0
$20 < L \leq 40$	$\frac{20+40}{2} = 30$	6
$40 < L \leq 60$	$\frac{40+60}{2} = 50$	8
$60 < L \leq 80$	$\frac{60+80}{2} = 70$	9
$80 < L \leq 100$	$\frac{80+100}{2} = 90$	12
$100 < L \leq 120$	$\frac{100+120}{2} = 110$	5
$120 < L \leq 140$	$\frac{120+140}{2} = 130$	0



Complete Exercise 1 on page 384.

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C. OGIVES

An ogive is a graph which represents the cumulative frequency of a set of data. The graph usually has an *s*-shape and is a smooth curve not joined by straight lines. Points are plotted using the upper limit of each interval while the graph is connected to the *x*-axis at the lower limit of the first class. In other words the *x*-intercept is not necessarily always at the origin. Another name for an ogive is a cumulative frequency plot.

Example 1: The following table shows the amount of time (in minutes) children aged between 6 and 10 spend watching television (TV) each day.

Daily time (T) spent watching TV (minutes)	Number of children
$0 \leq T < 20$	**21**
$20 \leq T < 40$	**48**
$40 \leq T < 60$	**56**
$60 \leq T < 80$	**32**
$80 \leq T < 100$	**10**
$100 \leq T < 120$	**5**

a) Draw a histogram to represent the data given in the table above.

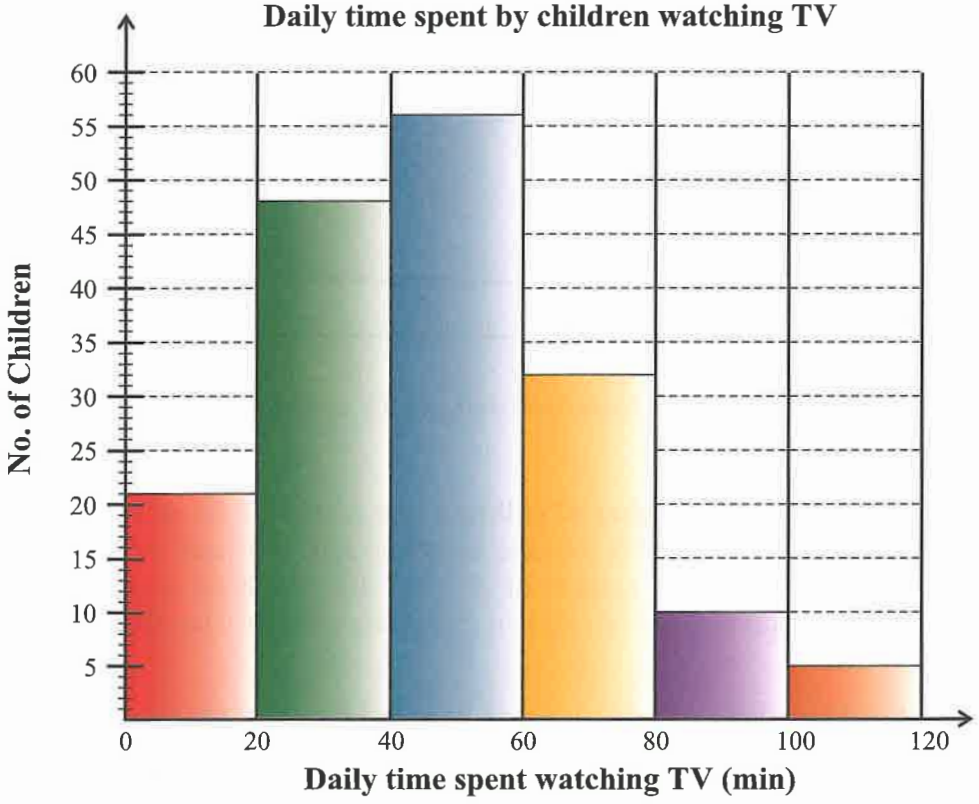
b) Draw an ogive to represent the above data.

c) Using the ogive, estimate the number of children that watch TV for an hour and half or less each day, as well as the number of children that watch TV for over an hour and a half each day.

d) Using your graph, determine the lower quartile, median and upper quartile.

e) Determine the estimated daily mean (average) amount of time that these children spend watching TV .

a)

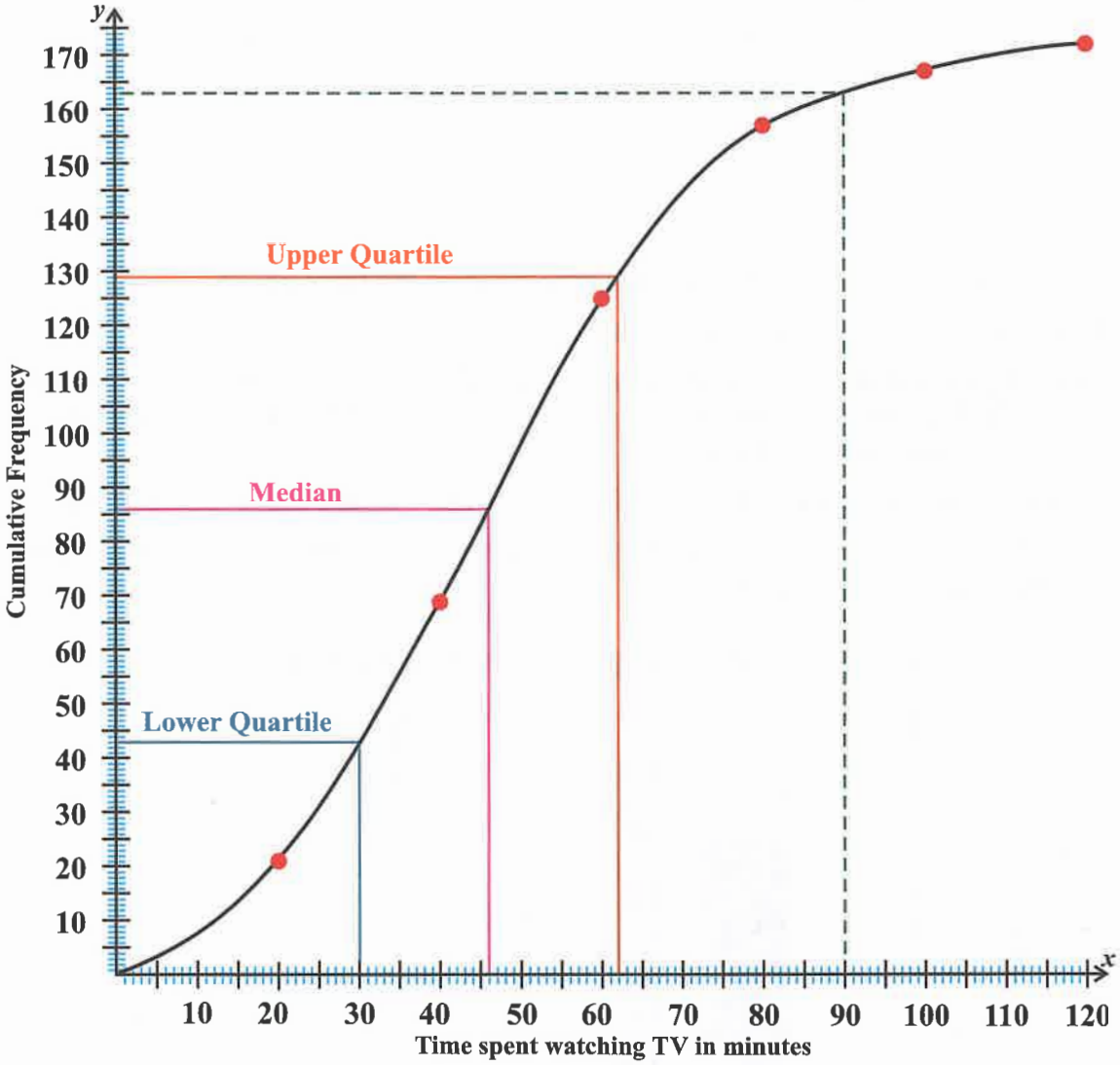


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b) To draw an ogive you have to add a cumulative frequency column to the table:

Time	Frequency	Cumulative Frequency
$0 \leq T < 20$	21	21
$20 \leq T < 40$	48	$69 \rightarrow 21+48$
$40 \leq T < 60$	56	$125 \rightarrow 69+56$
$60 \leq T < 80$	32	$157 \rightarrow 125+32$
$80 \leq T < 100$	10	$167 \rightarrow 157+10$
$100 \leq T < 120$	5	$172 \rightarrow 167+5$

Now plot the **cumulative frequency on the *y*-axis** and the **class intervals on the *x*-axis**. The *x*-intercept is at the lowest value of the smallest class (in this case 0) and you plot the **cumulative frequency against the upper limits** of the class intervals.



c) In order to estimate the **number of children** that watch TV for under an hour and a half, draw a dotted line at 90 minutes on the *x*-axis and read off the number of children on the *y*-axis. The number of children that watch for over an hour and a half will be the total number of children minus the number who watch for under an hour and a half.

$\therefore$ No. of children that watch for under an hour and a half ≈ 163

$\therefore$ No. of children that watch for over an hour and a half $\approx 172 - 163 = 9$

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d) The person who has the **lower quartile** time will be in the position which is a quarter of the cumulative frequency (total number of people). The **median position** will lie at half of the cumulative frequency and **upper quartile** will lie at three quarters of the cumulative frequency. Once these values have been found draw a line from the y-axis to the graph and then read the x-values at these points.
The x-values are the actual lower quartile, median, and upper quartile times.

$\therefore$ Calculate the y-value of the lower quartile, median and upper quartile

Lower quartile position: $172 \times \frac{1}{4} = 43$

$\therefore$ lower quartile $= 30$ (read off the graph)

Median position: $172 \times \frac{1}{2} = 86$

$\therefore$ median $= 46$ (read off the graph)

Upper quartile position: $172 \times \frac{3}{4} = 129$

$\therefore$ upper quartile $= 62$ (read off the graph)

e) In order to determine the **estimated mean,** calculate the midpoint of each interval and multiply it by the number of students in that interval (frequency). The sum of these values is then divided by the total of the cumulative frequency to obtain the estimated mean.

Time	Frequency (f)	Midpoint (x)	Frequency x midpoint ($f.x$)
$0 \leq T < 20$	21	10	210
$20 \leq T < 40$	48	30	1440
$40 \leq T < 60$	56	50	2800
$60 \leq T < 80$	32	70	2240
$80 \leq T < 100$	10	90	900
$100 \leq T < 120$	5	110	550
Sum	$\sum f = 172$		$\sum f.x = 8140$

$\therefore$ Estimated mean $= \frac{8140}{172} \rightarrow \frac{\sum f.x}{\sum f}$

$\therefore \bar{x} = 47$ minutes



Do not get confused between the MEAN and the MEDIAN. The mean is the average value of a data set, while the median is the middle value in an ordered set of data.



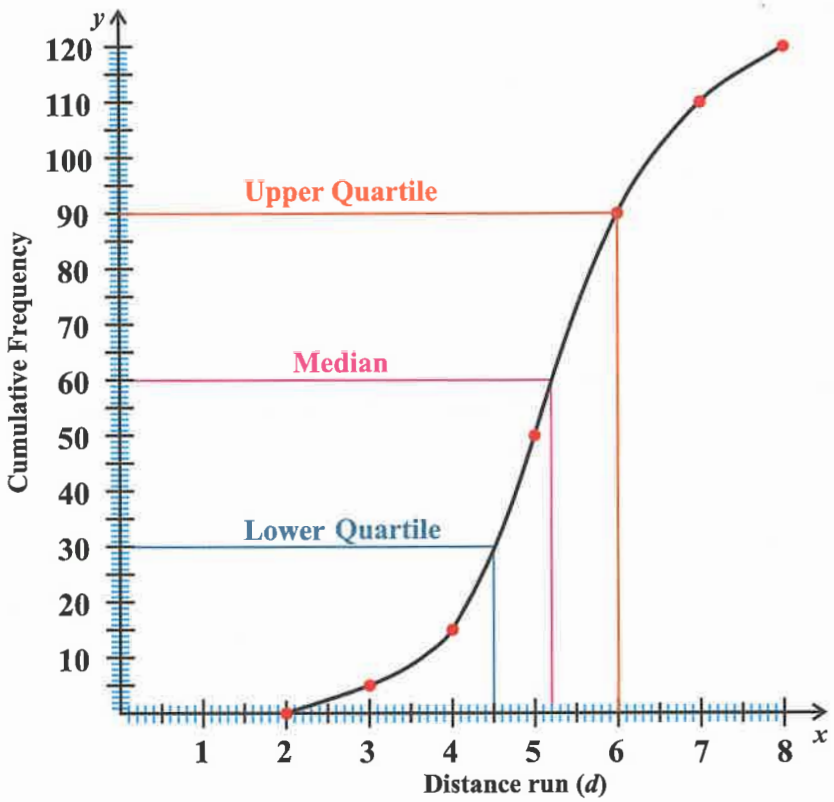
In order to determine the exact position of the median the formula shown below should be used. However, when working with ogives it is considered accurate enough to divide the cumulative frequency by 2.

$$\text{Position of median} = \frac{n+1}{2}$$

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Example 2: **The frequency table below shows the distances (d) in kilometres that a person training for a race runs in 6 months. Plot an ogive for the data and use it to determine the median and inter-quartile range.**

Distance (d)	Frequency	Cumulative Frequency
$2 \leq d < 3$	5	5
$3 \leq d < 4$	10	$15 \rightarrow 5+10$
$4 \leq d < 5$	35	$50 \rightarrow 15+35$
$5 \leq d < 6$	40	$90 \rightarrow 50+40$
$6 \leq d < 7$	20	$110 \rightarrow 90+20$
$7 \leq d < 8$	10	$120 \rightarrow 110+10$



Lower quartile position: $120 \times \frac{1}{4} = 30$

$\therefore$ lower quartile $= 4,5$ km (read off the graph)

Median position: $120 \times \frac{1}{2} = 60$

$\therefore$ median $= 5,2$ km (read off the graph)

Upper quartile position: $120 \times \frac{3}{4} = 90$

$\therefore$ upper quartile $= 6$ km (read off the graph)

You can now calculate the inter-quartile range:

Inter-quartile range $= Q_3 - Q_1 = 6 - 4,5 = 1,5$ km



, **Complete Exercise 2 on page 384.**

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D. VARIANCE AND STANDARD DEVIATION

Variance is a useful measure which is used to determine the standard deviation. The standard deviation is a representation of how much variation or dispersion there is around the mean. A low standard deviation shows that the data points lie close to the mean and the data set is generally consistent. A high standard deviation shows that the values are spread out over a large range and the data is not consistent.

You will be expected to calculate the variance and standard deviation manually as well as using a calculator.

Variance: $\sigma^2 = \frac{\sum(x-\bar{x})^2}{n}$ **Standard deviation:** $\sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$

Where: $x \rightarrow$ Data value given to you

$\bar{x} \rightarrow$ Mean

n $\rightarrow$ Sample size (number of data points in the sample)

Example 1: The table below shows the results of a Xhosa test of a Grade 11 class.

Mark obtained (%)	$(x-\bar{x})$	$(x-\bar{x})^2$
52		
44		
62		
66		
60		

a) Calculate the mean mark for the class.
b) Complete the table and use it to calculate the standard deviation of the marks.

a) Mean: $\bar{x} = \frac{52+44+62+66+60}{5} = 56{,}8$

b)

Mark obtained (%)	$(x-\bar{x})$	$(x-\bar{x})^2$
52	–4,8	23,04
44	–12,8	163,84
62	5,2	27,04
66	9,2	84,64
60	3,2	10,24
		308,8

$$\therefore \sigma = \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{308{,}8}{5}} = 7.86\,\%$$



The symbol $\sum$ means sum.

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Example 2: The table below shows the results of a Grade 11 class's Maths test.

52	44	62	66	60
57	95	78	71	62
100	69	62	72	73
55	32	83	78	80

a) Determine the mean mark for the class and the standard deviation.
b) Determine the percentage of learners whose marks lie within one standard deviation of the mean.

If you are not given a table to fill out as in the previous example use your calculator to determine the mean and standard deviation.

a) The table below shows how to calculate the standard deviation on a Casio *fx* 82ES or 82ZA. If you are using a different model follow the same table but press the buttons that correspond to the word next to the number shown.

	Button to press/method	Calculator display
1	**Mode**	**A list of various modes**
2	**2 (STAT)**	**A list of options in STAT mode**
3	**1 (1–VAR)**	**A table to input values**
4	Input each data value one at a time, pressing the = button after each entry.	
5	Once all the data is entered press **AC**	**0**
6	**Shift 1 (STAT)**	**A list of options to choose**
7	**4 (Var)**	1: n 2: $\bar{x}$ 3: σx 4: sx
8	Pressing **1** will display the number of data entries. Pressing **2** will display the mean and pressing **3** will display the standard deviation. (option 4 will not be used)	
9	**2 ($\bar{x}$) and then =**	**67,55**
10	To determine the standard deviation repeat steps 6,7 and 8 pressing **3** instead of **2**	
11	**3 (σx) and then =**	**15,75**

$\therefore$ The mean is 67,55 and the standard deviation is 15,75

b) One standard deviation lies between $\bar{x}-15,75$ and $\bar{x}+15,75$.

$\therefore$ one standard deviation lies between 51,8 and 83,3. By counting the number of marks that lie within this interval it can be seen that there are 16 marks within one standard deviation of the mean.

$\therefore \frac{16}{20}$ learners lie within one standard deviation of the mean.

$\therefore$ 80% of learners lie within one standard deviation.

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STANDARD DEVIATION WITH GROUPED DATA

In Grade 11 you will also be required to determine the standard deviation of grouped data. In most cases this will be performed using your calculator. In order to determine the standard deviation and mean of grouped data you have to add a frequency column to your calculator.

To add a frequency column follow the steps below:

1. Press **setup** (press **shift** and then **mode**)
2. Using the middle arrow keys scroll down to and press the number corresponding to **STAT**.
3. Press **1** (on).

You can now follow the same steps on the previous page except you will now have a frequency column in which you can enter the frequency of each data point.
You can leave your frequency column on for all your calculations.

Example: For the grouped data shown below, determine the mean and standard deviation.

Data point	Frequency
1	8
2	22
3	5
4	8
5	15
6	25
7	10

Using your calculator follow the steps on the previous page, entering the frequency of each point into the frequency column.
The mean of the above data set is: 4,24
The standard deviation of the above data set is: 1,98

Important notes on mean and standard deviation:

- **Adding or subtracting the same value to or from each point will not affect or change the value of the standard deviation of a data set.**
- **Adding or subtracting the same value to or from each point will increase or decrease the value of the mean by the amount added or subtracted.**

Complete Exercise 3 on page 385.

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E. SYMMETRICAL AND SKEWED DATA

It is important for you to be able to identify symmetrical and skewed data. If there are items with very large or very small values compared to the rest of the data set, they will cause the data set to be skewed.

There are three general rules that can be used to test if a data set is skewed:

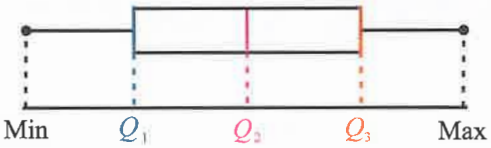
- If the **mean is the same as the median** the data is **normally distributed**.
- If the **mean is larger than the median** the data is **skewed to the right**.
- If the **mean is smaller than the median** the data is **skewed to the left**.

If the data is normally distributed it means that the data is symmetrical.

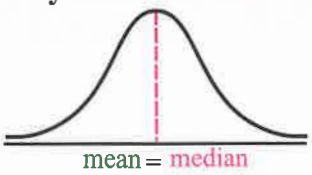
The diagrams below show symmetrical and skewed data. If the data has a tail from the median to the endpoint it is said to be skewed in the direction of the tail:

Symmetrical Data (normally distributed):

Symmetrical Data

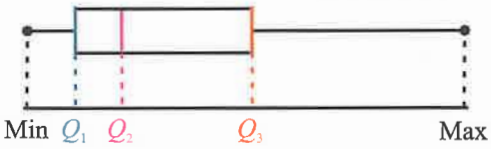


Symmetrical Data

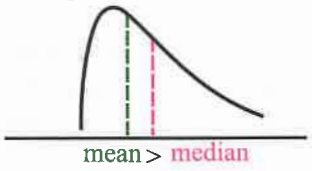


Right skew (positively skewed):

Right Skew (positively skewed)

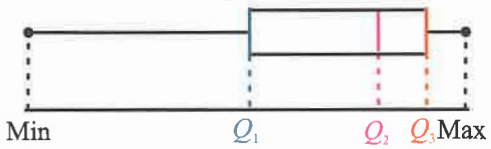


Right skewed

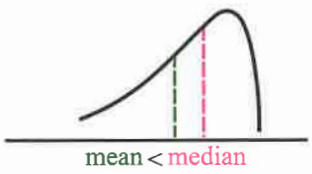


Left skew (negatively skewed):

Left Skew (negatively skewed)



Left skewed



If a data set is normally distributed:

- **Approximately 68% of the data falls within 1 standard deviation of the mean.**
- **Approximately 95% of the data falls within 2 standard deviations of the mean.**

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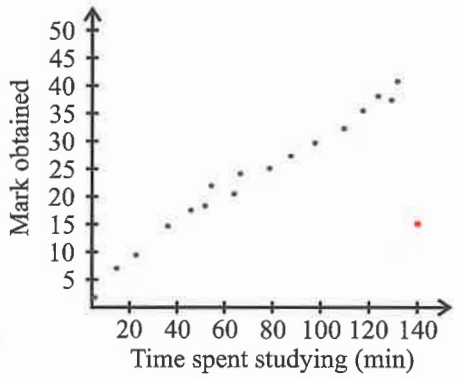
IDENTIFICATION OF OUTLIERS

An outlier is a data value which is much larger or smaller than the rest of the values in the data set. This value will not follow the general trend of the data.

A value is defined to be an outlier if it fulfils one of the conditions below:

- **The value is less than $Q_1 - (1{,}5 \times \text{IQR})$**
- **The value is greater than $Q_3 + (1{,}5 \times \text{IQR})$**

Outliers can also be seen from scatter plots of bivariate data (data which involves two variables). In a scatter plot a general trend can be seen from the plotted points. Points which do not follow the general trend are most likely outliers in the data set. The scatter plot below shows the amount of time that a Grade 11 class spent studying for a test and the marks they obtained out of 50.



In the scatter plot it can be seen that the marks follow a linear pattern except for the point $(140;15)$. This point clearly lies outside the trend of the general data and is an outlier. In Grade 12 you will deal with scatter plots in more detail. In Grade 11 you have to be able to identify outliers from scatter plots.

Example 1: The table below shows the Prime Interest rate in South Africa from 2006 up until 2012.

Year	2006	2007	2008	2009	2010	2011	2012
Rate	11	13	15	14	10	9	8,5

a) Determine the five number summary for the data in the table.
b) Draw a box and whisker plot for the data.
c) Comment on whether or not the data is skewed.
d) Are there outliers in the data?

a) You always have to write the data in order when dealing with a 5 number summary.

8,5	9	10	11	13	14	15

Minimum Value: 8,5

Median: Position $= \frac{n+1}{2} = \frac{7+1}{2} = 4 \therefore 11$ is the median

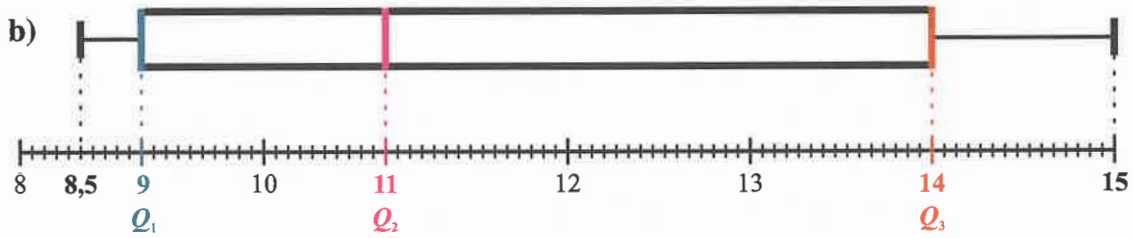
Lower Quartile: Position $= \frac{n+1}{2} = \frac{3+1}{2} = 2 \therefore 9$ is the lower quartile

Upper Quartile: Position = median positions $+ 2 = 6 \therefore 14$ is the upper quartile

Maximum Value: 15

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b)



c) It can be seen from the box and whisker plot that the data is skewed to the right.

d) $\text{IQR} = 14 - 9 = 5$

$Q_1 - (1{,}5 \times \text{IQR}) = 9 - (1{,}5 \times 5) = 1{,}5$ and $Q_3 + (1{,}5 \times \text{IQR}) = 14 + (1{,}5 \times 5) = 21{,}5$

There are no values less than 1,5 or greater than 21,5. Therefore, there are no outliers in the data set. Although the data is slightly skewed to the right there are no values that are high enough or low enough to be considered outliers.

Example 2: The table below shows the percentage distribution of the South African population by province in 2010.

EC	FS	Gauteng	KZN	Limpopo	Mpumalanga	NC	NW	WC
14	6	22	21	11	7	2	6	11

a) Determine the five number summary for the data in the table.
b) Draw a box and whisker plot for the data.
c) Are there any outliers in the data?

a) You always have to write the data in order when dealing with a 5 number summary.

2	6	6	7	11	11	14	21	22

Minimum Value: 2

Median: Position $= \frac{n+1}{2} = \frac{9+1}{2} = 5 \therefore 11$ is the median

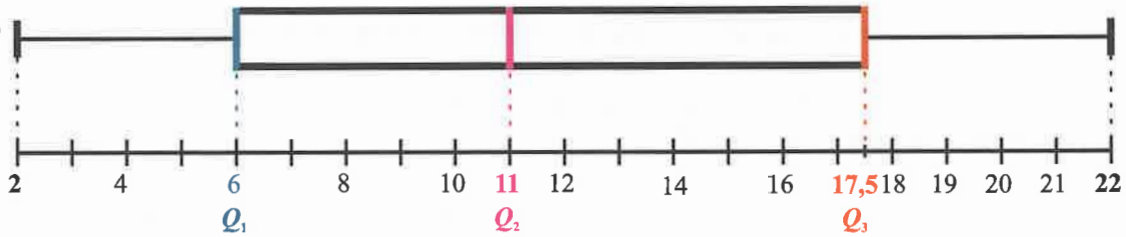
Lower Quartile: Position $= \frac{n+1}{2} = \frac{4+1}{2} = 2{,}5 \therefore \frac{6+6}{2} = 6$ is the lower quartile

Upper Quartile: Position = median position $+ 2{,}5 = 5 + 2{,}5 = 7{,}5$

$\therefore \frac{14+21}{2} = 17{,}5$ is the upper quartile

Maximum Value: 22

b)



c) $\text{IQR} = 17{,}5 - 6 = 11{,}5$

$Q_1 - (1{,}5 \times \text{IQR}) = 6 - (1{,}5 \times 11{,}5) = -11{,}25$

$Q_3 + (1{,}5 \times \text{IQR}) = 17{,}5 + (1{,}5 \times 11{,}5) = 34{,}75$

There are no values less than –11,5 or greater than 34,75. Therefore, there are no outliers in the data set.



Complete Exercise 4 on page 385 and 386 the Mixed Exercise on pages 386 and 387.

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CHAPTER 13 – CALCULATOR TIPS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
231 232 233 234	**A. Number Patterns** **B. Sketching Graphs** **C. Trigonometric Equations** **D. Determining Roots of an Equation**		• Use your calculator effectively to check solutions and save time in tests and exams.

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A. NUMBER PATTERNS

In grade 12 you will learn that your calculator can determine a line of best fit for bivariate data. For now it is sufficient for you to understand that if you enter points on a line for a linear, exponential or quadratic function, your calculator can determine the equation of the function that best fits the trend of the points.

Example 1: Determine the nth term of the linear sequence 5 ; 8 ; 11 ;...

As the sequence is linear, the equation has the form $T_n = bn + a \rightarrow y = mx + c$

This is the form of a straight line graph, where the term is a y-value and the term number (n) is the x-value.

In order to obtain the equation press:

Mode → STAT → A + BX.

In this form **X** is the same as ***n*** in the equation and **A** is the same as ***a*** and **B** is the same as ***b***.

Now enter the points of the sequence into the calculator. Remember that when $n = 1$; $T_n = 8 \rightarrow x = 1$; $y = 8$

$x \rightarrow n$	$y \rightarrow T_n$
1	5
2	8
3	11

Now press: **AC → Shift → STAT → REG → B →=**

$\therefore$ **B = 3**

Now press: **Shift → STAT → REG → A →=**

$\therefore$ **A = 2**

$\therefore T_n = 3n + 2$

Example 2: Determine the nth term of the quadratic sequence 6 ; 13 ; 24 ;...

As the sequence is quadratic $T_n = an^2 + bn + c$

In order to obtain the equation press:

Mode → STAT → _+cX^2.

In the calculator the equation is in the form $y = cx^2 + bx + a$. This means that the answer for ***a*** in the calculator is the answer for ***c*** in the equation $T_n = an^2 + bn + c$ and the answer for ***c*** in the calculator is the answer for ***a*** in the above formula

$x \rightarrow n$	$y \rightarrow T_n$
1	6
2	13
3	24

Now press: **AC → Shift → STAT → REG → C →=**

$\therefore a = 2 \rightarrow$ this is a in the equation $T_n = an^2 + bn + c$

Now press: **Shift → STAT → REG → B →=**

$\therefore b = 1$

Now press: **Shift → STAT → REG → A →=**

$\therefore c = 3 \rightarrow$ this is c in the equation $T_n = an^2 + bn + c$

$\therefore T_n = 2n^2 + 1n + 3$

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B. SKETCHING GRAPHS

It is possible to sketch any graph using the table method. Your calculator has a table mode which can draw up a table of points in order to sketch a graph. This is especially useful when drawing trigonometric functions.

Example 1: Use a calculator to draw a table of points that can be used to sketch the graph of $y = \sin(x + 30^\circ)$ where $x \in [-30^\circ; 330^\circ]$

In order to obtain a table press: **Mode → TABLE.**
Now type in the equation. You can obtain an x by pressing: **ALPHA → x.**

- The calculator will now ask you where on the x-axis to start. Choose -30°.
- The calculator will now ask you where on the x-axis to end. Choose 330°.
- You now have to choose the steps to increase in. Provided that there is no change in period choose steps of 90° for graphs of $\sin x$ and $\cos x$. For graphs of $\tan x$ choose steps of 45°.

From the calculator:

x	y	x	y
-30°	0	240°	-1
60°	1	330°	0
150°	0		

Example 2: Use a calculator to draw a table of points that can be used to sketch the graph of $y = \sin(2x)$ where $x \in [-360^\circ; 225^\circ]$

In order to obtain a table press: **Mode → TABLE.**
Now type in the equation. You can obtain an x by pressing: **ALPHA → x.**

- The calculator will now ask you where on the x-axis to start. Choose -360°.
- The calculator will now ask you where on the x-axis to end. Choose 225°.
- You now have to choose the steps to increase in. In this case $k = 2$, this means that the period of the graph is $\frac{360^\circ}{2} = 180^\circ$. You would usually increase in steps of 90°.

  As $k = 2$, increase in steps of $\frac{90^\circ}{k} = \frac{90^\circ}{2} = 45^\circ$.

From the calculator:

x	y	x	y
-360°	0	-45°	-1
-315°	1	0°	0
-270°	0	45°	1
-225°	-1	90°	0
-180°	0	135°	-1
-135°	1	180°	0
-90°	0	225°	1



If there is a change in period, increase in steps of $\frac{90^\circ}{k}$ for the graphs of $\sin x$ and $\cos x$. For the graph of $\tan x$ increase in steps of $\frac{45^\circ}{k}$.

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C. TRIGONOMETRIC EQUATIONS

When solving trigonometric equations in a specified domain it was seen that you had to substitute values for k in order to solve for the specific solution. Your calculator can draw up a table for various values of k.

Example : Solve for θ if $\sin\theta = \sin 40^\circ$ and $\theta \in [-360^\circ; 720^\circ]$

$\sin\theta = \sin 40^\circ$

$RA = 40^\circ$

$\therefore \theta = 40^\circ + 360^\circ k$ or $\theta = 180^\circ - 40^\circ + 360^\circ k\ ;\ k \in \mathbb{Z}$

$= 140^\circ + 360^\circ k$

You can now use your calculator to draw up a table of the values of θ for different values of k as shown below:
In order to obtain a table press: **Mode → TABLE.**

Now type in the equation $f(x) = 40^\circ + 360^\circ x$ → use an x instead of k.

- The calculator will now ask you where to start. A value of –3 is usually sufficient. However, if the value of θ lies within the specified domain when $k = \pm 3$ you have to test $k = \pm 4$.
- The calculator will now ask you where to end. A value of 3 is usually sufficient.
- You now have to choose the steps to increase in. Always choose 1.

Now repeat the process with the equation: $f(x) = 140^\circ + 360^\circ x$

	$\theta = 40^\circ + 360^\circ k$		$\theta = 140^\circ + 360^\circ k$	
x	$f(x)$		$f(x)$	
–3	–1040°	×	–940°	×
–2	–680°	×	–580°	×
–1	–320°	✓	–220°	✓
0	40°	✓	140°	✓
1	400°	✓	500°	✓
2	760°	×	860°	×
3	1120°	×	1220°	×

Now select the values of θ which lie in the given interval of $[-360^\circ; 720^\circ]$. Remember that in the calculator the value of θ is $f(x)$.

$\therefore \theta = -320^\circ$ or $\theta = 40^\circ$ or $\theta = 400^\circ$ or $\theta = -220^\circ$ or $\theta = 140^\circ$ or $\theta = 500^\circ$

this is also written as $\theta \in \{-320^\circ; -220^\circ; 40^\circ; 140^\circ; 400^\circ; 500^\circ\}$

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D. DETERMINING ROOTS OF AN EQUATION

Your calculator can also determine the roots of a quadratic equation. This functionality is only available on the Casio 991*fx*-ESPlus.

Example 1: Solve for x if $2x^2+7x+3=0$.

Press: **Mode → EQN → $aX^2+bX+c=0$.**

Now enter the values of a, b and c pressing the = button after each entry. Once all coefficients have been entered press the = button to obtain the first root and then press the = button again to obtain the second root.

$\therefore x=-\frac{1}{2}$ or $x=-3$

Note that you cannot simply write down the answer. You must work backwards from the answer as you did on page 41 to determine the factors showing all steps. This is shown below:

$2x^2+7x+3=0$

$\therefore (2x+1)(x+3)=0$

$\therefore x=-\frac{1}{2}$ or $x=-3$

Example 2: Solve for x if $2x^2-2x-4=0$.

If you can, always take out a HCF before using a calculator.

$\therefore 2\left(1x^2-1x-2\right)=0$

$\therefore 1x^2-1x-2=0$

Now Press: **Mode → EQN → $aX^2+bX+c=0$.**

Now enter the values of a, b and c pressing the = button after each entry. Once all coefficients have been entered press the = button to obtain the first root and then press the = button again to obtain the second root.

$\therefore x=-1$ or $x=2$

Now write down the full solution:

$\therefore 2\left(1x^2-1x-2\right)=0$

$\therefore 1x^2-1x-2=0$

$\therefore (x+1)(x-2)=0$

$\therefore x=-1$ or $x=2$



The functions of your calculator are not limited to what has been demonstrated in this section. Your calculator manual will show you all the functions that your calculator is capable of. Read through the manual and make sure you are familiar with your calculator.

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CHAPTER 14
MAKING THE MOST OF MATHS

WHAT YOU WILL LEARN IN THIS CHAPTER

PAGE	CONTENT	CAPS	ASSESSMENTS YOU WILL BE ABLE TO:
236 236 236 237	**A. Maths in the Classroom** **B. Exam Study Techniques** **C. The Morning of the Exam** **D. Exam Writing Techniques**		• Get the most out of mathematics in your classroom. • Study with confidence and use your time effectively. • Relax and walk into your exam with confidence. • Write your exam neatly and use your time efficiently.

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A. MATHS IN THE CLASSROOM

- Up to **80% of learning** should be achieved in the **classroom**. Try to sit as close as possible to the front of the classroom where you can see and hear properly.

- **Concentrate** and avoid the temptation to talk to your friends. Missing an explanation could result in a misunderstanding of an entire section.

- **Do not be shy!** If you don't understand an explanation that your teacher has given, put your hand up and say so. **Ask** your teacher to explain it again until you do understand. If you did not understand a concept, it is likely that the majority of your class is in the same position.

B. EXAM STUDY TECHNIQUES

- **Practising past exam papers** is the most helpful and practical form of revision. Past papers expose you to the required level of work, exam format and the necessary time limitations imposed on you.

- **Make sure you start studying early.** You should start with past papers at least two weeks before the exam. Make sure that as you do each question you check your answer. If you check your answers after you have completed the whole paper, you cannot remember what each question was about. This means that you do not learn from your mistakes. It is important that you revise each section before starting past papers.

- **In order to achieve the correct pace**, while timing yourself, do an entire past paper and only check your answers at the end of the paper. This should be done two to three days before your exam.

- Ensure the room that you are studying in is **brightly lit** and **properly ventilated**.

- Cellular phones can be your worst enemy whilst studying. **Turn your phone off** and check your messages only when you have finished studying.

C. THE MORNING OF THE EXAM

- Make sure that **everything you need for your exam is packed the night before** and ready to take in the morning. Ensure your **calculator is in the correct mode** with charged batteries. It is a good idea to take a spare calculator into your exam.

- **Sleep is very important**. Avoid going to bed too late or waking up very early to study. A maths test or exam requires a fresh brain.

- **Eat Breakfast!** Hunger affects concentration.

- **Avoid discussing work** before the exam. This is likely to cause confusion.

- You've worked hard and are **prepared** - now relax and enjoy the exam.

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D. EXAM WRITING TECHNIQUES

- **Read through the entire question paper before you start**. You can answer the questions in any order you like. Answer the questions that you are most confident in first. However, you must answer an entire question in the same section, don't answer parts of a question in different places in your answer book. If you are unsure of a certain part of the question, leave it out and come back to it.

- **If you leave out part of a question, make sure you leave enough space to answer it**. Always leave at least half a page open if the question is worth more than two marks. If you are not sure how much space will be needed, be safe and leave a full page.

- **Mark questions** that you have **left out** on the question paper, so that you remember to come back to them.

- **Neatness is crucial**. Keeping your solutions **neat** and **legible** during an exam may result in up to a 5% increase in your results compared to an untidy and illegible answer book. Neat workings also assist you, as it is easier to check answers and find mistakes. In addition, neatness helps to avoid silly mistakes. Do not start a new question at the end of a page.

- **Do not** attempt to **save space** in your exams! Cramped work is messy and difficult to mark. Leave at least one line between answers. Similarly, do not write complicated fractions on a single line.

- **Never cross out an answer, unless** you have already written another answer to replace it. Once you have what you feel is the correct answer, only then cross out the incorrect workings. If you leave two solutions without crossing one out only the first solution will be marked.

- **Do not use** any type of **correction fluid** (eg. tippex) in a test or exam.
  Rather **neatly cross out your answer** and carry on. Erasable pens should not be be used.

- **Follow the layout** shown in this handbook. Keep to the top to bottom layout and the exam will be a pleasure to write and mark.

- An answer book that has the final answer of each question **underlined with an arrow** is very pleasant for a teacher to mark. However, this can be very costly in terms of time. **If you are a slow or untidy worker, use a ruler** and neatly underline your answers with arrows, **only once you are finished** your test or exam and have **checked** through **your solutions**. Make sure that you don't cross out your answer when doing this.

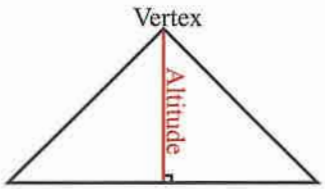
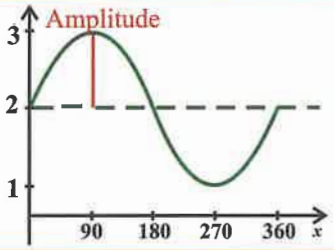
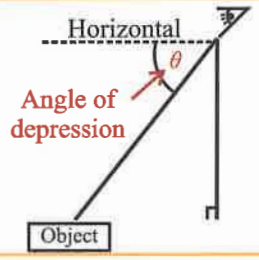
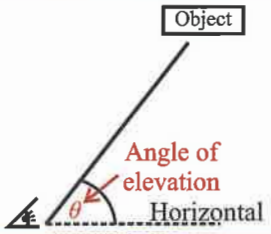
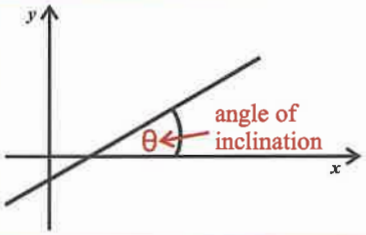
- Finally, **relax** and be **positive**. It is important that you **don't panic** at any stage during the exam.



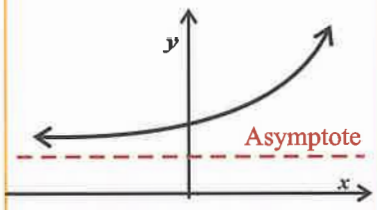
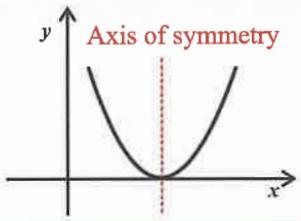
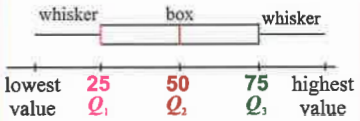
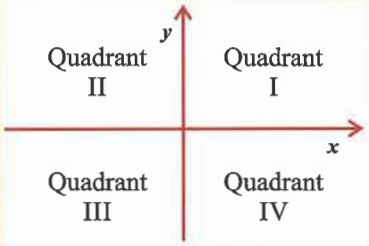
Good luck – You have earned it!

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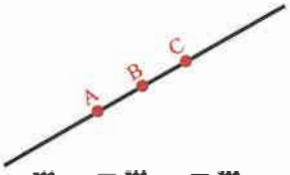
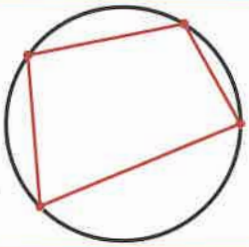
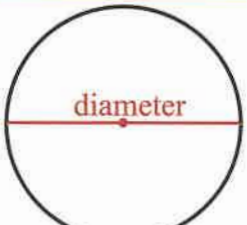
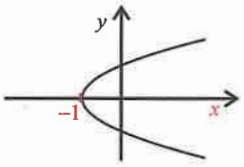
CHAPTER 15
GLOSSARY OF IMPORTANT TERMS AND SYMBOLS

Term	Explanation	Example (where possible)
altitude	A perpendicular line drawn from one side of a figure to the vertex of the other side.	
amplitude	The length of maximum deviation from the zero line. $\text{Amplitude} = \frac{y_{max} - y_{min}}{2}$	
angle of depression	The angle between the horizontal and the line of sight when looking down at an object.	
angle of elevation	The angle between the horizontal and the line of sight when looking up at an object.	
angle of inclination	The angle measured anticlockwise from the positive side of the x axis. It is always between 0° and 180°.	
arithmetic sequence	A sequence where the difference between any two sequential terms is constant.	In the sequence: 2, 4, 6... The constant difference d is $T_n - T_{n-1} = 2$.

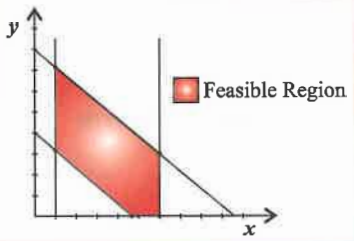
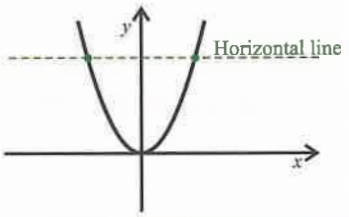
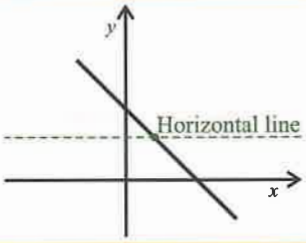
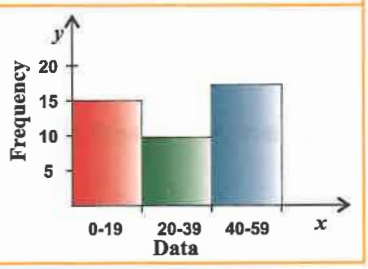
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Term	Explanation	Example (where possible)
asymptote	An imaginary line that a graph can never touch.	
axis of symmetry	A line drawn so that the graphs or parts of the graph on either side of the line are mirror images of each other.	
bivariate data	Data that is given in ordered pairs, an x and y coordinate. It is usually represented using a scatter plot.	The time a student spends studying and the mark that he or she obtains is an example of bivariate data.
box and whisker plots	A visual representation of the median (Q_2), lower quartile(Q_1), upper quartile (Q_3), as well as the highest and lowest values in a set of data, to show the spread around the median.	
Cartesian Plane	Four quadrants that are separated by an x and y axis	
circumference	The physical distance around a circle. Formula: $C = 2\pi r$ or $C = \pi d$	
coefficient	The number in front of a variable.	In the expression $2x^2 + 4$ the coefficient of x^2 is 2.

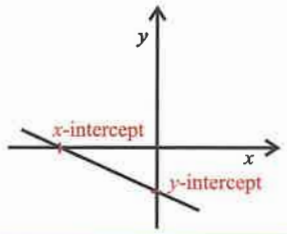
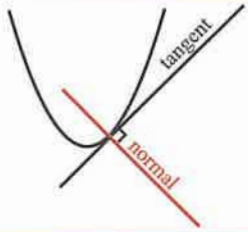
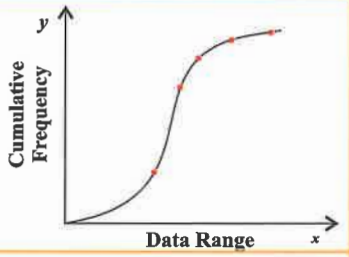
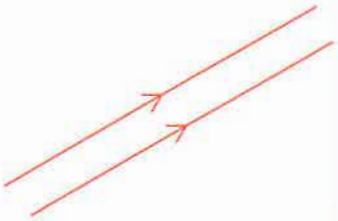
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Term	Explanation	Example (where possible)
collinear points	When 3 points lie on the same line they are collinear.	 $m_{AB} = m_{BC} = m_{AC}$
constant term	The term in an expression that does not contain any variables.	In the expression $2x^2 + bx + 4$ the constant term is 4.
cyclic quadrilateral	The vertices of a cyclic quadrilateral lie on the circumference of the same circle	
denominator	The number at the bottom of a fraction.	In the fraction $\frac{3}{x}$, x is the denominator.
diameter	A line through the centre of the circle that touches the circumference of the circle at two points.	
domain	The domain of a function is the possible x values that the function can have.	 In the above graph the domain is $x \geq -1$
effective interest rate	The equivalent annual interest rate that would be received due to the effects of compounding interest. $1 + i_{\text{eff}} = (1 + i_{\text{nom}})^n$	If the interest rate received is 10% p.a. compounded monthly, the effective rate is: $i_{\text{eff}} = \left(1 + {}^{0.1}/_{12}\right)^{12} - 1$

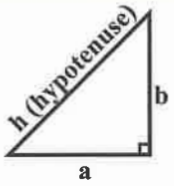
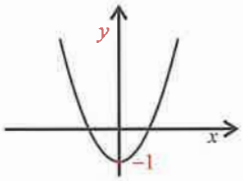
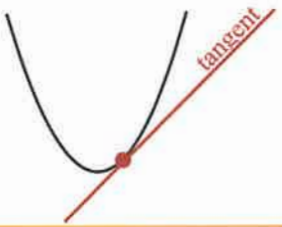
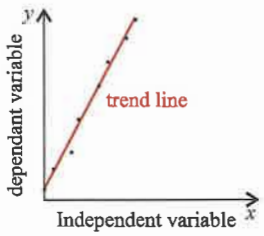
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Term	Explanation	Example (where possible)
exponent (and base)	The power that a number or variable is raised to.	In the term x^a, a is the exponent and x is the base.
feasible region	The area within a linear programming graph where all the constraints are satisfied.	
function (many-to-one)	An equation or graph where there are multiple x values for a single y value.	
function (one-to-one)	An equation or graph where there is a single y value for a single x value.	
geometric sequence	A sequence where the ratio of any two sequential terms is constant.	In the sequence: 4, 8, 16, ... the constant ratio is 2
gradient (m)	The rate of change of y with respect x. $m = \frac{y_2 - y_1}{x_2 - x_1}$	The average gradient between the points (5;2) and (10;5) is: $m = \frac{5-2}{10-5} = \frac{3}{5}$
histogram	A graph without gaps that uses rectangles to represent frequency.	

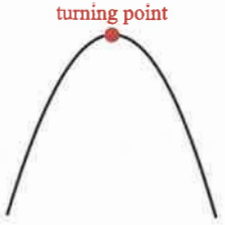
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Term	Explanation	Example (where possible)
intercept (x or y)	The point where a line cuts the x-axis (x-intercept) and y-axis (y-intercept).	
normal	A line that is perpendicular to a tangent at the point of contact.	
numerator	The number or expression at the top of a fraction.	$\frac{x+5}{2}$ $x+5$ is the numerator and 2 is the denominator
ogive	A graph where the data range (x) is plotted against the cumulative frequency (y).	
outlier (of a data set)	A value in a set of data that is much higher or lower than any of the other values in the data set.	2, 100, 125, 126, 130 In the above set of data points 2, is an outlier.
parallel lines (//)	A value in a set of data that is much higher or lower than any of the other values in the data set.	
period	The number of degrees it takes for a trigonometric function to complete one cycle.	Graphs of sine and cosine have a period of 360° whilst tan has a period of 180°

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Term	Explanation	Example (where possible)
Pythagoras (Theorem of Pythagoras)	In a 90° triangle the (hypotenuse)^2 = sum of the square of the opposite two sides.	 $h^2 = a^2 + b^2$
range	The possible y-values that a graph or function can have.	 In the above graph the range is $y \geq -1$
reciprocal	The reciprocal of a number is one divided by the number. A number multiplied by its reciprocal is 1.	The reciprocal of x is $\frac{1}{x}$
root (of an equation)	The value of a variable that satisfies an equation.	The root of $x-5=0$ is 5
surd form	An irrational number that is written with a root symbol (usually in its simplest form).	$\sqrt{50}$ written in simplest surd form is $5\sqrt{2}$
tangent	A straight line that touches a graph at a single point. The gradient of the tangent is equal to the gradient of the graph at the point of contact.	
trend line (line of best fit) (regression line)	A line that shows the general shape of a scatter plot, thus showing the overall trend of the data.	

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Term	Explanation	Example (where possible)
turning point	A point on a curve where the gradient at the point is equal to zero. At the TP the derivative of the function equals zero.	
quadratic formula	Used to solve for the roots of any quadratic equation.	The roots of $ax^2+bx+c=0$ are: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

GLOSSARY OF IMPORTANT SYMBOLS

Symbol	Meaning	Symbol	Meaning
$\sqrt{}$	square root	$\in$	element (of a set)
$\sqrt[n]{}$	n^{th} root	$\neq$	not equal to
$\mathbb{N}$	natural numbers	$\approx$	approximately equal to
$\mathbb{N}_0$	whole numbers	$A > B$	A is greater than B
$\mathbb{Z}$	integers	$A < B$	A is less than B
$\mathbb{Q}$	rational numbers	$A \geq B$	A is greater than or equal to B
$\mathbb{Q}'$	irrational numbers	$A \leq B$	A is less than or equal to B
$\mathbb{R}$	real numbers	$\therefore$	therefore
$\cap$	intersection (and)	$\cup$	Union (or)

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MATHS HANDBOOK and STUDY GUIDE™

EXERCISES & SOLUTIONS



The exercises contained in this section are designed for reinforcement of the material covered in each chapter of the book. At the end of the exercises for each relevant chapter, learners will find a mixed exercise. The mixed exercises are designed to test all of the work in the relevant chapters and are of the same standard that learners would expect to find in a test or exam. Learners are strongly encouraged to make sure that they fully understand all questions within the mixed exercises.

The exercises in this section contain worked through solutions. These are merely suggested solutions and are not necessarily the only methods that can be used to arrive at a correct answer. If there is any doubt in the method which learners have used, they are encouraged to check their methodology with their teachers.

Worked through solutions are provided in order for learners to check their answers and understand errors which they have made. Very little will be gained from viewing solutions before an attempt has been made by the learner to solve the problems themselves. Learners should only look at solutions after a true attempt of the problem has been made.

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Chapter 1– The Number System, Exponents and Surds

The Number System

Exercise 1:

1) Complete the table below. The first row has been completed for you.

Number	$\mathbb{N}$	$\mathbb{N}_0$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}'$	Non Real
5	✓	✓	✓	✓	×	×
0						
-7						
$(-3)^2$						
$\sqrt{-4}$						
π						
$\frac{-5+2\sqrt{3}}{6}$						
$\frac{-10}{-2}$						
$\sqrt[3]{-8}$						

2) State whether the following numbers are real or non-real.

a) $\frac{4-\sqrt{(4)(2)-12}}{2}$

b) $\left(\sqrt{3}-\sqrt{-16}\right)^2$

c) $\sqrt{3}\times\sqrt{2}$

d) $\left(2\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-\sqrt{-2}\right)$

Grade 10 Revision

Exercise 2:

1) Simplify the following without a calculator. Leave your answers with positive exponents:

a) $\left(\frac{16}{x^{-4}}\right)^{\frac{3}{4}}$

b) $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

c) $\left(\frac{25}{4}\right)^{-\frac{3}{2}}$

d) $(0{,}008)^{-\frac{2}{3}}$

2) Simplify the following without a calculator. Leave your answers with positive exponents:

a) $\frac{63^{1-n}.7^{n-1}.81^{-1}}{4^{n+2}.36^{-n-1}}$

b) $\frac{5^n.10^{2n-1}.2^{4n+1}}{20^{3n}}$

c) $\frac{10^{x+2}.8^x.25^{x-1}}{16^{x+1}.2.125^{x+1}}$

d) $\frac{9^{2a+1}.6^{2a-3}}{2.3^{5a-2}.3^a.4^{a-2}}$

e) $\frac{\sqrt{3}.\sqrt[4]{12}.\sqrt{8}}{\sqrt[4]{27}}$

f) $\frac{\sqrt[3]{ab}.\sqrt[3]{a^4b^2}.a^{-1/3}}{a^{-2/3}}$

3) Simplify the following without a calculator. Leave your answers with positive exponents:

a) $\frac{2^x+2^{x+1}}{2^x-2^{x+2}}$

b) $\frac{2.3^{x+1}+9.3^{x-3}}{4.3^{x-1}-81.3^{x-4}}$

c) $\frac{m^{x+2}-3m^{x+1}}{3m^{x-2}-m^{x-1}}$

Simplification of Surds

Exercise 3:

1) Simplify the following fully without the use of a calculator:

a) $\sqrt{8}\times\sqrt{6}$

b) $\left(3\sqrt{7}\right)^2$

c) $\left(2\sqrt{2}-\sqrt{3}\right)^2$

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d) $\sqrt{16+9}$

e) $\sqrt{16}+\sqrt{9}$

f) $\sqrt{3^{-1}}\times\sqrt{27}$

g) $\sqrt{27}+\sqrt{48}$

h) $\left(\sqrt{8x}-\sqrt{12x}\right)\left(\sqrt{8x}+\sqrt{12x}\right)$

i) $\frac{}{2+2\sqrt{20}}$

j) $\frac{10+\sqrt{50}}{5}$

k) $\frac{\sqrt{45}+\sqrt{125}}{2\sqrt{20}+\sqrt{80}}$

l) $\frac{\sqrt{28}+\sqrt{63}}{\sqrt{63}+\sqrt{175}}$

m) $\left(\sqrt{45}+\sqrt{20}\right)^2$

n) $\sqrt{\frac{}{16^2}+\frac{}{\theta^2}+\frac{}{7a}}$

o) $\left(3+\sqrt{3}+\sqrt{5}\right)\left(3-\sqrt{3}-\sqrt{5}\right)$

2) **Rationalise the denominators in each of the following surd expressions:**

a) $\frac{12}{\sqrt{3}}$ b) $\frac{1-\sqrt{5}}{\sqrt{5}}$ c) $\frac{8}{\sqrt{2}+2}$

3) **The area of a rectangle is 20m^2. If it has a breadth of $5+\sqrt{5}$ m, determine its length. Write your answer in the form $a+b\sqrt{c}$.**

Simple Surd Equations

Exercise 4:

1) **Solve for x in the following equations:**

a) $\sqrt{x+3}=10$

b) $\sqrt{x+2}=\sqrt{17}$

c) $\sqrt{8-x}=-2$

d) $\sqrt{3x-5}+8=9$

e) $2\sqrt{\frac{x}{2}+3}+4=12$

f) $3\sqrt{\sqrt{x}+3}=9$

2)

a) **A rectangular fence has a perimeter of 35 metres. It has a width of 5 metres and a length of $\sqrt{2x+3}$ metres. Determine the value of x.**

b) **A circular garden has a radius of $\sqrt{x+5}$ m. In order to cover the entire garden with fertilizer 35π m^2 of fertilizer is required. Determine the value of x.**

Equations with Rational Exponents

Exercise 5: Solve for x in the following equations

a) $\left(\frac{1}{2}\right)^x=4$

b) $4^{2x+1}=\frac{1}{8}$

c) $2^{x-1}=64$

d) $5^{x-2}=\frac{1}{5^x}$

e) $9\times81^x=27$

f) $7\times9^x=7$

g) $5^{2x+3}=125^{x+5}$

h) $2^{2x+2}=(0,5)^{2x-4}$

i) $25.2^x=4.5^x$

j) $7^{x-2}=2^{x-2}$

k) $3x^{\frac{2}{3}}=27$

l) $p^{2x}.p^{-x+3}=\sqrt[3]{p^2}$

m) $\frac{1}{2}x^{\frac{4}{5}}=8$

n) $2^{x+1}+2^{x+2}=96$

o) $9x^{\frac{1}{2}}-3=0$

p) $3\sqrt[3]{x}=24$

q) $\left(x^{\frac{1}{4}}-5\right)\left(x^{\frac{2}{3}}-4\right)=0$

r) $x^{\frac{4}{3}}=\frac{16}{81}$

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Mixed Exercise:

1) **Classify the following numbers as rational, irrational or non real.**

a) $1,452.....$

b) $1,23232323...$

c) $\pi+\sqrt{-4}$

d) $\frac{22}{7}$

e) $\sqrt[3]{-27}$

f) $\sqrt[7]{-8}$

2) **Simplify the following expressions without a calculator, leaving all answers with positive exponents:**

a) $\left(3x^{-1}\right)^{-3}$

b) $\frac{\sqrt{x}}{x^{-3/2}}$

c) $\left(\frac{x^2}{y^2}\right)^{p+q}\left(\frac{y^2}{x^2}\right)^{q-p}$

d) $\left(\frac{x^{-2}}{2y^0}\right)^{-3}\times\sqrt[3]{x^9}$

e) $\frac{8^x.4^{x+1}}{2^{5x+3}}$

f) $\frac{5+2\sqrt{50}}{5}$

g) $\frac{4+\sqrt{12}}{\left(1+\sqrt{3}\right)^2}$

h) $\frac{\sqrt{5}}{\left(6+\sqrt{5}\right)\left(6-\sqrt{5}\right)}$

i) $\frac{\sqrt[3]{x}\times\sqrt{y^3}}{x^{-2/3}y^{1/2}}$

j) $\left(2\sqrt{3}-\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)$

k) $\left(a^2b+ab^2\right)^0$

3) **Solve for x in the following equations:**

a) $\sqrt{x+2}=4$

b) $\sqrt{7+\sqrt{x}}=3$

c) $2\sqrt{8-x}=2$

4) **If $p+\frac{1}{p}=q$, express $p^2+2+\frac{1}{p^2}$ in terms of q.**

5) **If x and y are rational numbers and $\sqrt{x}+\sqrt{y}=\sqrt{7+\sqrt{48}}$, determine a possible value of:**

a) $x+y$

b) xy

6) **If the area of a circle is 100 cm^2, determine:**

a) The length of the radius in simplest surd form.

b) The circumference of the circle.

7) **If an equation is in the form $ax^2+bx+c=0$ then $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$. Using the information above, solve for x in the following equations. Leave answers in simplest surd form.**

a) $x^2+3x-6=0$

b) $-2x^2-3x+6=0$

8) **Determine the value of $\frac{\sqrt{10^{1009}}}{\sqrt{10^{1011}}-\sqrt{10^{1007}}}$ without the use of a calculator.**

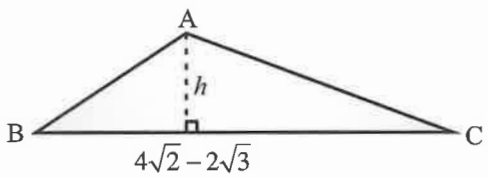
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9) **Evaluate the following expressions and leave denominators rational where applicable:**

a) $\sqrt{\frac{2^{2020}+2^{2019}}{2^{2019}-2^{2017}}}$

b) $\sqrt{\frac{3^{120}+3^{121}}{3^{122}-3^{121}}}$

10) **The triangle below has an area of 20 cm^2 and a base of $4\sqrt{2}-2\sqrt{3}$ cm. Determine the height (h) of the triangle in simplest surd form. Write your answer with a rational denominator.**



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Chapter 1– The Number System, Exponents and Surds (Memo)

The Number System

Exercise 1:

1)

Number	$\mathbb{N}$	$\mathbb{N}_0$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}'$	Non Real
5	✓	✓	✓	✓	×	×
0	×	✓	✓	✓	×	×
-7	×	×	✓	✓	×	×
$(-3)^2$	✓	✓	✓	✓	×	×
$\sqrt{-4}$	×	×	×	×	×	✓
π	×	×	×	×	✓	×
$\frac{-5+2\sqrt{3}}{6}$	×	×	×	×	✓	×
$\frac{-10}{-2}$	✓	✓	✓	✓	×	×
$\sqrt[3]{-8}$	×	×	✓	✓	×	×

2)

a) Non real

b) Non real

c) Real

d) Real

Grade 10 Revision

Exercise 2:

1)

a) $\left(\frac{16}{x^{-4}}\right)^{\frac{3}{4}}$

$=\left(\frac{2^4}{x^{-4}}\right)^{\frac{3}{4}}$

$=\frac{2^3}{x^{-3}}$

$=8x^3$

b) $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

$=\left(\frac{2^3}{3^3}\right)^{-\frac{4}{3}}$

$=\frac{2^{-4}}{3^{-4}}$

$=\frac{81}{16}$

c) $\left(\frac{25}{4}\right)^{-\frac{3}{2}}$

$=\left(\frac{5^2}{2^2}\right)^{-\frac{3}{2}}$

$=\frac{5^{-3}}{2^{-3}}$

$=\frac{8}{125}$

d) $(0{,}008)^{-\frac{2}{3}}$

$=\left(\frac{8}{1000}\right)^{-\frac{2}{3}}$

$=\left(\frac{2^3}{10^3}\right)^{-\frac{2}{3}}$

$=\frac{2^{-2}}{10^{-2}}$

$=\frac{100}{4}$

$=25$

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2)

a) $\dfrac{63^{1-n}.7^{n-1}.81^{-1}}{4^{n+2}.36^{-n-1}}$

$=\dfrac{\left(3^2.7\right)^{1-n}.7^{n-1}.\left(3^4\right)^{-1}}{\left(2^2\right)^{n+2}.\left(3^2.2^2\right)^{-n-1}}$

$=\dfrac{3^{2-2n}.7^{1-n}.7^{n-1}.3^{-4}}{2^{2n+4}.3^{-2n-2}.2^{-2n-2}}$

$=3^{2-2n-4+2n+2}.7^{1-n+n-1}.2^{-2n-4+2n+2}$

$=3^0.7^0.2^{-2}$

$=\dfrac{1}{4}$

b) $\dfrac{5^n.10^{2n-1}.2^{4n+1}}{20^{3n}}$

$=\dfrac{5^n.(5.2)^{2n-1}.2^{4n+1}}{\left(5.2^2\right)^{3n}}$

$=\dfrac{5^n.5^{2n-1}.2^{2n-1}.2^{4n+1}}{5^{3n}.2^{6n}}$

$=5^{n+2n-1-3n}.2^{2n-1+4n+1-6n}$

$=5^{-1}.2^0$

$=\dfrac{1}{5}$

c) $\dfrac{10^{x+2}.8^x.25^{x-1}}{16^{x+1}.2.125^{x+1}}$

$=\dfrac{(5.2)^{x+2}.\left(2^3\right)^x.\left(5^2\right)^{x-1}}{\left(2^4\right)^{x+1}.2^1.\left(5^3\right)^{x+1}}$

$=\dfrac{5^{x+2}.2^{x+2}.2^{3x}.5^{2x-2}}{2^{4x+4}.2^1.5^{3x+3}}$

$=5^{x+2+2x-2-3x-3}.2^{x+2+3x-4x-4-1}$

$=5^{-3}.2^{-3}$

$=\dfrac{1}{5^3.2^3}$

$=\dfrac{1}{1000}$

d) $\dfrac{9^{2a+1}.6^{2a-3}}{2.3^{5a-2}.3^a.4^{a-2}}$

$=\dfrac{\left(3^2\right)^{2a+1}.(2.3)^{2a-3}}{2^1.3^{5a-2}.3^a.\left(2^2\right)^{a-2}}$

$=\dfrac{3^{4a+2}.2^{2a-3}.3^{2a-3}}{2^1.3^{5a-2}.3^a.2^{2a-4}}$

$=3^{4a+2+2a-3-5a+2-a}.2^{2a-3-1-2a+4}$

$=3^1.2^0$

$=3$

e) $\dfrac{\sqrt{3}.\sqrt[4]{12}.\sqrt{8}}{\sqrt[4]{27}}$

$=\dfrac{3^{1/2}.\left(2^2.3\right)^{1/4}.\left(2^3\right)^{1/2}}{\left(3^3\right)^{1/4}}$

$=\dfrac{3^{1/2}.2^{1/2}.3^{1/4}.2^{3/2}}{3^{3/4}}$

$=3^{1/2+1/4-3/4}.2^{1/2+3/2}$

$=3^0.2^2$

$=4$

f) $\dfrac{\sqrt[3]{ab}.\sqrt[3]{a^4b^2}.a^{-1/3}}{a^{-2/3}}$

$=\dfrac{(ab)^{1/3}.\left(a^4b^2\right)^{1/3}.a^{-1/3}}{a^{-2/3}}$

$=\dfrac{a^{1/3}.b^{1/3}.a^{4/3}b^{2/3}.a^{-1/3}}{a^{-2/3}}$

$=a^{1/3+4/3-1/3+2/3}.b^{1/3+2/3}$

$=a^2b$

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3)

a) $\dfrac{2^x+2^{x+1}}{2^x-2^{x+2}}$

$=\dfrac{2^x+2^x.2^1}{2^x-2^x.2^2}$

$=\dfrac{2^x(1+2)}{2^x(1-2^2)}$

$=-1$

b) $\dfrac{2.3^{x+1}+9.3^{x-3}}{4.3^{x-1}-81.3^{x-4}}$

$=\dfrac{2.3^x.3^1+9.3^x.3^{-3}}{4.3^x.3^{-1}-81.3^x.3^{-4}}$

$=\dfrac{3^x\left(2.3^1+9.3^{-3}\right)}{3^x\left(4.3^{-1}-81.3^{-4}\right)}$

$=\dfrac{6+\frac{9}{27}}{\frac{4}{3}-\frac{81}{81}}$

$=19$

c) $\dfrac{m^{x+2}-3m^{x+1}}{3m^{x-2}-m^{x-1}}$

$=\dfrac{m^x.m^2-3.m^x.m^1}{3.m^x.m^{-2}-m^x.m^{-1}}$

$=\dfrac{m^x\left(m^2-3m\right)}{m^x\left(3m^{-2}-m^{-1}\right)}$

$=\dfrac{m(m-3)}{\left(\frac{3}{m^2}-\frac{1}{m}\right)}$

$=\dfrac{m(m-3)}{1}\div\dfrac{3-m}{m^2}$

$=\dfrac{m(m-3)}{1}\times\dfrac{m^2}{-(m-3)}$

$=-m^3$

Simplification of Surds

Exercise 3:

1)

a) $\sqrt{8}\times\sqrt{6}$

$=\sqrt{48}$

$=\sqrt{16\times3}$

$=4\sqrt{3}$

b) $\left(3\sqrt{7}\right)^2$

$=9\times7$

$=63$

c) $\left(2\sqrt{2}-\sqrt{3}\right)^2$

$=8-4\sqrt{6}+3$

$=11-4\sqrt{6}$

d) $\sqrt{16+9}$

$=\sqrt{25}$

$=5$

e) $\sqrt{16}+\sqrt{9}$

$=4+3$

$=7$

f) $\sqrt{3^{-1}}\times\sqrt{27}$

$=\sqrt{9}$

$=3$

g) $\sqrt{27}+\sqrt{48}$

$=3\sqrt{3}+4\sqrt{3}$

$=7\sqrt{3}$

h) $\left(\sqrt{8x}-\sqrt{12x}\right)\left(\sqrt{8x}+\sqrt{12x}\right)$

$=8x-12x$

$=-4x$

i) $\dfrac{2+\sqrt{20}}{2}$

$=\dfrac{2+2\sqrt{5}}{2}$

$=\dfrac{2\left(1+\sqrt{5}\right)}{2}$

$=1+\sqrt{5}$

j) $\dfrac{10+\sqrt{50}}{5}$

$=\dfrac{10+5\sqrt{2}}{5}$

$=\dfrac{5\left(2+\sqrt{2}\right)}{5}$

$=2+\sqrt{2}$

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k) $\dfrac{\sqrt{45}+\sqrt{125}}{2\sqrt{20}+\sqrt{80}}$

$=\dfrac{3\sqrt{5}+5\sqrt{5}}{4\sqrt{5}+4\sqrt{5}}$

$=\dfrac{8\sqrt{5}}{8\sqrt{5}}$

$=1$

l) $\dfrac{\sqrt{28}+\sqrt{63}}{\sqrt{63}+\sqrt{175}}$

$=\dfrac{2\sqrt{7}+3\sqrt{7}}{3\sqrt{7}+5\sqrt{7}}$

$=\dfrac{5\cancel{\sqrt{7}}}{8\cancel{\sqrt{7}}}$

$=\dfrac{5}{8}$

m) $\left(\sqrt{45}+\sqrt{20}\right)^2$

$=\left(3\sqrt{5}+2\sqrt{5}\right)^2$

$=\left(5\sqrt{5}\right)^2$

$=125$

n) $\sqrt{\dfrac{a^2}{16}+\dfrac{a^2}{9}}+\dfrac{7a}{12}$

$=\sqrt{\dfrac{9a^2+16a^2}{144}}+\dfrac{7a}{12}$

$=\sqrt{\dfrac{25a^2}{144}}+\dfrac{7a}{12}$

$=\dfrac{5a}{12}+\dfrac{7a}{12}$

$=\dfrac{12a}{12}$

$=a$

o) $\left(3+\sqrt{3}+\sqrt{5}\right)\left(3-\sqrt{3}-\sqrt{5}\right)$

$=\left[3+\left(\sqrt{3}+\sqrt{5}\right)\right]\left[3-\left(\sqrt{3}+\sqrt{5}\right)\right]$

$=9-\left(\sqrt{3}+\sqrt{5}\right)^2$

$=9-\left(3+2\sqrt{15}+5\right)$

$=1-2\sqrt{15}$

2)

a) $\dfrac{12}{\sqrt{3}}$

$=\dfrac{12}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$

$\dfrac{12\sqrt{3}}{3}$

$=4\sqrt{3}$

b) $\dfrac{1-\sqrt{5}}{\sqrt{5}}$

$=\dfrac{1-\sqrt{5}}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}$

$=\dfrac{\sqrt{5}-5}{5}$

c) $\dfrac{8}{\sqrt{2}+2}$

$=\dfrac{8}{\sqrt{2}+2}\times\dfrac{\sqrt{2}-2}{\sqrt{2}-2}$

$=\dfrac{8\sqrt{2}-16}{-2}$

$=\dfrac{8\left(\sqrt{2}-2\right)}{-2}$

$=8-4\sqrt{2}$

3)

$A=l\times B$

$\therefore 20=l\left(5+\sqrt{5}\right)$

$\therefore l=\dfrac{20}{5+\sqrt{5}}$

$=\dfrac{20}{5+\sqrt{5}}\times\dfrac{5-\sqrt{5}}{5-\sqrt{5}}$

$=\dfrac{100-20\sqrt{5}}{20}$

$=\dfrac{\cancel{20}\left(5-\sqrt{5}\right)}{\cancel{20}}$

$=5-\sqrt{5}$ metres

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Simple Surd Equations

Exercise 4:

1)

a) $\sqrt{x+3}=10$

$\therefore x+3=100$

$\therefore x=97$

b) $\sqrt{x+2}=\sqrt{17}$

$\therefore x+2=17$

$\therefore x=15$

c) $\sqrt{8-x}=-2$

$\therefore$ No Solution as $\sqrt{8-x}\geq 0$

d) $\sqrt{3x-5}+8=9$

$\therefore \sqrt{3x-5}=1$

$\therefore 3x-5=1$

$\therefore 3x=6$

$\therefore x=2$

e) $2\sqrt{\frac{x}{2}+3}+4=12$

$\therefore 2\sqrt{\frac{x}{2}+3}=8$

$\therefore \sqrt{\frac{x}{2}+3}=4$

$\therefore \frac{x}{2}+3=16$

$\therefore \frac{x}{2}=13$

$\therefore x=26$

f) $3\sqrt{\sqrt{x}+3}=9$

$\therefore 9\left(\sqrt{x}+3\right)=81$

$\therefore \sqrt{x}+3=9$

$\therefore \sqrt{x}=6$

$\therefore x=36$

2)

a) $2\sqrt{2x+3}+2(5)=35$

$\therefore 4(2x+3)=625$

$\therefore 2x+3=\frac{625}{4}$

$\therefore 2x=\frac{613}{4}$

$\therefore x=\frac{613}{8}$

$\therefore x=76,63$

b) $A_{\bigcirc}=\pi r^2$

$\therefore 35\pi=\pi\left(\sqrt{x+5}\right)^2$

$\therefore 35\pi=\pi(x+5)$

$\therefore 35=x+5$

$\therefore x=30$

Equations with Rational Exponents

Exercise 5:

a) $\left(\frac{1}{2}\right)^x=4$

$\therefore 2^{-x}=2^2$

$\therefore x=-2$

b) $4^{2x+1}=\frac{1}{8}$

$\therefore 2^{4x+2}=2^{-3}$

$\therefore 4x+2=-3$

$\therefore x=-\frac{5}{4}$

c) $2^{x-1}=64$

$\therefore 2^{x-1}=2^6$

$\therefore x-1=6$

$\therefore x=7$

d) $5^{x-2}=\frac{1}{5^x}$

$\therefore 5^{x-2}=5^{-x}$

$\therefore x-2=-x$

$\therefore x=1$

e) $9\times 81^x=27$

$\therefore 3^{4x}=3^1$

$\therefore 4x=1$

$\therefore x=\frac{1}{4}$

f) $7\times 9^x=7$

$\therefore 3^{2x}=1$

$\therefore 3^{2x}=3^0$

$\therefore x=0$

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g) $5^{2x+3}=125^{x+5}$

$\therefore 5^{2x+3}=5^{3x+15}$

$\therefore 2x+3=3x+15$

$\therefore x=-12$

h) $2^{2x+2}=(0,5)^{2x-4}$

$\therefore 2^{2x+2}=\left(2^{-1}\right)^{2x-4}$

$\therefore 2^{2x+2}=2^{-2x+4}$

$\therefore 2x+2=-2x+4$

$\therefore x=\frac{1}{2}$

i) $25.2^x=4.5^x$

$\therefore 5^2.2^x=2^2.5^x$

$\therefore \left(\frac{2}{5}\right)^x=\left(\frac{2}{5}\right)^2$

$\therefore x=2$

j) $7^{x-2}=2^{x-2}$

$\therefore x-2=0$

$\therefore x=2$

k) $3x^{\frac{2}{3}}=27$

$\therefore x^{\frac{2}{3}}=9$

$\therefore x=\left(3^2\right)^{\frac{3}{2}}$

$\therefore x=\pm 3^3=\pm 27$

l) $p^{2x}.p^{-x+3}=\sqrt[3]{p^2}$

$\therefore p^{2x-x+3}=p^{\frac{2}{3}}$

$\therefore x+3=\frac{2}{3}$

$\therefore x=-\frac{7}{3}$

m) $\frac{1}{2}x^{\frac{4}{5}}=8$

$\therefore x^{\frac{4}{5}}=2^4$

$\therefore x=\left(2^4\right)^{\frac{5}{4}}$

$\therefore x=\pm 2^5=\pm 32$

n) $2^{x+1}+2^{x+2}=96$

$\therefore 2^x.2^1+2^x.2^2=96$

$\therefore 2^x\left(2^1+2^2\right)=96$

$\therefore 2^x=16$

$\therefore 2^x=2^4$

$\therefore x=4$

o) $9x^{\frac{1}{2}}-3=0$

$\therefore 9x^{\frac{1}{2}}=3$

$\therefore x^{\frac{1}{2}}=3^{-1}$

$\therefore x=3^{-2}$

$\therefore x=\frac{1}{9}$

p) $3\sqrt[3]{x}=24$

$\therefore x^{\frac{1}{3}}=2^3$

$\therefore x=\left(2^3\right)^3$

$\therefore x=2^9=512$

q) $\left(x^{\frac{1}{4}}-5\right)\left(x^{\frac{2}{3}}-4\right)=0$

$\therefore x^{\frac{1}{4}}=5$ or $x^{\frac{2}{3}}=2^2$

$\therefore x=5^4$ or $x=\pm\left(2^2\right)^{\frac{3}{2}}$

$\therefore x=625$ or $x=\pm 8$

r) $x^{\frac{4}{3}}=\frac{16}{81}$

$\therefore x^{\frac{4}{3}}=\frac{2^4}{3^4}$

$\therefore x=\pm\left(\frac{2^4}{3^4}\right)^{\frac{3}{4}}$

$\therefore x=\pm\frac{2^3}{3^3}$

$\therefore x=\pm\frac{8}{27}$

Mixed Exercise:

1)

a) Irrational

b) Rational

c) Non real

d) Rational

e) Rational

f) Irrational

2)

a) $\left(3x^{-1}\right)^{-3}$

$=3^{-3}x^3$

$=\frac{x^3}{27}$

b) $\frac{\sqrt{x}}{x^{-\frac{3}{2}}}$

$=x^{\frac{1}{2}}.x^{\frac{3}{2}}$

$=x^2$

c) $\left(\frac{x^2}{y^2}\right)^{p+q}\left(\frac{y^2}{x^2}\right)^{q-p}$

$=\frac{x^{2p+2q}}{y^{2p+2q}}\times\frac{y^{2q-2p}}{x^{2q-2p}}$

$=x^{2p+2q-2q+2p}.y^{2q-2p-2p-2q}$

$=x^{4p}.y^{-4p}$

$=\frac{x^{4p}}{y^{4p}}$

d) $\left(\frac{x^{-2}}{2y^0}\right)^{-3}\times\sqrt[3]{x^9}$

$=\frac{x^6}{2^{-3}}\times x^{\frac{9}{3}}$

$=8x^6\times x^3$

$=8x^9$

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e) $\dfrac{8^x.4^{x+1}}{2^{5x+3}}$

$=\dfrac{2^{3x}\times\left(2^2\right)^{x+1}}{2^{5x+3}}$

$=\dfrac{2^{3x}.2^{2x+2}}{2^{5x+3}}$

$=2^{3x+2x+2-5x-3}$

$=2^{-1}=\frac{1}{2}$

f) $\dfrac{5+2\sqrt{50}}{5}$

$=\dfrac{5+10\sqrt{2}}{5}$

$=\dfrac{\cancel{5}\left(1+2\sqrt{2}\right)}{\cancel{5}}$

$=1+2\sqrt{2}$

g) $\dfrac{4+\sqrt{12}}{\left(1+\sqrt{3}\right)^2}$

$=\dfrac{4+2\sqrt{3}}{1+2\sqrt{3}+3}$

$=\dfrac{4+2\sqrt{3}}{4+2\sqrt{3}}$

$=1$

h) $\dfrac{\sqrt{5}}{\left(6+\sqrt{5}\right)\left(6-\sqrt{5}\right)}$

$=\dfrac{\sqrt{5}}{36-5}$

$=\dfrac{\sqrt{5}}{31}$

i) $\dfrac{\sqrt[3]{x}\times\sqrt{y^3}}{x^{-\frac{2}{3}}y^{\frac{1}{2}}}$

$=\dfrac{x^{\frac{1}{3}}y^{\frac{3}{2}}}{x^{-\frac{2}{3}}y^{\frac{1}{2}}}$

$=x^{\frac{1}{3}+\frac{2}{3}}.y^{\frac{3}{2}-\frac{1}{2}}$

$=xy$

j) $\left(2\sqrt{3}-\dfrac{1}{\sqrt{3}}\right)\left(\sqrt{3}+\dfrac{1}{\sqrt{3}}\right)$

$=6+2-1-\dfrac{1}{3}$

$=\dfrac{20}{3}$

k) $\left(a^2b+ab^2\right)^0$

$=1$

3)

a) $\sqrt{x+2}=4$

$\therefore x+2=16$

$\therefore x=14$

Check: $\sqrt{14+2}=4$

b) $\sqrt{7+\sqrt{x}}=3$

$\therefore 7+\sqrt{x}=9$

$\therefore \sqrt{x}=2$

$\therefore x=4$

Check: $\sqrt{7+\sqrt{4}}=3$

c) $2\sqrt{8-x}=2$

$\therefore \sqrt{8-x}=1$

$\therefore 8-x=1$

$\therefore x=7$

Check: $2\sqrt{8-7}=2$

4) $\left(p+\dfrac{1}{p}\right)^2=q^2$

$p^2+2+\dfrac{1}{p^2}=q^2$

5)

a) $\left(\sqrt{x}+\sqrt{y}\right)^2=\left(\sqrt{7+\sqrt{48}}\right)^2$

$\therefore x+2\sqrt{xy}+y=7+\sqrt{48}$

$\therefore x+y+2\sqrt{xy}=7+\sqrt{48}$

$\therefore x+y=7$

b) $x+y+2\sqrt{xy}=7+\sqrt{4\times12}$

$\therefore x+y+2\sqrt{xy}=7+2\sqrt{12}$

$\therefore xy=12$

6)

a) $A_{\bigcirc}=\pi r^2$

$\therefore 100=\pi r^2$

$\therefore r^2=\dfrac{100}{\pi}$

$\therefore r=\dfrac{10}{\sqrt{\pi}}\text{ cm}\rightarrow\dfrac{10\sqrt{\pi}}{\pi}$

b) Circumference $=2\pi r$

$=2\pi\left(\dfrac{10}{\sqrt{\pi}}\right)$

$=\dfrac{20\pi}{\sqrt{\pi}}\text{cm}$

$==20\sqrt{\pi}\text{ cm}$

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7)

a) $x^2+3x-6=0$

$$\therefore x=\frac{-3+\sqrt{(3)^2-4(1)(-6)}}{2(1)} \text{ or } x=\frac{-3-\sqrt{(3)^2-4(1)(-6)}}{2(1)}$$

$$\therefore x=\frac{-3+\sqrt{33}}{2} \text{ or } x=\frac{-3-\sqrt{33}}{2}$$

b) $-2x^2-3x+6=0$

$$\therefore x=\frac{-(-3)+\sqrt{(-3)^2-4(-2)(6)}}{2(-2)} \text{ or } x=\frac{-(-3)-\sqrt{(-3)^2-4(-2)(6)}}{2(-2)}$$

$$\therefore x=\frac{-3+\sqrt{57}}{4} \text{ or } x=\frac{-3-\sqrt{57}}{4}$$

8)

$$\frac{\sqrt{10^{1009}}}{\sqrt{10^{1011}}-\sqrt{10^{1007}}}$$

$$=\frac{\sqrt{10^2.10^{1007}}}{\sqrt{10^4.10^{1007}}-\sqrt{10^{1007}}}$$

$$=\frac{10\sqrt{10^{1007}}}{10^2\sqrt{10^{1007}}-\sqrt{10^{1007}}}$$

$$=\frac{10\cancel{\sqrt{10^{1007}}}}{\cancel{\sqrt{10^{1007}}}\left(10^2-1\right)}=\frac{10}{99}$$

9)

a)

$$\sqrt{\frac{2^{2020}+2^{2019}}{2^{2019}-2^{2017}}}$$

$$=\sqrt{\frac{2^{2017}\left(2^3+2^2\right)}{2^{2017}\left(2^2-1\right)}}$$

$$=\sqrt{\frac{12}{3}}$$

$$=2$$

b)

$$\sqrt{\frac{3^{120}+3^{121}}{3^{122}-3^{121}}}$$

$$=\sqrt{\frac{3^{120}(1+3)}{3^{120}\left(3^2-3^1\right)}}$$

$$=\sqrt{\frac{4}{6}}$$

$$=\frac{\sqrt{4}}{\sqrt{6}}=\frac{2}{\sqrt{6}}=\frac{2}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}=\frac{2\sqrt{6}}{6}=\frac{\sqrt{6}}{3}$$

10) $A_{\triangle}=\frac{1}{2}\text{b.}\perp\text{h}$

$$\therefore 20=\frac{1}{2}\left(4\sqrt{2}-2\sqrt{3}\right)(h)$$

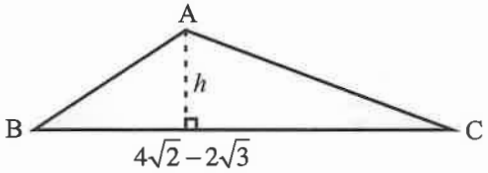
$$\therefore 40=\left(4\sqrt{2}-2\sqrt{3}\right)(h)$$

$$\therefore h=\frac{40}{4\sqrt{2}-2\sqrt{3}} \rightarrow \text{rationalise the denominator}$$

$$\therefore h=\frac{40}{4\sqrt{2}-2\sqrt{3}}\times\frac{4\sqrt{2}+2\sqrt{3}}{4\sqrt{2}+2\sqrt{3}}$$

$$\therefore h=\frac{40\left(4\sqrt{2}+2\sqrt{3}\right)}{20}$$

$$\therefore h=8\sqrt{2}+4\sqrt{3} \text{ cm}$$



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Chapter 2– Equations and Inequalities

Grade 10 Revision

Exercise 1:

1) Factorise the following expressions:

a) $x^2-2x-15$

b) x^2+x-12

c) $x^2-8x+12$

d) x^2-6x+9

e) $4x^2-16x+16$

f) $x^2+10x+25$

g) $2x^2-5x-12$

h) $6x^2+13x-5$

i) $6x^2+5x-4$

j) $a^3-a^2b+b^3-ab^2$

k) $x^2-a^2-2ab-b^2$

l) $2x^3-3x^2-8x+12$

2) Simplify the following expressions:

a) $\frac{4}{x+1}+\frac{8}{x-2}$

b) $\frac{1}{x-2}+\frac{2}{(x-1)^2}-\frac{1}{x-1}$

c) $\frac{x^2+x-6}{x^2-6x+5}\div\frac{x^2+5x+6}{x^2-3x-10}$

d) $\frac{x^2+5x+6}{x^2-9}\div\frac{x^2-4}{x^2-5x+6}$

e) $\frac{1}{x-3}+\frac{2}{x+1}-\frac{3}{3-x}$

f) $\frac{x}{x-2}-\frac{1}{x+2}+\frac{8}{4-x^2}$

g) $\frac{5}{x+3}-\frac{2}{3-x}+\frac{3-5x}{x^2-9}$

Quadratic Equations

Exercise 2:

1) Solve for x in the following equations:

a) $x^2-5x+6=0$

b) $x^2+3x=0$

c) $(x-5)(x+12)=0$

d) $(x-5)(x+12)=-70$

e) $4x^2-12x+10=1$

f) $-x^3-8x^2-15x=0$

2) Solve for the unknown variable in the following equations:

a) $(y+3)^2=y^2+6y+9$

b) $x^2+16=0$

Equations with Fractions

Exercise 3:

1) Solve for x in the following equations:

a) $x+\frac{18}{x}=11$

b) $2x+4=\frac{6}{2x-1}$

c) $\frac{2}{x+1}-\frac{6}{2-x}=\frac{2x+4}{x+1}$

d) $\frac{4x-14}{x+4}-\frac{2x-4}{x+2}=0$

e) $\frac{2x}{x-1}+\frac{1}{x+1}=1$

f) $x-1+\frac{15}{x+2}=5$

g) $\frac{3x-24}{9}+\frac{9}{3x-6}=0$

h) $\frac{3}{x+3}+\frac{2x}{2-x}=1-\frac{5x+10}{x^2+x-6}$

i) $\frac{8}{y+1}+\frac{2y-6}{y^2+3y+2}=-\frac{10}{4-y^2}$

2) Solve for the unknown variable in each of the following equations:

a) $\frac{3a-1}{a^2-2a-3}+\frac{3a}{a+1}+\frac{2}{3-a}=0$

b) $\frac{5y+4}{y+2}+\frac{2y+3}{1-y}=\frac{2y^2-17}{y^2+y-2}$

c) $\frac{6p^2-53p+10}{p^2-3p-10}-\frac{3p}{5-p}-\frac{8p}{p+2}=0$

d) $2-\frac{t+3}{2-t}+\frac{2t+6}{t^2+t-6}=0$

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Equations with Substitution

Exercise 4: Solve for the unknown variable in each of the following equations:

a) $\left(x^2-x\right)^2-18\left(x^2-x\right)+72=0$

b) $\left(x^2+3x\right)^2-8\left(x^2+3x\right)-20=0$

c) $x^2-3x+4=-\dfrac{4}{x^2-3x}$

d) $\left(q^2+2q\right)^2-2\left(q^2+2q\right)-3=0$

e) $\dfrac{p^2+5p}{2}+5=-\dfrac{12}{p^2+5p}$

f) $\left(x^2-x\right)^2-8x^2+8x+12=0$

Quadratic Surd Equations

Exercise 5: Solve for the x in each of the following equations:

a) $\sqrt{x}+x=0$

b) $x=\sqrt{x+6}$

c) $2x-\sqrt{32-8x}=0$

d) $x+2=\sqrt{9x+18}$

e) $x-4=\sqrt{56-5x}$

f) $x-7=\sqrt{2x+1}$

g) $\sqrt{22-7x}-x=-4$

h) $4\sqrt{x-3}-3=2x-9$

i) $\sqrt{x-5}-1=\dfrac{6}{\sqrt{x-5}}$

j) $\sqrt{2x-1}-\dfrac{3}{\sqrt{2x-1}}=-2$

k) $\sqrt{12+x}=\sqrt{x}+3$

l) $2\sqrt{4-x}=2\sqrt{x}-4$

Exponential Equations

Exercise 6: Solve for the unknown variable in each of the following equations:

a) $7^{-2a}=7^{2-3a}$

b) $2^{3m}.2^{-m}=2^{-2m}$

c) $5^{-3x}.5^{x}=\dfrac{1}{5}$

d) $\dfrac{81^{3y+2}}{243^{-y}}=3^4$

e) $5^{2x-4}=1$

f) $3^{2x-4}=1/9$

g) $4^{x}-2^{x+2}=0$

h) $9^{x}-3^{x}=6$

i) $3^{2x+1}-28.3^{x}=-9$

Simultaneous Equations

Exercise 7:

1) Solve for x and y in each of the following equations:

a) $y+6x=4$ and $2y=5x^2-22x+8$

b) $2x-y=8$ and $y=x^2+4x-23$

c) $y+x=6$ and $x^2+y^2=36$

d) $2x-3y=2$ and $x^2-2y^2=8$

e) $y-2=2(x-1)^2$ and $y-x=2$

f) $y-x=-6$ and $2x^2+5xy=-2y^2$

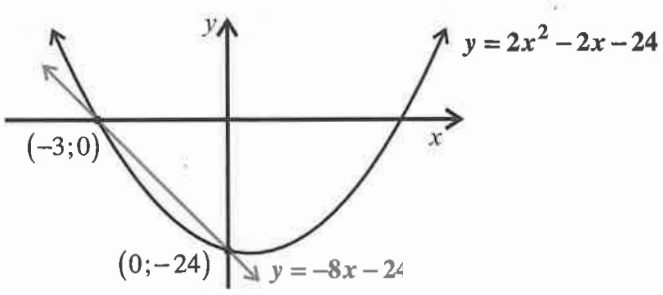
g) $y-x=5$ and $xy-3y=8x-20$

h) $2^{x+y}=256$ and $xy+3x+2y=36$

i) $5^{x-2y-1}=1$ and $x^2+2xy-2x-4y=0$

j) $(x+5)^2+(y-3)^2=49$ and $y+x=-9$

2) Given the diagram below solve the simultaneous equations of $y=-8x-24$ and $y=2x^2-2x-24$.



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Completing the Square

Exercise 8:

1) Complete the square on the following quadratic expressions:

a) $y = x^2 + 6x - 8$

b) $y = x^2 - 4x + 9$

c) $y = x^2 - 12x + 2$

d) $y = x^2 - 8x$

e) $y = 2x^2 - 4x + 6$

f) $y = -3x^2 - 6x + 12$

g) $y = -2x^2 - 3x + 5$

h) $y = 3x^2 + 10x - 2$

2) Solve for x by completing the square on the following quadratic equations:

a) $x^2 - 6x - 5 = 0$

b) $x^2 + 2x - 7 = 0$

c) $2x^2 + 3x - 8 = 0$

d) $-5x^2 + 15x + 1 = 0$

e) $x^2 + 2bx + 3c = 0$

f) $x^2 - 2x - 3c = 0$

g) $x^2 + 2x + 3 = 0$

h) $ax^2 + 3bx - 18 = 0$

Applications of Completing the Square

Exercise 9: Complete the square to determine the maximum or minimum value of:

a) $x^2 - 2x - 10$

b) $-x^2 - 8x + 3$

c) $x^2 - 12x + 40$

d) $-2x^2 - 12x + 10$

The Quadratic Formula and Nature of Roots

Exercise 10:

1) Solve for x in the following equations correct to 1 decimal place:

a) $4x^2 + 5x - 3 = 0$

b) $7x^2 - 3x - 2 = 0$

c) $-3x^2 - 4x + 5 = 0$

d) $2x^2 - 6x - 12 = 0$

2) Given that $(x - 2{,}5)\left(x - \sqrt{-1}\right)(x + 2)\left(x - \sqrt{4}\right)\left(x + \sqrt{3}\right) = 0$, solve for x if:

a) $x \in \mathbb{Z}$

b) $x \in \mathbb{R}$

c) $x \in \mathbb{Q}'$

d) is non-real

3) Given that $2x^2(x-2) + 5x(x-2) - 5(x-2) = 0$, solve for x if:

a) $x \in \mathbb{N}$

b) $x \in \mathbb{Q}'$

4) Determine the nature of roots of the equations below without solving the equations:

a) $14x^2 - 6x + 8 = 0$

b) $9x^2 + 15x + 3 = 0$

c) $50x^2 + 40x + 8 = 0$

d) $-25x^2 - 10x - 1 = 0$

Inequalities

Exercise 11:

1) Solve for x in the following inequalities:

a) $x^2 - 4 \geq 0$

b) $x^2 - x - 6 > 0$

c) $x^2 - 3x - 10 \leq 0$

d) $x^2 + 9x + 18 \leq 0$

e) $-x^2 + 2x + 15 < 0$

f) $\dfrac{(x-3)(x+2)}{x-8} \leq 0$

g) $\dfrac{(x-3)}{x+2} + 1 \leq 0$

h) $\dfrac{(x+1)(x-3)}{x-5} \geq 1$

i) $\dfrac{(x-2)(x+1)}{x-4} - 2 \geq 0$

j) $\dfrac{(x-2)^2(x+5)}{x+1} < 0$

k) $\dfrac{(x-1)^2(x-5)}{x+5} > 0$

l) $\dfrac{x^2 + 5x + 11}{x-2} + 1 \geq 0$

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2) **Solve the following inequalities:**

a) $(x-25)^2 \leq 0$

b) $(a+3)^2 > 0$

c) $(p+5)^2 \geq 0$

d) $(q+3)^2 + 5 < 0$

3) **Solve for x in the following inequalities** (all answers should be in simplest surd form where applicable)**:**

a) $x^2 - 3x - 5 \geq 0$

b) $2x^2 - 2x + 8 \leq 0$

c) $3x^2 + 6x - 7 \leq 0$

d) $3x^2 - 5x + 4 > 0$

4) **For what values of x is the expression $\frac{\sqrt{x^2-4x-12}}{x+10}$**

a) Undefined? b) Non-Real? c) Real?

Mathematical Modelling

Exercise 12:

1) Thabo purchased two types of cable, type A and B. He purchased 3 metres more of type A then he did of type B and spent a total of R150 on type A and R144 on type B. How many metres of each type did he purchase if type B is R2 more per metre than type A?

2) Mrs. Mottiar runs a day-care centre and pays R60 for milk powder at Rx per kilogram. The next time she orders milk powder she finds the price has increased by R5 per kilogram. She still pays R60 for her order but this time receives 2 kilograms less than she did on her previous order. What was the original price per kilogram and how much did she originally purchase?

3) Joanne and Lerato sell homemade chocolate chip muffins at a school fun day. They spend R250 making the muffins and sell each muffin for R8 more than it cost them to make. If they were left with 2 muffins at the end of the fun day and they made a total profit of R374, how much did each muffin cost them to make?

4) Mrs. Modise is hosting a party. She uses blue and gold ribbon for decorations. Blue ribbon costs R1 less per metre than gold ribbon and she buys 10 metres more blue ribbon than gold. Determine the number of metres of each type of ribbon bought and the cost per metre of each ribbon if she spent a total of R80 on blue ribbon and R90 on gold ribbon.

Exercise 13:

1) A cyclist competes in a 100km race. He is unhappy with his time and calculates that he can improve his time by 1 hour if he can increase his average cycling speed by 5km/h in the next race. What was his average speed in the race?

2) A paramedic travels 30km from a hospital to an emergency. On the way back he travels 30km/h slower and the journey takes him 5 minutes longer. What was his speed to the emergency?

3) A mountain bike trail consists of two parts. Part one consists of a 20km stretch on a good sand road. Part two consists of 12km of off-road terrain. A novice mountain biker takes 5 hours to complete the trail. If his speed on part one is 2,5 times greater than his speed on part two, how long did each part take him?

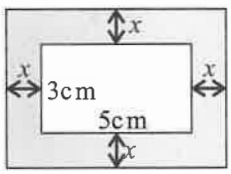
4) Jerry drives from his house to drop his children off at school in traffic. On his return journey there is no traffic and Jerry can travel faster. If he travels to the school at 40 km/h and he returns at 60 km/h what was his average speed for the entire journey?(Hint: Average speed is total distance divided by total time.)

Exercise 14:

1) A rectangle has a perimeter of 10 cm and an area of 6 cm^2. Determine the dimensions of the rectangle.

2) A rectangular garden has an area of 40 m^2. If its length is increased by 3 m and the width is increased by 2 m the area of the garden will double. Determine the dimensions of the original garden.

3) A 5 cm × 3 cm picture that is to be framed is first placed on a cardboard backing. The backing gives the picture a border of x cm all round. If the area of border is 48 cm^2, determine the width of the border.



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Exercise 15:

1) Joshua can paint a $50\,m^2$ room in eight hours on his own. Ridwaan is a more experienced painter and can complete the job in half the time. How long would it take for them to paint a $50m^2$ room working together?

2) Two elements heat an oven to $850^\circ C$. The larger element can heat the oven on its own in 3 hours and the smaller element can heat the oven in 6 hours. How long would they take to heat the oven operating together?

3) A cargo ship uses two engines. If the ship runs both motors it can complete its journey in 40 days. If only the main engine was used, the ship would complete its journey in 60 days less than if the smaller engine was used on its own. How long would the ship take to complete its journey if only the smaller engine was used?

4) Two machines are used to pack large containers. Working together they can pack a container in 6 hours and 40 minutes. Working on its own, one machine will take 3 hours longer than the other to pack the container. How long would the faster machine take packing on its own?

Mixed Exercise:

1) **Simplify the following:**

a) $\dfrac{16(x+2)-x^2(x+2)}{(x+2)(x+4)}$

b) $\dfrac{5}{3x-3}+\dfrac{2x+1}{1-x^2}+\dfrac{1}{2x+2}$

c) $\dfrac{x^2+xy}{x-y}\div\dfrac{x^2-y^2}{y-x}$

d) $\dfrac{a}{a-1}+\dfrac{12}{1-a^2}-\dfrac{6}{a+1}$

e) $\dfrac{2x+2}{1-x-2x^2}\div\dfrac{4-8x}{4x^2-4x+1}$

f) $\dfrac{p^2-5p+6}{1-p^2}\times\dfrac{p^2+2p-3}{9-p^2}$

2) **Solve the following equations and inequalities:**

a) $2\left(x^2+6\right)+11x=0$

b) $(x-2)\left(x^2-5\right)\left(x^2+9\right)=0$

c) $\dfrac{6}{x+1}-\dfrac{3x}{1-x^2}=\dfrac{2x}{x-1}$

d) $\left(x^2-5x+2\right)^2-\left(x^2-5x\right)=8$

e) $x^2+6x-\dfrac{35}{x^2+6x}=2$

f) $\dfrac{x}{x-2}=\dfrac{1}{x-3}-\dfrac{2}{2-x}$

g) $\left(x^2+4\right)(x-1)\geq 0$

h) $x^2-2x+7\geq 0$

i) $\dfrac{\left(x^2+1\right)(3-2x)}{3x+2}\leq 0$

j) $\dfrac{2x^2+3x-2}{3x}\leq 0$

3) **Solve the following surd equations:**

a) $1-2x+\sqrt{5x-1}=0$

b) $x+5=\sqrt{3-3x}$

4) **Determine the value of x and y in the following simultaneous equations:**

a) $x-2y=-1$ and
$x^2-xy+y^2=7$

b) $2^{y-3x}=1/16$ and
$x^2+xy=24$

5) **Solve for x correct to one decimal place:**

a) $-3x^2-7x+12=0$

b) $3x^2+5x+1=0$

c) $4x^2+7x+2=0$

d) $8x^2+3x-2=0$

6) **By completing the square in the following expressions determine:**

a) The minimum value of x^2-2x

b) The maximum value of $-2x^2+12x-17$

c) The maximum value of $\dfrac{2}{x^2-6x+10}$

d) The minimum value of $\dfrac{6}{-x^2+8x-13}$

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7) **For which values of x is the expression $\sqrt{\frac{2x}{(x+3)(x-2)}}$:**

a) **Real?** b) **Non-real?** c) **Undefined?**

8) **Determine the value/s of k if $x^2+2kx+5k+6$ is a perfect square.**

9) **If the current price of filtered tap water increases by 5c per litre, one litre less of water can be purchased for R52,80. What is the present price of filtered tap water?**

10) **Jane and Sipho are training for a cycling race. Jane lives in Pretoria and Sipho lives in Johannesburg 66km apart. During their training they cycle towards each other and meet exactly halfway. Jane cycles 1km/h slower than Sipho and has to leave 18 minutes earlier in order to meet halfway. At what speed does Sipho cycle?**

11) **Determine the value/s of p for which $4x^2-4x=2p$ will have real roots.**

12) **Determine the value/s of x for which the roots of $3y^2-2yx+3=0$ will be non-real.**

13) **Determine the nature of roots of $(p+2)x^2+8x+(2-p)=0$.**

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Grade 10 Revision

Exercise 1:

1)

a) $x^2-2x-15$
$=(x-5)(x+3)$

b) x^2+x-12
$=(x+4)(x-3)$

c) $x^2-8x+12$
$=(x-6)(x-2)$

d) x^2-6x+9
$=(x-3)^2$

e) $4x^2-16x+16$
$=4\left(x^2-4x+4\right)$
$=4(x-2)^2$

f) $x^2+10x+25$
$=(x+5)^2$

g) $2x^2-5x-12$
$=(2x+3)(x-4)$

h) $6x^2+13x-5$
$=(3x-1)(2x+5)$

i) $6x^2+5x-4$
$=(3x+4)(2x-1)$

j) $a^3-a^2b+b^3-ab^2$
$=a^2(a-b)-b^2(a-b)$
$=(a-b)\left(a^2-b^2\right)$
$=(a-b)(a-b)(a+b)$

k) $x^2-a^2-2ab-b^2$
$=x^2-\left(a^2+2ab+b^2\right)$
$=x^2-(a+b)^2$
$=(x-a-b)(x+a+b)$

l) $2x^3-3x^2-8x+12$
$=x^2(2x-3)-4(2x-3)$
$=(2x-3)\left(x^2-4\right)$
$=(2x-3)(x-2)(x+2)$

2)

a) $\dfrac{4}{x+1}+\dfrac{8}{x-2}$
$=\dfrac{4(x-2)+8(x+1)}{(x+1)(x-2)}$
$=\dfrac{4x-8+8x+8}{(x+1)(x-2)}$
$=\dfrac{12x}{(x+1)(x-2)}$

b) $\dfrac{1}{x-2}+\dfrac{2}{(x-1)^2}-\dfrac{1}{x-1}$
$=\dfrac{1(x-1)^2+2(x-2)-(x-1)(x-2)}{(x-2)(x-1)^2}$
$=\dfrac{x^2-2x+1+2x-4-\left(x^2-3x+2\right)}{(x-2)(x-1)^2}$
$=\dfrac{x^2-3-x^2+3x-2}{(x-2)(x-1)^2}$
$=\dfrac{3x-5}{(x-2)(x-1)^2}$

c) $\dfrac{x^2+x-6}{x^2-6x+5}\div\dfrac{x^2+5x+6}{x^2-3x-10}$
$=\dfrac{x^2+x-6}{x^2-6x+5}\times\dfrac{x^2-3x-10}{x^2+5x+6}$
$=\dfrac{\cancel{(x+3)}(x-2)}{\cancel{(x-5)}(x-1)}\times\dfrac{\cancel{(x-5)}\cancel{(x+2)}}{\cancel{(x+3)}\cancel{(x+2)}}$
$=\dfrac{x-2}{x-1}$

d) $\dfrac{x^2+5x+6}{x^2-9}\div\dfrac{x^2-4}{x^2-5x+6}$
$=\dfrac{x^2+5x+6}{x^2-9}\times\dfrac{x^2-5x+6}{x^2-4}$
$=\dfrac{\cancel{(x+3)}\cancel{(x+2)}}{\cancel{(x+3)}\cancel{(x-3)}}\times\dfrac{\cancel{(x-3)}\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-2)}}$
$=1$

e) $\dfrac{1}{x-3}+\dfrac{2}{x+1}-\dfrac{3}{3-x}$
$=\dfrac{1}{x-3}+\dfrac{2}{x+1}+\dfrac{3}{x-3}$
$=\dfrac{1(x+1)+2(x-3)+3(x+1)}{(x+1)(x-3)}$
$=\dfrac{x+1+2x-6+3x+3}{(x+1)(x-3)}$
$=\dfrac{6x-2}{(x+1)(x-3)}$

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f) $\frac{x}{x-2}-\frac{1}{x+2}+\frac{8}{4-x^2}$

$=\frac{x}{x-2}-\frac{1}{x+2}-\frac{8}{x^2-4}$

$=\frac{x}{x-2}-\frac{1}{x+2}-\frac{8}{(x-2)(x+2)}$

$=\frac{x(x+2)-1(x-2)-8}{(x-2)(x+2)}$

$=\frac{x^2+2x-x+2-8}{(x-2)(x+2)}$

$=\frac{x^2+x-6}{(x-2)(x+2)}$

$=\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$

$=\frac{x+3}{x+2}$

g) $\frac{5}{x+3}-\frac{2}{3-x}+\frac{3-5x}{x^2-9}$

$=\frac{5}{x+3}+\frac{2}{x-3}+\frac{3-5x}{(x-3)(x+3)}$

$=\frac{5(x-3)+2(x+3)+3-5x}{(x-3)(x+3)}$

$=\frac{5x-15+2x+6+3-5x}{(x-3)(x+3)}$

$=\frac{2x-6}{(x-3)(x+3)}$

$=\frac{2\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$

$=\frac{2}{x+3}$

Exercise 2:

1)

a) $x^2-5x+6=0$

$\therefore (x-3)(x-2)=0$

$\therefore x=3 \text{ or } x=2$

b) $x^2+3x=0$

$\therefore x(x+3)=0$

$\therefore x=0 \text{ or } x=-3$

c) $(x-5)(x+12)=0$

$\therefore x=5 \text{ or } x=-12$

d) $(x-5)(x+12)=-70$

$\therefore x^2+7x-60+70=0$

$\therefore x^2+7x+10=0$

$\therefore (x+5)(x+2)=0$

$\therefore x=-5 \text{ or } x=-2$

e) $4x^2-12x+10=1$

$\therefore 4x^2-12x+9=0$

$\therefore (2x-3)^2=0$

$\therefore x=\frac{3}{2}$

f) $-x^3-8x^2-15x=0$

$\therefore x^3+8x^2+15x=0$

$\therefore x\left(x^2+8x+15\right)$

$\therefore x(x+5)(x+3)=0$

$\therefore x=0 \text{ or } x=-5 \text{ or } x=-3$

2)

a) $(y+3)^2=y^2+6y+9$

$\therefore y^2+6y+9=y^2+6y+9$

$\therefore 0=0$

$\therefore y\in\mathbb{R}$

b) $x^2+16=0$

$\therefore x^2=-16$

$\therefore x=\pm\sqrt{-16}$

$\therefore$ No real solution

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Exercise 3:

1)

a) $x+\frac{18}{x}=11$

$\therefore x^2+18=11x,\ x\neq 0$

$\therefore x^2-11x+18=0$

$\therefore (x-2)(x-9)=0$

$\therefore x=2 \text{ or } x=9$

b) $2x+4=\frac{6}{2x-1}$

$\therefore (2x+4)(2x-1)=6,\ x\neq\frac{1}{2}$

$\therefore 4x^2+6x-4=6$

$\therefore 4x^2+6x-10=0$

$\therefore 2x^2+3x-5=0$

$\therefore (2x+5)(x-1)=0$

$\therefore x=-\frac{5}{2} \text{ or } x=1$

c) $\frac{2}{x+1}-\frac{6}{2-x}=\frac{2x+4}{x+1}$

$\therefore \frac{2}{x+1}+\frac{6}{x-2}=\frac{2x+4}{x+1},\ x\neq -1 \text{ and } x\neq 2$

$\therefore 2(x-2)+6(x+1)=(2x+4)(x-2)$

$\therefore 2x-4+6x+6=2x^2-8$

$\therefore 2x^2-8x-10=0$

$\therefore x^2-4x-5=0$

$\therefore (x-5)(x+1)=0$

$\therefore x=5 \quad (x\neq -1)$

d) $\frac{4x-14}{x+4}-\frac{2x-4}{x+2}=0, x\neq -4 \text{ and } x\neq -2$

$\therefore (4x-14)(x+2)-(2x-4)(x+4)=0$

$\therefore 4x^2-6x-28-\left(2x^2+4x-16\right)=0$

$\therefore 2x^2-10x-12=0$

$\therefore x^2-5x-6=0$

$\therefore (x-6)(x+1)=0$

$\therefore x=6 \text{ or } x=-1$

e) $\frac{2x}{x-1}+\frac{1}{x+1}=1$

$\therefore 2x(x+1)+1(x-1)=1(x+1)(x-1),\ x\neq\pm 1$

$\therefore 2x^2+2x+x-1=x^2-1$

$\therefore x^2+3x=0$

$\therefore x(x+3)=0$

$\therefore x=0 \text{ or } x=-3$

f) $x-1+\frac{15}{x+2}=5,\ x\neq -2$

$\therefore (x-1)(x+2)+15=5(x+2)$

$\therefore x^2+x-2+15-5x-10=0$

$\therefore x^2-4x+3=0$

$\therefore (x-3)(x-1)=0$

$\therefore x=3 \text{ or } x=1$

g) $\frac{3x-24}{9}+\frac{9}{3x-6}=0,\ x\neq 2$

$\therefore (3x-24)(3x-6)+81=0$

$\therefore 9x^2-90x+225=0$

$\therefore x^2-10x+25=0$

$\therefore (x-5)^2=0$

$\therefore x=5$

h) $\frac{3}{x+3}+\frac{2x}{2-x}=1-\frac{5x+10}{x^2+x-6},\ x\neq -3 \text{ and } x\neq 2$

$\therefore \frac{3}{x+3}-\frac{2x}{x-2}=1-\frac{5x+10}{(x+3)(x-2)}$

$\therefore 3(x-2)-2x(x+3)=1(x+3)(x-2)-(5x+10)$

$\therefore 3x-6-2x^2-6x=x^2+x-6-5x-10$

$\therefore -2x^2-3x-6=x^2-4x-16$

$\therefore 3x^2-x-10=0$

$\therefore (3x+5)(x-2)=0$

$\therefore x=-\frac{5}{3} \quad (x\neq 2)$

i) $\frac{8}{y+1}+\frac{2y-6}{y^2+3y+2}=-\frac{10}{4-y^2},\ y\neq -1 \text{ and } y\neq\pm 2$

$\therefore \frac{8}{y+1}+\frac{2y-6}{(y+2)(y+1)}=\frac{10}{y^2-4}$

$\therefore \frac{8}{y+1}+\frac{2y-6}{(y+2)(y+1)}=\frac{10}{(y-2)(y+2)}$

$\therefore 8(y+2)(y-2)+(2y-6)(y-2)=10(y+1)$

$\therefore 8\left(y^2-4\right)+2y^2-10y+12=10y+10$

$\therefore 8y^2-32+2y^2-10y+12-10y-10=0$

$\therefore 10y^2-20y-30=0$

$\therefore y^2-2y-3=0$

$\therefore (y-3)(y+1)=0$

$\therefore y=3 \quad (y\neq -1)$

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2)

a) $\dfrac{3a-1}{a^2-2a-3}+\dfrac{3a}{a+1}+\dfrac{2}{3-a}=0$

$\therefore \dfrac{3a-1}{(a-3)(a+1)}+\dfrac{3a}{a+1}-\dfrac{2}{a-3}=0,\ a\neq 3 \text{ and } a\neq -1$

$\therefore 3a-1+3a(a-3)-2(a+1)=0$

$\therefore 3a-1+3a^2-9a-2a-2=0$

$\therefore 3a^2-8a-3=0$

$\therefore (3a+1)(a-3)=0$

$\therefore a=-\dfrac{1}{3}\ \ (a\neq 3)$

b) $\dfrac{5y+4}{y+2}+\dfrac{2y+3}{1-y}=\dfrac{2y^2-17}{y^2+y-2}$

$\therefore \dfrac{5y+4}{y+2}-\dfrac{2y+3}{y-1}-\dfrac{2y^2-17}{(y+2)(y-1)}=0,\ y\neq -2 \text{ and } y\neq 1$

$\therefore (5y+4)(y-1)-(2y+3)(y+2)-\left(2y^2-17\right)=0$

$\therefore 5y^2-y-4-\left(2y^2+7y+6\right)-2y^2+17=0$

$\therefore 3y^2-y+13-2y^2-7y-6=0$

$\therefore y^2-8y+7=0$

$\therefore (y-7)(y-1)=0$

$\therefore y=7\ \ (y\neq 1)$

c) $\dfrac{6p^2-53p+10}{p^2-3p-10}-\dfrac{3p}{5-p}-\dfrac{8p}{p+2}=0$

$\therefore \dfrac{6p^2-53p+10}{(p-5)(p+2)}+\dfrac{3p}{p-5}-\dfrac{8p}{p+2}=0,\ p\neq 5 \text{ and } p\neq -2$

$\therefore 6p^2-53p+10+3p(p+2)-8p(p-5)=0$

$\therefore 6p^2-53p+10+3p^2+6p-8p^2+40p=0$

$\therefore p^2-7p+10=0$

$\therefore (p-2)(p-5)=0$

$\therefore p=2\ \ (p\neq 5)$

d) $2-\dfrac{t+3}{2-t}+\dfrac{2t+6}{t^2+t-6}=0$

$\therefore 2+\dfrac{t+3}{t-2}+\dfrac{2t+6}{(t+3)(t-2)}=0,\ t\neq -3 \text{ and } t\neq 2$

$\therefore 2(t+3)(t-2)+(t+3)^2+2t+6=0$

$\therefore 2t^2+2t-12+t^2+6t+9+2t+6=0$

$\therefore 3t^2+10t+3=0$

$\therefore (3t+1)(t+3)=0$

$\therefore t=-\dfrac{1}{3}\ \ (t\neq -3)$

Equations with Substitution

Exercise 4:

a) $\left(x^2-x\right)^2-18\left(x^2-x\right)+72=0$

Let $k=x^2-x$

$\therefore k^2-18k+72=0$

$\therefore (k-12)(k-6)=0$

$\therefore k=12 \text{ or } k=6$

$\therefore x^2-x=12 \text{ or } x^2-x=6$

$\therefore x^2-x-12=0 \text{ or } x^2-x-6=0$

$\therefore (x-4)(x+3)=0 \text{ or } (x-3)(x+2)=0$

$\therefore x=4 \text{ or } x=-3 \text{ or } x=3 \text{ or } x=-2$

b) $\left(x^2+3x\right)^2-8\left(x^2+3x\right)-20=0$

Let $k=x^2+3x$

$\therefore k^2-8k-20=0$

$\therefore (k-10)(k+2)=0$

$\therefore k=10 \text{ or } k=-2$

$\therefore x^2+3x=10 \text{ or } x^2+3x=-2$

$\therefore x^2+3x-10=0 \text{ or } x^2+3x+2=0$

$\therefore (x+5)(x-2)=0 \text{ or } (x+2)(x+1)=0$

$\therefore x=-5 \text{ or } x=2 \text{ or } x=-2 \text{ or } x=-1$

c) $x^2-3x+4=-\dfrac{4}{x^2-3x}$

let $k=x^2-3x$

$\therefore k+4=-\dfrac{4}{k},\ \ k\neq 0\ (x\neq 0 \text{ and } x\neq 3)$

$\therefore k^2+4k+4=0$

$\therefore (k+2)^2=0$

$\therefore k=-2$

$\therefore x^2-3x=-2$

$\therefore x^2-3x+2=0$

$\therefore (x-2)(x-1)=0$

$\therefore x=2 \text{ or } x=1$

d) $\left(q^2+2q\right)^2-2\left(q^2+2q\right)-3=0$

Let $k=q^2+2q$

$\therefore k^2-2k-3=0$

$\therefore (k-3)(k+1)=0$

$\therefore k=3 \text{ or } k=-1$

$\therefore q^2+2q=3 \text{ or } q^2+2q=-1$

$\therefore q^2+2q-3=0 \text{ or } q^2+2q+1=0$

$\therefore (q+3)(q-1)=0 \text{ or } (q+1)(q+1)=0$

$\therefore q=-3 \text{ or } q=1 \text{ or } q=-1$

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e) $\frac{p^2+5p}{2}+5=-\frac{12}{p^2+5p}$

let $k=p^2+5p$

$\therefore \frac{k}{2}+5=-\frac{12}{k}, \quad k\neq 0\ (p\neq 0 \text{ and } p\neq -5)$

$\therefore k^2+10k+24=0$

$\therefore (k+4)(k+6)=0$

$\therefore k=-4 \text{ or } k=-6$

$\therefore p^2+5p=-4 \text{ or } p^2+5p=-6$

$\therefore p^2+5p+4=0 \text{ or } p^2+5p+6=0$

$\therefore (p+1)(p+4)=0 \text{ or } (p+2)(p+3)=0$

$\therefore p=-1 \text{ or } p=-4 \text{ or } p=-2 \text{ or } p=-3$

f) $\left(x^2-x\right)^2-8x^2+8x+12=0$

$\therefore \left(x^2-x\right)^2-8\left(x^2-x\right)+12=0$

let $k=x^2-x$

$\therefore k^2-8k+12=0$

$\therefore (k-6)(k-2)=0$

$\therefore k=6 \text{ or } k=2$

$\therefore x^2-x=6 \text{ or } x^2-x=2$

$\therefore x^2-x-6=0 \text{ or } x^2-x-2=0$

$\therefore (x-3)(x+2)=0 \text{ or } (x-2)(x+1)=0$

$\therefore x=3 \text{ or } x=-2 \text{ or } x=2 \text{ or } x=-1$

Quadratic Surd Equations

Exercise 5:

a) $\sqrt{x}+x=0$

$\therefore \sqrt{x}=-x$

$\therefore x=x^2$

$\therefore x^2-x=0$

$\therefore x(x-1)=0$

$\therefore x=0 \quad (x\neq 1)$

b) $x=\sqrt{x+6}$

$\therefore x^2=x+6$

$\therefore x^2-x-6=0$

$\therefore (x-3)(x+2)=0$

$\therefore x=3 \quad (x\neq -2)$

c) $2x-\sqrt{32-8x}=0$

$\therefore 2x=\sqrt{32-8x}$

$\therefore 4x^2=32-8x$

$\therefore 4x^2+8x-32=0$

$\therefore x^2+2x-8=0$

$\therefore (x-2)(x+4)=0$

$\therefore x=2 \quad (x\neq -4)$

d) $x+2=\sqrt{9x+18}$

$\therefore x^2+4x+4=9x+18$

$\therefore x^2-5x-14=0$

$\therefore (x-7)(x+2)=0$

$\therefore x=7 \text{ or } x=-2$

e) $x-4=\sqrt{56-5x}$

$\therefore x^2-8x+16=56-5x$

$\therefore x^2-3x-40=0$

$\therefore (x-8)(x+5)=0$

$\therefore x=8 \quad (x\neq -5)$

f) $x-7=\sqrt{2x+1}$

$\therefore x^2-14x+49=2x+1$

$\therefore x^2-16x+48=0$

$\therefore (x-12)(x-4)=0$

$\therefore x=12 \quad (x\neq 4)$

g) $\sqrt{22-7x}-x=-4$

$\therefore \sqrt{22-7x}=x-4$

$\therefore 22-7x=x^2-8x+16$

$\therefore x^2-x-6=0$

$\therefore (x-3)(x+2)=0$

$\therefore$ No solution on checking

h) $4\sqrt{x-3}-3=2x-9$

$\therefore 4\sqrt{x-3}=2x-6$

$\therefore 16(x-3)=4x^2-24x+36$

$\therefore 16x-48=4x^2-24x+36$

$\therefore 4x^2-40x+84=0$

$\therefore x^2-10x+21=0$

$\therefore (x-7)(x-3)=0$

$\therefore x=7 \text{ or } x=3$

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i) $\sqrt{x-5}-1=\dfrac{6}{\sqrt{x-5}}$

Let $k=\sqrt{x-5}$

$\therefore k-1=\dfrac{6}{k} \quad (k>0)$

$\therefore k^2-k-6=0$

$\therefore (k-3)(k+2)=0$

$\therefore k=3 \text{ or } k=-2$

$\therefore \sqrt{x-5}=3 \text{ or } \sqrt{x-5}=-2 \rightarrow \text{no solution}$

$\therefore x-5=9$

$\therefore x=14$

j) $\sqrt{2x-1}-\dfrac{3}{\sqrt{2x-1}}=-2$

Let $k=\sqrt{2x-1}$

$\therefore k-\dfrac{3}{k}=-2, (k>0)$

$\therefore k^2+2k-3=0$

$\therefore (k-1)(k+3)=0$

$\therefore k=1 \text{ or } k=-3$

$\therefore \sqrt{2x-1}=1 \text{ or } \sqrt{2x-1}=-3 \rightarrow \text{N/A}$

$\therefore 2x-1=1$

$\therefore x=1$

k) $\sqrt{12+x}=\sqrt{x}+3$

$\therefore 12+x=x+6\sqrt{x}+9$

$\therefore 3=6\sqrt{x}$

$\therefore \dfrac{1}{2}=\sqrt{x}$

$\therefore x=\dfrac{1}{4}$

l) $2\sqrt{4-x}=2\sqrt{x}-4$

$\therefore 4(4-x)=4x-16\sqrt{x}+16$

$\therefore 16-4x-4x-16=-16\sqrt{x}$

$\therefore -8x=-16\sqrt{x}$

$\therefore 64x^2=256x$

$\therefore 64x^2-256x=0$

$\therefore x^2-4x=0$

$\therefore x(x-4)=0$

$\therefore x=4 \quad (x\neq 0)$

Exponential Equations

Exercise 6:

a) $7^{-2a}=7^{2-3a}$

$\therefore -2a=2-3a$

$\therefore a=2$

b) $2^{3m}.2^{-m}=2^{-2m}$

$\therefore 2^{3m-m}=2^{-2m}$

$\therefore 3m-m=-2m$

$\therefore m=0$

c) $5^{-3x}.5^{x}=\dfrac{1}{5}$

$\therefore 5^{-2x}=5^{-1}$

$\therefore -2x=-1$

$\therefore x=1/2$

d) $\dfrac{81^{3y+2}}{243^{-y}}=3^4$

$\therefore \dfrac{\left(3^4\right)^{3y+2}}{\left(3^5\right)^{-y}}=3^4$

$\therefore 3^{12y+8+5y}=3^4$

$\therefore 17y+8=4$

$\therefore y=-4/17$

e) $5^{2x-4}=1$

$\therefore 5^{2x-4}=5^0$

$\therefore 2x-4=0$

$\therefore 2x=4$

$\therefore x=2$

f) $3^{2x-4}=1/9$

$\therefore 3^{2x-4}=3^{-2}$

$\therefore 2x-4=-2$

$\therefore 2x=2$

$\therefore x=1$

g) $4^x-2^{x+2}=0$

$\therefore 2^{2x}-2^x.2^2=0$

Let $k=2^x$

$\therefore k^2-4k=0$

$\therefore k(k-4)=0$

$\therefore k=0 \text{ or } k=4$

$\therefore 2^x=0 \text{ or } 2^x=2^2$

$\therefore x=2 \quad (2^x\neq 0)$

h) $9^x-3^x=6$

$\therefore 3^{2x}-3^x-6=0$

Let $k=3^x$

$\therefore k^2-k-6=0$

$\therefore (k-3)(k+2)=0$

$\therefore k=3 \text{ or } k=-2$

$\therefore 3^x=3^1 \text{ or } 3^x=-2$

$\therefore x=1 \quad (3^x\neq -2)$

i) $3^{2x+1}-28.3^x=-9$

$\therefore 3.3^{2x}-28.3^x+9=0$

Let $k=3^x$

$\therefore 3k^2-28k+9=0$

$\therefore (3k-1)(k-9)=0$

$\therefore k=\dfrac{1}{3} \text{ or } k=9$

$\therefore 3^x=\dfrac{1}{3} \text{ or } 3^x=9$

$\therefore 3^x=3^{-1} \text{ or } 3^x=3^2$

$\therefore x=-1 \text{ or } x=2$

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Exercise 7:

1)

a) $y+6x=4 \rightarrow [1]$

$2y=5x^2-22x+8 \rightarrow [2]$

from $[1]$: $y=-6x+4 \rightarrow [3]$

$[3]$ into $[2]$:

$\therefore 2(-6x+4)=5x^2-22x+8$

$\therefore -12x+8=5x^2-22x+8$

$\therefore 5x^2-10x=0$

$\therefore 5x(x-2)=0$

$\therefore x=0$ or $x=2 \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore y=4$ or $y=-8$

$\therefore x=0$ and $y=4$ or $x=2$ and $y=-8$

b) $2x-y=8 \rightarrow [1]$

$y=x^2+4x-23 \rightarrow [2]$

from $[1]$: $y=2x-8 \rightarrow [3]$

$[3]$ into $[2]$:

$\therefore 2x-8=x^2+4x-23$

$\therefore x^2+2x-15=0$

$\therefore (x-3)(x+5)=0$

$\therefore x=3$ or $x=-5 \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore y=-2$ or $y=-18$

$\therefore x=3$ and $y=-2$ or $x=-5$ and $y=-18$

c) $y+x=6 \rightarrow [1]$

$x^2+y^2=36 \rightarrow [2]$

from $[1]$: $x=6-y \rightarrow [3]$

$[3]$ into $[2]$:

$\therefore (6-y)^2+y^2=36$

$\therefore 36-12y+y^2+y^2=36$

$\therefore 2y^2-12y=0$

$\therefore 2y(y-6)=0$

$\therefore y=0$ or $y=6 \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore x=6$ or $x=0$

$\therefore x=6$ and $y=0$ or $x=0$ and $y=6$

d) $2x-3y=2 \rightarrow [1]$

$x^2-2y^2=8 \rightarrow$

from $[1]$: $x=\dfrac{3y+2}{2} \rightarrow [3]$

$[3]$ into $[2]$:

$\therefore \left(\dfrac{3y+2}{2}\right)^2-2y^2=8$

$\therefore \dfrac{9y^2+12y+4}{4}-2y^2=8$

$\therefore 9y^2+12y+4-8y^2-32=0$

$\therefore y^2+12y-28=0$

$\therefore (y-2)(y+14)=0$

$\therefore y=2$ or $y=-14 \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore x=4$ or $x=-20$

$\therefore x=4$ and $y=2$ or $x=-20$ and $y=-14$

e) $y-2=2(x-1)^2 \rightarrow [1]$

$y-x=2 \rightarrow [2]$

from $[1]$: $y=x+2 \rightarrow [3]$

$[3]$ into $[2]$:

$\therefore (x+2)-2=2(x-1)^2$

$\therefore x+2-2=2\left(x^2-2x+1\right)$

$\therefore x=2x^2-4x+2$

$\therefore 2x^2-5x+2=0$

$\therefore (x-2)(2x-1)=0$

$\therefore x=2$ or $x=\frac{1}{2} \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore y=4$ or $y=\frac{5}{2}$

$\therefore x=2$ and $y=4$ or $x=\frac{1}{2}$ and $y=\frac{5}{2}$

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f) $y-x=-6 \rightarrow [1]$

$2x^2+5xy=-2y^2 \rightarrow [2]$

from [1]: $y=x-6 \rightarrow [3]$

[3] into [2]:

$\therefore 2x^2+5x(x-6)=-2(x-6)^2$

$\therefore 2x^2+5x^2-30x=-2\left(x^2-12x+36\right)$

$\therefore 7x^2-30x=-2x^2+24x-72$

$\therefore 9x^2-54x+72=0$

$\therefore x^2-6x+8=0$

$\therefore (x-4)(x-2)=0$

$\therefore x=4$ or $x=2 \rightarrow [4]$

[4] into [3]:

$\therefore y=-2$ or $y=-4$

$\therefore x=4$ and $y=-2$ or $x=2$ and $y=-4$

g) $y-x=5 \rightarrow [1]$

$xy-3y=8x-20 \rightarrow [2]$

from [1]: $y=x+5 \rightarrow [3]$

[3] into [2]:

$\therefore x(x+5)-3(x+5)=8x-20$

$\therefore x^2+5x-3x-15=8x-20$

$\therefore x^2-6x+5=0$

$\therefore (x-5)(x-1)=0$

$\therefore x=5$ or $x=1 \rightarrow [4]$

[4] into [3]:

$\therefore y=10$ or $y=6$

$\therefore x=5$ and $y=10$ or $x=1$ and $y=6$

h) $2^{x+y}=256 \rightarrow [1]$

$xy+3x+2y=36 \rightarrow [2]$

from [1]: $2^{x+y}=256$

$\therefore 2^{x+y}=2^8$

$\therefore x+y=8$

$\therefore y=8-x \rightarrow [3]$

[3] into [2]:

$\therefore x(8-x)+3x+2(8-x)=36$

$\therefore 8x-x^2+3x+16-2x=36$

$\therefore x^2-9x+20=0$

$\therefore (x-5)(x-4)=0$

$\therefore x=5$ or $x=4 \rightarrow [4]$

[4] into [3]:

$\therefore y=3$ or $y=4$

$\therefore x=5$ and $y=3$ or $x=4$ and $y=4$

i) $5^{x-2y-1}=1 \rightarrow [1]$

$x^2+2xy-2x-4y=0 \rightarrow [2]$

from [1]: $5^{x-2y-1}=5^0$

$\therefore x-2y-1=0$

$\therefore x=2y+1 \rightarrow [3]$

[3] into [2]:

$\therefore (2y+1)^2+2(2y+1)y-2(2y+1)-4y=0$

$\therefore 4y^2+4y+1+4y^2+2y-4y-2-4y=0$

$\therefore 8y^2-2y-1=0$

$\therefore (2y-1)(4y+1)=0$

$\therefore y=\frac{1}{2}$ or $y=-\frac{1}{4} \rightarrow [4]$

[4] into [3]:

$\therefore x=2$ or $x=\frac{1}{2}$

$\therefore x=2$ and $y=\frac{1}{2}$ or $x=\frac{1}{2}$ and $y=-\frac{1}{4}$

j) $(x+5)^2+(y-3)^2=49 \rightarrow [1]$

$y+x=-9 \rightarrow [2]$

from [2]: $y=-x-9 \rightarrow [3]$

[3] into [1]:

$\therefore (x+5)^2+(-x-9-3)^2=49$

$\therefore (x+5)^2+(-x-12)^2=49$

$\therefore x^2+10x+25+x^2+24x+144-49=0$

$\therefore 2x^2+34x+120=0$

$\therefore x^2+17x+60=0$

$\therefore (x+5)(x+12)=0$

$\therefore x=-5$ or $x=-12 \rightarrow [4]$

[4] into [3]:

$\therefore y=-4$ or $y=3$

$\therefore x=-5$ and $y=-4$ or $x=-12$ and $y=3$

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2) The solutions to simultaneous equations represents the points of intersections of the two graphs. Therefore, the solution to the equations are the x and y-values of intersection.

$\therefore x=-3$ and $y=0$ or $x=0$ and $y=-24$.

$\therefore$ Points of intersection are $(-3;0)$ and $(0;-24)$

Completing the square

Exercise 8:

1)

a) $y=x^2+6x-8$

$\therefore y=\left[x^2+6x+(3)^2\right]-(3)^2-8$

$\therefore y=(x+3)^2-17$

b) $y=x^2-4x+9$

$\therefore y=\left[x^2-4x+(-2)^2\right]-(-2)^2+9$

$\therefore y=(x-2)^2+5$

c) $y=x^2-12x+2$

$\therefore y=\left[x^2-12x+(-6)^2\right]-(-6)^2+2$

$\therefore y=(x-6)^2-34$

d) $y=x^2-8x$

$\therefore y=\left[x^2-8x+(-4)^2\right]-(-4)^2$

$\therefore y=(x-4)^2-16$

e) $y=2x^2-4x+6$

$\therefore y=2\left[x^2-2x+3\right]$

$\therefore y=2\left[\left(x^2-2x+(-1)^2\right)-(-1)^2+3\right]$

$\therefore y=2\left[(x-1)^2+2\right]$

$\therefore y=2(x-1)^2+4$

f) $y=-3x^2-6x+12$

$\therefore y=-3\left[x^2+2x-4\right]$

$\therefore y=-3\left[\left(x^2+2x+(1)^2\right)-(1)^2-4\right]$

$\therefore y=-3\left[(x+1)^2-5\right]$

$\therefore y=-3(x+1)^2+15$

g) $y=-2x^2-3x+5$

$\therefore y=-2\left[x^2+\frac{3}{2}x-\frac{5}{2}\right]$

$\therefore y=-2\left[\left(x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2\right)-\left(\frac{3}{4}\right)^2-\frac{5}{2}\right]$

$\therefore y=-2\left[\left(x+\frac{3}{4}\right)^2-\frac{49}{16}\right]$

$\therefore y=-2\left(x+\frac{3}{4}\right)^2+\frac{49}{8}$

h) $y=3x^2+10x-2$

$\therefore y=3\left[x^2+\frac{10}{3}x-\frac{2}{3}\right]$

$\therefore y=3\left[\left(x^2+\frac{10}{3}x+\left(\frac{10}{6}\right)^2\right)-\left(\frac{10}{6}\right)^2-\frac{2}{3}\right]$

$\therefore y=3\left[\left(x+\frac{10}{6}\right)^2-\frac{31}{9}\right]$

$\therefore y=3\left(x+\frac{10}{6}\right)^2-\frac{31}{3}$

2)

a) $x^2-6x-5=0$

$\therefore x^2-6x=5$

$\therefore x^2-6x+(-3)^2=5+(-3)^2$

$\therefore (x-3)^2=14$

$\therefore x-3=\pm\sqrt{14}$

$\therefore x=3\pm\sqrt{14}$

b) $x^2+2x-7=0$

$\therefore x^2+2x=7$

$\therefore x^2+2x+(1)^2=7+(1)^2$

$\therefore (x+1)^2=8$

$\therefore x+1=\pm\sqrt{8}$

$\therefore x=-1\pm2\sqrt{2}$

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c) $2x^2+3x-8=0$

$\therefore 2x^2+3x=8$

$\therefore x^2+\frac{3}{2}x=4$

$\therefore x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2=4+\left(\frac{3}{4}\right)^2$

$\therefore \left(x+\frac{3}{4}\right)^2=\frac{73}{16}$

$\therefore x+\frac{3}{4}=\pm\frac{\sqrt{73}}{4}$

$\therefore x=\frac{-3\pm\sqrt{73}}{4}$

d) $-5x^2+15x+1=0$

$\therefore -5x^2+15x=-1$

$\therefore x^2-3x=\frac{1}{5}$

$\therefore x^2-3x+\left(-\frac{3}{2}\right)^2=\frac{1}{5}+\left(-\frac{3}{2}\right)^2$

$\therefore \left(x-\frac{3}{2}\right)^2=\frac{49}{20}$

$\therefore x-\frac{3}{2}=\pm\frac{7}{\sqrt{20}}$

$\therefore x=\frac{3}{2}\pm\frac{7}{\sqrt{20}}$

e) $x^2+2bx+3c=0$

$\therefore x^2+2bx=-3c$

$\therefore x^2+2bx+(b)^2=-3c+(b)^2$

$\therefore (x+b)^2=-3c+b^2$

$\therefore x+b=\pm\sqrt{b^2-3c}$

$\therefore x=-b\pm\sqrt{b^2-3c}$

f) $x^2-2x-3c=0$

$\therefore x^2-2x=3c$

$\therefore x^2-2x+(-1)^2=3c+(-1)^2$

$\therefore (x-1)^2=3c+1$

$\therefore x-1=\pm\sqrt{3c+1}$

$\therefore x=1\pm\sqrt{3c+1}$

g) $x^2+2x+3=0$

$\therefore x^2+2x=-3$

$\therefore x^2+2x+(1)^2=-3+(1)^2$

$\therefore (x+1)^2=-2$

$\therefore x+1=\pm\sqrt{-2}$

$\therefore$ No real solution

h) $ax^2+3bx-18=0$

$\therefore ax^2+3bx=18$

$\therefore x^2+\frac{3b}{a}x=\frac{18}{a}$

$\therefore x^2+\frac{3b}{a}x+\left(\frac{3b}{2a}\right)^2=\frac{18}{a}+\left(\frac{3b}{2a}\right)^2$

$\therefore \left(x+\frac{3b}{2a}\right)^2=\frac{72a+9b^2}{4a^2}$

$\therefore x+\frac{3b}{2a}=\pm\frac{\sqrt{72a+9b^2}}{2a}$

$\therefore x=\frac{-3b\pm\sqrt{9\left(8a+b^2\right)}}{2a}$

$\therefore x=\frac{-3b\pm3\sqrt{8a+b^2}}{2a}$

$\therefore x=\frac{3\left(-b\pm\sqrt{8a+b^2}\right)}{2a}$

Applications of Completing the Square

Exercise 9:

a) $x^2-2x-10$

$=\left[x^2-2x+(-1)^2\right]-(-1)^2-10$

$=(x-1)^2-11$

$\therefore$ Minimum value of -11

b) $-x^2-8x+3$

$=-\left[x^2+8x-3\right]$

$=-\left[\left(x^2+8x+(4)^2\right)-(4)^2-3\right]$

$=-(x+4)^2+19$

$\therefore$ Maximum value of 19

c) $x^2-12x+40$

$=\left[x^2-12x+(-6)^2\right]-(-6)^2+40$

$=(x-6)^2+4$

$\therefore$ Minimum value of 4

d) $-2x^2-12x+10$

$=-2\left[x^2+6x-5\right]$

$=-2\left[\left(x^2+6x+(3)^2\right)-(3)^2-5\right]$

$=-2(x+3)^2+28$

$\therefore$ Maximum value of 28

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The Quadratic Formula and Nature of Roots

Exercise 10:

1)

a) $x=\dfrac{-5\pm\sqrt{(5)^2-4(4)(-3)}}{2(4)}=\dfrac{-5\pm\sqrt{73}}{8}$

$\therefore x=0,4 \text{ or } x=-1,7$

b) $x=\dfrac{-(-3)\pm\sqrt{(-3)^2-4(7)(-2)}}{2(7)}=\dfrac{3\pm\sqrt{65}}{14}$

$\therefore x=0,8 \text{ or } x=-0,4$

c) $x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(-3)(5)}}{2(-3)}=-\dfrac{2\pm\sqrt{19}}{3}$

$\therefore x=0,8 \text{ or } x=-2,1$

d) $x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(2)(-12)}}{2(2)}=\dfrac{3\pm\sqrt{33}}{2}$

$\therefore x=4,4 \text{ or } x=-1,4$

2) $x=2,5 \text{ or } x=\sqrt{-1} \text{ or } x=-2 \text{ or } x=2 \text{ or } x=-\sqrt{3}$

a) $x=2 \text{ or } x=-2$

b) $x=2,5 \text{ or } x=-2 \text{ or } x=2 \text{ or } x=-\sqrt{3}$

c) $x=-\sqrt{3}$

d) $x=\sqrt{-1}$

3) $2x^2(x-2)+5x(x-2)-5(x-2)=0$

$\therefore (x-2)\left(2x^2+5x-5\right)=0$

$\therefore x=2 \text{ or } x=\dfrac{-5\pm\sqrt{5^2-4(2)(-5)}}{2(2)}=\dfrac{-5\pm\sqrt{65}}{4}$

a) $x=2$

b) $x=\dfrac{-5\pm\sqrt{65}}{4}$

4)

a) $\Delta=(-6)^2-4(14)(8)=-412$

$\therefore$ Roots are non-real.

b) $\Delta=(15)^2-4(9)(3)=117$

$\therefore$ Roots are real, unequal and irrational.

c) $\Delta=(40)^2-4(50)(8)=0$

$\therefore$ Roots are real and equal.

d) $\Delta=(-10)^2-4(-25)(-1)=0$

$\therefore$ Roots are real and equal.

Inequalities

Exercise 11:

1)

a) $x^2-4\geq 0$

$\therefore (x-2)(x+2)\geq 0$

$\therefore$ CV: $x=\pm 2$



$\therefore x\leq -2 \text{ or } x\geq 2$

also written as $x\in(-\infty;-2]$ or $[2;\infty)$

b) $x^2-x-6>0$

$\therefore (x+2)(x-3)>0$

$\therefore$ CV: $x=-2 \text{ or } x=3$



$\therefore x<-2 \text{ or } x>3$

also written as $x\in(-\infty;-2)$ or $(3;\infty)$

c) $x^2-3x-10\leq 0$

$\therefore (x+2)(x-5)\leq 0$

$\therefore$ CV: $x=-2 \text{ or } x=5$



$\therefore -2\leq x\leq 5$

also written as $x\in[-2;5]$

d) $x^2+9x+18\leq 0$

$\therefore (x+6)(x+3)\leq 0$

$\therefore$ CV: $x=-6 \text{ or } x=-3$



$\therefore -6\leq x\leq -3$

also written as $x\in[-6;-3]$

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e) $-x^2+2x+15<0$

$\therefore x^2-2x-15>0$

$\therefore (x+3)(x-5)>0$

$\therefore$ CV: $x=-3$ or $x=5$

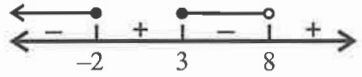


$\therefore x<-3$ or $x>5$

also written as $x\in(-\infty;-3)$ or $(5;\infty)$

f) $\dfrac{(x-3)(x+2)}{x-8}\le 0$

$\therefore$ CV: $x=3$ or $x=-2$ or $x=8$



$\therefore x\le -2$ or $3\le x<8$

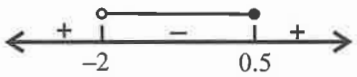
also written as $x\in(-\infty;-2]$ or $[3;8)$

g) $\dfrac{(x-3)}{x+2}+1\le 0$

$\therefore \dfrac{x-3+x+2}{x+2}\le 0$

$\therefore \dfrac{2x-1}{x+2}\le 0$

$\therefore$ CV: $x=-2$ or $x=\frac{1}{2}$



$\therefore -2<x\le \frac{1}{2}$

also written as $x\in\left(-2;\frac{1}{2}\right]$

h) $\dfrac{(x+1)(x-3)}{x-5}\ge 1$

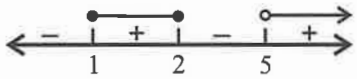
$\therefore \dfrac{x^2-2x-3}{x-5}-1\ge 0$

$\therefore \dfrac{x^2-2x-3-x+5}{x-5}\ge 0$

$\therefore \dfrac{x^2-3x+2}{x-5}\ge 0$

$\therefore \dfrac{(x-2)(x-1)}{x-5}\ge 0$

$\therefore$ CV: $x=1$ or $x=2$ or $x=5$



$\therefore 1\le x\le 2$ or $x>5$

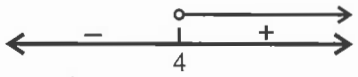
also written as $x\in[1;2]$ or $(5;\infty)$

i) $\dfrac{(x-2)(x+1)}{x-4}-2\ge 0$

$\therefore \dfrac{x^2-x-2-2x+8}{x-4}\ge 0$

$\therefore \dfrac{x^2-3x+6}{x-4}\ge 0$

$\therefore$ CV: $x=4$

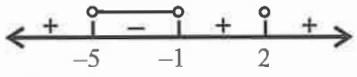


$\therefore x>4$

also written as $x\in(4;\infty)$

j) $\dfrac{(x-2)^2(x+5)}{x+1}<0$

$\therefore$ CV: $x=-5$ or $x=-1$ or $x=2$

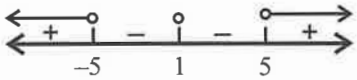


$\therefore -5<x<-1$

also written as $x\in(-5;-1)$

k) $\dfrac{(x-1)^2(x-5)}{x+5}>0$

$\therefore$ CV: $x=\pm 5$ or $x=1$



$\therefore x<-5$ or $x>5$

also written as $x\in(-\infty;-5)$ or $(5;\infty)$

l) $\dfrac{x^2+5x+11}{x-2}+1\ge 0$

$\therefore \dfrac{x^2+5x+11+x-2}{x-2}\ge 0$

$\therefore \dfrac{x^2+6x+9}{x-2}\ge 0$

$\therefore \dfrac{(x+3)^2}{x-2}\ge 0$

$\therefore$ CV: $x=-3$ or $x=2$



$\therefore x>2$ or $x=-3$

also written as $x\in(2;\infty)$ or $x=-3$

2)

a) $x=25$

b) $a\in\mathbb{R},\ a\neq -3$

c) $p\in\mathbb{R}$

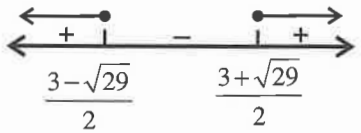
d) No Solution

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3)

a) $x^2-3x-5\geq 0$

$\therefore$ CV: $x=\dfrac{3+\sqrt{29}}{2}$ or $x=\dfrac{3-\sqrt{29}}{2}$



$\therefore x\leq\dfrac{3-\sqrt{29}}{2}$ or $x\geq\dfrac{3+\sqrt{29}}{2}$

This can also be written as:

$x\in\left(-\infty;\dfrac{3-\sqrt{29}}{2}\right]$ or $\left[\dfrac{3+\sqrt{29}}{2};\infty\right)$

b) $2x^2-2x+8\leq 0$

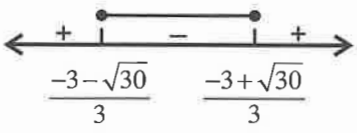
$\therefore 2\left[x^2-x+\left(-\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)^2+4\right]\leq 0$

$\therefore 2\left(x-\frac{1}{2}\right)^2+\frac{15}{2}\leq 0$

$\therefore$ No Solution since minimum value is $\frac{15}{2}$

c) $3x^2+6x-7\leq 0$

$\therefore$ CV: $x=\dfrac{-3+\sqrt{30}}{3}$ or $x=\dfrac{-3-\sqrt{30}}{3}$



$\therefore\dfrac{-3-\sqrt{30}}{3}\leq x\leq\dfrac{-3+\sqrt{30}}{3}$

This can also be written as:

$x\in\left[\dfrac{-3-\sqrt{30}}{3};\dfrac{-3+\sqrt{30}}{3}\right]$

d) $3x^2-5x+4>0$

$\therefore 3\left[x^2-\frac{5}{3}x+\left(-\frac{5}{6}\right)^2-\left(-\frac{5}{6}\right)^2+\frac{4}{3}\right]>0$

$\therefore 3\left(x-\frac{5}{6}\right)^2+\frac{23}{12}>0$

$\therefore x\in\mathbb{R}$

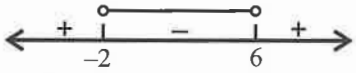
4)

a) $x=-10$

b) $x^2-4x-12<0,\ x\neq -10$

$\therefore (x-6)(x+2)<0$

$\therefore$ CV: $x=-2$ or $x=6$



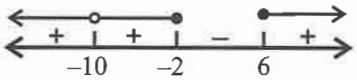
$\therefore -2<x<6$

also written as $x\in(-2;6)$

c) $x^2-4x-12\geq 0,\ x\neq -10$

$\therefore (x-6)(x+2)\geq 0$

$\therefore$ CV: $x=-2$ or $x=6$



$\therefore x<-10$ or $-10<x\leq -2$ or $x\geq 6$

This can also be written as:

$x\in(-\infty;-10]$ or $(-10;-2]$ or $[6;\infty)$

Mathematical Modelling

Exercise 12:

1)

Cable Type	Number of metres bought	Price per metre	Total Cost
A	$x+3$	$\dfrac{150}{x+3}$	150
B	x	$\dfrac{144}{x}$	144

$\dfrac{144}{x}-\dfrac{150}{x+3}=2\rightarrow$ the difference in price between Type A and B is R2

$\therefore 144(x+3)-150x=2x(x+3)$

$\therefore 144x+432-150x=2x^2+6x$

$\therefore 2x^2+12x-432=0$

$\therefore x^2+6x-216=0$

$\therefore (x-12)(x+18)=0$

$\therefore x=12$ or $x=-18\rightarrow$ N/A

$\therefore$ He purchased 12 metres of type B and 15 metres of type A

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2)

Price	Number of kilogram bought	Price per kilogram	Total Cost
Original	$\frac{60}{x}$	x	60
After increase	$\frac{60}{x+5}$	$x+5$	60

$\frac{60}{x}-\frac{60}{x+5}=2 \rightarrow$ 2 kilograms less were bought for the same price

$\therefore 60(x+5)-60x=2x(x+5)$

$\therefore 60x+300-60x=2x^2+10x$

$\therefore 2x^2+10x-300=0$

$\therefore x^2+5x-150=0$

$\therefore (x-10)(x+15)=0$

$\therefore x=10$ or $x=-15 \rightarrow$ N/A

$\therefore$ She purchased 6 kilograms at R10 per kilogram.

3)

Situation	Number of muffins	Price per muffin	Total Cost
Baked	x	$\frac{250}{x}$	250
Sold	$x-2$	$\frac{624}{x-2}$	624

$\frac{624}{x-2}-\frac{250}{x}=8 \rightarrow$ Selling price $-$ cost price $=8$

$\therefore 624x-250(x-2)=8x(x-2)$

$\therefore 624x-250x+500=8x^2-16x$

$\therefore 8x^2-390x-500=0$

$\therefore 4x^2-195x-250=0$

$\therefore (x-50)(4x+5)=0$

$\therefore x=50$ or $x=-5/4 \rightarrow$ N/A

$\therefore$ The cost to make each muffin was $\frac{250}{50}=$ R5 per muffin.

4)

Type of Ribbon	Number of metres	Cost per metre	Total Cost
Blue	$\frac{80}{x}$	x	80
Gold	$\frac{90}{x+1}$	$x+1$	90

$\frac{80}{x}-\frac{90}{x+1}=10 \rightarrow 10$ metres more of blue ribbon is bought than of gold

$\therefore 80(x+1)-90x=10x(x+1)$

$\therefore 80x+80-90x=10x^2+10x$

$\therefore 10x^2+20x-80=0$

$\therefore x^2+2x-8=0$

$\therefore (x-2)(x+4)=0$

$\therefore x=2$ or $x=-4 \rightarrow$ N/A

$\therefore$ 40 m of blue ribbon was bought at a cost of R2/m and 30 m of gold ribbon was bought at a cost of R3/m.

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Exercise 13:

1)

Situation	Speed	Distance	Time
Original Speed	x	100	$\frac{100}{x}$
Increased Speed	$x+5$	100	$\frac{100}{x+5}$

$\frac{100}{x}-\frac{100}{x+5}=1 \rightarrow$ Longer time – shorter time = time saved

$\therefore 100(x+5)-100x=x(x+5)$

$\therefore 100x+500-100x=x^2+5x$

$\therefore x^2+5x-500=0$

$\therefore (x-20)(x+25)=0$

$\therefore x=20$ or $x=-25 \rightarrow$ N/A

$\therefore$ He cycled at 20 km/h in the race.

2)

Situation	Speed	Distance	Time
To emergency	x	30	$\frac{30}{x}$
Return Journey	$x-30$	30	$\frac{30}{x-30}$

$\frac{30}{x-30}-\frac{30}{x}=\frac{5}{60} \rightarrow$ time saved has to be in hours as speed is in kilometres per **hour**

$\therefore 30(60)x-30(60)(x-30)=5x(x-30)$

$\therefore 1800x-1800x+54000=5x^2-150x$

$\therefore 5x^2-150x-54000=0$

$\therefore x^2-30x-10800=0$

$\therefore (x-120)(x+90)=0$

$\therefore x=120$ or $x=-90 \rightarrow$ N/A

$\therefore$ He drove to the emergency at 120 km/h.

3)

Situation	Speed	Distance	Time
Part One	$2,5x$	20	$\frac{20}{2,5x}$
Part Two	x	12	$\frac{12}{x}$

$\frac{20}{2,5x}+\frac{12}{x}=5 \rightarrow$ total time taken was 5 hours

$\therefore 20+12(2,5)=5(2,5x)$

$\therefore 50=12,5x$

$\therefore x=4$

$\therefore$ Part 1 took 2 hours $\left(\frac{20}{2,5\times 4}\right)$ and Part 2 took 3 hours $\left(\frac{12}{4}\right)$.

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4)

Situation	Speed	Distance	Time
To School	40	x	$\frac{x}{40}$
Return Journey	60	x	$\frac{x}{60}$

$$\text{Average speed} = \frac{x+x}{\frac{x}{40}+\frac{x}{60}} \rightarrow \frac{\text{total distance}}{\text{total time}}$$

$$= 2x \div \left(\frac{60x+40x}{2400}\right)$$

$$= 2\cancel{x} \times \frac{2400}{100\cancel{x}}$$

$$= 48\,\text{km/h}$$

Exercise 14:

1) $2x+2y=10 \rightarrow [1]$

$xy=6 \rightarrow [2]$

From $[1]$: $x=5-y \rightarrow [3]$

$[3]$ into $[2]$

$\therefore (5-y)y=6$

$\therefore 5y-y^2=6$

$\therefore y^2-5y+6=0$

$\therefore (y-3)(y-2)=0$

$\therefore y=3$ or $y=2 \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore x=2$ or $x=3$

$\therefore$ The dimensions are $3\,\text{cm} \times 2\,\text{cm}$

2) $xy=40 \rightarrow [1]$

$(x+3)(y+2)=80 \rightarrow [2]$

From $[1]$: $x=\frac{40}{y} \rightarrow [3]$

$[3]$ into $[2]$

$\therefore \left(\frac{40}{y}+3\right)(y+2)=80$

$\therefore 40+\frac{80}{y}+3y+6=80$

$\therefore 40y+80+3y^2+6y=80y$

$\therefore 3y^2-34y+80=0$

$\therefore (y-8)(3y-10)=0$

$\therefore y=8$ or $y=\frac{10}{3} \rightarrow [4]$

$[4]$ into $[3]$:

$\therefore x=5$ or $x=12$

$\therefore$ The dimensions are $8\,\text{m} \times 5\,\text{m}$ or $\frac{10}{3}\,\text{m} \times 12\,\text{m}$

3) $\text{Area}_{\text{border}} = \text{Area}_{\text{board}} - \text{Area}_{\text{picture}}$

$\therefore (5+2x)(3+2x)-(5)(3)=48$

$\therefore 15+16x+4x^2-15=48$

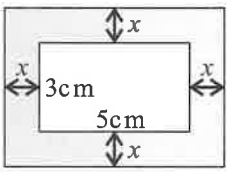
$\therefore 4x^2+16x-48=0$

$\therefore x^2+4x-12=0$

$\therefore (x-2)(x+6)=0$

$\therefore x=2$ or $x=-6 \rightarrow$ N/A

$\therefore$ The border is $2\,\text{cm}$ in width.



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Exercise 15:

1) $\frac{1}{8}+\frac{1}{4}=\frac{1}{x}$

$\therefore x+2x=8$

$\therefore 3x=8$

$\therefore x=8/3$ hours working together.

2) $\frac{1}{3}+\frac{1}{6}=\frac{1}{x}$

$\therefore 2x+x=6$

$\therefore 3x=6$

$\therefore x=2$ hours operating together.

3) $\frac{1}{x}+\frac{1}{x-60}=\frac{1}{40}$

$\therefore 40(x-60)+40x=x(x-60)$

$\therefore 40x-2400+40x=x^2-60x$

$\therefore x^2-140x+2400=0$

$\therefore (x-120)(x-20)=0$

$\therefore x=120$ or $x=20\rightarrow$ N/A

$\therefore$ It would take 120 days using the smaller engine only.

4) $\frac{1}{x}+\frac{1}{x+3}=\frac{1}{\frac{20}{3}}$

$\therefore \frac{1}{x}+\frac{1}{x+3}=\frac{3}{20}$

$\therefore 20(x+3)+20x=3x(x+3)$

$\therefore 20x+60+20x=3x^2+9x$

$\therefore 3x^2-31x-60=0$

$\therefore (x-12)(3x+5)=0$

$\therefore x=12$ or $x=-\frac{5}{3}\rightarrow$ N/A

$\therefore$ The faster machine takes 12 hours on its own.

Mixed Exercise:

1)

a) $\frac{16(x+2)-x^2(x+2)}{(x+2)(x+4)}$

$=\frac{(x+2)(16-x^2)}{(x+2)(x+4)}$

$=\frac{(4-x)(4+x)}{x+4}$

$=4-x$

b) $\frac{5}{3x-3}+\frac{2x+1}{1-x^2}+\frac{1}{2x+2}$

$=\frac{5}{3(x-1)}-\frac{2x+1}{(x-1)(x+1)}+\frac{1}{2(x+1)}$

$=\frac{5(2)(x+1)-6(2x+1)+3(x-1)}{6(x-1)(x+1)}$

$=\frac{10x+10-12x-6+3x-3}{6(x-1)(x+1)}$

$=\frac{x+1}{6(x-1)(x+1)}$

$=\frac{1}{6x-6}$

c) $\frac{x^2+xy}{x-y}\div\frac{x^2-y^2}{y-x}$

$=\frac{x(x+y)}{(x-y)}\times\frac{-(x-y)}{(x-y)(x+y)}$

$=-\frac{x}{x-y}$

d) $\frac{a}{a-1}+\frac{12}{1-a^2}-\frac{6}{a+1}$

$=\frac{a}{a-1}-\frac{12}{(a-1)(a+1)}-\frac{6}{a+1}$

$=\frac{a(a+1)-12-6(a-1)}{(a-1)(a+1)}$

$=\frac{a^2+a-12-6a+6}{(a-1)(a+1)}$

$=\frac{a^2-5a-6}{(a-1)(a+1)}$

$=\frac{(a-6)(a+1)}{(a-1)(a+1)}$

$=\frac{a-6}{a-1}$

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e) $\dfrac{2x+2}{1-x-2x^2} \div \dfrac{4-8x}{4x^2-4x+1}$

$= \dfrac{2(x+1)}{-(2x^2+x-1)} \times \dfrac{(2x-1)(2x-1)}{4(1-2x)}$

$= \dfrac{2\cancel{(x+1)}}{-\cancel{(2x-1)}\cancel{(x+1)}} \times \dfrac{\cancel{(2x-1)}\cancel{(2x-1)}}{-4\cancel{(2x-1)}}$

$= 1/2$

f) $\dfrac{p^2-5p+6}{1-p^2} \times \dfrac{p^2+2p-3}{9-p^2}$

$= \dfrac{\cancel{(p-3)}(p-2)}{-\cancel{(p-1)}(p+1)} \times \dfrac{\cancel{(p+3)}\cancel{(p-1)}}{-\cancel{(p-3)}\cancel{(p+3)}}$

$= \dfrac{p-2}{p+1}$

2)

a) $2(x^2+6)+11x=0$

$\therefore 2x^2+11x+12=0$

$\therefore (2x+3)(x+4)=0$

$\therefore x=-3/2$ or $x=-4$

b) $(x-2)(x^2-5)(x^2+9)=0$

$\therefore x=2$ or $x^2=5$ or $x^2=-9 \rightarrow$ N/A

$\therefore x=2$ or $x=\pm\sqrt{5}$

c) $\dfrac{6}{x+1}-\dfrac{3x}{1-x^2}=\dfrac{2x}{x-1} \rightarrow x \neq \pm 1$

$\therefore \dfrac{6}{x+1}+\dfrac{3x}{(x-1)(x+1)}-\dfrac{2x}{x-1}=0$

$\therefore 6(x-1)+3x-2x(x+1)=0$

$\therefore 6x-6+3x-2x^2-2x=0$

$\therefore 2x^2-7x+6=0$

$\therefore (2x-3)(x-2)=0$

$\therefore x=3/2$ or $x=2$

d) $(x^2-5x+2)^2-(x^2-5x)=8$

Let $k=x^2-5x$

$\therefore (k+2)^2-k-8=0$

$\therefore k^2+4k+4-k-8=0$

$\therefore k^2+3k-4=0$

$\therefore (k+4)(k-1)=0$

$\therefore k=-4$ or $k=1$

$\therefore x^2-5x+4=0$ or $x^2-5x-1=0$

$\therefore (x-4)(x-1)=0$ or $x=\dfrac{5\pm\sqrt{29}}{2}$

$\therefore x=4$ or $x=1$ or $x=\dfrac{5\pm\sqrt{29}}{2}$

e) $x^2+6x-\dfrac{35}{x^2+6x}=2$

Let $k=x^2+6x$, $x\neq 0$ and $x\neq -6$

$\therefore k-35/k=2$

$\therefore k^2-2k-35=0$

$\therefore (k-7)(k+5)=0$

$\therefore k=7$ or $k=-5$

$\therefore x^2+6x-7=0$ or $x^2+6x+5=0$

$\therefore (x-1)(x+7)=0$ or $(x+1)(x+5)=0$

$\therefore x=1$ or $x=-7$ or $x=-1$ or $x=-5$

f) $\dfrac{x}{x-2}=\dfrac{1}{x-3}-\dfrac{2}{2-x} \rightarrow x\neq 2$ and $x\neq 3$

$\therefore \dfrac{x}{x-2}=\dfrac{1}{x-3}+\dfrac{2}{x-2}$

$\therefore \dfrac{x}{x-2}-\dfrac{1}{x-3}-\dfrac{2}{x-2}=0$

$\therefore x(x-3)-1(x-2)-2(x-3)=0$

$\therefore x^2-3x-x+2-2x+6=0$

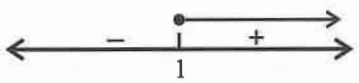
$\therefore x^2-6x+8=0$

$\therefore (x-4)(x-2)=0$

$\therefore x=4$ $(x\neq 2)$

g) $(x^2+4)(x-1)\geq 0$

$\therefore$ CV: $x=1$



$\therefore x\geq 1$

also written as $x\in[1;\infty)$

h) $x^2-2x+7\geq 0$

$\therefore \left[x^2-2x+(-1)^2\right]-(-1)^2+7\geq 0$

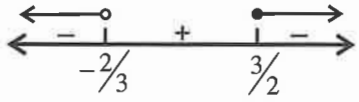
$\therefore (x-1)^2+6\geq 0$

$\therefore x\in\mathbb{R}$

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i) $\dfrac{(x^2+1)(3-2x)}{3x+2} \le 0$

$\therefore$ CV: $x = 3/2$ or $x = -2/3$



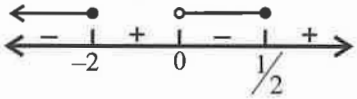
$\therefore x < -2/3$ or $x \ge 3/2$

also written as $x \in (-\infty; -2/3)$ or $[3/2; \infty)$

j) $\dfrac{2x^2+3x-2}{3x} \le 0$

$\therefore \dfrac{(2x-1)(x+2)}{3x} \le 0$

$\therefore$ CV: $x = 1/2$ or $x = -2$ or $x = 0$



$\therefore x \le -2$ or $0 < x \le 1/2$

also written as $x \in (-\infty; -2]$ or $(0; 1/2]$

3)

a) $1-2x+\sqrt{5x-1} = 0$

$\therefore 1-2x = -\sqrt{5x-1}$

$\therefore 1-4x+4x^2 = 5x-1$

$\therefore 4x^2-9x+2 = 0$

$\therefore (x-2)(4x-1) = 0$

$\therefore x = 2 \quad (x \ne 1/4)$

b) $x+5 = \sqrt{3-3x}$

$\therefore x^2+10x+25 = 3-3x$

$\therefore x^2+13x+22 = 0$

$\therefore (x+2)(x+11) = 0$

$\therefore x = -2 \quad (x \ne -11)$

4)

a) $x-2y = -1 \rightarrow \boxed{1}$

$x^2-xy+y^2 = 7 \rightarrow \boxed{2}$

from $\boxed{1}$: $x = 2y-1 \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{2}$:

$(2y-1)^2-(2y-1)y+y^2 = 7$

$\therefore 4y^2-4y+1-2y^2+y+y^2 = 7$

$\therefore 3y^2-3y-6 = 0$

$\therefore (y-2)(y+1) = 0$

$\therefore y = 2$ or $y = -1 \rightarrow \boxed{4}$

$\boxed{4}$ into $\boxed{3}$:

$\therefore x = 3$ or $x = -3$

$\therefore x = 3$ and $y = 2$ or $x = -3$ and $y = -1$

b) $2^{y-3x} = 1/16 \rightarrow \boxed{1}$

$x^2+xy = 24 \rightarrow \boxed{2}$

from $\boxed{1}$: $2^{y-3x} = 2^{-4}$

$\therefore y-3x = -4$

$\therefore y = 3x-4 \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{2}$:

$x^2+x(3x-4) = 24$

$\therefore x^2+3x^2-4x = 24$

$\therefore 4x^2-4x-24 = 0$

$\therefore (x+2)(x-3) = 0$

$\therefore x = -2$ or $x = 3 \rightarrow \boxed{4}$

$\boxed{4}$ into $\boxed{3}$:

$\therefore y = -10$ or $y = 5$

$\therefore x = -2$ and $y = -10$ or $x = 3$ and $y = 5$

5)

a) $-3x^2-7x+12 = 0$

$\therefore x = \dfrac{-(-7) \pm \sqrt{(-7)^2-4(-3)(12)}}{2(-3)}$

$\therefore x = 1{,}1$ or $x = -3{,}5$

b) $3x^2+5x+1 = 0$

$\therefore x = \dfrac{-5 \pm \sqrt{(5)^2-4(3)(1)}}{2(3)}$

$\therefore x = -0{,}2$ or $x = -1{,}4$

c) $4x^2+7x+2 = 0$

$\therefore x = \dfrac{-7 \pm \sqrt{(7)^2-4(4)(2)}}{2(4)}$

$\therefore x = -1{,}4$ or $x = -0{,}4$

d) $8x^2+3x-2 = 0$

$\therefore x = \dfrac{-3 \pm \sqrt{(3)^2-4(8)(-2)}}{2(8)}$

$\therefore x = 0{,}3$ or $x = -0{,}7$

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6)

a) x^2-2x

$=\left[x^2-2x+(-1)^2\right]-(-1)^2$

$=(x-1)^2-1$

$\therefore$ minimum value of -1.

b) $-2x^2+12x-17$

$=-2\left[x^2-6x+{}^{17}/_{2}\right]$

$=-2\left[x^2-6x+(-3)^2-(-3)^2+{}^{17}/_{2}\right]$

$=-2\left[(x-3)^2-{}^{1}/_{2}\right]$

$=-2(x-3)^2+1$

$\therefore$ maximum value of 1.

c) $\dfrac{2}{x^2-6x+10}\rightarrow$ max when denominator is a min

$\therefore$ complete the square on: $x^2-6x+10$

$=\left[x^2-6x+(-3)^2\right]-(-3)^2+10$

$=(x-3)^2+1$

$\therefore$ maximum value of $\dfrac{2}{1}=2$.

d) $\dfrac{6}{-x^2+8x-13}\rightarrow$ min when denominator is a max

$\therefore$ complete the square on: $-x^2+8x-13$

$=-\left[x^2-8x+13\right]$

$=-\left[x^2-8x+(-4)^2-(-4)^2+13\right]$

$=-\left[(x-4)^2-3\right]$

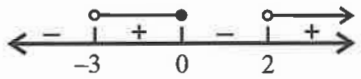
$=-(x-4)^2+3$

$\therefore$ minimum value of $\dfrac{6}{3}=2$.

7)

a) $\dfrac{2x}{(x+3)(x-2)}\geq 0$

CV: $x=-3$ or $x=0$ or $x=2$

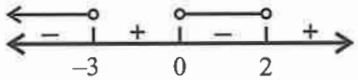


$\therefore -3<x\leq 0$ or $x>2$

also written as $x\in(-3;0]$ or $(2;\infty)$

b) $\dfrac{2x}{(x+3)(x-2)}<0$

CV: $x=-3$ or $x=0$ or $x=2$



$\therefore x<-3$ or $0<x<2$

also written as $x\in(-\infty;-3)$ or $(0;2)$

c) $x=-3$ or $x=2$

8) To be a perfect square $\left({}^{2k}/_{2}\right)^2=5k+6$

$\therefore k^2-5k-6=0$

$\therefore (k-6)(k+1)=0$

$\therefore k=6$ or $k=-1$

9)

Situation	Price per litre	Number of litres	Total Cost (in cents)
Before increase	x	$\dfrac{5280}{x}$	5280
After increase	$x+5$	$\dfrac{5280}{x+5}$	5280

$\dfrac{5280}{x}-\dfrac{5280}{x+5}=1\rightarrow 1$ litre less can be purchased

$\therefore 5280(x+5)-5280x=x(x+5)$

$\therefore x^2+5x-26400=0$

$\therefore (x-160)(x+165)=0$

$\therefore x=160$ or $x=-165\rightarrow$ N/A

$\therefore$ The current price is R1,60 per litre.

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10)

Cyclist	Speed	Distance	Time
Sipho	x	33	$\frac{33}{x}$
Jane	$x-1$	33	$\frac{33}{x-1}$

$\frac{33}{x-1}-\frac{33}{x}=\frac{18}{60}\rightarrow$ time must be in hours

$\therefore 33(60)(x)-33(60)(x-1)=18x(x-1)$

$\therefore 1980x-1980x+1980=18x^2-18x$

$\therefore 18x^2-18x-1980=0$

$\therefore x^2-x-110=0$

$\therefore (x-11)(x+10)=0$

$\therefore x=11$ or $x=-10\rightarrow$ N/A

$\therefore$ Sipho cycles at 11km/h.

11) $4x^2-4x-2p=0$

$\Delta=(-4)^2-4(4)(-2p)$

$=16+32p$

For real roots $\Delta\geq 0$

$\therefore 16+32p\geq 0$

$\therefore 32p\geq -16$

$\therefore p\geq -\frac{1}{2}$

12) $3y^2-2yx+3=0$

$\Delta=(-2x)^2-4(3)(3)$

$=4x^2-36$

For non-real roots $\Delta<0$

$\therefore 4x^2-36<0$

$\therefore 4\left(x^2-9\right)<0\rightarrow$ divide both sides by 4

$\therefore (x-3)(x+3)<0$

CV: $x=3$ or $x=-3$



$\therefore -3<x<3$

13) $(p+2)x^2+8x+(2-p)=0$

$\Delta=(8)^2-4(p+2)(2-p)$

$=64-4\left(-p^2+4\right)$

$=64+4p^2-16$

$=4p^2+48$

$4p^2\geq 0$

$\therefore 4p^2+48>0$

$\therefore$ roots are real and unequal.

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Arithmetic (Linear) Number Patterns

Exercise 1: For each of the sequences below:

a) **Write down the next 3 terms in the sequence.**
b) **Determine a formula for the n^{th} term of the sequence.**
c) **Hence, determine the 50^{th} term in each sequence.**

1) $-4;-2;0;\ldots$
2) $2;5;8;\ldots$
3) $29;21;13;\ldots$
4) $\frac{2}{3};\frac{4}{3};2;\ldots$
5) $5;\frac{19}{4};\frac{9}{2};\ldots$
6) $\frac{32}{3};\frac{34}{3};12;\ldots$

Exercise 2:

1) **Given the number pattern 12; 6; 0;... determine:**
   a) A formula for the n^{th} term in the sequence.
   b) Which term of the sequence will be –72.

2) **Angie is planning a party and calculates that it will cost her R440 for decorations. She also works out that in order to cater for her guests it will cost R15 per person. You may assume that at least one person will attend the party.**
   a) Determine a general formula that Angie can use to calculate the cost of having n people at her party.
   b) How much would Angie have to spend to have 35 people at her party?
   c) If Angie can afford to spend R1180, how many people can she have at her party?

3) **A Supermarket makes a display of tinned peas by stacking the tins in rows on top of each other. There are 22 tins in the 14^{th} row from the bottom and 49 tins in the 5^{th} row from the bottom. The number of tins in each row form an arithmetic sequence.**
   a) Determine a formula for the number of tins in any row of the display.
   b) Which row will have 10 tins of peas in it?

Quadratic Number Patterns

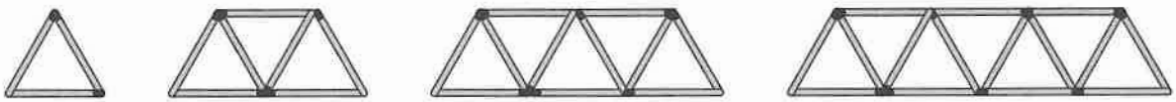
Exercise 3: For each of the quadratic sequences below:

a) **Determine a formula for the n^{th} term of the sequence.**
b) **Hence, determine the 25^{th} term in each sequence.**

1) $7;16;29;46;\ldots$
2) $1;18;41;70;\ldots$
3) $4;10;18;28;\ldots$
4) $\frac{5}{2};14;\frac{67}{2};61;\ldots$
5) $-3;13;53;117;\ldots$
6) $-\frac{9}{2};-\frac{7}{2};\frac{7}{2};\frac{33}{2};\ldots$

Mixed Exercise:

1) **The following pattern is formed using matchsticks. Determine the number of matchsticks that will be in the 30^{th} pattern of the sequence.**



3 7 11 15

2) **A pattern is formed from black and grey tiles as shown below. Determine the tiles that will be required for the 18^{th} figure of the sequence.**

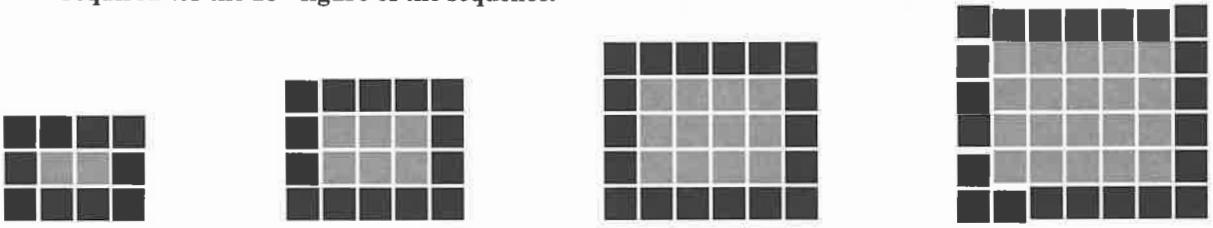


Figure 1 Figure 2 Figure 3 Figure 4

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3) **A triangular pattern is formed from a single dot by adding a row of dots to each subsequent triangle as shown below. Determine which figure would contain 28 dots.**

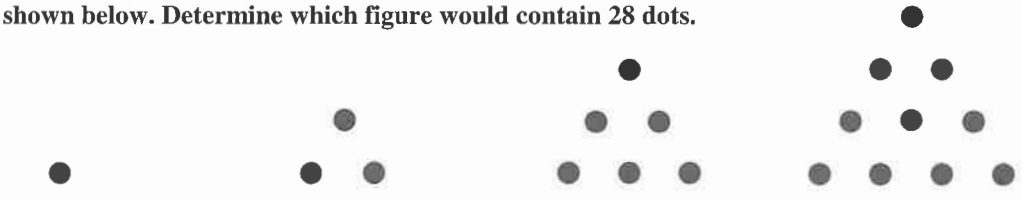


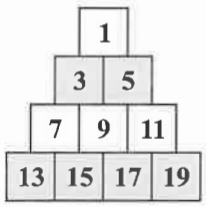
Figure 1 Figure 2 Figure 3 Figure 4

4) **Determine the 12th term of the following sequences:** (hint: the sequence may not be linear or quadratic)

a) 1 ; 4 ; 9 ; 16 ; 25 ...

b) 1 ; 8 ; 27 ; 64 ; 125 ...

5) **Determine the sum of the 20th row in the figure below.**



6) **Given the sequence 575 ; 528 ; 483 ; 440 ; ... determine:**

a) A formula for the nth term of the sequence.

b) Which term/s of the sequence would have a value of zero?

c) The term that has the lowest value in the sequence.

7) **Given that the sequence x ; 16 ; 32 ; 54 ; y ; 116 ; 156 ; z ; ... is quadratic, determine:**

a) The values of x , y and z.

b) A formula for the nth term of the sequence.

8) **If the 8th, 9th and 10th terms of a quadratic sequence are -15 ; -8 ; 1 , find a formula for the nth term.**

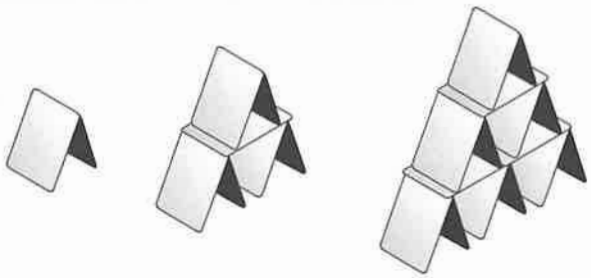
9) **Anne decides to build honeycombs out of stacks of glued ice-cream sticks for a biology project. The pattern shows the formation of the honeycombs. Each wall contains 10 ice-cream sticks glued together. Complete the table below and determine a formula for the number of ice-cream sticks required for the nth step as well as the number of honeycombs in the nth step.**



Step 1 Step 2 Step 3 Step 4

Step	1	2	3	4	5
Honeycombs	1			10	
Walls	6			42	

10) **The diagrams below shows the formation of a card tower.**



No. of Storeys	1	2	3	4	5	y
No. of Cards	2	7	15	26	x	155

a) Determine the values of x and y in the table above.

b) A deck of playing cards contains 54 cards, how many storeys can be built with a single deck?

c) If each playing card is 9cm in length and the tower forms equilateral triangles, determine the maximum height of a tower formed by a single deck of cards.

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Chapter 3– Number Patterns (Memo)

Arithmetic (Linear) Number Patterns

Exercise 1:

1*a*) $2;4;6$

b) $T_n = -4+(n-1)(2)$

c) $T_{50} = -4+(49)(2) = 94$

2*a*) $11;14;17$

b) $T_n = 2+(n-1)(3)$

c) $T_{50} = 2+(49)(3) = 149$

3*a*) $5;-3;-11$

b) $T_n = 29+(n-1)(-8)$

c) $T_{50} = 29+(49)(-8) = -363$

4*a*) $\frac{8}{3};\frac{10}{3};4$

b) $T_n = \frac{2}{3}+(n-1)\left(\frac{2}{3}\right)$

c) $T_{50} = \frac{2}{3}+(49)\left(\frac{2}{3}\right) = \frac{100}{3}$

5*a*) $\frac{17}{4};4;\frac{15}{4}$

b) $T_n = 5+(n-1)\left(-\frac{1}{4}\right)$

c) $T_{50} = 5+(49)\left(-\frac{1}{4}\right) = -\frac{29}{4}$

6*a*) $\frac{38}{3};\frac{40}{3};14$

b) $T_n = \frac{32}{3}+(n-1)\left(\frac{2}{3}\right)$

c) $T_{50} = \frac{32}{3}+(49)\left(\frac{2}{3}\right) = \frac{130}{3}$

Exercise 2:

1)

a) $T_n = 12+(n-1)(-6)$
$= 12-6n+6$
$= -6n+18$

b) $-72 = 12+(n-1)(-6)$
$\therefore -84 = -6(n-1)$
$\therefore 14 = n-1$
$\therefore n = 15$

2)

a) $T_n = 455+(n-1)(15)$

b) $T_{35} = 455+(34)(15)$
$= \text{R}965$

c) $1180 = 455+(n-1)(15)$
$\therefore 725 = 15(n-1)$
$\therefore 48,33 = n-1$
$\therefore n = 49,33$
$\therefore$ She can have 49 people at her party.

3)

a) $T_{14} = 22$ and $T_5 = 49$
$\therefore 22 = a+13d \rightarrow \boxed{1}$ and $49 = a+4d \rightarrow \boxed{2}$
$\boxed{2}-\boxed{1}$:
$27 = -9d$
$\therefore d = -3 \rightarrow \boxed{3}$
$\boxed{3}$ into $\boxed{1}$
$22 = a+13(-3)$
$\therefore a = 61$
$\therefore T_n = 61+(n-1)(-3)$

b) $T_n = 10$
$\therefore 10 = 61+(n-1)(-3)$
$\therefore -51 = (n-1)(-3)$
$\therefore 17 = n-1$
$\therefore n = 18$
$\therefore$ There will be 10 tins in the 18^{th} row.

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Exercise 3:

1a) $7 \searrow 16 \searrow 29 \searrow 46$
$9 \searrow 13 \searrow 17$
$4 \quad 4$

$a = 2 \rightarrow (2a = 4)$

$\therefore T_n = 2n^2 + bn + c$

$T_1: 5 = b + c \rightarrow \boxed{1}$

$T_2: 8 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$3 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 2$

$\therefore T_n = 2n^2 + 3n + 2$

b) $T_{25} = 2(25)^2 + 3(25) + 2 = 1327$

2a) $1 \searrow 18 \searrow 41 \searrow 70$
$17 \searrow 23 \searrow 29$
$6 \quad 6$

$a = 3 \rightarrow (2a = 6)$

$\therefore T_n = 3n^2 + bn + c$

$T_1: -2 = b + c \rightarrow \boxed{1}$

$T_2: 6 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$8 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = -10$

$\therefore T_n = 3n^2 + 8n - 10$

b) $T_{25} = 3(25)^2 + 8(25) - 10 = 2065$

3a) $4 \searrow 10 \searrow 18 \searrow 28$
$6 \searrow 8 \searrow 10$
$2 \quad 2$

$a = 1 \rightarrow (2a = 2)$

$\therefore T_n = n^2 + bn + c$

$T_1: 3 = b + c \rightarrow \boxed{1}$

$T_2: 6 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$3 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = n^2 + 3n$

b) $T_{25} = (25)^2 + 3(25) = 700$

4a) $\frac{5}{2} \searrow 14 \searrow \frac{67}{2} \searrow 61$
$\frac{23}{2} \searrow \frac{39}{2} \searrow \frac{55}{2}$
$8 \quad 8$

$a = 4 \rightarrow (2a = 8)$

$\therefore T_n = 4n^2 + bn + c$

$-\frac{3}{2} = b + c \rightarrow \boxed{1}$

$-2 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$-\frac{1}{2} = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = -1$

$\therefore T_n = 4n^2 - \frac{1}{2}n - 1$

b) $T_{25} = 4(25)^2 - 0{,}5(25) - 1 = 2486{,}5$

5a) $-3 \searrow 13 \searrow 53 \searrow 117$
$16 \searrow 40 \searrow 64$
$24 \quad 24$

$a = 12 \rightarrow (2a = 24)$

$\therefore T_n = 12n^2 + bn + c$

$-15 = b + c \rightarrow \boxed{1}$

$-35 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$-20 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 5$

$\therefore T_n = 12n^2 - 20n + 5$

b) $T_{25} = (25)^2 - 20(25) + 5 = 7005$

6a) $-\frac{9}{2} \searrow -\frac{7}{2} \searrow \frac{7}{2} \searrow \frac{33}{2}$
$1 \searrow 7 \searrow 13$
$6 \quad 6$

$a = 3 \rightarrow (2a = 6)$

$\therefore T_n = 3n^2 + bn + c$

$-\frac{15}{2} = b + c \rightarrow \boxed{1}$

$-\frac{31}{2} = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$-8 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = \frac{1}{2}$

$\therefore T_n = 3n^2 - 8n + \frac{1}{2}$

b) $T_{25} = 3(25)^2 - 8(25) + 0{,}5 = 1675{,}5$

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Mixed Exercise:

1) $3;7;11;15;\ \ldots \rightarrow$ arithmetic $(d=4)$

$\therefore T_n = 3+(n-1)(4)$

$\therefore T_{30} = 3+29(4)$

$\therefore T_{30} = 119$ matchsticks

2) **Black tile pattern:** $10;14;18;22;\ \ldots \rightarrow$ arithmetic $(d=4)$

Grey tile pattern: $2;6;12;20;\ \ldots \rightarrow$ Quadratic (second difference $=2$)

Black tiles: $T_n = 10+(n-1)(4)$

$\therefore T_{18} = 10+17(4)$

$= 78$ tiles

Grey tiles: $T_n = an^2+bn+c$

$a=1 \rightarrow (2a=2)$

$\therefore T_n = n^2+bn+c$

$1 = b+c \rightarrow \boxed{1}$

$2 = 2b+c \rightarrow \boxed{2}$

$\boxed{2}-\boxed{1}$:

$1 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = n^2+n$

$\therefore T_{18} = (18)^2+18 = 342$ tiles

$\therefore$ Total tiles: $78+342 = 420$ tiles

3) $1;3;6;10;\ \ldots \rightarrow$ Quadratic (second difference $=1$)

$a = \frac{1}{2} \rightarrow (2a=1)$

$\therefore T_n = \frac{1}{2}n^2+bn+c$

$\frac{1}{2} = b+c \rightarrow \boxed{1}$

$1 = 2b+c \rightarrow \boxed{2}$

$\boxed{2}-\boxed{1}$:

$\frac{1}{2} = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = \frac{1}{2}n^2+\frac{1}{2}n$

$\therefore 28 = \frac{1}{2}n^2+\frac{1}{2}n$

$\therefore 56 = n^2+n$

$\therefore n^2+n-56 = 0$

$\therefore (n-7)(n+8) = 0$

$\therefore n = 7$ or $n = -8 \rightarrow$ N/A as you cannot have a negative number of terms

$\therefore$ The 7^{th} figure would contain 28 dots

4)

a) $1;4;9;16;25\ldots$

This is a pattern of perfect squares:

$\therefore T_n = n^2$

$\therefore T_{12} = 144$

b) $1;8;27;64;125\ldots$

This is a pattern of perfect cubes:

$\therefore T_n = n^3$

$\therefore T_{12} = 1728$

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5) Sum of row $1=1$; Sum of row $2=8$; Sum of row $2=27$; Sum of row $4=64$

Sequence: $1;8;27;64;\ldots$

This is a pattern of perfect cubes:

$\therefore T_n = n^3$

$\therefore T_{20} = 8000$

6) $575;528;483;440;\ldots \rightarrow$ Quadratic $(\text{second difference} = 2)$

a) $a=1 \rightarrow (2a=2)$

$\therefore T_n = n^2 + bn + c$

$574 = b + c \rightarrow \boxed{1}$

$524 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$-50 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 624$

$\therefore T_n = n^2 - 50n + 624$

b) $n^2 - 50n + 624 = 0$

$\therefore (n-24)(n-26) = 0$

$\therefore n = 24$ or $n = 26$

$\therefore T_{24} = 0$ and $T_{26} = 0$

c) As the 24th and 26th term of the sequence is 0 the 25th term has to be negative. This will be the only negative term in the sequence. Therefore the 25th term will have the lowest value.

7) **In this sequence it can be seen that the second difference is 6. The 1st difference between each term can then be determined.**

a) $x \searrow\swarrow 16 \searrow\swarrow 32 \searrow\swarrow 54 \searrow\swarrow y \searrow\swarrow 116 \searrow\swarrow 156 \searrow\swarrow z$

$10 \searrow\swarrow 16 \searrow\swarrow 22 \searrow\swarrow 28 \searrow\swarrow 34 \searrow\swarrow 40 \searrow\swarrow 46$

$6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6$

$x = 16 - 10$ and $y = 54 + 28$ and $z = 156 + 46$

$= 6$ $= 82$ $= 202$

b) $a=3 \rightarrow (2a=6)$

$\therefore T_n = 3n^2 + bn + c$

$3 = b + c \rightarrow \boxed{1}$

$4 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$1 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 2$

$\therefore T_n = 3n^2 + n + 2$

8) $\underset{T_8}{\ldots;-15} \searrow\swarrow \underset{T_9}{-8} \searrow\swarrow \underset{T_{10}}{1};\ldots$

$7 \searrow\swarrow 9$

2

$a=1 \rightarrow (2a=2)$

$\therefore T_n = n^2 + bn + c$

$-79 = 8b + c \rightarrow \boxed{1}$

$-89 = 9b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$-10 = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 1$

$\therefore T_n = n^2 - 10n + 1$

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9)

Step	1	2	3	4	5
Honeycombs	1	3	6	10	15
Walls	6	15	27	42	60

Honeycombs: $1;3;6;10;15;\ \ldots \rightarrow$ Quadratic $(\text{second difference} = 1)$

Walls: $6;15;27;42;60;\ \ldots \rightarrow$ Quadratic $(\text{second difference} = 3)$

Honeycombs : $1;3;6;10;15;\ \ldots$

Sequence is quadratic $(\text{second difference} = 1)$

$a = \frac{1}{2} \rightarrow (2a = 2)$

$\therefore T_n = \frac{1}{2}n^2 + bn + c$

$\frac{1}{2} = b + c \rightarrow \boxed{1}$

$1 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\frac{1}{2} = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = \frac{1}{2}n^2 + \frac{1}{2}n \rightarrow$ number of honeycombs

Walls : $6;15;27;42;60;\ \ldots$

Sequence is quadratic $(\text{second difference} = 3)$

$a = \frac{3}{2} \rightarrow (2a = 3)$

$\therefore T_n = \frac{3}{2}n^2 + bn + c$

$\frac{9}{2} = b + c \rightarrow \boxed{1}$

$9 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\frac{9}{2} = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = \frac{3}{2}n^2 + \frac{9}{2}n \rightarrow$ number of walls

but each wall contains 10 ice-cream sticks

$\therefore$ the nth term for the number of ice-cream sticks is:

$T_n = 15n^2 + 45n$

10) Determine the general equation for the nth term of the sequence:

$2 \searrow 7 \searrow 15 \searrow 26 \searrow 40 \searrow 57$

$5 \searrow 8 \searrow 11 \searrow 14 \searrow 17$

$3 \quad 3 \quad 3 \quad 3$

$a = \frac{3}{2} \rightarrow (2a = 3)$

$\therefore T_n = \frac{3}{2}n^2 + bn + c$

$\frac{1}{2} = b + c \rightarrow \boxed{1}$

$1 = 2b + c \rightarrow \boxed{2}$

$\boxed{2} - \boxed{1}$:

$\frac{1}{2} = b \rightarrow \boxed{3}$

$\boxed{3}$ into $\boxed{1}$:

$\therefore c = 0$

$\therefore T_n = \frac{3}{2}n^2 + \frac{1}{2}n$

a) $x = T_5 = 40$ cards

y:

$155 = \frac{3}{2}n^2 + \frac{1}{2}n$

$\therefore 3n^2 + n - 310 = 0$

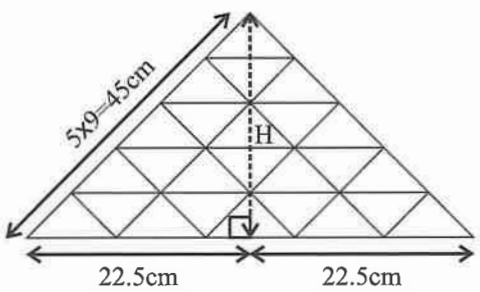
$\therefore (n - 10)(3n + 31) = 0$

$\therefore n = 10$ or $n = -\frac{31}{3} \rightarrow$ N/A

$\therefore y = 10$ stories

b) From the above pattern 5 stories is the most that can be built from a single deck. $(T_6 = 57)$

c)



$H^2 = (45)^2 - (22{,}5)^2$

$\therefore H = 38{,}97\,\text{cm}$

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Grade 10 Revision

Exercise 1:

1) Determine the lengths of the line segments joining the following points:

a) $A(2;-5)$ and $B(-1;-8)$

b) $P(5;5)$ and $Q(-2;3)$

c) $S(10;-6)$ and $R(-3;2)$

d) $M(-1;-5)$ and $N(1;-10)$

2) Determine the midpoints of the line segments joining the following points:

a) $A(2;-5)$ and $B(-1;-8)$

b) $P(5;5)$ and $Q(-2;3)$

c) $S(10;-6)$ and $R(-3;2)$

d) $M(-1;-5)$ and $N(1;-10)$

3) Determine the gradients of the line segments joining the following points:

a) $A(2;-5)$ and $B(-1;-8)$

b) $P(5;5)$ and $Q(-2;3)$

4) In triangle ABC, with $A(4;6)$, $B(x;y)$ and $C(-4;-6)$, $P(5;1)$ is the midpoint of AB. Determine:

a) The coordinates of point B.

b) The perimeter of the triangle.

c) If the triangle is right-angled.

5) Determine two possible values of x if AB is $\sqrt{185}$ units and A is the point $(4;6)$ and B is the point $(x;-5)$.

6) Determine if AB and CD are parallel, perpendicular or neither given the points:

a) $A(-8;-4)$ and $B(2;4)$ and $C(-4;0)$ and $D(-20;20)$

b) $A(5;6)$ and $B(4;2)$ and $C(8;24)$ and $D(4;8)$

7) Determine whether or not the points $A(-5;-7)$ and $B(4;11)$ and $C(28;59)$ are collinear.

8) If A is the point $A(x;3)$ and B is the point $B(-2;y)$ determine the value/s of :

a) x if AD is perpendicular to the line $y=-3x+6$ and D is the point $D(4;2)$.

b) x and y if the midpoint of AB is $(1;6)$.

Finding the equation of a straight line

Exercise 2:

1) Determine the equation of the line which has the gradient given below and passes through the given point.

a) $m=-2$ and point $(3;8)$

b) $m=4$ and point $(-5;3)$

c) $m=-\frac{1}{3}$ and point $(3;1)$

2) Determine the equation of the line which:

a) Passes through the points $(-2;5)$ and $(-8;12)$.

b) Is parallel to $3y-12x=-9$ and passes through the point $(2;2)$.

c) Is perpendicular to $4y-x-8=0$ and passes through the point $(-2;6)$.

d) Is a vertical line through the point $(3;5)$.

e) Is a horizontal line through the point $(-8;-12)$.

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Exercise 3:

1) **Correct to 1 decimal place, determine the angle of inclination of the straight lines which pass through the points:**

   a) $(1;1)$ and $(4;7)$

   b) $(5;4)$ and $(1;8)$

   c) $(6;4)$ and $(-8;-3)$

   d) $(-1;5)$ and $(1;-3)$

   e) $(-2;3)$ and $(-3;6)$

   f) $(7;-7)$ and $(8;7)$

2) **Determine the gradient of a straight line with an angle of inclination of:**

   a) $30°$

   b) $45°$

   c) $60°$

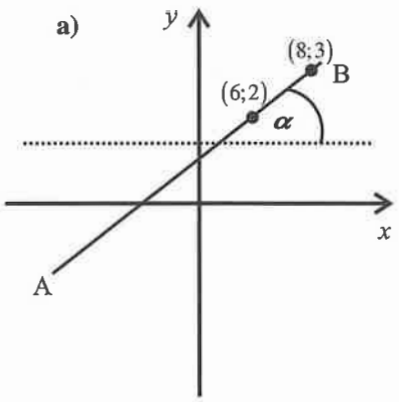
   d) $150°$

   e) $0°$

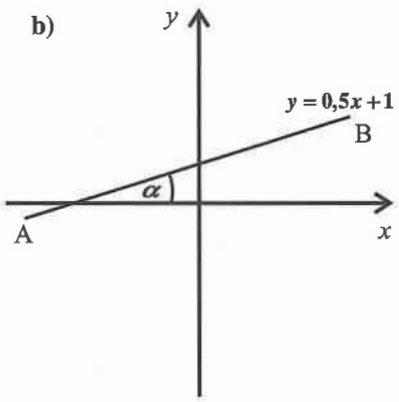
   f) $180°$

3) **Determine the magnitude (size) of angle α in each of the following diagrams. Give your answer correct to 1 decimal place. You may assume that all dotted lines are parallel to the x-axis.**

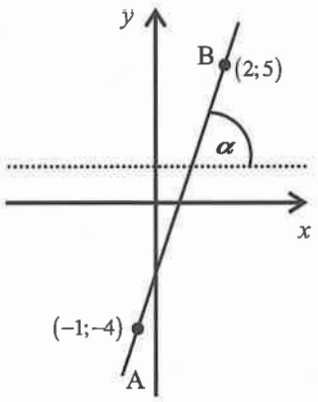
a)



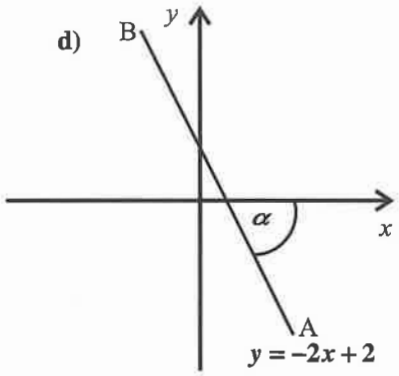
b)



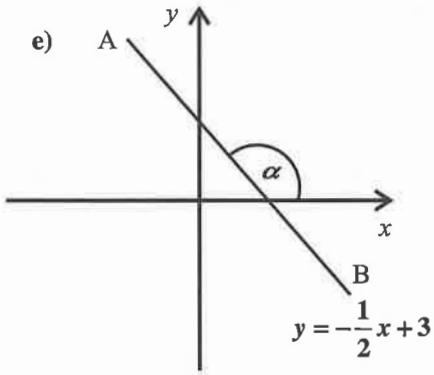
c)



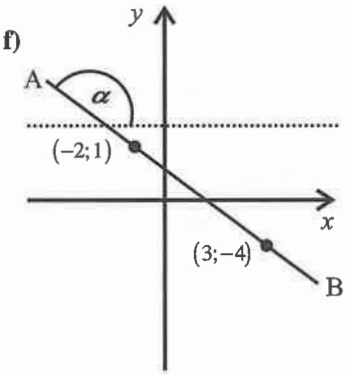
d)



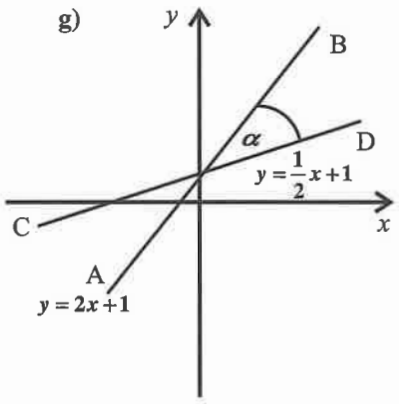
e)



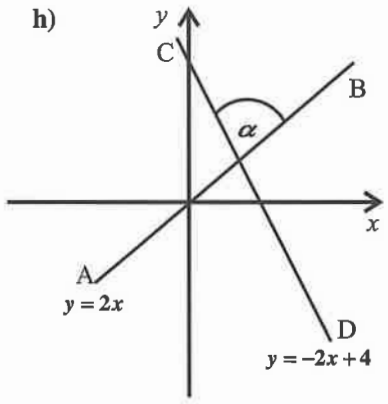
f)



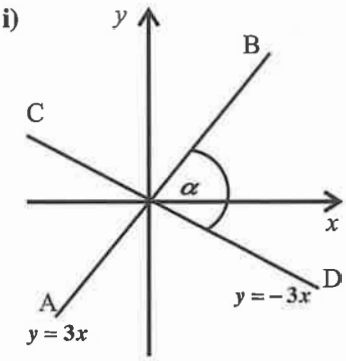
g)



h)



i)



Mixed Exercise:

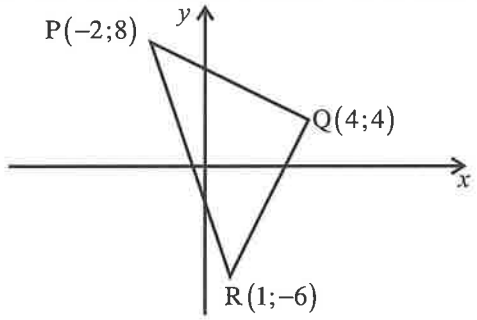
1) **If a quadrilateral has vertices A(–2;2), B(–3;–2), C(3;–1) and D(4;3), prove that:**

   a) ABCD is parallelogram.

   b) ABCD is not a rhombus or a square.

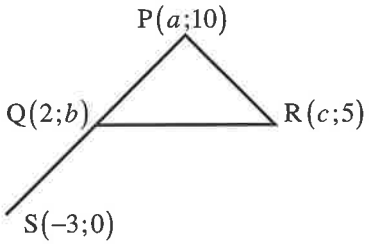
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2) **The diagram below shows triangle PQR with vertices $P(-2;8), Q(4;4)$ and $R(1;-6)$. Determine the following, leaving your solutions in simplest surd form where necessary.**



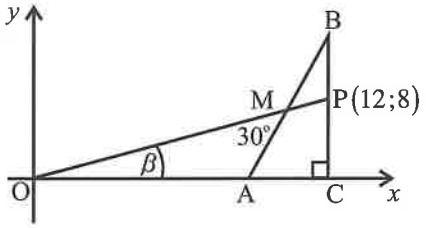
a) The perimeter of triangle PQR correct to 1 decimal place.
b) The gradients of PR and QR.
c) The angle of inclination of PR and QR
d) Hence determine angle $P\hat{R}Q$.

3) **Points $P(a;10)$, $Q(2;b)$ and $R(c;5)$ of triangle PQR are shown below. PQ is produced its own length to point $S(-3;0)$. It is given that PQ = PR. The diagram is not drawn to scale.**



a) Determine the values of a and b.
b) Hence, determine the value/s of c.
c) Determine the equation of PS.

4) **In the diagram below, M is the midpoint of line segment OP. The diagram is not drawn to scale.**



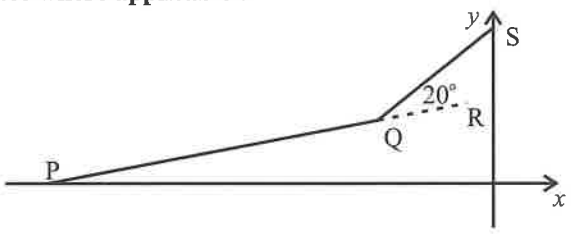
a) Determine the coordinates of M.
b) Determine the gradient of OP.
c) Determine the value of β, the angle of inclination of OP.
d) Determine the gradient of AB correct to one decimal place.
e) What is the area of triangle POC?

5) **Given that $P(-2;6)$, $Q(14;2)$ and $R(x;4)$ are points in the Cartesian plane, calculate the values/s of x, if:**

a) QR is perpendicular to the x-axis.
b) R is the midpoint of PQ.
c) The gradient of QR is equal to 2.
d) The length of QR is $2\sqrt{2}$ units.

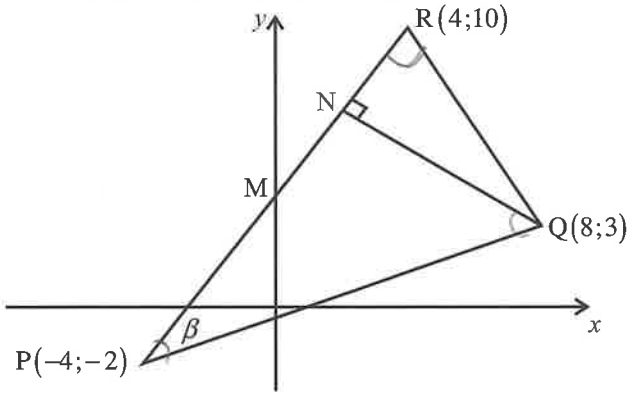
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6) **In the diagram below QS is a straight line with equation $2y-4x=4$ and $S\hat{Q}R=20°$. Determine, correct to one decimal place where applicable :**



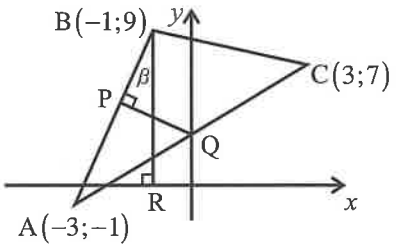
a) The angle of inclination of QS.
b) The gradient of line PQ.

7) **$P(-4;-2)$, $Q(8;3)$ and $R(4;10)$ are the vertices of triangle PQR. QN is perpendicular to PR and M is the midpoint of PR. The diagram is not drawn to scale.**



a) Determine the length of PR.
b) Determine the coordinates of M the midpoint of PR.
c) Calculate the gradient of PR.
d) Hence, determine the equation of QN.
e) Calculate the area of triangle PQR if N is the point $(2;7)$.
f) Determine the size of angle β correct to one decimal place.

8) **$A(-3;-1)$, $B(-1;9)$ and $C(3;7)$ are the vertices of triangle ABC. AB⊥PQ and BR is perpendicular to the x-axis.**



a) Determine the coordinates of Q, the midpoint of AC.
b) Determine the equation of AB.
c) Hence, determine the equation of PQ.
d) Determine θ_{AB}, the angle of inclination of AB.
e) Hence, determine the value of β if $\beta = A\hat{B}R$.

9) **The equation of a line can be written as $(5-2p)x+(p+2)y=10$.**

a) Rewrite the above equation in the form of $y=mx+c$.
b) Determine the equation of the line if the line is parallel to $y=3x+2$.
c) Determine the value of p if the line passes through the point $(-4;8)$.
d) Determine the value of p if the line is parallel to the y-axis.
e) Determine the value of p if the line is parallel to the x-axis.

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Grade 10 Revision

Exercise 1:

1)

a) $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore AB = \sqrt{(-1-2)^2 + (-8+5)^2}$

$\therefore AB = \sqrt{18} = 3\sqrt{2}$ units

b) $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore PQ = \sqrt{(-2-5)^2 + (3-5)^2}$

$\therefore PQ = \sqrt{53}$ units

c) $SR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore SR = \sqrt{(-3-10)^2 + (2+6)^2}$

$\therefore SR = \sqrt{233}$ units

d) $MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore MN = \sqrt{(1+1)^2 + (-10+5)^2}$

$\therefore MN = \sqrt{29}$ units

2)

a) $M\left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

$= M\left(\frac{-1+2}{2}; \frac{-8-5}{2}\right)$

$= M\left(\frac{1}{2}; -\frac{13}{2}\right)$

b) $M\left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

$= M\left(\frac{-2+5}{2}; \frac{3+5}{2}\right)$

$= M\left(\frac{3}{2}; 4\right)$

c) $M\left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

$= M\left(\frac{-3+10}{2}; \frac{2-6}{2}\right)$

$= M\left(\frac{7}{2}; -2\right)$

d) $M\left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

$= M\left(\frac{1-1}{2}; \frac{-10-5}{2}\right)$

$= M\left(0; \frac{-15}{2}\right)$

3)

a) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{AB} = \frac{-8+5}{-1-2}$

$\therefore m_{AB} = 1$

b) $m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{PQ} = \frac{3-5}{-2-5}$

$\therefore m_{PQ} = \frac{2}{7}$

4)

a) P is the midpoint of AB

$\therefore \frac{x_A + x_B}{2} = 5$ and $\frac{y_A + y_B}{2} = 1$

$\therefore 4 + x_B = 10$ and $6 + y_B = 2$

$\therefore x_B = 6$ and $y_B = -4$

$\therefore$ B is the point $(6; -4)$

b) Perimeter = AB + BC + AC

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore AB = \sqrt{(6-4)^2 + (-4-6)^2}$

$\therefore AB = 2\sqrt{26}$

$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore BC = \sqrt{(6+4)^2 + (-4+6)^2}$

$\therefore BC = 2\sqrt{26}$

$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore AC = \sqrt{(-4-4)^2 + (-6-6)^2}$

$\therefore AC = 4\sqrt{13}$

$\therefore$ Perimeter $= 2\sqrt{26} + 2\sqrt{26} + 4\sqrt{13} = 34{,}82$ units

c) For the triangle to be right angled : $AC^2 = AB^2 + BC^2 \rightarrow$ Pythagoras

$AC^2 = 208$ units

$AB^2 + BC^2 = 104 + 104 = 208$ units

$\therefore \Delta ABC$ is right angled at B

The properties of perpendicular lines could also have been used to prove that $\hat{B}$ is a right angle.

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5) $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore AB^2 = (x-4)^2 + (-5-6)^2$

$\therefore 185 = x^2 - 8x + 16 + 121$

$\therefore x^2 - 8x - 48 = 0$

$\therefore (x-12)(x+4) = 0$

$\therefore x = 12 \quad \text{or} \quad x = -4$

6)

a) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ $\qquad$ $m_{CD} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{AB} = \frac{4+4}{2+8}$ $\qquad$ $\therefore m_{CD} = \frac{20-0}{-20+4}$

$\therefore m_{AB} = \frac{4}{5}$ $\qquad$ $\therefore m_{CD} = -\frac{5}{4}$

$\therefore m_{AB} \times m_{CD} = -1$

$\therefore AB \perp CD$

b) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ $\qquad$ $m_{CD} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{AB} = \frac{2-6}{4-5}$ $\qquad$ $\therefore m_{CD} = \frac{8-24}{4-8}$

$\therefore m_{AB} = 4$ $\qquad$ $\therefore m_{CD} = 4$

$\therefore m_{AB} = m_{CD}$

$\therefore AB \,//\, CD$

7)

a) $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{AB} = \frac{11+7}{4+5}$

$\therefore m_{AB} = 2$

$\therefore$ the points are collinear as $m_{AB} = m_{BC}$

b) $m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore m_{BC} = \frac{59-11}{28-4}$

$\therefore m_{BC} = 2$

8)

a) $m_{AD} = \frac{1}{3}$

$\therefore \frac{1}{3} = \frac{3-2}{x-4}$

$\therefore x - 4 = 3$

$\therefore x = 7$

b) $\frac{x-2}{2} = 1$ and $\frac{3+y}{2} = 6$

$\therefore x - 2 = 2$ and $3 + y = 12$

$\therefore x = 4$ and $y = 9$

Finding the equation of a straight line

Exercise 2:

1)

a) $y = mx + c$

$\therefore y = -2x + c$

sub $(3;8)$

$\therefore 8 = -2(3) + c$

$\therefore c = 14$

$\therefore y = -2x + 14$

b) $y = mx + c$

$\therefore y = 4x + c$

sub $(-5;3)$

$\therefore 3 = 4(-5) + c$

$\therefore c = 23$

$\therefore y = 4x + 23$

or

a) $y - y_1 = m(x - x_1)$

$\therefore y - y_1 = -2(x - x_1)$

sub $(3;8)$

$\therefore y - 8 = -2(x - 3)$

$\therefore y - 8 = -2x + 6$

$\therefore y = -2x + 14$

b) $y - y_1 = m(x - x_1)$

$\therefore y - y_1 = 4(x - x_1)$

sub $(-5;3)$

$\therefore y - 3 = 4(x + 5)$

$\therefore y - 3 = 4x + 20$

$\therefore y = 4x + 23$

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c) $y = mx + c$

$\therefore y = -\frac{1}{3}x + c$

sub $(3;1)$

$\therefore 1 = -\frac{1}{3}(3) + c$

$\therefore c = 2$

$\therefore y = -\frac{1}{3}x + 2$

or

c) $y - y_1 = m(x - x_1)$

$\therefore y - y_1 = -\frac{1}{3}(x - x_1)$

sub $(3;1)$

$\therefore y - 1 = -\frac{1}{3}(x - 3)$

$\therefore y - 1 = -\frac{1}{3}x + 1$

$\therefore y = -\frac{1}{3}x + 2$

2)

a) $y = mx + c$

$m = \frac{12-5}{-8+2} = -\frac{7}{6}$

$\therefore y = -\frac{7}{6}x + c$

sub $(-2;5) \rightarrow (-8;12)$ could also be used

$\therefore 5 = -\frac{7}{6}(-2) + c$

$\therefore c = \frac{8}{3}$

$\therefore y = -\frac{7}{6}x + \frac{8}{3}$

or

a) $y - y_1 = m(x - x_1)$

$m = \frac{12-5}{-8+2} = -\frac{7}{6}$

$\therefore y - y_1 = -\frac{7}{6}(x - x_1)$

sub $(-2;5) \rightarrow (-8;12)$ could also be used

$\therefore y - 5 = -\frac{7}{6}(x + 2)$

$\therefore y - 5 = -\frac{7}{6} - \frac{7}{3}$

$\therefore y = -\frac{7}{6}x + \frac{8}{3}$

b) $3y - 12x = -9$

$\therefore y = 4x - 3$

$\therefore m_{\text{line}} = 4$

$\therefore y = 4x + c$

sub $(2;2)$

$\therefore 2 = 4(2) + c$

$\therefore c = -6$

$\therefore y = 4x - 6$

b) $3y - 12x = -9$

$\therefore y = 4x - 3$

$\therefore m_{\text{line}} = 4$

$\therefore y - y_1 = 4(x - x_1)$

sub $(2;2)$

$\therefore y - 2 = 4(x - 2)$

$\therefore y - 2 = 4x - 8$

$\therefore y = 4x - 6$

c) $4y - x - 8 = 0$

$\therefore y = \frac{1}{4}x + 2$

$\therefore m_{\text{line}} = -4$

$\therefore y = -4x + c$

sub $(-2;6)$

$\therefore 6 = -4(-2) + c$

$\therefore c = -2$

$\therefore y = -4x - 2$

c) $4y - x - 8 = 0$

$\therefore y = \frac{1}{4}x + 2$

$\therefore m_{\text{line}} = -4$

$\therefore y - y_1 = -4(x - x_1)$

sub $(-2;6)$

$\therefore y - 6 = -4(x + 2)$

$\therefore y - 6 = -4x - 8$

$\therefore y = -4x - 2$

d) A vertical line has the equation of $x =$ *constant*

$\therefore x = 3$

e) A horizontal line has the equation of $y =$ *constant*

$\therefore y = -12$

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Exercise 3:

1)

a) $m=\frac{7-1}{4-1}=2$

$\therefore \tan\theta=2$

$\therefore \theta=63,4^\circ$

b) $m=\frac{8-4}{1-5}=-1$

$\therefore \tan\theta=-1$

$\therefore \theta=-45^\circ+180^\circ=135,0^\circ$

c) $m=\frac{-3-4}{-8-6}=\frac{1}{2}$

$\therefore \tan\theta=\frac{1}{2}$

$\therefore \theta=26,6^\circ$

d) $m=\frac{-3-5}{1+1}=-4$

$\therefore \tan\theta=-4$

$\therefore \theta=-75,96^\circ+180^\circ=104,0^\circ$

e) $m=\frac{6-3}{-3+2}=-3$

$\therefore \tan\theta=-3$

$\therefore \theta=-71,57^\circ+180^\circ=108,4^\circ$

f) $m=\frac{7+7}{8-7}=14$

$\therefore \tan\theta=14$

$\therefore \theta=85,9^\circ$

2)

a) $\tan 30^\circ=\frac{\sqrt{3}}{3}$

b) $\tan 45^\circ=1$

c) $\tan 60^\circ=\sqrt{3}$

d) $\tan 150^\circ=-\frac{\sqrt{3}}{3}$

e) $\tan 0^\circ=0$

e) $\tan 180^\circ=0$

3)

a) $\therefore m_{AB}=\frac{3-2}{8-6}=\frac{1}{2}$

$\tan\alpha=m_{AB}$

$\therefore \tan\alpha=\frac{1}{2}$

$\therefore \alpha=26,6^\circ$

b) $\tan\alpha=0,5$

$\therefore \alpha=26,6^\circ$

c) $\therefore m_{AB}=\frac{5+4}{2+1}=3$

$\tan\alpha=3$

$\therefore \alpha=71,6^\circ$

d) $\therefore \tan\alpha=-2$

$\therefore \alpha=-63,43^\circ$

$\therefore \alpha=63,4^\circ$

e) $\therefore \tan\alpha=-\frac{1}{2}$

$\therefore \alpha=-26,57^\circ+180^\circ$

$\therefore \alpha=153,4^\circ$

f) $\therefore m_{AB}=\frac{-4-1}{3+2}=-1$

$\therefore \tan\alpha=-1$

$\therefore \alpha=-45^\circ+180^\circ$

$\therefore \alpha=135^\circ$

g) $\alpha=\theta_{AB}-\theta_{CD}$

$\tan\theta_{AB}=2$

$\therefore \theta_{AB}=63,43^\circ$

$\tan\theta_{CD}=\frac{1}{2}$

$\therefore \theta_{CD}=26,57^\circ$

$\therefore \alpha=63,43^\circ-26,57^\circ$

$\therefore \alpha=36,9^\circ$

h) $\alpha=\theta_{CD}-\theta_{AB}$

$\tan\theta_{CD}=-2$

$\therefore \theta_{CD}=-63,43^\circ+180^\circ$

$\therefore \theta_{CD}=116,57^\circ$

$\tan\theta_{AB}=2$

$\therefore \theta_{AB}=63,43^\circ$

$\therefore \alpha=116,57^\circ-63,43^\circ$

$\therefore \alpha=53,1^\circ$

i) $\alpha=\theta_{AB}+\theta_{CD}$

$\tan\theta_{AB}=3$

$\therefore \theta_{AB}=71,57^\circ$

$\tan\theta_{CD}=-3$

$\therefore \theta_{CD}=-71,57^\circ$

$\therefore \alpha=71,57^\circ+71,57^\circ$

$\therefore \alpha=143,1^\circ$

Mixed Exercise:

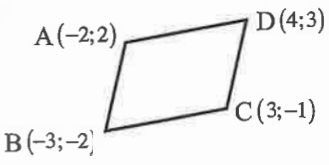
1)

a) $M_{AC}=\left(\frac{3-2}{2};\frac{-1+2}{2}\right)$

$=\left(\frac{1}{2};\frac{1}{2}\right)$

$M_{BD}=\left(\frac{-3+4}{2};\frac{-2+3}{2}\right)$

$=\left(\frac{1}{2};\frac{1}{2}\right)$



$\therefore$ Midpoint of AC = Midpoint of BD

$\therefore$ ABCD is a parm (diags bisect each other)

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b) Diagonals of a rhombus and square intersect at 90°

$m_{AC} = \frac{-1-2}{3+2} = -\frac{3}{5}$ and $m_{BD} = \frac{3+2}{4+3} = \frac{5}{7}$

$m_{AC} \times m_{BD} = -\frac{3}{5} \times \frac{5}{7} = -\frac{3}{7}$

$\therefore m_{AC} \times m_{BD} \neq -1$

$\therefore$ ABCD is not a rhombus or a square as diagonals do not intersect at 90°.

2)

a) Perimeter = PQ + QR + PR

$PQ = \sqrt{(4-8)^2 + (4+2)^2}$

$\therefore PQ = 2\sqrt{13}$ units

$QR = \sqrt{(-6-4)^2 + (1-4)^2}$

$\therefore QR = \sqrt{109}$ units

$PR = \sqrt{(-6-8)^2 + (1+2)^2}$

$\therefore PR = \sqrt{205}$ units

$\therefore$ Perimeter $= 2\sqrt{13} + \sqrt{109} + \sqrt{205} = 32,0$ units

b) $m_{PR} = \frac{y_r - y_p}{x_r - x_p} = \frac{-6-8}{1+2} = -\frac{14}{3}$

$m_{QR} = \frac{y_r - y_q}{x_r - x_q} = \frac{-6-4}{1-4} = \frac{10}{3}$

c) $\tan\theta_{PR} = -\frac{14}{3}$

$\therefore \theta_{PR} = -77,91° + 180° = 102,09°$

$\tan\theta_{QR} = \frac{10}{3}$

$\therefore \theta_{PR} = 73,3°$

d) $P\hat{R}Q = \theta_{PR} - \theta_{QR}$

$= 102,09° - 73,3°$

$= 28,79°$

3) In this question it is important to note that it is given that PQ = PR = QS.

a) $\frac{a-3}{2} = 2$ and $\frac{10-0}{2} = b$

$\therefore a = 7$ $\therefore b = 5$

b) $PQ^2 = (7-2)^2 + (10-5)^2$

$= 50$

$\therefore PR^2 = 50$

$\therefore 50 = (c-7)^2 + (5-10)^2$

$\therefore 50 = c^2 - 14c + 49 + 25$

$\therefore c^2 - 14c + 24 = 0$

$\therefore (c-12)(c-2) = 0$

$\therefore c = 12$ or $c = 2 \rightarrow$ N/A

c) $m_{PS} = \frac{y_p - y_s}{x_p - x_s} = \frac{10-0}{7+3} = 1$

$\therefore y = 1x + c$

sub $(-3;0) \rightarrow$ points P or Q could be used

$\therefore 0 = -3 + c$

$\therefore c = 3$

$\therefore y = x + 3$

or

b) QR is a vertical line as the y-values of Q and R are the same.

$\therefore$ The x-value of the midpoint of QR is 7

$\therefore \frac{2+c}{2} = 7$

$\therefore c = 12$

c) $m_{PS} = \frac{y_p - y_s}{x_p - x_s} = \frac{10-0}{7+3} = 1$

$\therefore y - y_1 = 1(x - x_1)$

sub $(-3;0) \rightarrow$ points P or Q could be used

$\therefore y - 0 = x + 3$

$\therefore y = x + 3$

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4)

a) $M=\left(\frac{12+0}{2};\frac{8+0}{2}\right)=(6;4)$

b) $m_{OP}=\frac{y_p-y_o}{x_p-x_o}=\frac{8-0}{12-0}=\frac{2}{3}$

c) $\tan\beta=2/3$

$\therefore \beta=33,69°$

d) $\theta_{AB}=\beta+30° \rightarrow$ ext. angle of a triangle

$\therefore \theta_{AB}=33,69°+30°=63,69°$

$m_{AB}=\tan\theta_{AB}$

$\therefore m_{AB}=\tan 63,69°=2,0$

e) $A_{\blacktriangle}=1/2\times \text{base}\times \perp \text{height}$

$\therefore A_{\blacktriangle}=1/2(12)(8)$

$\therefore A_{\blacktriangle}=48 \text{ units}^2$

5)

a) $x=14$

b) $x=\frac{14-2}{2}$

$\therefore x=6$

c) $m_{QR}=\frac{y_r-y_q}{x_r-x_q}$

$\therefore 2=\frac{4-2}{x-14}$

$\therefore x-14=1$

$\therefore x=15$

d) $QR^2=(x-14)^2+(4-2)^2$

$\therefore 8=x^2-28x+196+4$

$\therefore x^2-28x+192=0$

$\therefore (x-16)(x-12)=0$

$\therefore x=16 \text{ or } x=12$

6)

a) $2y=4x+4$

$\therefore y=2x+2$

$\tan\theta_{QS}=2$

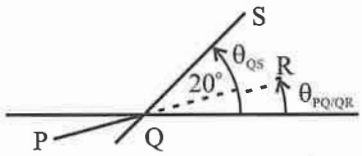
$\therefore \theta_{QS}=63,4°$

b) $20°=\theta_{QS}-\theta_{PQ}$

$\therefore \theta_{PQ}=63,4°-20°$

$\therefore \theta_{PQ}=43,4°$

$\therefore m_{PQ}=\tan\theta_{PQ}=\tan 43,4°=0,9$



7)

a) $PR=\sqrt{(10+2)^2+(4+4)^2}$

$\therefore AB=4\sqrt{13} \text{ units}$

b) $M=\left(\frac{4-4}{2};\frac{10-2}{2}\right)=(0;4)$

c) $m_{PR}=\frac{10+2}{4+4}=\frac{3}{2}$

d) $m_{QN}=-\frac{2}{3} \rightarrow QN \perp PR$

$\therefore y=-\frac{2}{3}x+c$

Sub $Q(8;3)$

$\therefore 3=-\frac{2}{3}(8)+c$

$\therefore c=\frac{25}{3}$

$\therefore y=-\frac{2}{3}x+\frac{25}{3}$

e) $QN=\sqrt{(8-2)^2+(3-7)^2}=2\sqrt{13} \text{ units}$

$A_{PQR}=1/2\times PR\times QN$

$\therefore A_{PQR}=1/2\left(4\sqrt{13}\right)\left(2\sqrt{13}\right)$

$\therefore A_{PQR}=52 \text{ units}^2$

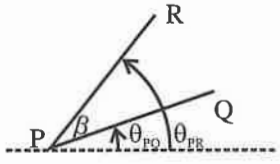
f) $m_{PQ}=\frac{3+2}{8+4}=\frac{5}{12}$

$\tan\theta_{PQ}=\frac{5}{12}$ and $\tan\theta_{PR}=\frac{3}{2}$

$\therefore \theta_{PQ}=22,62°$ and $\theta_{PR}=56,31°$

$\therefore \beta=\theta_{PR}-\theta_{PQ}$

$\therefore \beta=56,31°-22,62°=33,7°$



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8)

a) $Q=\left(\frac{3-3}{2};\frac{7-1}{2}\right)=(0;3)$

b) $m_{AB}=\frac{9+1}{-1+3}=5$

$\therefore y=5x+c$

Sub $A(-3;-1)\rightarrow B(-1;9)$ can also be used

$\therefore -1=5(-3)+c$

$\therefore c=14$

$\therefore y=5x+14$

c) $m_{PQ}=-\frac{1}{5}\rightarrow PQ\perp AB$

$\therefore y=-\frac{1}{5}x+c$

Sub $Q(0;3)$

$\therefore 3=-\frac{1}{5}(0)+c$

$\therefore c=3$

$\therefore y=-\frac{1}{5}x+3$

d) $\tan\theta_{AB}=5$

$\therefore \theta_{AB}=78,69^\circ$

e) $\beta=90^\circ-\theta_{AB}\rightarrow$ angels in a triangle

$\therefore \beta=90^\circ-78,69^\circ$

$\therefore \beta=11,31^\circ$

9)

a) $(p+2)y=-(5-2p)x+10$

$\therefore y=\left(\frac{2p-5}{p+2}\right)x+\frac{10}{p+2}$

b) $\frac{2p-5}{p+2}=3$

$\therefore 2p-5=3(p+2)$

$\therefore 2p-5=3p+6$

$\therefore p=-11$

$\therefore y=\left[\frac{2(-11)-5}{-11+2}\right]x+\frac{10}{-11+2}$

$\therefore y=3x-\frac{10}{9}$

c) $y=\left(\frac{2p-5}{p+2}\right)x+\frac{10}{p+2}$

Sub $(-4;8)$

$\therefore 8=\left(\frac{2p-5}{p+2}\right)(-4)+\frac{10}{p+2}$

$\therefore 8(p+2)=-8p+20+10$

$\therefore 8p+16=-8p+30$

$\therefore 16p=14$

$\therefore p=\frac{14}{16}=\frac{7}{8}$

d) gradient is undefined

$\therefore p+2=0$

$\therefore p=-2$

e) gradient $=0$

$\therefore 2p-5=0$

$\therefore p=5/2$

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Sketching the Parabola

Exercise 1:

1) Sketch the graphs of the following parabolas showing the turning point and intercepts with the axes:

a) $y=2(x-3)^2-2$

b) $f(x)=-(x+2)^2+4$

c) $h(x)=-(x-3)^2+5$

d) $p(x)=\frac{1}{2}(x+2)^2+2$

e) $m(x)=-(x+2)^2-1$

f) $f:x\to(x+3)^2-4$

2) Sketch the graphs of the following parabolas showing the turning point and intercepts with the axes:

a) $y=x^2+4x+3$

b) $f(x)=-x^2+2x+3$

c) $f(x)=-2x^2+6x$

d) $h(x)=-2x^2+8x+24$

e) $y=-\frac{9}{2}x^2+27x+\frac{27}{2}$

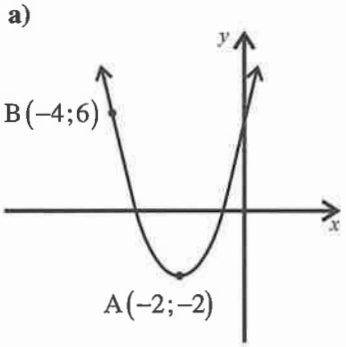
f) $y=-\frac{1}{2}x^2+2x+2$

Determining the Equation of a Parabola

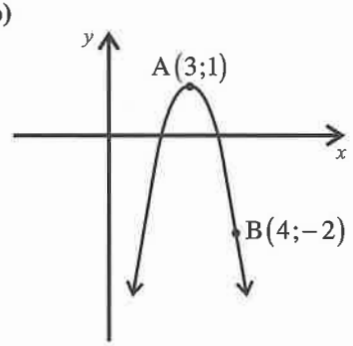
Exercise 2:

1) Determine the equations of the parabolas below. In each of the graphs, A is the turning point and B is a point on the curve.

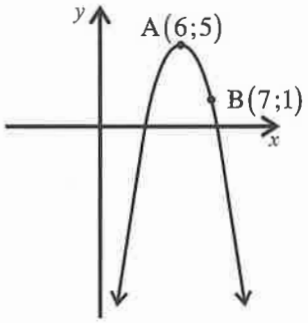
a)



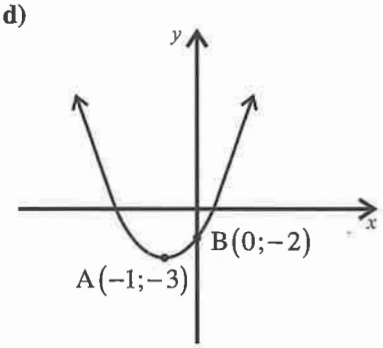
b)



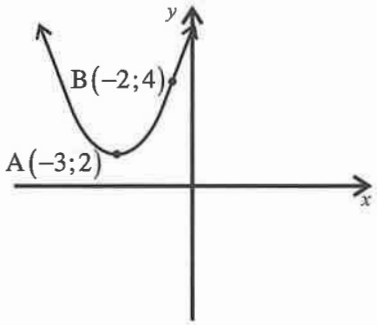
c)



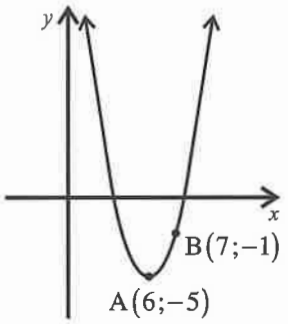
d)



e)

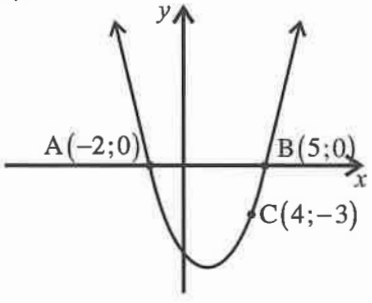


f)

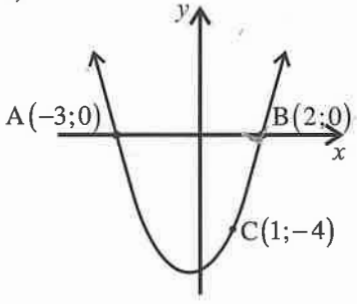


2) Determine the equations of the parabolas below. In each of the graphs, A and B are the *x*-intercepts and C is a point on the curve.

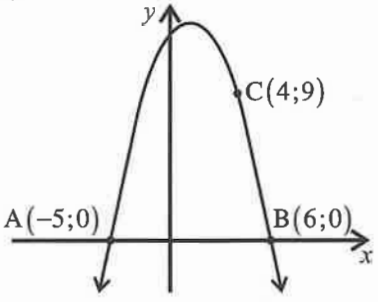
a)



b)



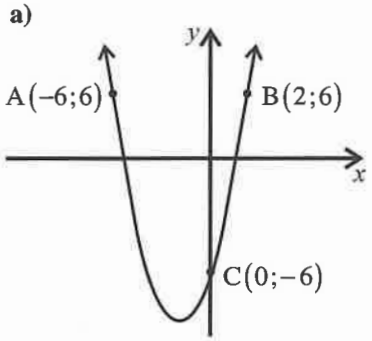
c)



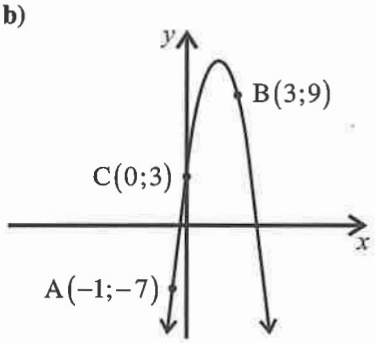
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3) Determine the equations of the parabolas below. In each of the graphs A and B are points on the curve and C is the y-intercept.

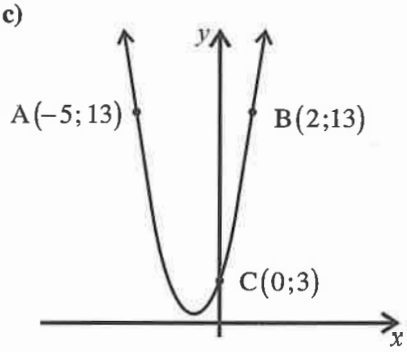
a)



b)



c)



Sketching and Finding the Equation of the Hyperbola

Exercise 3:

1) Draw a neat sketch of the graphs below showing all intercepts with the axes, asymptotes, and both axes of symmetry.

a) $g(x)=\frac{-4}{x+2}-2$

b) $f:x\to-\frac{1}{x+1}-2$

c) $f(x)=\frac{9}{x+3}-1$

d) $y=\frac{6}{x-1}-3$

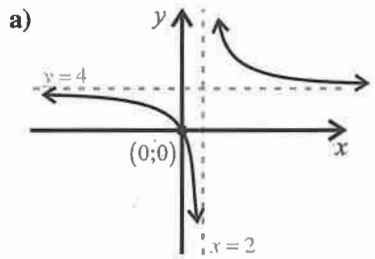
e) $y=\frac{-6}{x-1}+3$

f) $y=\frac{-4}{x-2}-1$

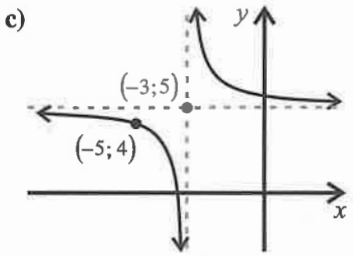
2) For each of the graphs in question 1, determine the coordinates of the points of intersections of the axes of symmetry and the graphs.

Exercise 4: Determine the equations of the hyperbolas shown below:

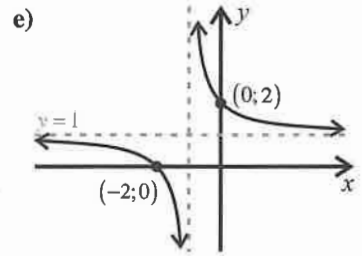
a)



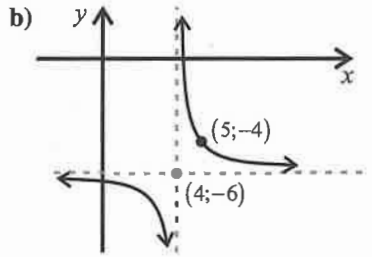
c)



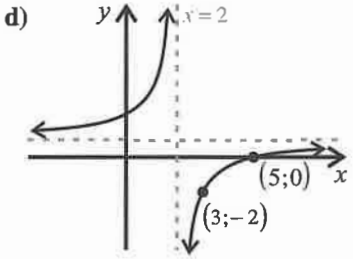
e)



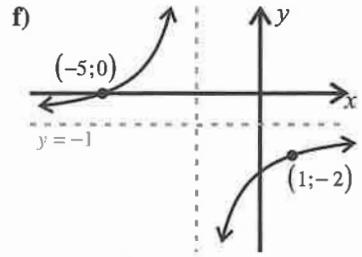
b)



d)



f)



Sketching and Finding the Equation of the Exponential Function

Exercise 5: Sketch the graphs below showing all intercepts with the axes and asymptotes.

a) $y=2^{x+1}+2$

b) $f(x)=-(2)^{x+1}-2$

c) $f:x\to 2^{-x+1}+2$

d) $f(x)=-\left(\frac{1}{2}\right)^{x+2}+4$

e) $f(x)=\left(\frac{1}{2}\right)^{x+1}-2$

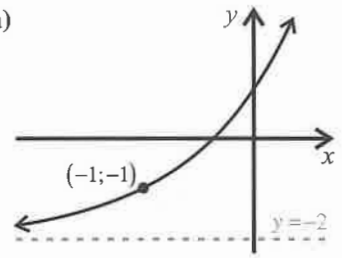
f) $y=4(2)^{x-2}+1$

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Exercise 6:

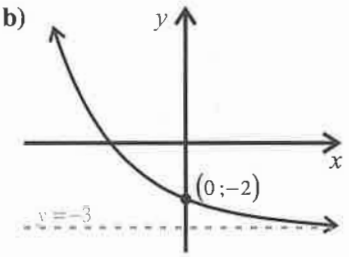
In each of the graphs below, solve for the values of b, p or q and write down the equation of the graph.

a)



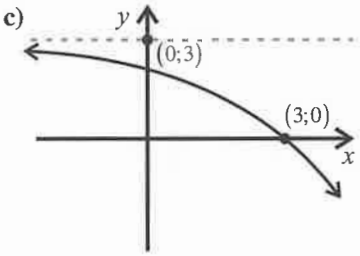
$y = 3^{x-p} + q$

b)



$f(x) = 2(b)^{x+1} + q$

c)



$h: x \to -2(b)^{x-2} + q$

Translations and Reflections of Graphs

Exercise 7:

1) **The graph of $f(x) = x^2 + 2x + 5$ is given. Write down the new equation of $h(x)$ and describe the translation of $f(x)$ if:**

   a) $h(x) = f(x-2)$

   b) $h(x) = f(x) - 2$

   c) $h(x) = -f(x)$

   d) $h(x) = f(-x)$

2) **The graph of $f(x) = \frac{4}{x} + 1$ is given. Write down the new equation of the graph if:**

   a) The graph is shifted 2 units down.

   b) The graph is shifted 2 units left.

   c) The graph is reflected about the y-axis.

   d) The graph is reflected about the x-axis.

   e) The graph is shifted 2 units right and 1 unit down.

   f) The graph is shifted 2 units left and 1 unit up.

3) **The parabola $f(x)$ has turning point $A(-3;2)$ and point $B(-1;-6)$ lies on the graph.**

   a) Determine the equation of the parabola in the form $f(x) = a(x-p)^2 + q$.

   b) If the parabola is shifted 2 units to the right and 1 unit down, determine the new turning point.

   c) If $g(x) = f(x-1)$, what is the y-intercept of $g(x)$?

   d) If $g(x) = -f(x) + 1$, what is the turning point of $g(x)$?

Graph Interpretation

Exercise 8:

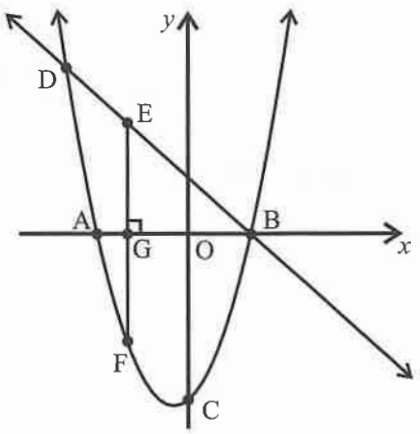
1) **Draw a rough sketch of the graph of $f(x) = ax^2 + bx + c$ if:**

   a) $a < 0; b > 0$ **and** $c > 0$

   b) $a < 0; b < 0$ **and** $c > 0$

   c) $a > 0; b > 0$ **and** $c < 0$

2) **The parabola alongside has the equation $y = 2x^2 + x - 3$ and the straight line DB is parallel to the line $y = -x$. Points A and B are the x-intercepts of the parabola and C is the y-intercept. The graphs intersect at points D and B. Determine the:**



   a) Coordinates of A, B and C.

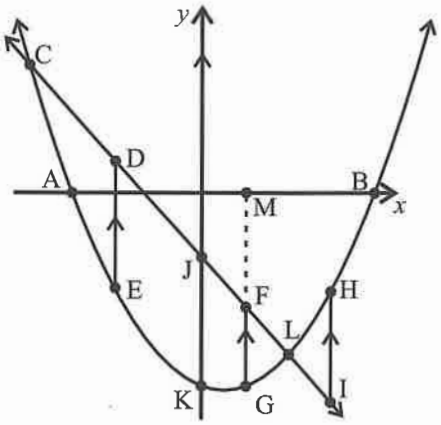
   b) Equation of line DB.

   c) Coordinates of D.

   d) Length of EF if OG is 1 unit.

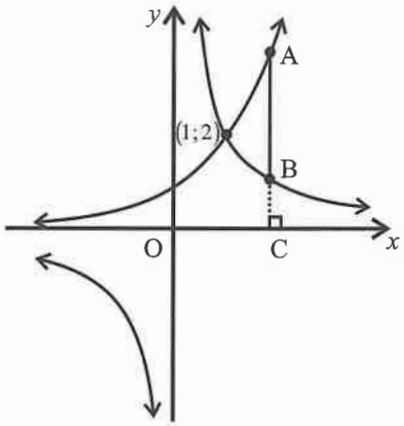
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3) **The graphs of $f(x)=x^2-x-12$ and $g(x)=-3x-4$ are sketched alongside. The graphs intersect at C and L. Determine:**



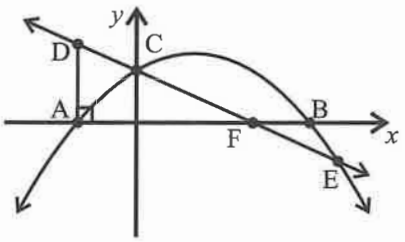
a) The length of JK.
b) The coordinates of C and L.
c) The coordinates of A and B.
d) The length of HI if the x-coordinate of H is 3.
e) The length of AM if FG = 5 units.

4) **The graphs of $f(x)=a^x$ and $g(x)=\frac{k}{x}, x\neq 0$ are sketched alongside.**
The graphs intersect at the point $(1;2)$. Determine:



a) The values of a and k.
b) The length of AB if OC = 2 units.
c) The equation of the reflection of $y=f(x)$ in the y-axis.
d) The equation of the reflection of $y=g(x)$ in the x-axis.

5) **The graphs of $f(x)=-x^2+2x+3$ and $g(x)=-\frac{3}{2}x+3$ are sketched alongside. Determine:**

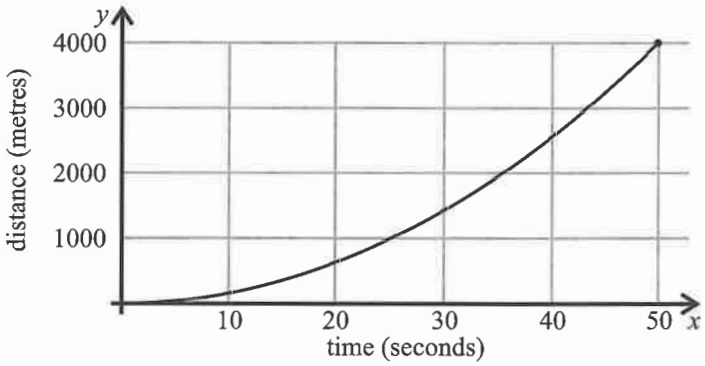


a) The coordinates of A, B and C.
b) The length of DA.
c) The coordinates of E and F.
d) The values of x for which $f(x).g(x)\leq 0$.
e) The turning point of $f(x)$.
f) The values of k for which $-x^2+2x+3=k$ will have two positive roots.

Real World Applications and Average Gradient

Exercise 9:

1) **The graph below represents the take off distance of a commercial plane. The graph van be represented by the equation $y=ax^2$.**



a) Determine the value of a.
b) What distance has the plane covered after 30 seconds?
c) How long, to the nearest second, does it take the plane to cover a distance of 2500 metres?
d) What is the average speed of the plane during the take off? Give your answer in metres per second (m/s).

Hint: speed $=\frac{\text{distance}}{\text{time}}$. This means that speed is the gradient of a distance-time graph.

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2) **The equation $h(t) = -5{,}1(x-2)^2 + 21$ represents the height of a cricket ball after it has been hit.**

a) Sketch the graph of height of the cricket ball (in metres) against time (in seconds) in the air.
b) How high above the ground was the cricket ball when it was hit?
c) How long was the ball in the air for?
d) What was the maximum height above the ground that the ball reached?
e) How long after the ball was hit did it reach its maximum height?

Trigonometric Functions

Exercise 10:

1) **On separate axes, sketch the graphs of the following functions if $x \in [-180°; 360°]$.**

a) $f(x) = 0.5\cos x + 1$

b) $f(x) = 2\tan x - 2$

c) $h(x) = 2\sin(x - 30°) + 1$

d) $y = 2\tan(x + 45°) - 1$

e) $h(x) = 3\sin(2x) + 1$

f) $y = \frac{1}{2}\cos(2x) + 1$

g) $y = -2\cos(x + 30°)$

h) $y = -2\sin(0{,}5x)$

2) **The equations of 6 trigonometric functions are given below. Match the equations to graph shown.**

a) $y = -\tan(x + 45°)$

b) $y = \frac{1}{2}\cos(3x)$

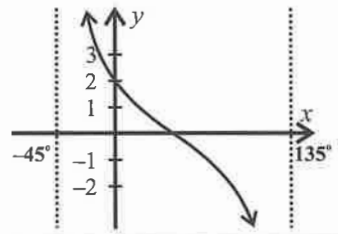
c) $y = \frac{1}{2}\cos(x - 60°)$

d) $y = -2\tan(x - 45°)$

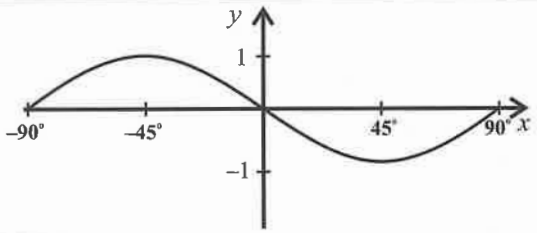
e) $y = -\sin(2x)$

f) $y = \sin(x + 90°)$

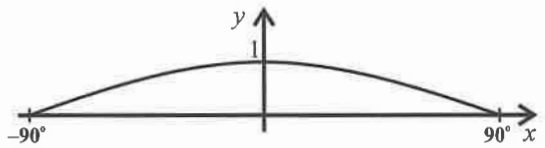
Graph 1



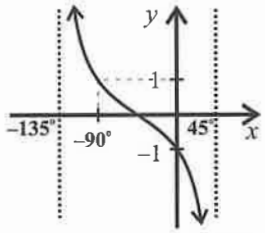
Graph 2



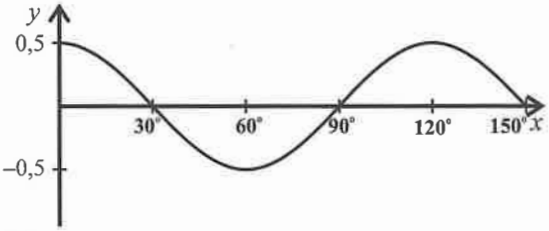
Graph 3



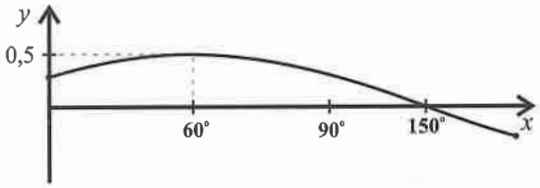
Graph 4



Graph 5

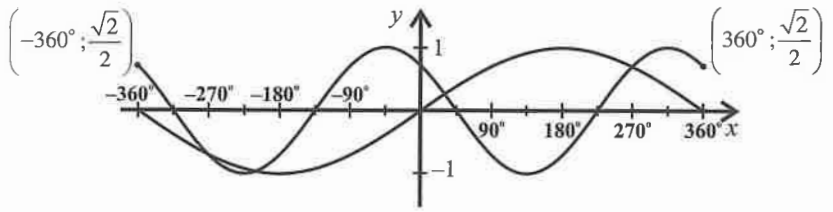


Graph 6



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3) **The graphs of $f(x)=\sin(kx)$ and $h(x)=\cos(x+p)$ are sketched below where $x\in[-360°;360°]$:**



a) What is the period of f?

b) Determine the values of k and p.

c) If $g(x)=f(x)+1$ what is the range of $g(x)$?

4) **If $x\in[-90°;360°]$, on the same set of axes sketch the graphs of $f(x)=2\sin(x)$ and $g(x)=-\sin(x-90°)$.**

a) What is the amplitude of $f(x)$?

b) What is the period of $g(x)$?

c) Using your graph, determine for which values of x: $f(x).g(x)<0$.

Mixed Exercise:

1) **The point $(-1;2)$ lies on the graph of $f(x)=3^{x-p}+q$ and the point $(2;-7)$ lies on its asymptote.**

a) Determine the equation of f.

b) Determine the value of k correct to 2 decimal places if $(k;6)$ is a point on the graph of f.

c) Determine the equation of $g(x)$ if g is a reflection of f about the y-axis.

d) Describe in words how the graph of $h(x)$ could be obtained from $f(x)$ if $h(x)=-3^{x+3}+7$.

2) **The equation of a hyperbola is $f(x)=\frac{3}{x}$. Write down the equation of $g(x)$ if $g(x)$ is obtained by transforming $f(x)$ according to the following conditions:**

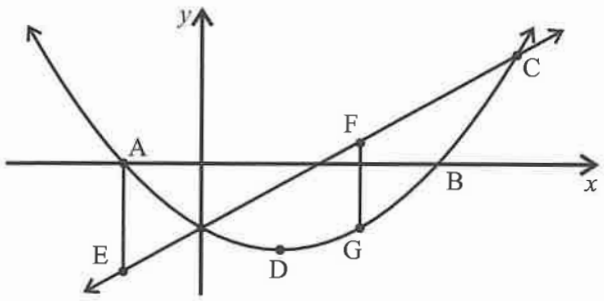
a) A shift of 2 units vertically up.

b) A shift of 5 units right.

c) The horizontal asymptote is $y=-2$ and the vertical asymptote is $x=-1$.

d) The asymptotes intersect at the point $(5;-4)$.

3) **The graphs of $f(x)=2x^2-4x-6$ and $g(x)=4x-6$ are sketched below:**



a) Determine the coordinates of A and B, the x-intercepts of f.

b) Determine the coordinates of C, a point of intersection of the graphs.

c) Determine the length of line AE.

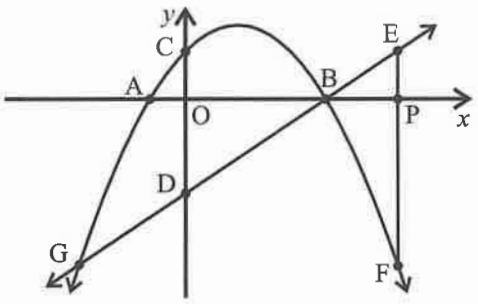
d) Determine the coordinates of F and G if FG is 8 units in length.

e) Determine the coordinates of D, the turning point of f.

f) Determine the values of k for which $k=2x^2-4x-6$ has non-real roots.

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4) **The graphs of $f(x)=-\frac{1}{2}x^2+\frac{3}{2}x+2$ and $g(x)=x-4$ are sketched below:**



a) Determine the coordinates of A and B, the x-intercepts of f.

b) Determine the length of CD.

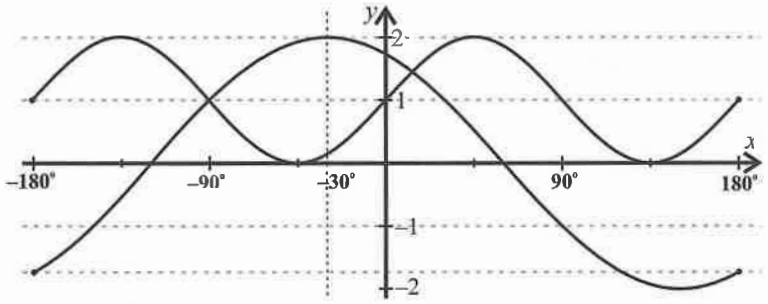
c) If EF is 9 units in length and is parallel to the y-axis, determine the length of OP.

d) Determine the coordinates of G, a point of intersection of the graphs.

e) For which values of x is $f(x)\leq g(x)$?

f) For which values of x is $\frac{f(x)}{g(x)}\leq 0$?

5) **The graphs of $f(x)=a\cos(x-p)$ and $g(x)=\sin(kx)+q$ for $x\in[-180°;180°]$.**



a) Determine the values of a, p, k and q.

b) If the graph of h is obtained by shifting the graph of f $30°$ to the right, determine the equation of h.

c) If $x\in[-180°;0°]$ determine the values of x for which $f(x)<g(x)$?

6) **If $x\in[-180°;180°]$, sketch the graphs of $f(x)=\tan(x)$ and $g(x)=-\cos(x)$ on the same axes.**

a) What is the period of f and g.

b) What is the domain of f and the range of g.

c) If $x\in[0°;180°]$ for which value/s of x is $f(x)\geq g(x)$?

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Sketching the Parabola

Exercise 1:

1)

a) $y=2(x-3)^2-2$

$a>0 \rightarrow$ happy face

TP: $(3;-2)$

y-int: $y=2(0-3)^2-2=16$

$\therefore$ y-int: $(0;16)$

x-int: $0=2\left(x^2-6x+9\right)-2$

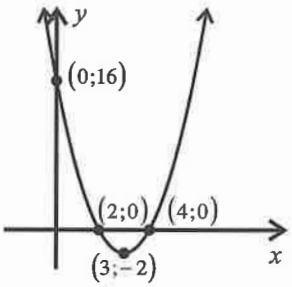
$\therefore 2x^2-12x+16=0$

$\therefore x^2-6x+8=0$

$\therefore (x-2)(x-4)=0$

$\therefore x=2$ or $x=4$

$\therefore$ x-int: $(2;0)$ or $(4;0)$



b) $f(x)=-(x+2)^2+4$

$a<0 \rightarrow$ sad face

TP: $(-2;4)$

y-int: $y=-(0+2)^2+4=0$

$\therefore$ y-int: $(0;0)$

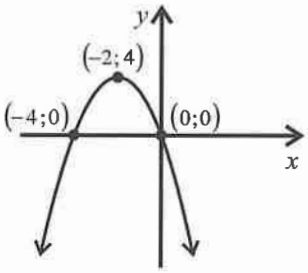
x-int: $0=-(x+2)^2+4$

$\therefore -x^2-4x=0$

$\therefore -x(x+4)=0$

$\therefore x=0$ or $x=-4$

$\therefore$ x-int: $(0;0)$ or $(-4;0)$



c) $h(x)=-(x-3)^2+5$

$a<0 \rightarrow$ sad face

TP: $(3;5)$

y-int: $y=-(0-3)^2+5=-4$

$\therefore$ y-int: $(0;-4)$

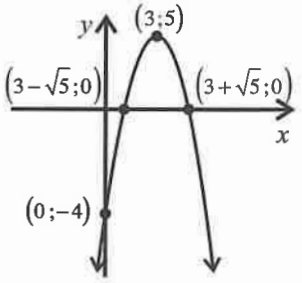
x-int: $0=-\left(x^2-6x+9\right)+5$

$\therefore 0=-x^2+6x-4$

$\therefore x=\dfrac{-6\pm\sqrt{(6)^2-4(-1)(-4)}}{2(-1)}$

$\therefore x=3-\sqrt{5}$ or $x=3+\sqrt{5}$

$\therefore$ x-int: $\left(3-\sqrt{5};0\right)$ or $\left(3+\sqrt{5};0\right)$



d) $p(x)=\dfrac{1}{2}(x+2)^2+2$

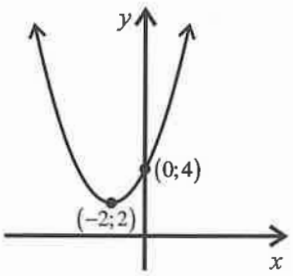
$a>0 \rightarrow$ happy face

TP: $(-2;2)$

y-int: $y=\frac{1}{2}(0+2)^2+2=4$

$\therefore$ y-int: $(0;4)$

x-int: none $(a>0$ and $q>0)$



e) $m(x)=-(x+2)^2-1$

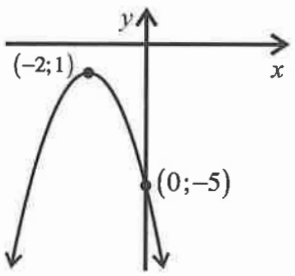
$a<0 \rightarrow$ sad face

TP: $(-2;-1)$

y-int: $y=-(0+2)^2-1=-5$

$\therefore$ y-int: $(0;-5)$

x-int: none $(a<0$ and $q<0)$



f) $f:x\rightarrow(x+3)^2-4$

$a>0 \rightarrow$ happy face

TP: $(-3;-4)$

y-int: $y=(0+3)^2-4=5$

$\therefore$ y-int: $(0;5)$

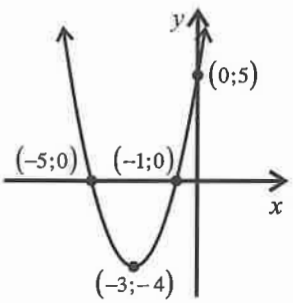
x-int: $0=(x+3)^2-4$

$\therefore x^2+6x+5=0$

$\therefore (x+1)(x+5)=0$

$\therefore x=-1$ or $x=-5$

$\therefore$ x-int: $(-1;0)$ or $(-5;0)$



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2)

a) $y=x^2+4x+3$

$a>0 \rightarrow$ happy face

TP: $x=-\frac{4}{2}=-2$

$\therefore y=(-2)^2+4(-2)+3=-1$

$\therefore$ TP: $(-2;-1)$

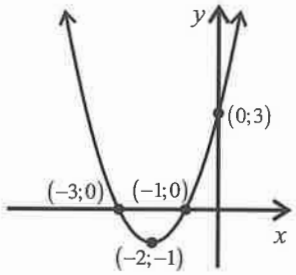
y-int: $(0;3)$

x-int: $0=x^2+4x+3$

$\therefore (x+1)(x+3)=0$

$\therefore x=-1$ or $x=-3$

$\therefore x$-int: $(-1;0)$ or $(-3;0)$



b) $f(x)=-x^2+2x+3$

$a<0 \rightarrow$ sad face

TP: $x=-\frac{2}{2(-1)}=1$

$f(1)=-(1)^2+2(1)+3=4$

$\therefore$ TP: $(1;4)$

y-int: $(0;3)$

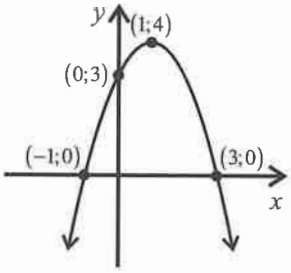
x-int: $0=-x^2+2x+3$

$\therefore x^2-2x-3=0$

$\therefore (x+1)(x-3)=0$

$\therefore x=-1$ or $x=3$

$\therefore x$-int: $(-1;0)$ or $(3;0)$



c) $f(x)=-2x^2+6x$

$a<0 \rightarrow$ sad face

TP: $x=-\frac{6}{2(-2)}=\frac{3}{2}$

$f\left(\frac{3}{2}\right)=-2\left(\frac{3}{2}\right)^2+6\left(\frac{3}{2}\right)=\frac{9}{2}$

$\therefore$ TP: $\left(\frac{3}{2};\frac{9}{2}\right)$

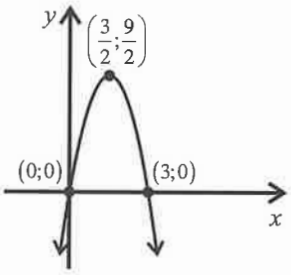
y-int: $(0;0)$

x-int: $0=-2x^2+6x$

$\therefore -2x(x-3)$

$\therefore x=0$ or $x=3$

$\therefore x$-int: $(0;0)$ or $(3;0)$



d) $h(x)=-2x^2+8x+24$

$a<0 \rightarrow$ sad face

TP: $x=-\frac{8}{2(-2)}=2$

$\therefore y=-2(2)^2+8(2)+24=32$

$\therefore$ TP: $(2;32)$

y-int: $(0;24)$

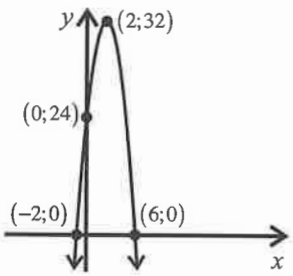
x-int: $0=-2x^2+8x+24$

$\therefore 0=-2\left(x^2-4x-12\right)$

$\therefore (x+2)(x-6)=0$

$\therefore x=-2$ or $x=6$

$\therefore x$-int: $(-2;0)$ or $(6;0)$



e) $y=-\frac{9}{2}x^2+27x+\frac{27}{2}$

$a<0 \rightarrow$ sad face

TP: $x=-\frac{27}{2(-9/2)}=3$

$\therefore y=-\frac{9}{2}(3)^2+27(3)+\frac{27}{2}=54$

$\therefore$ TP: $(3;54)$

y-int: $\left(0;\frac{27}{2}\right)$

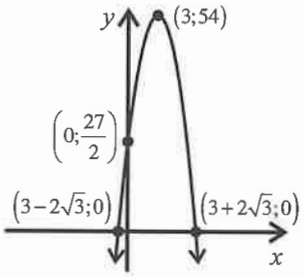
x-int: $0=-\frac{9}{2}x^2+27x+\frac{27}{2}$

$\therefore 0=-9x^2+54x+27$

$\therefore x=\frac{-54\pm\sqrt{(54)^2-4(-9)(27)}}{2(-9)}$

$\therefore x=3-2\sqrt{3}$ or $x=3+2\sqrt{3}$

$\therefore x$-int: $\left(3-2\sqrt{3};0\right)$ or $\left(3+2\sqrt{3};0\right)$



f) $y=-\frac{1}{2}x^2+2x+2$

$a<0 \rightarrow$ sad face

TP: $x=-\frac{2}{2(-1/2)}=2$

$\therefore y=-\frac{1}{2}(2)^2+2(2)+2=4$

$\therefore$ TP: $(2;4)$

y-int: $(0;2)$

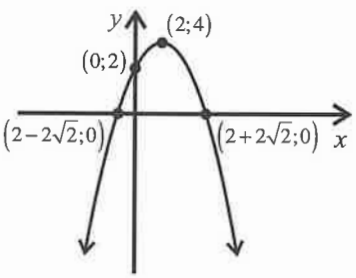
x-int: $0=-\frac{1}{2}x^2+2x+2$

$\therefore 0=x^2-4x-4$

$\therefore x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-4)}}{2(1)}$

$\therefore x=2-2\sqrt{2}$ or $x=2+2\sqrt{2}$

$\therefore x$-int: $\left(2-2\sqrt{2};0\right)$ or $\left(2+2\sqrt{2};0\right)$



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Exercise 2:

1)

a) $y=a(x+2)^2-2$

Sub $B(-4;6)$

$\therefore 6=a(-4+2)^2-2$

$\therefore 8=4a$

$\therefore a=2$

$\therefore y=2(x+2)^2-2$

$\therefore y=2\left(x^2+4x+4\right)-2$

$\therefore y=2x^2+8x+6$

b) $y=a(x-3)^2+1$

Sub $B(4;-2)$

$\therefore -2=a(4-3)^2+1$

$\therefore -3=a$

$\therefore y=-3(x-3)^2+1$

$\therefore y=-3\left(x^2-6x+9\right)+1$

$\therefore y=-3x^2+18x-26$

c) $y=a(x-6)^2+5$

Sub $B(7;1)$

$\therefore 1=a(7-6)^2+5$

$\therefore -4=a$

$\therefore y=-4(x-6)^2+5$

$\therefore y=-4\left(x^2-12x+36\right)+5$

$\therefore y=-4x^2+48x-139$

d) $y=a(x+1)^2-3$

Sub $B(0;-2)$

$\therefore -2=a(0+1)^2-3$

$\therefore a=1$

$\therefore y=1(x+1)^2-3$

$\therefore y=x^2+2x-2$

e) $y=a(x+3)^2+2$

Sub $B(-2;4)$

$\therefore 4=a(-2+3)^2+2$

$\therefore 2=a$

$\therefore y=2(x+3)^2+2$

$\therefore y=2\left(x^2+6x+9\right)+2$

$\therefore y=2x^2+12x+20$

f) $y=a(x-6)^2-5$

Sub $B(7;-1)$

$\therefore -1=a(7-6)^2-5$

$\therefore 4=a$

$\therefore y=4(x-6)^2-5$

$\therefore y=4\left(x^2-12x+36\right)-5$

$\therefore y=4x^2-48x+139$

2)

a) $y=a(x+2)(x-5)$

Sub $C(4;-3)$

$\therefore -3=a(4+2)(4-5)$

$\therefore -3=a(-6)$

$\therefore a=\frac{1}{2}$

$\therefore y=\frac{1}{2}(x+2)(x-5)$

$\therefore y=\frac{1}{2}x^2-\frac{3}{2}x-5$

b) $y=a(x+3)(x-2)$

Sub $C(1;-4)$

$\therefore -4=a(1+3)(1-2)$

$\therefore -4=a(-4)$

$\therefore a=1$

$\therefore y=1(x+3)(x-2)$

$\therefore y=x^2+x-6$

c) $y=a(x+5)(x-6)$

Sub $(4;9)$

$\therefore 9=a(4+5)(4-6)$

$\therefore 9=a(-18)$

$\therefore a=-\frac{1}{2}$

$\therefore y=-\frac{1}{2}(x+5)(x-6)$

$\therefore y=-\frac{1}{2}x^2+\frac{1}{2}x+15$

3)

a) $y=ax^2+bx-6$

Sub $A(-6;6)$

$\therefore 6=a(-6)^2+b(-6)-6$

$\therefore b=6a-2 \rightarrow \boxed{1}$

Sub $B(2;6)$

$\therefore 6=a(2)^2+b(2)-6$

$\therefore 12=4a+2b \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$12=4a+2(6a-2)$

$\therefore 16=16a$

$a=1 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$b=6(1)-3=4$

$\therefore y=x^2+4x-6$

b) $y=ax^2+bx+3$

Sub $A(-1;-7)$

$\therefore -7=a(-1)^2+b(-1)+3$

$\therefore b=a+10 \rightarrow \boxed{1}$

Sub $B(3;9)$

$\therefore 9=a(3)^2+b(3)+3$

$\therefore 6=9a+3b \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore 6=9a+3(a+10)$

$\therefore 6=12a+30$

$a=-2 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$b=-2+10=8$

$\therefore y=-2x^2+8x+3$

c) $y=ax^2+bx+3$

Sub $A(-5;13)$

$\therefore 13=a(-5)^2+b(-5)+3$

$\therefore b=5a-2 \rightarrow \boxed{1}$

Sub $B(2;13)$

$\therefore 13=a(2)^2+b(2)+3$

$\therefore 10=4a+2b \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore 10=4a+2(5a-2)$

$\therefore 14=14a$

$a=1 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$b=5(1)-2=3$

$\therefore y=x^2+3x+3$

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Exercise 3:

1)

a) $g(x)=\dfrac{-4}{x+2}-2$

$a<0 \rightarrow 2^{nd}$ and 4^{th} quadrants

asymptotes: $x=-2$ and $y=-2$

x-int: $0=\dfrac{-4}{x+2}-2$

$\therefore 2=\dfrac{-4}{x+2}$

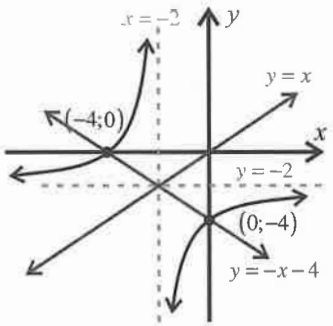
$\therefore 2x+4=-4$

$\therefore x=-4 \rightarrow (-4;0)$

y-int: $y=\dfrac{-4}{0+2}-2=-4$

$\therefore$ y-int: $(0;-4)$

AS: $y=x$ and $y=-x-4$



b) $f:x \rightarrow -\dfrac{1}{x+1}-2$

$a<0 \rightarrow 2^{nd}$ and 4^{th} quadrants

asymptotes: $x=-1$ and $y=-2$

x-int: $0=\dfrac{-1}{x+1}-2$

$\therefore 2=\dfrac{-1}{x+1}$

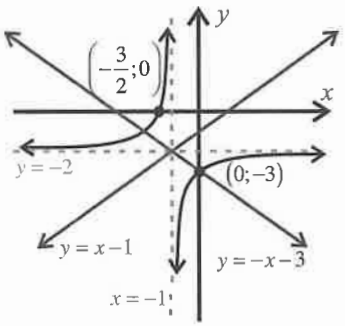
$\therefore 2x+2=-1$

$\therefore x=-3/2 \rightarrow (-3/2;0)$

y-int: $y=\dfrac{-1}{0+1}-2=-3$

$\therefore$ y-int: $(0;-3)$

AS: $y=x-1$ and $y=-x-3$



c) $f(x)=\dfrac{9}{x+3}-1$

$a>0 \rightarrow 1^{st}$ and 3^{rd} quadrants

asymptotes: $x=-3$ and $y=-1$

x-int: $0=\dfrac{9}{x+3}-1$

$\therefore 1=\dfrac{9}{x+3}$

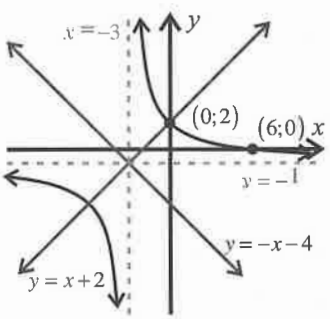
$\therefore x+3=9$

$\therefore x=6 \rightarrow (6;0)$

y-int: $y=\dfrac{9}{0+3}-1=2$

$\therefore$ y-int: $(0;2)$

AS: $y=x+2$ and $y=-x-4$



d) $y=\dfrac{6}{x-1}-3$

$a>0 \rightarrow 1^{st}$ and 3^{rd} quadrants

asymptotes: $x=1$ and $y=-3$

x-int: $0=\dfrac{6}{x-1}-3$

$\therefore 3=\dfrac{6}{x-1}$

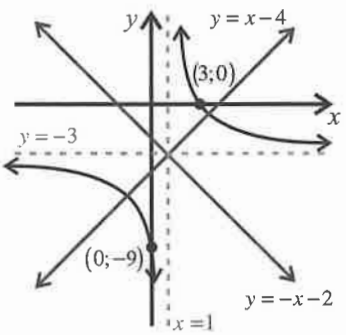
$\therefore 3x-3=6$

$\therefore x=3 \rightarrow (3;0)$

y-int: $y=\dfrac{6}{0-1}-3=-9$

$\therefore$ y-int: $(0;-9)$

AS: $y=x-4$ and $y=-x-2$



e) $y=\dfrac{-6}{x-1}+3$

$a<0 \rightarrow 2^{nd}$ and 4^{th} quadrants

asymptotes: $x=1$ and $y=3$

x-int: $0=\dfrac{-6}{x-1}+3$

$\therefore -3=\dfrac{6}{x-1}$

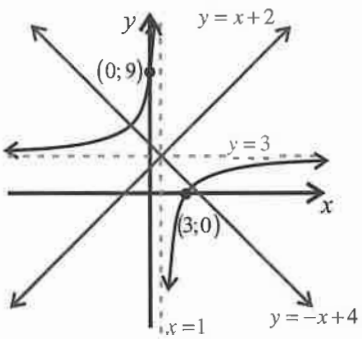
$\therefore -3x+3=-6$

$\therefore x=3 \rightarrow (3;0)$

y-int: $y=\dfrac{-6}{0-1}+3=9$

$\therefore$ y-int: $(0;9)$

AS: $y=x+2$ and $y=-x+4$



f) $y=\dfrac{-4}{x-2}-1$

$a<0 \rightarrow 2^{nd}$ and 4^{th} quadrants

asymptotes: $x=2$ and $y=-1$

x-int: $0=\dfrac{-4}{x-2}-1$

$\therefore 1=\dfrac{-4}{x-2}$

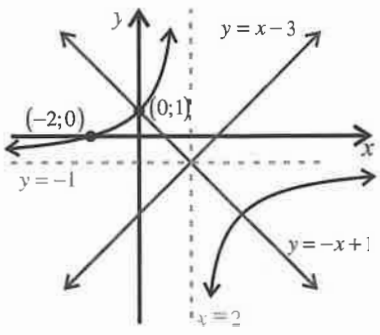
$\therefore x-2=-4$

$\therefore x=-2 \rightarrow (-2;0)$

y-int: $y=\dfrac{-4}{0-2}-1=1$

$\therefore$ y-int: $(0;1)$

AS: $y=x-3$ and $y=-x+1$



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2)

a) $a<0$; $p=-2$ and $q=-2$

Points of intersection:

$=\left(-\sqrt{4}-2;\sqrt{4}-2\right)$ and $\left(\sqrt{4}-2;-\sqrt{4}-2\right)$

$=(-4;0)$ and $(0;-4)$

b) $a<0$; $p=-1$ and $q=-2$

Points of intersection:

$\left(-\sqrt{1}-1;\sqrt{1}-2\right)$ and $\left(\sqrt{1}-1;-\sqrt{1}-2\right)$

$=(-2;-1)$ and $(0;-3)$

c) $a>0$; $p=-3$ and $q=-1$

Points of intersection:

$\left(\sqrt{9}-3;\sqrt{9}-1\right)$ and $\left(-\sqrt{9}-3;-\sqrt{9}-1\right)$

$=(0;2)$ and $(-6;-4)$

d) $a>0$; $p=1$ and $q=-3$

Points of intersection:

$\left(\sqrt{6}+1;\sqrt{6}-3\right)$ and $\left(-\sqrt{6}+1;-\sqrt{6}-3\right)$

e) $a<0$; $p=1$ and $q=3$

Points of intersection:

$\left(-\sqrt{6}+1;\sqrt{6}+3\right)$ and $\left(\sqrt{6}+1;-\sqrt{6}+3\right)$

f) $a<0$; $p=2$ and $q=-1$

Points of intersection:

$\left(-\sqrt{4}+2;\sqrt{4}-1\right)$ and $\left(\sqrt{4}+2;-\sqrt{4}-1\right)$

$=(0;1)$ and $(4;-3)$

Exercise 4:

a) $y=\frac{a}{x-2}+4$

Sub $(0;0)$

$\therefore 0=\frac{a}{0-2}+4$

$\therefore -4=\frac{a}{-2}$

$\therefore a=8$

$\therefore y=\frac{8}{x-2}+4$

b) $y=\frac{a}{x-4}-6$

Sub $(5;-4)$

$\therefore -4=\frac{a}{5-4}-6$

$\therefore 2=\frac{a}{1}$

$\therefore a=2$

$\therefore y=\frac{2}{x-4}-6$

c) $y=\frac{a}{x+3}+5$

Sub $(-5;4)$

$\therefore 4=\frac{a}{-5+3}+5$

$\therefore -1=\frac{a}{-2}$

$\therefore a=2$

$\therefore y=\frac{2}{x+3}+5$

d) $y=\frac{a}{x-2}+q$

Sub $(5;0)$

$\therefore 0=\frac{a}{5-2}+q$

$\therefore -q=\frac{a}{3}$

$\therefore a=-3q \rightarrow \boxed{1}$

Sub $(3;-2)$

$-2=\frac{a}{3-2}+q$

$\therefore -2=a+q \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore -2=-3q+q$

$\therefore q=1 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$\therefore a=-3$

$\therefore y=\frac{-3}{x-2}+1$

e) $y=\frac{a}{x-p}+1$

Sub $(0;2)$

$\therefore 2=\frac{a}{0-p}+1$

$\therefore 1=\frac{a}{-p}$

$\therefore a=-p \rightarrow \boxed{1}$

Sub $(-2;0)$

$0=\frac{a}{-2-p}+1$

$\therefore -1=\frac{a}{-2-p}$

$\therefore a=p+2 \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore p+2=-p$

$\therefore p=-1 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$\therefore a=1$

$\therefore y=\frac{1}{x+1}+1$

f) $y=\frac{a}{x-p}-1$

Sub $(-5;0)$

$\therefore 0=\frac{a}{-5-p}-1$

$\therefore 1=\frac{a}{-5-p}$

$\therefore a=-5-p \rightarrow \boxed{1}$

Sub $(1;-2)$

$-2=\frac{a}{1-p}-1$

$\therefore -1=\frac{a}{1-p}$

$\therefore a=p-1 \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$\therefore -5-p=p-1$

$\therefore p=-2 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$\therefore a=-3$

$\therefore y=\frac{-3}{x+2}-1$

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Exercise 5:

a) $y = 2^{x+1} + 2$

$a > 0 \rightarrow$ above the asymptote

$a > 0$ and $b > 0 \rightarrow$ increasing

y-int: let $x = 0$

$\therefore y = 2^{0+1} + 2 = 4$

$\therefore$ y-int: $(0;4)$

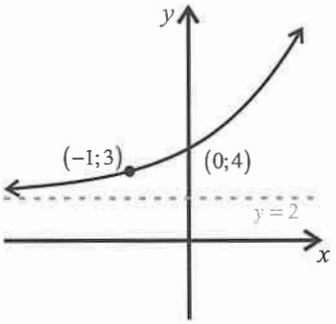
horizontal asymptote: $y = 2$

x-int: none $\rightarrow a > 0$ and $q > 0$

additional point: Sub $x = -1$

$\therefore y = 2^0 + 2 = 3$

$\therefore$ additional point: $(-1;3)$



b) $f(x) = -(2)^{x+1} - 2$

$a < 0 \rightarrow$ below the asymptote

$a < 0$ and $b > 0 \rightarrow$ decreasing

y-int: let $x = 0$

$\therefore f(x) = -(2)^{0+1} - 2 = -4$

$\therefore$ y-int: $(0;-4)$

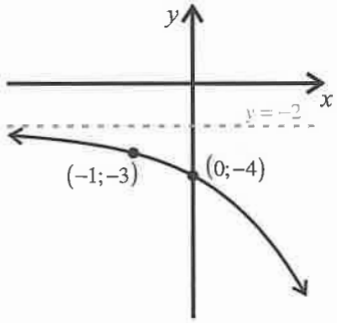
horizontal asymptote: $y = -2$

x-int: none $\rightarrow a < 0$ and $q < 0$

additional point: Sub $x = -1$

$\therefore f(x) = -(2)^0 - 2 = -3$

$\therefore$ additional point: $(-1;-3)$



c) $f: x \rightarrow 2^{-x+1} + 2$

$\therefore f: x \rightarrow \left(\frac{1}{2}\right)^{x-1} + 2$

$a > 0 \rightarrow$ above the asymptote

$a > 0$ and $0 < b < 1 \rightarrow$ decreasing

y-int: let $x = 0$

$\therefore y = \left(\frac{1}{2}\right)^{0-1} + 2$

$\therefore$ y-int: $(0;4)$

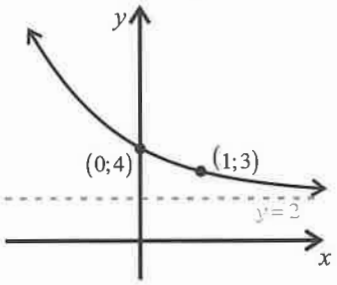
horizontal asymptote: $y = 2$

x-int: none $\rightarrow a > 0$ and $q > 0$

additional point: Sub $x = 1$

$\therefore y = \left(\frac{1}{2}\right)^{1-1} + 2 = 3$

$\therefore$ additional point: $(1;3)$



d) $f(x) = -\left(\frac{1}{2}\right)^{x+2} + 4$

$a < 0 \rightarrow$ below the asymptote

$a < 0$ and $0 < b < 1 \rightarrow$ increasing

y-int: let $x = 0$

$\therefore f(x) = -\left(\frac{1}{2}\right)^{0+2} + 4 = 15/4$

$\therefore$ y-int: $\left(0;15/4\right)$

horizontal asymptote: $y = 4$

x-int: let $y = 0$

$\therefore 0 = -\left(\frac{1}{2}\right)^{x+2} + 4$

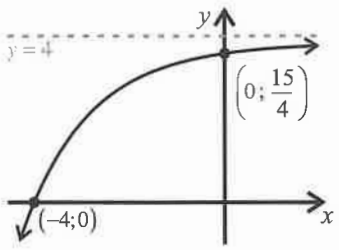
$\therefore \left(\frac{1}{2}\right)^{x+2} = 4$

$\therefore 2^{-x-2} = 2^2$

$\therefore -x - 2 = 2$

$\therefore x = -4$

$\therefore$ x-int: $(-4;0)$



e) $f(x) = \left(\frac{1}{2}\right)^{x+1} - 2$

$a > 0 \rightarrow$ above the asymptote

$a > 0$ and $0 < b < 1 \rightarrow$ decreasing

y-int: let $x = 0$

$\therefore f(x) = \left(\frac{1}{2}\right)^{0+1} - 2$

$\therefore$ y-int: $\left(0;-3/2\right)$

horizontal asymptote: $y = -2$

x-int: let $y = 0$

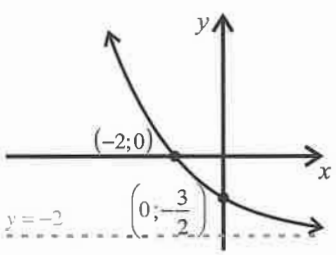
$\therefore 0 = \left(\frac{1}{2}\right)^{x+1} - 2$

$\therefore 2^1 = 2^{-x-1}$

$\therefore 1 = -x - 1$

$\therefore x = -2$

$\therefore$ x-int: $(-2;0)$



f) $y = 4(2)^{x-2} + 1$

$a > 0 \rightarrow$ above the asymptote

$a > 0$ and $b > 0 \rightarrow$ increasing

y-int: let $x = 0$

$\therefore y = 4(2)^{0-2} + 1 = 2$

$\therefore$ y-int: $(0;2)$

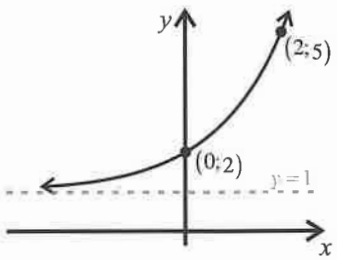
horizontal asymptote: $y = 1$

x-int: none $\rightarrow a > 0$ and $q > 0$

additional point: Sub $x = 2$

$\therefore y = 4(2)^{2-2} + 1$

$\therefore$ additional point: $(2;5)$



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Exercise 6:

a) $y=3^{x-p}-2$

Subst $(-1;-1)$

$\therefore -1=3^{-1-p}-2$

$\therefore 1=3^{-1-p}$

$\therefore 3^0=3^{-1-p}$

$\therefore 0=-1-p$

$\therefore p=-1$

$\therefore y=3^{x+1}-2$

b) $f(x)=2(b)^{x+1}-3$

Subst $(0;-2)$

$\therefore -2=2b^{0+1}-3$

$\therefore 1=2b$

$\therefore b=\frac{1}{2}$

$\therefore f(x)=2\left(\frac{1}{2}\right)^{x+1}-3$

c) $h(x)=-2(b)^{x-2}+3$

Subst $(3;0)$

$\therefore 0=-2(b)^{3-2}+3$

$\therefore 2b=3$

$\therefore b=\frac{3}{2}$

$\therefore h:x\to -2\left(\frac{3}{2}\right)^{x-2}+3$

Translations and Reflections of Graphs

Exercise 7:

1)

a) $h(x)=f(x-2)$

$\therefore h(x)=(x-2)^2+2(x-2)+5$

$=x^2-4x+4+2x-4+5$

$=x^2-2x+5$

The graph has been shifted 2 units to the right.

b) $h(x)=f(x)-2$

$\therefore h(x)=x^2+2x+5-2$

$=x^2+2x+3$

The graph has been shifted 2 units down.

c) $h(x)=-f(x)$

$\therefore h(x)=-\left(x^2+2x+5\right)$

$=-x^2-2x-5$

The graph has been reflected over the x-axis.

d) $h(x)=f(-x)$

$\therefore h(x)=(-x)^2+2(-x)+5$

$=x^2-2x+5$

The graph has been reflected over the y-axis.

2)

a) $f(x)-2=\frac{4}{x}+1-2$

$=\frac{4}{x}-1$

b) $f(x+2)=\frac{4}{x+2}+1$

c) $f(-x)=\frac{4}{-x}+1$

$=-\frac{4}{x}+1$

d) $-f(x)=-\left(\frac{4}{x}+1\right)=-\frac{4}{x}-1$

e) $f(x-2)-1=\frac{4}{x-2}+1-1$

$=\frac{4}{x-2}$

f) $f(x+2)+1=\frac{4}{x+2}+1+1$

$=\frac{4}{x+2}+2$

3)

a) $y=a(x+3)^2+2$

Sub $(-1;-6)$

$\therefore -6=a(-1+3)^2+2$

$\therefore -8=4a$

$\therefore a=-2$

$\therefore f(x)=-2(x+3)^2+2$

b) $(-1;1)$

c) The graph of $f(x)$ is shifted one unit right.

This means that point B will lie on the y-axis.

$\therefore$ The y-intercept of $g(x)$ is $(0;-6)$

d) The graph of $f(x)$ is reflected over the x-axis.

and shifted 1 unit up

$\therefore$ The turning point of $g(x)$ is $(-3;-1)$

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Exercise 8:

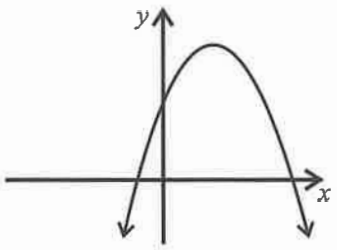
1)

a) AS: $x=\frac{-b}{2a}\rightarrow\frac{-(+)}{-}\rightarrow+$

$\therefore x$ value of the TP is positive

$c>0\rightarrow y$-int is positive

$a<0\rightarrow$ sad face

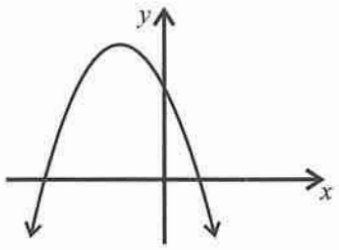


b) AS: $x=\frac{-b}{2a}\rightarrow\frac{-(-)}{-}\rightarrow-$

$\therefore x$ value of the TP is negative

$c>0\rightarrow y$-int is positive

$a<0\rightarrow$ sad face

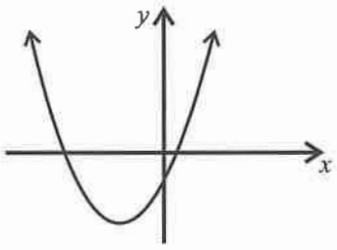


c) AS: $x=\frac{-b}{2a}\rightarrow\frac{-(+)}{+}\rightarrow-$

$\therefore x$ value of the TP is negative

$c<0\rightarrow y$-int is negative

$a>0\rightarrow$ happy face



2)

a) C is the point $(0;-3)$

x-int: let $y=0$

$\therefore 2x^2+x-3=0$

$\therefore (2x+3)(x-1)=0$

$\therefore x=-\frac{3}{2}$ or $x=1$

$\therefore x$-int's: $A\left(-\frac{3}{2};0\right)$ and $B(1;0)$

b) $y=mx+c$

$\therefore y=-x+c\rightarrow DB\,//\,y=-x$

Sub $B(1;0)$

$\therefore 0=-1+c$

$\therefore c=1$

$\therefore y=-x+1$

c) $2x^2+x-3=-x+1$

$\therefore 2x^2+2x-4=0$

$\therefore x^2+x-2=0$

$\therefore (x+2)(x-1)=0$

$\therefore x=-2$ or $x=1\rightarrow$ N/A

Sub $x=-2$ into $y=-x+1$

$\therefore y=-(-2)+1=3$

$\therefore$ D is the point $(-2;3)$

d) $EF=-x+1-\left(2x^2+x-3\right)$

$=-x+1-2x^2-x+3$

$=-2x^2-2x+4$

Sub $x=-1$

$\therefore EF=-2(-1)^2-2(-1)+4$

$\therefore EF=4$ units

3)

a) $JK=y_{top}-y_{bot}=y_J-y_K$

$\therefore JK=-4-(-12)$

$\therefore JK=8$ units

b) $x^2-x-12=-3x-4$

$\therefore x^2+2x-8=0$

$\therefore (x+4)(x-2)=0$

$\therefore x=-4$ or $x=2$

Sub these points into $g(x)$

$g(-4)=8$ and $g(2)=-10$

$\therefore$ C is the point $(-4;8)$ and L is $(2;-10)$

c) $x^2-x-12=0$

$\therefore (x+3)(x-4)=0$

$\therefore x=-3$ or $x=4$

$\therefore$ A is the point $(-3;0)$ and B is $(4;0)$

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d) $\text{HI} = x^2 - x - 12 - (-3x - 4)$

$= x^2 + 2x - 8$

Sub $x = 3$

$\therefore \text{HI} = 3^2 + 2(3) - 8$

$\therefore \text{HI} = 7$ units

4)

a) Sub $(1;2)$ into $f(x)$ and $g(x)$

$\therefore 2 = a^1$ and $2 = \frac{k}{1}$

$\therefore a = 2$ and $k = 2$

c) $y = 2^{-x} \rightarrow y = \left(\frac{1}{2}\right)^x$

5)

a) C is the point $(0;3)$

x-int: let $y = 0$

$\therefore 0 = -x^2 + 2x + 3$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x+1)(x-3) = 0$

$\therefore x = -1$ or $x = 3$

The points are: $\text{A}(-1;0)$ and $\text{B}(3;0)$

b) $\text{DA} = -\frac{3}{2}x + 3 - \left(-x^2 + 2x + 3\right)$

$= -\frac{3}{2}x + 3 + x^2 - 2x - 3$

$= x^2 - \frac{7}{2}x$

Sub $x = -1$

$\therefore \text{DA} = (-1)^2 - \frac{7}{2}(-1)$

$= 4{,}5$ units

e) $\text{FG} = -3x - 4 - \left(x^2 - x - 12\right)$

$\therefore 5 = -x^2 - 2x + 8$

$\therefore x^2 + 2x - 3 = 0$

$\therefore (x-1)(x+3) = 0$

$\therefore x = 1$ or $x = -3 \rightarrow$ N/A

$\therefore \text{AM} = x_{right} - x_{left} = x_{\text{M}} - x_{\text{A}}$

$\therefore \text{AM} = 1 - (-3)$

$\therefore \text{AM} = 4$ units

b) $\text{AB} = f(x) - g(x)$

$= 2^x - \frac{2}{x}$

Sub $x = 2$

$\therefore \text{AB} = 2^2 - \frac{2}{2}$

$\therefore \text{AB} = 3$ units

d) $y = -\frac{2}{x}; \; x \neq 0$

c) E: $g(x) = f(x)$

$\therefore -\frac{3}{2}x + 3 = -x^2 + 2x + 3$

$\therefore x^2 - \frac{7}{2}x = 0$

$\therefore x\left(x - \frac{7}{2}\right) = 0$

$\therefore x = \frac{7}{2}$ or $x = 0 \rightarrow$ N/A

$g\left(\frac{7}{2}\right) = -\frac{3}{2}\left(\frac{7}{2}\right) + 3 = -\frac{9}{4}$

$\therefore$ E is the point $\left(\frac{7}{2};-\frac{9}{4}\right)$

F: $g(x) = 0$

$\therefore 0 = -\frac{3}{2}x + 3$

$\therefore 0 = -3x + 6$

$\therefore x = 2$

$\therefore$ F is the point $(2;0)$

d) $x \leq -1$ or $2 \leq x \leq 3$

Also written as:

$x \in (-\infty;-1]$ or $[2;3]$

e) $x = \frac{-b}{2a} = \frac{-2}{2(-1)} = 1$

$f(1) = -(1)^2 + 2(1) + 3 = 4$

$\therefore$ TP is $(1;4)$

f) $-x^2 + 2x + 3 - k = 0$

$\therefore 3 < k < 4 \rightarrow$ also written as $k \in (3;4)$

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Real World Applications and Average Gradient

Exercise 9:

1)

a) Sub the point $(50;4000)$

$\therefore 4000 = a(50)^2$

$\therefore a = 1,6$

b) Sub $x = 30$ into $y = 1,6x^2$

$\therefore y = 1,6(30)^2$

$\therefore y = 1440$ metres

c) Sub $y = 2500$

$\therefore 2500 = 1,6x^2$

$\therefore x^2 = 1562,5$

$\therefore x = 39,53$ or $x \neq -39,53$

$\therefore x = 40$ seconds

d) $m = \frac{4000-0}{50-0} = 80$

$\therefore$ Average take off speed is 80 m/s.

2)

a) $h(t) = -5,1(x-2)^2 + 21$

$a < 0 \rightarrow$ sad face

TP: $(2;21)$

y-int: $y = -5,1(0-2)^2 + 21 = 0,6$

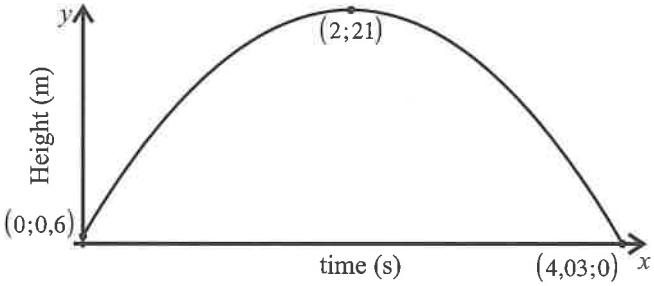
x-int: $0 = -5,1(x-2)^2 + 21$

$\therefore 5,1(x-2)^2 = 21$

$\therefore (x-2)^2 = 70/17$

$\therefore x - 2 = \pm\sqrt{70/17}$

$\therefore x = 4,03$ or $x = -0,03 \rightarrow$ N/A



b) The ball is hit when $t = 0$

$h(0) = 0,6$ metres

c) 4,03 seconds

d) The TP represents the maximum height

$\therefore$ Max height is 21 metres.

e) The TP represents the maximum height

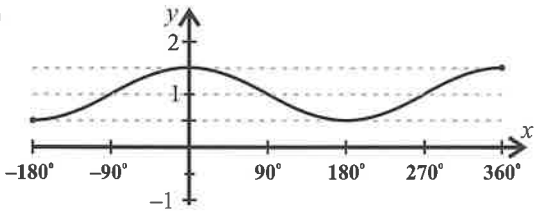
$\therefore$ Max height is reached after 2 seconds.

Trigonometric Functions

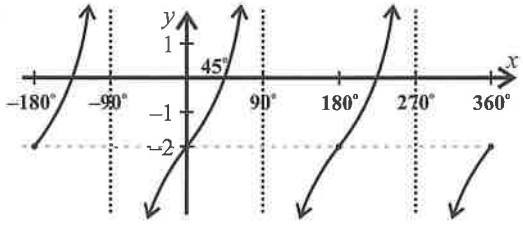
Exercise 10:

1)

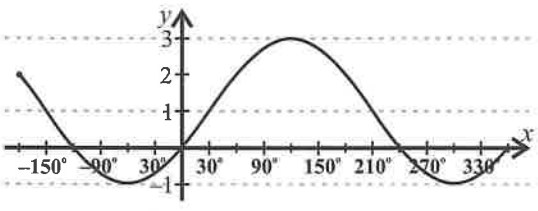
a)



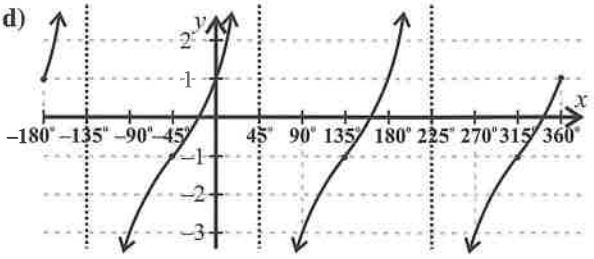
b)



c)

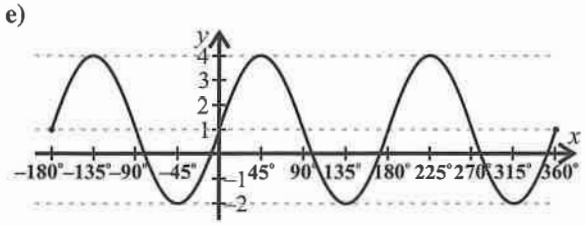


d)

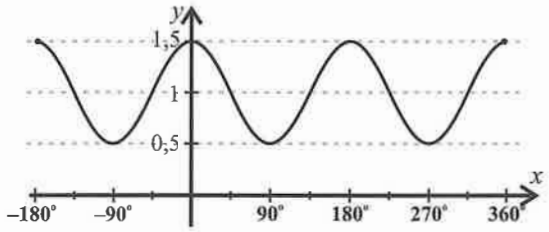


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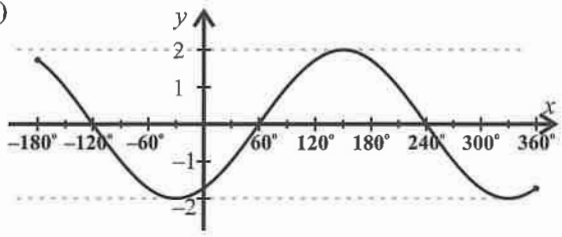
e)



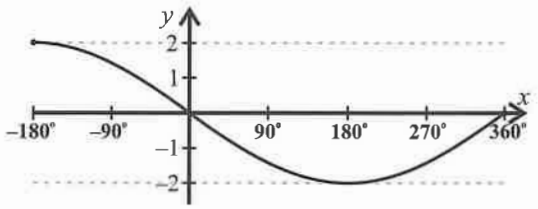
f)



g)



h)



2)

a) Graph 4
b) Graph 5
c) Graph 6
d) Graph 1
e) Graph 2
f) Graph 3

3)

a) $f(x)$ completes half a cycle in $360°$

$\therefore$ The period of $f(x)$ is $2 \times 360° = 720°$

b) $\frac{360°}{k} = 720°$

$\therefore k = \frac{1}{2}$

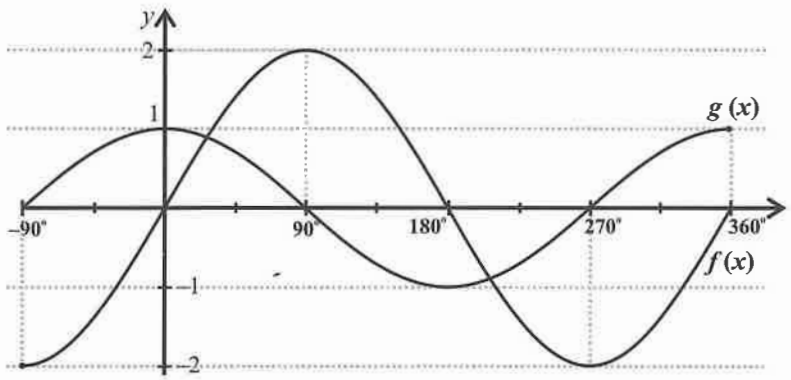
The graph of $h(x)$ is a cos graph shifted $45°$ left.

$\therefore p = 45°$

c) The range of $f(x)$ is $y \in [-1;1]$

$\therefore$ The range of $g(x)$ is $y \in [0;2]$

4)



a) 2

b) $360°$

c) This occurs where one graph is positive and the other is negative

$\therefore -90° < x < 0°$ or $90° < x < 180°$ or $270° < x < 360°$

This is also written as $x \in (-90°;0°)$ or $(90°;180°)$ or $(270°;360°)$

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Mixed Exercise:

1)

a) $f(x)=3^{x-p}-7$

Sub the point $(-1;2)$

$\therefore 2=3^{-1-p}-7$

$\therefore 9=3^{-1-p}$

$\therefore 3^2=3^{-1-p}$

$\therefore 2=-1-p$

$\therefore p=-3$

$\therefore f(x)=3^{x+3}-7$

b) Sub in the point $(k;6)$

$\therefore 6=3^{k+3}-7$

$\therefore 13=3^{k+3}$

use trial and error on the calculator

$\therefore k+3=2,34$

$\therefore k=-0,66$

c) All the x-values change sign:

$\therefore g(x)=3^{-x+3}-7$

d) It can be seen that the y-value of $f(x)$ has changed sign.

$\therefore h(x)$ is a reflection of $f(x)$ about the x-axis.

2)

a) $g(x)=\frac{3}{x}+2$

b) $g(x)=\frac{3}{x-5}$

c) $g(x)=\frac{3}{x+1}-2$

d) $g(x)=\frac{3}{x-5}-4$

3)

a) x-int: let $y=0$

$\therefore 0=2x^2-4x-6$

$\therefore 0=x^2-2x-3$

$\therefore 0=(x+1)(x-3)$

$\therefore x=-1$ or $x=3$

$\therefore A(-1;0)$ and $B(3;0)$

b) At C: $f(x)=g(x)$

$\therefore 2x^2-4x-6=4x-6$

$\therefore 2x^2-8x=0$

$\therefore 2x(x-4)=0$

$\therefore x=4$ or $x=0 \rightarrow$ N/A

$\therefore g(4)=4(4)-6=10 \rightarrow f(4)$ can also be used

$\therefore$ C is the point $(4;10)$

c) $AE=f(x)-g(x)$

$=2x^2-4x-6-(4x-6)$

$=2x^2-8x$

At A the x coordinate is -1

$\therefore AE=2(-1)^2-8(-1)=10$ units

d) $FG=g(x)-f(x)$

$\therefore 8=4x-6-\left(2x^2-4x-6\right)$

$\therefore 8=4x-6-2x^2+4x+6$

$\therefore 2x^2-8x+8=0$

$\therefore x^2-4x+4=0$

$\therefore (x-2)^2=0$

$\therefore x=2$

$f(2)=-6$ and $g(2)=2$

$F(2;2)$ and $G(2;-6)$

e) $x=\frac{-b}{2a}=\frac{-(-4)}{2(2)}=1$

$f(1)=2(1)^2-4(1)-6=-8$

D is the point $(1;-8)$

f) $0=2x^2-4x-6-k$

the graph has to be shifted up by more than 8 units

$\therefore k<-8$

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4)

a) x-int: let $y=0$

$\therefore 0=-\frac{1}{2}x^2+\frac{3}{2}x+2$

$\therefore 0=x^2-3x-4$

$\therefore 0=(x+1)(x-4)$

$\therefore x=-1$ or $x=4$

$\therefore A(-1;0)$ and $B(4;0)$

b) $CD=y_{top}-y_{bot}=y_C-y_D$

$\therefore CD=2-(-4)=6$ units

c) $EF=g(x)-f(x)$

$\therefore 9=x-4-\left(-\frac{1}{2}x^2+\frac{3}{2}x+2\right)$

$\therefore 9=x-4+\frac{1}{2}x^2-\frac{3}{2}x-2$

$\therefore 18=2x-8+x^2-3x-4$

$\therefore 0=x^2-x-30$

$\therefore 0=(x-6)(x+5)$

$\therefore x=6$ or $x=-5 \rightarrow$ N/A

$\therefore OP=6$ units

d) At g: $f(x)=g(x)$

$\therefore x-4=-\frac{1}{2}x^2+\frac{3}{2}x+2$

$\therefore 2x-8=-x^2+3x+4$

$\therefore x^2-x-12=0$

$\therefore (x+3)(x-4)=0$

$\therefore x=-3$ or $x=4 \rightarrow$ N/A

$\therefore g(-3)=-3-4=-7 \rightarrow f(4)$ can also be used

$\therefore$ G is the point $(-3;-7)$

e) $x\le -3$ or $x\ge 4$

This can also be written as:

$x\in(-\infty;-3]$ or $[4;\infty)$

f) This occurs where one graph is positive and the other negative:

$\therefore x\ge -1$ but $x\neq 4$ as $g(x)$ is 0 when $x=4$

This can also be written as: $x\in[-1;\infty)$ but $x\neq 4$

5)

a) $p=-30^\circ$

$a=2$

$k=2$

$q=1$

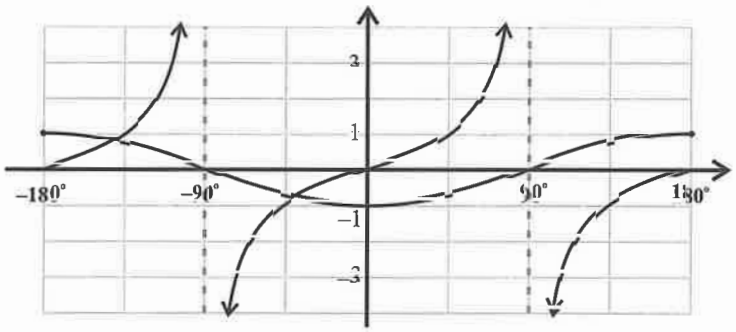
b) $h(x)=2\cos(x+30^\circ-30^\circ)$

$=2\cos x$

c) $-180^\circ\le x<-90^\circ$

This can also be written as: $x\in[-180^\circ;-90^\circ)$

6)



a) f: 180°

g: 360°

b) $f: x\in[-180^\circ;180^\circ]$ but $x\neq\pm 90^\circ$

$g: y\in[-1;1]$

c) $0^\circ\le x<90^\circ$

This is also written as $x\in[0^\circ;90^\circ)$

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Grade 10 Revision

Exercise 1:

1) Determine which quadrant the terminal arm would lie in for each of the following cases:

a) $\cos\theta > 0$ and $\tan\theta < 0$

b) $\sin\theta < 0$ and $\theta \in [90^\circ; 270^\circ]$

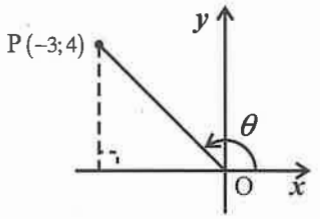
c) $\tan x > 0$ and $\sin x > 0$

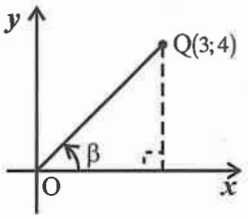
d) $\tan\theta < 0$ and $\theta \in [180^\circ; 360^\circ]$

e) $\sin x > 0$ and $90^\circ \le x \le 360^\circ$

f) $\sin\beta < 0$ and $\cos\beta < 0$

2) Using the diagrams below determine the value of:





a) $\sin\theta$

b) $\sin\beta$

c) $\cos\theta$

d) $\cos\beta$

e) $\tan\theta$

f) $\tan\beta$

3) If $3\sin\theta = 2$ and $\cos\theta < 0$, with the aid of a diagram determine the value of:

a) $\tan\theta$

b) $\dfrac{\sin\theta}{\cos\theta}$

c) $\sin^2\theta + \cos^2\theta$

4) Without the use of a calculator, determine the value of:

a) $\sin^2 30^\circ$

b) $\sqrt{1-\cos^2 30^\circ}$

c) $\cos^2 30^\circ$

d) $\sqrt{1-\sin^2 30^\circ}$

e) $\tan 60^\circ$

f) $\dfrac{\sin 60^\circ}{\cos 60^\circ}$

Identities

Exercise 2:

1) Simplify the following expressions as far as possible:

a) $\dfrac{\sin^2 x + \sin^2 x\cos x}{1+\cos x}$

b) $\left(1-\sin^2\theta\right)\left(1+\tan^2\theta\right)$

c) $\dfrac{4\sin^2 x + 4\cos^2 x}{-2\sin^2 x - 2\cos^2 x}$

d) $\tan x\cos x\sin x - 1$

e) $(\sin\theta - \cos\theta)^2 + 2\cos\theta\sin\theta$

f) $\dfrac{1}{\cos^2 y} - \tan^2 y + 1$

g) $\tan^2\theta - \dfrac{1}{\cos^2\theta}$ \\

h) $\left(\dfrac{1}{\cos^2\theta} - 1\right)\left(\dfrac{1}{\sin^2\theta} - 1\right)$

2) Prove the following identities:

a) $(1+\tan^2 x)\cos^2 x = 1$

b) $\sin^2 x + \dfrac{\sin x\cos x}{\tan x} = 1$

c) $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = \dfrac{2\tan x}{\cos x}$

d) $\dfrac{\tan^2\alpha}{1+\tan^2\alpha} = \sin^2\alpha$

e) $1 + \dfrac{2\cos x(1+\cos x)}{\sin^2 x} = \dfrac{1+\cos x}{1-\cos x}$

f) $\dfrac{1-2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$

g) $\dfrac{2\cos^2\theta + \cos\theta - 1}{\sin^2\theta} = \dfrac{2\cos\theta - 1}{1-\cos\theta}$

h) $\dfrac{2\cos^2\beta + 5\cos\beta - 3}{2\sin\beta\cos\beta - \sin\beta} = \dfrac{\cos\beta + 3}{\sin\beta}$

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i) $\dfrac{\cos^2\theta-\cos\theta-\sin^2\theta}{2\sin\theta\cos\theta+\sin\theta}=\dfrac{\cos\theta-1}{\sin\theta}$

j) $\dfrac{1-\sin x}{\cos x}=\dfrac{\cos x}{1+\sin x}$

k) $\dfrac{5\cos^2\theta-\cos\theta\sin\theta-3}{2\cos^2\theta-3\cos\theta\sin\theta}=\dfrac{\cos\theta+\sin\theta}{\cos\theta}$

l) $\dfrac{2\sin^2 x+\sin x-1}{\cos^2 x}=\dfrac{2\sin x-1}{1-\sin x}$

m) $\dfrac{1-\cos x}{\sin x}=\dfrac{\sin x}{1+\cos x}$

n) $\dfrac{1}{\sin x-1}=\dfrac{1}{\sin x+1}-\dfrac{2}{\cos^2 x}$

o) $\dfrac{\cos x}{1+\sin x}=\sqrt{\dfrac{1-\sin x}{1+\sin x}}$

p) $\left(\tan x+\dfrac{1}{\tan x}\right)\left(1-\cos^2 x\right)=\tan x$

Reduction Formula

Exercise 3:

1) Simplify the following expressions as far as possible:

a) $\dfrac{2\sin\left(180^\circ-x\right)\cos\left(360^\circ-x\right)}{3\sin\left(180^\circ+x\right)\cos\left(180^\circ-x\right)}$

b) $1+\dfrac{2\sin^2\left(360^\circ-\theta\right)}{\cos^2\left(180^\circ+\theta\right)}\times\dfrac{1}{\tan^2\left(180^\circ-\theta\right)}$

c) $\dfrac{\cos\left(180^\circ+x\right)\sin\left(360^\circ-x\right)\tan\left(180^\circ-x\right)}{\sin\left(180^\circ+x\right)\sin\left(360^\circ-x\right)}$

d) $\dfrac{\cos^2\left(180^\circ+\theta\right)\sin\left(180^\circ-\theta\right)\tan\left(180^\circ+\theta\right)}{\sin\left(360^\circ-\theta\right)\sin\left(180^\circ+\theta\right)}$

2) Simplify the following expressions as far as possible:

a) $\left[\sin(-x)+\cos(-x)\right]^2+2\sin x\sin\left(90^\circ-x\right)$

b) $\sin\left(90^\circ+B\right)\cos B+\dfrac{\sin^2\left(180^\circ+B\right)\tan(-B)}{\tan\left(360^\circ-B\right)}$

c) $\dfrac{1-\cos\left(90^\circ+\theta\right)}{\cos^2\left(180^\circ-\theta\right)}$

d) $\dfrac{\cos\left(90^\circ+\theta\right)}{\cos\left(360^\circ-\theta\right)\tan\left(360^\circ-\theta\right)}$

e) $\dfrac{\sin\left(\theta-180^\circ\right)}{\tan\left(360^\circ-\theta\right)\sin\left(90^\circ-\theta\right)}-\cos\beta\sin\left(90^\circ-\beta\right)$

f) $\dfrac{\left[\tan^2\theta-\cos^2\left(90^\circ+\theta\right)\right]\left[\dfrac{1}{\tan^2\left(90^\circ-\theta\right)}+1\right]}{\tan^2\left(180^\circ+\theta\right)}$

Exercise 4:

1) Evaluate the following expressions without the use of a calculator:

a) $\tan 240^\circ$

b) $\sin 300^\circ$

c) $\cos 420^\circ$

d) $\sin 210^\circ-\cos 300^\circ$

e) $\tan 240^\circ-\cos\left(-30^\circ\right)$

f) $\sin\left(-150^\circ\right)+\cos\left(120^\circ\right)$

2) Determine the value of the following expressions without the use of a calculator:

a) $\cos 300^\circ-\sin 150^\circ$

b) $\dfrac{\sin 300^\circ}{\tan 240^\circ}$

c) $\dfrac{\sin\left(-60^\circ\right)\tan\left(210^\circ\right)}{\cos^2 135^\circ}$

d) $\sqrt{2}\tan 150^\circ\cos 225^\circ\sin 240^\circ$

e) $\dfrac{\cos 240^\circ+\sin^2\left(-45^\circ\right)\cos 150^\circ}{-\sin 300^\circ+\tan\left(-135^\circ\right)}$

f) $\dfrac{\sin 25^\circ\cos 135^\circ\tan 675^\circ}{\sin 315^\circ\cos 65^\circ}$

g) $\dfrac{\sin 120^\circ\sin 202^\circ\tan 68^\circ}{\sin\left(-68^\circ\right)\tan 420^\circ}$

h) $\dfrac{\sin^2\left(225^\circ\right)-\tan(-\theta)+\cos\left(90^\circ-\theta\right)}{\sin(-\theta)+\cos\left(-120^\circ\right)+\tan\left(-180^\circ-\theta\right)}$

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3) If $\sin 56^\circ = t$ express the following in terms of t:

a) $\sin 236^\circ$ c) $\tan 124^\circ$ e) $\sin 34^\circ$

b) $\cos 304^\circ$ d) $\sin 776^\circ$ f) $\cos 146^\circ$

Trigonometric Equations

Exercise 5:

1) Solve the following equations correct to one decimal place where necessary:

a) $\sin\theta = 0,35$

b) $\tan 2\theta = 5$

c) $\cos\left(2\theta + 10^\circ\right) = 0,866$

d) $\cos\left(x + 20^\circ\right) = \sin 70^\circ$

2) Solve the following equations correct to one decimal place where necessary:

a) $\sin^2 x - \sin x \cos x = 0$

b) $2\sin x \tan x - \tan x = 0$

c) $8\cos 2\theta = \dfrac{2}{\cos 2\theta}$

d) $\sin x \cos x - 3\cos x = 0$

e) $\sin\left(\theta - 20^\circ\right) = \cos\theta$

f) $3\sin^2 x - 5\sin x = 2$

g) $\cos^2\theta - \sin\theta - 1 = 0$

h) $3\sin^2\theta - \sin\theta\cos\theta - 1 = 0$

i) $\sin^2\theta - 7\sin\theta\cos\theta + 2 = 0$

j) $-\cos\theta = \sin\left(\theta - 10^\circ\right)$

k) $\cos 2x = -\cos x$

l) $\cos x = \tan(-x)$

3) Solve the equations in question 2 if their domain is restricted as given below. Each restriction corresponds to the relevant question in question 2.

a) $x \in \left[-150^\circ; 200^\circ\right]$

b) $x \in \left[0^\circ; 180^\circ\right]$

c) $\theta \in \left[0^\circ; 120^\circ\right]$

d) $x \in \left[0^\circ; 180^\circ\right]$

e) $\theta \in \left[-360^\circ; 180^\circ\right]$

f) $\theta \in \left[-60^\circ; 300^\circ\right]$

g) $x \in \left[0^\circ; 360^\circ\right]$

h) $x \in \left[-180^\circ; 90^\circ\right]$

i) $\theta \in \left[-180^\circ; 180^\circ\right]$

j) $x \in \left[-360^\circ; 180^\circ\right]$

k) $\theta \in \left[-360^\circ; 0^\circ\right]$

l) $\theta \in \left[0^\circ; 500^\circ\right]$

Restrictions

Exercise 6:

1) For which values of θ are the following identities undefined?

a) $\dfrac{2\sin^3\theta + 2\sin\theta\cos^2\theta}{\cos\theta} = 2\tan\theta$

b) $\left(\tan^2\theta - \sin^2\theta\right)\left(1 + \dfrac{\cos^2\theta}{\sin^2\theta}\right) = \tan^2\theta$

Mixed Exercise:

1) Simplify the following without the use of a calculator:

a) $\sqrt{(1-\cos x)(1+\cos x)}$

b) $\dfrac{\cos\left(360^\circ - x\right)\sin\left(180^\circ + x\right)\cos\left(90^\circ - x\right)}{\sin\left(180^\circ - x\right)\tan\left(180^\circ - x\right)}$

c) $\dfrac{\tan 150^\circ \cos 135^\circ \cos 100^\circ}{\sin 585^\circ \cos 150^\circ \sin 170^\circ}$

d) $\tan\left(x - 180^\circ\right)\cos\left(900^\circ + x\right)\left[\sin(-x) - \dfrac{\cos^2\left(180^\circ - x\right)}{\cos\left(x - 90^\circ\right)}\right]$

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e) $3\sin^2 150^\circ + 2\sin^2 225^\circ - \cos 360^\circ - 2\cos^2 330^\circ$

f) $\dfrac{\tan(-311^\circ)\sin(-143^\circ)}{\cos 307^\circ \tan 229^\circ}$

g) $\dfrac{\sin(-\theta)\sin(360^\circ+\theta)+\cos(90^\circ+\theta)}{-\sin(360^\circ-\theta)-\tan(315^\circ)}$

2) Prove the following identities without the use of a calculator

a) $\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}\right)^2 = \dfrac{1-\cos x}{1+\cos x}$

b) $\dfrac{\sin x}{1-\cos x}-\dfrac{\sin x}{1+\cos x}=\dfrac{2}{\tan x}$

3) Given the identity $\sin^2\theta+\cos^2\theta+\tan^2\theta=\dfrac{1}{\cos^2\theta}$:

a) Prove that the identity above is true.

b) Hence, without using a calculator determine the value of $\sin^2 240^\circ+\cos^2 240^\circ+\tan^2 240^\circ$.

c) With reason say whether or not $\sqrt{\sin^2 240^\circ+\cos^2 240^\circ+\tan^2 240^\circ}=\dfrac{1}{\cos 240^\circ}$ is a valid statement.

4) Given the identity $\left(\dfrac{1}{\cos\theta}+\tan\theta\right)\left(\dfrac{1}{\cos\theta}-\tan\theta\right)=1$:

a) Prove that the identity above is true.

b) Hence, determine the value of $\dfrac{1}{\cos\theta}+\tan\theta$ if $\dfrac{1}{\cos\theta}-\tan\theta=2$.

5) Determine the solution/s to the following equations correct to 1 decimal place.

a) $\cos\left(\frac{x}{2}+15^\circ\right)=\tan 12^\circ$ and $x\in[-270^\circ;540^\circ]$

b) $7\cos^2\theta+\sin\theta\cos\theta-6=0$

6) Given the expression $\dfrac{\sin\theta\cos\theta\tan\theta}{1-\cos^2\theta}$:

a) Simplify the above expression as far as possible.

b) Hence, solve the equation $\tan A=\dfrac{\sin\theta\cos\theta\tan\theta}{1-\cos^2\theta}$ if $A\in[0^\circ;360^\circ]$.

7) For which values of x is the expression $\dfrac{3}{\cos^2 x-2\cos x}$ is undefined?

8) If $\theta\in[-180^\circ;180^\circ]$, determine the value of θ if $\tan 5\theta=\tan\theta$.

9) If $6\cos\theta=5$ and $\sin\theta<0$, without a calculator and with aid of a diagram determine the value of:

a) $\sin\theta$

b) $\tan^2\theta\cos\theta$

c) $\cos\alpha$ if $\alpha+\theta=90^\circ$

10) If $3\cos\theta+k=0$ and $k<0$ and it is also given that $\sin(180^\circ+\theta)>0$ determine which quadrant θ lies in.

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11) If $\theta \in \left[0°;270°\right]$, draw the functions of $f(\theta)=\cos 2\theta$ and $g(\theta)=\sin\left(\theta-45°\right)$ on the same set of axes showing all intercepts with the axes.

a) For which values of θ is $f(\theta)=g(\theta)$.

b) For which values of θ is $f(\theta)\leq g(\theta)$.

c) For which values of θ is $\frac{g(\theta)}{f(\theta)}\leq 0$.

d) For which values of θ is $g(\theta)-f(\theta)=1$.

12) Using your knowledge of trigonometry, determine the value of:

a) $\cos 0° + \cos 1° + \cos 2° + ... + \cos 178° + \cos 179° + \cos 180°$

b) $\sin^2 1° + \sin^2 2° + \sin^2 3° + ... + \sin^2 88° + \sin^2 89° + \sin^2 90°$

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Chapter 6– Trigonometry (Memo)

Grade 10 Revision

Exercise 1:

1)

a) 4^{th} quadrant

b) 3^{rd} quadrant

c) 1^{st} quadrant

d) 4^{th} quadrant

e) 2^{nd} quadrant

f) 3^{rd} quadrant

2) The first step in this problem is to use Pythagoras to solve for the lengths of OP and OQ.

$$OP^2 = (-3)^2 + (4)^2$$
$$= 25$$
$$\therefore OP = 5 \text{ units}$$

$$OQ^2 = (3)^2 + (4)^2$$
$$= 25$$
$$\therefore OQ = 5 \text{ units}$$

a) $\sin\theta = \frac{4}{5} \rightarrow \frac{y}{r}$

b) $\sin\beta = \frac{4}{5} \rightarrow \frac{y}{r}$

c) $\cos\theta = \frac{-3}{5} \rightarrow \frac{x}{r}$

d) $\cos\beta = \frac{3}{5} \rightarrow \frac{x}{r}$

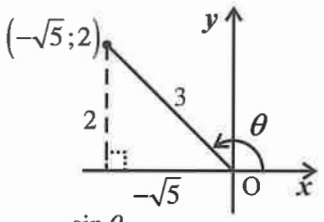
e) $\tan\theta = \frac{4}{-3} \rightarrow \frac{y}{x}$

f) $\tan\beta = \frac{4}{3} \rightarrow \frac{y}{x}$

3) $\sin\theta = \frac{2}{3} \rightarrow \frac{y}{r}$

$x^2 = 3^2 - 2^2$

$\therefore x = -\sqrt{5} \rightarrow 2^{nd}$ quadrant



a) $\tan\theta$

$= \frac{2}{-\sqrt{5}}$

$= -\frac{2}{\sqrt{5}}$

b) $\frac{\sin\theta}{\cos\theta}$

$= \frac{2/3}{-\sqrt{5}/3}$

$= \frac{2}{-\sqrt{5}} \rightarrow \frac{2}{-\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

c) $\sin^2\theta + \cos^2\theta$

$= \left(\frac{2}{3}\right)^2 + \left(\frac{-\sqrt{5}}{3}\right)^2$

$= \frac{4}{9} + \frac{5}{9}$

$= 1$

4)

a) $\sin^2 30°$

$= \left(\frac{1}{2}\right)^2$

$= \frac{1}{4}$

b) $\sqrt{1-\cos^2 30°}$

$= \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{1-\frac{3}{4}}$

$= \sqrt{\frac{1}{4}} = \frac{1}{2}$

c) $\cos^2 30°$

$= \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{3}{4}$

d) $\sqrt{1-\sin^2 30°}$

$= \sqrt{1-\left(\frac{1}{2}\right)^2}$

$= \sqrt{1-\frac{1}{4}}$

$= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

e) $\tan 60°$

$= \frac{\sqrt{3}}{1}$

$= \sqrt{3}$

f) $\frac{\sin 60°}{\cos 60°}$

$= \frac{\sqrt{3}/2}{1/2}$

$= \frac{\sqrt{3}}{2} \div \frac{1}{2}$

$= \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

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Exercise 2:

1)

a) $\dfrac{\sin^2 x+\sin^2 x\cos x}{1+\cos x}$

$=\dfrac{\sin^2 x\cancel{(1+\cos x)}}{\cancel{1+\cos x}}$

$=\sin^2 x$

b) $(1-\sin^2\theta)(1+\tan^2\theta)$

$=(1-\sin^2\theta)\left(1+\dfrac{\sin^2\theta}{\cos^2\theta}\right)$

$=\left(\cancel{\cos^2\theta}\right)\left(\dfrac{\cos^2\theta+\sin^2\theta}{\cancel{\cos^2\theta}}\right)$

$=\cos^2\theta+\sin^2\theta=1$

c) $\dfrac{4\sin^2 x+4\cos^2 x}{-2\sin^2 x-2\cos^2 x}$

$=\dfrac{4(\sin^2 x+\cos^2 x)}{-2(\sin^2 x+\cos^2 x)}$

$=-2$

d) $\tan x\cos x\sin x-1$

$=\left(\dfrac{\sin x}{\cancel{\cos x}}\right)\left(\cancel{\cos x}\right)(\sin x)-1$

$=\sin^2 x-1$

$=-(1-\sin^2 x)$

$=-\cos^2 x$

e) $(\sin\theta-\cos\theta)^2+2\cos\theta\sin\theta$

$=\sin^2\theta-2\cos\theta\sin\theta+\cos^2\theta+2\cos\theta\sin\theta$

$=\sin^2\theta+\cos^2\theta$

$=1$

f) $\dfrac{1}{\cos^2 y}-\tan^2 y+1$

$=\dfrac{1}{\cos^2 y}-\dfrac{\sin^2 y}{\cos^2 y}+1$

$=\dfrac{1-\sin^2 y+\cos^2 y}{\cos^2 y}$

$=\dfrac{\cos^2 y+\cos^2 y}{\cos^2 y}$

$=\dfrac{2\cos^2 y}{\cos^2 y}=2$

g) $\tan^2\theta-\dfrac{1}{\cos^2\theta}$

$=\dfrac{\sin^2\theta}{\cos^2\theta}-\dfrac{1}{\cos^2\theta}$

$=\dfrac{\sin^2\theta-1}{\cos^2\theta}$

$=\dfrac{-(1-\sin^2\theta)}{\cos^2\theta}$

$=\dfrac{-\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}=-1$

h) $\left(\dfrac{1}{\cos^2\theta}-1\right)\left(\dfrac{1}{\sin^2\theta}-1\right)$

$=\left(\dfrac{1-\cos^2\theta}{\cos^2\theta}\right)\left(\dfrac{1-\sin^2\theta}{\sin^2\theta}\right)$

$=\dfrac{1-\sin^2\theta-\cos^2\theta+\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta}$

$=\dfrac{1-(\sin^2\theta+\cos^2\theta)+\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta}$

$=\dfrac{1-1+\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta}=1$

2)

a) $\text{LHS}=\left(1+\dfrac{\sin^2 x}{\cos^2 x}\right)(\cos^2 x)$

$=\left(\dfrac{\cos^2 x+\sin^2 x}{\cancel{\cos^2 x}}\right)\left(\cancel{\cos^2 x}\right)$

$=1=\text{RHS}$

b) $\text{LHS}=\sin^2 x+\left(\sin x\cos x\div\dfrac{\sin x}{\cos x}\right)$

$=\sin^2 x+\left(\cancel{\sin x}\cos x\times\dfrac{\cos x}{\cancel{\sin x}}\right)$

$=\sin^2 x+\cos^2 x$

$=1=\text{RHS}$

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c) $\text{LHS} = \dfrac{1+\sin x-(1-\sin x)}{(1-\sin x)(1+\sin x)}$

$= \dfrac{2\sin x}{1-\sin^2 x}$

$= \dfrac{2\sin x}{\cos^2 x}$

$= \dfrac{2\sin x}{(\cos x)(\cos x)}$

$= \dfrac{2\tan x}{\cos x} = \text{RHS}$

d) $\dfrac{\tan^2\alpha}{1+\tan^2\alpha} = \sin^2\alpha$

$\text{LHS} = \dfrac{\sin^2\alpha}{\cos^2\alpha} \div \left(1+\dfrac{\sin^2\alpha}{\cos^2\alpha}\right)$

$= \dfrac{\sin^2\alpha}{\cos^2\alpha} \div \left(\dfrac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}\right)$

$= \dfrac{\sin^2\alpha}{\cancel{\cos^2\alpha}} \times \left(\dfrac{\cancel{\cos^2\alpha}}{1}\right)$

$= \sin^2\alpha = \text{RHS}$

e) $\text{LHS} = 1+\dfrac{2\cos x(1+\cos x)}{1-\cos^2 x}$

$= 1+\dfrac{2\cos x(1+\cos x)}{(1-\cos x)(1+\cos x)}$

$= 1+\dfrac{2\cos x}{1-\cos x}$

$= \dfrac{1-\cos x+2\cos x}{1-\cos x}$

$= \dfrac{1+\cos x}{1-\cos x} = \text{RHS}$

f) $\text{LHS} = \dfrac{\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta}{\sin^2\theta-\cos^2\theta}$

$= \dfrac{(\sin\theta-\cos\theta)^2}{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}$

$= \dfrac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta} = \text{RHS}$

g) $\text{LHS} = \dfrac{(2\cos\theta-1)(\cos\theta+1)}{1-\cos^2\theta}$

$= \dfrac{(2\cos\theta-1)\,\cancel{(\cos\theta+1)}}{(1-\cos\theta)\,\cancel{(1+\cos\theta)}}$

$= \dfrac{2\cos\theta-1}{1-\cos\theta} = \text{RHS}$

h) $\text{LHS} = \dfrac{2\cos^2\beta+5\cos\beta-3}{2\sin\beta\cos\beta-\sin\beta}$

$= \dfrac{\cancel{(2\cos\beta-1)}\,(\cos\beta+3)}{\sin\beta\,\cancel{(2\cos\beta-1)}}$

$= \dfrac{\cos\beta+3}{\sin\beta} = \text{RHS}$

i) $\text{LHS} = \dfrac{\cos^2\theta-\cos\theta-\left(1-\cos^2\theta\right)}{\sin\theta(2\cos\theta+1)}$

$= \dfrac{2\cos^2\theta-\cos\theta-1}{\sin\theta(2\cos\theta+1)}$

$= \dfrac{\cancel{(2\cos\theta+1)}\,(\cos\theta-1)}{\sin\theta\,\cancel{(2\cos\theta+1)}}$

$= \dfrac{\cos\theta-1}{\sin\theta} = \text{RHS}$

j) $\text{LHS} = \dfrac{1-\sin x}{\cos x} \times \dfrac{1+\sin x}{1+\sin x}$

$= \dfrac{1-\sin^2 x}{\cos x(1+\sin x)}$

$= \dfrac{\cos^2 x}{\cos x(1+\sin x)}$

$= \dfrac{\cos x}{1+\sin x} = \text{RHS}$

k) $\text{LHS} = \dfrac{5\cos^2\theta-\cos\theta\sin\theta-3\left(\sin^2\theta+\cos^2\theta\right)}{\cos\theta(2\cos\theta-3\sin\theta)}$

$= \dfrac{2\cos^2\theta-\cos\theta\sin\theta-3\sin^2\theta}{\cos\theta(2\cos\theta-3\sin\theta)}$

$= \dfrac{(2\cos\theta-3\sin\theta)(\cos\theta+\sin\theta)}{\cos\theta(2\cos\theta-3\sin\theta)}$

$= \dfrac{\cos\theta+\sin\theta}{\cos\theta} = \text{RHS}$

l) $\text{LHS} = \dfrac{(2\sin x-1)(\sin x+1)}{1-\sin^2 x}$

$= \dfrac{(2\sin x-1)\,\cancel{(\sin x+1)}}{(1-\sin x)\,\cancel{(1+\sin x)}}$

$= \dfrac{2\sin x-1}{1-\sin x} = \text{RHS}$

m) $\text{LHS} = \dfrac{1-\cos x}{\sin x} \times \dfrac{1+\cos x}{1+\cos x}$

$= \dfrac{1-\cos^2 x}{\sin x(1+\cos x)}$

$= \dfrac{\sin^2 x}{\sin x(1+\cos x)}$

$= \dfrac{\sin x}{1+\cos x} = \text{RHS}$

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n) $\text{RHS} = \dfrac{\cos^2 x - 2\sin x - 2}{\cos^2 x(\sin x + 1)}$

$= \dfrac{1 - \sin^2 x - 2\sin x - 2}{\cos^2 x(\sin x + 1)}$

$= \dfrac{-\sin^2 x - 2\sin x - 1}{\cos^2 x(\sin x + 1)}$

$= \dfrac{-(\sin^2 x + 2\sin x + 1)}{\cos^2 x(\sin x + 1)}$

$= \dfrac{-(\sin x + 1)^2}{\cos^2 x(\sin x + 1)}$

$= \dfrac{-(\sin x + 1)}{1 - \sin^2 x}$

$= \dfrac{-(\sin x + 1)}{(1 - \sin x)(1 + \sin x)}$

$= \dfrac{-1}{1 - \sin x} = \dfrac{1}{\sin x - 1} = \text{LHS}$

o) $\text{RHS} = \sqrt{\dfrac{1 - \sin x}{1 + \sin x}} \times \sqrt{\dfrac{1 + \sin x}{1 + \sin x}}$

$= \sqrt{\dfrac{1 - \sin^2 x}{(1 + \sin x)^2}}$

$= \dfrac{\sqrt{\cos^2 x}}{1 + \sin x}$

$= \dfrac{\cos x}{1 + \sin x} = \text{LHS}$

p) $\text{LHS} = \left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right)(1 - \cos^2 x)$

$= \left(\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)(1 - \cos^2 x)$

$= \left(\dfrac{1}{\sin x \cos x}\right)(\sin^2 x)$

$= \dfrac{\sin x}{\cos x} = \tan x = \text{RHS}$

Reduction Formula

Exercise 3:

1)

a) $\dfrac{2\sin(180° - x)\cos(360° - x)}{3\sin(180° + x)\cos(180° - x)}$

$= \dfrac{2(\sin x)(\cos x)}{3(-\sin x)(-\cos x)}$

$= \dfrac{2}{3}$

b) $1 + \dfrac{2\sin^2(360 - \theta)}{\cos^2(180° + \theta)} \times \dfrac{1}{\tan^2(180° - \theta)}$

$= 1 + \dfrac{2\sin^2\theta}{\cos^2\theta} \times \dfrac{1}{\tan^2\theta}$

$= 1 + \dfrac{2\sin^2\theta}{\cos^2\theta} \times \dfrac{\cos^2\theta}{\sin^2\theta}$

$= 1 + 2 = 3$

c) $\dfrac{\cos(180° + x)\sin(360° - x)\tan(180° - x)}{\sin(180° + x)\sin(360° - x)}$

$= \dfrac{(-\cos x)(-\sin x)(-\tan x)}{(-\sin x)(-\sin x)}$

$= \dfrac{(\cos x)\left(\dfrac{\sin x}{\cos x}\right)}{(-\sin x)}$

$= -1$

d) $\dfrac{\cos^2(180° + \theta)\sin(180° - \theta)\tan(180° + \theta)}{\sin(360° - \theta)\sin(180° + \theta)}$

$= \dfrac{(\cos^2\theta)(\sin\theta)(\tan\theta)}{(-\sin\theta)(-\sin\theta)}$

$= \dfrac{(\cos^2\theta)\left(\dfrac{\sin\theta}{\cos\theta}\right)}{(\sin\theta)}$

$= \cos\theta$

2)

a) $[\sin(-x) + \cos(-x)]^2 + 2\sin x \sin(90° - x)$

$= (-\sin x + \cos x)^2 + 2\sin x \cos x$

$= \sin^2 x - 2\sin x \cos x + \cos x^2 + 2\sin x \cos x$

$= \sin^2 x + \cos^2 x = 1$

b) $\sin(90° + B)\cos B + \dfrac{\sin^2(180° + B)\tan(-B)}{\tan(360° - B)}$

$= \cos B \cos B + \dfrac{\sin^2 B(-\tan B)}{-\tan B}$

$= \cos^2 B + \sin^2 B = 1$

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c) $\dfrac{1-\cos\left(90^\circ+\theta\right)}{\cos^2\left(180^\circ-\theta\right)}$

$=\dfrac{1+\sin\theta}{\cos^2\theta}$

$=\dfrac{1+\sin\theta}{1-\sin^2\theta}$

$=\dfrac{1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$

$=\dfrac{1}{1-\sin\theta}$

d) $\dfrac{\cos\left(90^\circ+\theta\right)}{\cos\left(360^\circ-\theta\right)\tan\left(360^\circ-\theta\right)}$

$=\dfrac{-\sin\theta}{\cos\theta(-\tan\theta)}$

$=\dfrac{-\cancel{\sin\theta}}{\cancel{\cos\theta}\left(-\dfrac{\cancel{\sin\theta}}{\cancel{\cos\theta}}\right)}$

$=1$

e) $\dfrac{\sin\left(\theta-180^\circ\right)}{\tan\left(360^\circ-\theta\right)\sin\left(90^\circ-\theta\right)}-\cos\beta\sin\left(90^\circ-\beta\right)$

$=\dfrac{-\sin\theta}{-\tan\theta\cos\theta}-\cos\beta\cos\beta$

$=\dfrac{-\cancel{\sin\theta}}{\left(-\dfrac{\cancel{\sin\theta}}{\cancel{\cos\theta}}\right)\cancel{\cos\theta}}-\cos^2\beta$

$=1-\cos^2\beta=\sin^2\beta$

f) $\dfrac{\left[\tan^2\theta-\cos^2\left(90^\circ+\theta\right)\right]\left[\dfrac{1}{\tan^2\left(90^\circ-\theta\right)}+1\right]}{\tan^2\left(180^\circ+\theta\right)}$

$=\dfrac{\left(\tan^2\theta-\sin^2\theta\right)\left[\dfrac{\cos^2\left(90^\circ-\theta\right)}{\sin^2\left(90^\circ-\theta\right)}+1\right]}{\tan^2\theta}$

$=\dfrac{\left(\dfrac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta\right)\left(\dfrac{\sin^2\theta}{\cos^2\theta}+1\right)}{\tan^2\theta}$

$=\dfrac{\left(\dfrac{\sin^2\theta-\sin^2\theta\cos^2\theta}{\cos^2\theta}\right)\left(\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right)}{\dfrac{\sin^2\theta}{\cos^2\theta}}$

$=\dfrac{\cancel{\sin^2\theta}\left(1-\cos^2\theta\right)}{\cancel{\cos^2\theta}\left(\cos^2\theta\right)}\times\dfrac{\cancel{\cos^2\theta}}{\sin^2\theta}$

$=\dfrac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$

Exercise 4:

1)

a) $\tan 240^\circ$

$=\tan\left(180^\circ+60^\circ\right)$

$=\tan 60^\circ$

$=\sqrt{3}$

b) $\sin 300^\circ$

$=\sin\left(360^\circ-60^\circ\right)$

$=-\sin 60^\circ$

$=-\sqrt{3}/2$

c) $\cos 420^\circ$

$=\cos\left(420^\circ-360^\circ\right)$

$=\cos 60^\circ$

$=1/2$

d) $\sin 210^\circ-\cos 300^\circ$

$=\sin\left(180^\circ+30^\circ\right)-\cos\left(360^\circ-60^\circ\right)$

$=-\sin 30^\circ-\cos 60^\circ$

$=-1/2-1/2$

$=-1$

e) $\tan 240^\circ-\cos\left(-30^\circ\right)$

$=\tan\left(180^\circ+60^\circ\right)-\cos\left(30^\circ\right)$

$=\tan 60^\circ-\cos 30^\circ$

$=\sqrt{3}-\sqrt{3}/2$

$=\sqrt{3}/2$

f) $\sin\left(-150^\circ\right)+\cos\left(120^\circ\right)$

$=-\sin\left(180^\circ-30^\circ\right)+\cos\left(180^\circ-60^\circ\right)$

$=-\sin 30^\circ-\cos 60^\circ$

$=-1/2-1/2$

$=-1$

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2)

a) $\cos(300^\circ) - \sin(150^\circ)$

$= \cos 60^\circ - \sin 30^\circ$

$= \frac{1}{2} - \frac{1}{2}$

$= 0$

b) $\dfrac{\sin 300^\circ}{\tan 240^\circ}$

$= \dfrac{-\sin 60^\circ}{\tan 60^\circ}$

$= \dfrac{-\frac{\sqrt{3}}{2}}{\sqrt{3}} = -\frac{1}{2}$

c) $\dfrac{\sin(-60^\circ)\tan(210^\circ)}{\cos^2 135^\circ}$

$= \dfrac{(-\sin 60^\circ)(\tan 30^\circ)}{(-\cos 45^\circ)^2}$

$= \dfrac{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{3}}\right)}{\left(-\frac{\sqrt{2}}{2}\right)^2} = -1$

d) $\sqrt{2}\tan 150^\circ \cos 225^\circ \sin 240^\circ$

$= \sqrt{2}(-\tan 30^\circ)(-\cos 45^\circ)(-\sin 60^\circ)$

$= -\sqrt{2}\left(\frac{1}{\cancel{\sqrt{3}}}\right)\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\cancel{\sqrt{3}}}{2}\right)$

$= -\frac{2}{4} = -\frac{1}{2}$

e) $\dfrac{\cos 240^\circ + \sin^2(-45^\circ)\cos 150^\circ}{-\sin 300^\circ + \tan(-135^\circ)}$

$= \dfrac{(-\cos 60^\circ) + (-\sin 45^\circ)^2(-\cos 30^\circ)}{-(-\sin 60^\circ) - (-\tan 45^\circ)}$

$= \dfrac{-\frac{1}{2} + \left(-\frac{\sqrt{2}}{2}\right)^2\left(-\frac{\sqrt{3}}{2}\right)}{-\left(-\frac{\sqrt{3}}{2}\right) - (-1)}$

$= \dfrac{-\frac{1}{2} - \frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} + 1} = \dfrac{-\frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)}{\frac{\sqrt{3}}{2} + 1} = -\frac{1}{2}$

f) $\dfrac{\sin 25^\circ \cos 135^\circ \tan 675^\circ}{\sin 315^\circ \cos 65^\circ}$

$= \dfrac{(\sin 25^\circ)(-\cos 45^\circ)(\tan 315^\circ)}{(-\sin 45^\circ)(\cos 65^\circ)}$

$= \dfrac{\cancel{(\sin 25^\circ)}\left(-\frac{\sqrt{2}}{2}\right)(-\tan 45^\circ)}{\cancel{\left(-\frac{\sqrt{2}}{2}\right)}\cancel{(\sin 25^\circ)}}$

$= -1$

g) $\dfrac{\sin 120^\circ \sin 202^\circ \tan 68^\circ}{\sin(-68^\circ)\tan 420^\circ}$

$= \dfrac{(\sin 60^\circ)(-\sin 22^\circ)\left(\frac{\sin 68^\circ}{\cos 68^\circ}\right)}{(-\sin 68^\circ)(\tan 60^\circ)}$

$= \dfrac{\left(\frac{\sqrt{3}}{2}\right)\cancel{(-\sin 22^\circ)}\left(\frac{\cancel{\cos 22^\circ}}{\cancel{\sin 22^\circ}}\right)}{\cancel{(-\cos 22^\circ)}\left(\cancel{\sqrt{3}}\right)}$

$= \frac{1}{2}$

h) $\dfrac{\sin^2(225^\circ) - \tan(-\theta) + \cos(90^\circ - \theta)}{\sin(-\theta) + \cos(-120^\circ) + \tan(-180^\circ - \theta)}$

$= \dfrac{\sin^2 45^\circ + \tan\theta + \sin\theta}{-\sin\theta - \cos 60^\circ - \tan\theta}$

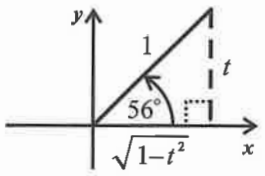
$= \dfrac{\left(\frac{\sqrt{2}}{2}\right)^2 + \tan\theta + \sin\theta}{-\sin\theta - \frac{1}{2} - \tan\theta}$

$= \dfrac{\frac{1}{2} + \tan\theta + \sin\theta}{-(\sin\theta + \frac{1}{2} + \tan\theta)}$

$= -1$

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3) The first step in this type of question is always to draw a diagram as shown below:



Calculation of x: $\sin 56° = \frac{t}{1} \rightarrow \frac{y}{r}$

$x^2 + y^2 = r^2$

$\therefore x^2 = r^2 - y^2$

$\therefore x = \sqrt{1-t^2}$

a) $\sin 236°$

$= \sin(180° + 56°)$

$= -\sin 56°$

$= -t$

b) $\cos 304°$

$= \cos(360° - 56°)$

$= \cos 56°$

$= \sqrt{1-t^2}$

c) $\tan 124°$

$= \tan(180° - 56°)$

$= -\tan 56°$

$= -\frac{t}{\sqrt{1-t^2}}$

d) $\sin 776°$

$= \sin(56° + 720°)$

$= \sin 56°$

$= t$

e) $\sin 34°$

$= \sin(90° - 56°)$

$= \cos 56°$

$= \sqrt{1-t^2}$

f) $\cos 146°$

$= \cos(90° + 56°)$

$= -\sin 56°$

$= -t$

Trigonometric Equations

Exercise 5:

1)

a) $\sin\theta = 0,35$

$\therefore \text{RA} = 20,487...°$

$\therefore \theta = 20,5° + 360°k$

or $\theta = 159,5° + 360°k\ ;\ k \in \mathbb{Z}$

b) $\tan 2\theta = 5$

$\therefore \text{RA} = 78,690...°$

$\therefore 2\theta = 78,690...° + 180°k\ ;\ k \in \mathbb{Z}$

$\therefore \theta = 39,3° + 90°k$

c) $\cos(2\theta + 10°) = 0,866$

$\therefore \text{RA} = 30,002...°$

$\therefore 2\theta + 10° = \pm(30,002...°) + 360°k\ ;\ k \in \mathbb{Z}$

$\therefore 2\theta = 20,002...° + 360°k$ or $2\theta = -40,002...° + 360°k$

$\therefore \theta = 10° + 180°k$ or $\theta = -20° + 180°k$

d) $\cos(x + 20°) = \sin 70°$

$\therefore \text{RA} = 20°$

$\therefore x + 20° = \pm 20° + 360°k\ ;\ k \in \mathbb{Z}$

$\therefore x = 0° + 360°k$ or $x = -40° + 360°k$

2)

a) $\sin^2 x - \sin x \cos x = 0$

$\therefore \sin x(\sin x - \cos x) = 0$

$\therefore \sin x = 0$ or $\sin x = \cos x$

$\therefore \sin x = 0$ or $\tan x = 1$

$\therefore \text{RA} = 0°$ or $\text{RA} = 45°$

$\therefore x = 0° + 360°k$ or $x = 180° + 360°k$ or $x = 45° + 180°k\ ;\ k \in \mathbb{Z}$

b) $2\sin x \tan x - \tan x = 0$

$\therefore \tan x(2\sin x - 1) = 0$

$\therefore \tan x = 0$ or $\sin x = \frac{1}{2}$

$\therefore \text{RA} = 0°$ or $\text{RA} = 30°$

$\therefore x = 180°k$ or $x = 30° + 360°k$ or $x = 150° + 360°k\ ;\ k \in \mathbb{Z}$

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c) $8\cos 2\theta = \dfrac{2}{\cos 2\theta}$

$\therefore \cos^2 2\theta = 1/4$

$\therefore \cos 2\theta = \pm 1/2$

$\therefore \text{RA} = 60^\circ \quad \text{or} \quad \text{RA} = 120^\circ$

$\therefore 2\theta = \pm 60^\circ + 360^\circ k$

$\therefore \theta = \pm 30^\circ + 180^\circ k \quad \text{or}$

or

$2\theta = \pm 120^\circ + 360^\circ k$

$\therefore \theta = \pm 60^\circ + 180^\circ k \; ; k \in \mathbb{Z}$

d) $\sin x \cos x - 3\cos x = 0$

$\therefore \cos x(\sin x - 3) = 0$

$\therefore \cos x = 0 \quad \text{or} \quad \sin x = 3 \rightarrow \text{No solution}$

$\therefore \text{RA} = 90^\circ$

$\therefore x = \pm 90^\circ + 360^\circ k \; ; k \in \mathbb{Z}$

e) $\sin(\theta - 20^\circ) = \cos\theta$

$\therefore \sin(\theta - 20^\circ) = \sin(90^\circ - \theta)$

$\therefore \text{RA} = 90^\circ - \theta$

$\therefore \theta - 20^\circ = 90^\circ - \theta + 360^\circ k ; k \in \mathbb{Z}$

$\therefore 2\theta = 110^\circ + 360^\circ k$

$\therefore \theta = 55^\circ + 180^\circ k$

or

$\theta - 20^\circ = 180 - (90^\circ - \theta) + 360^\circ k$

$\therefore \theta - 20^\circ = 90 + \theta + 360^\circ k$

$\therefore 0 = 110^\circ + 360^\circ k \rightarrow \text{no solution}$

f) $3\sin^2 x - 5\sin x = 2$

$\therefore 3\sin^2 x - 5\sin x - 2 = 0$

$\therefore (3\sin x + 1)(\sin x - 2) = 0$

$\therefore \sin x = -1/3 \quad \text{or} \quad \sin x = 2 \rightarrow \text{no solution}$

$\therefore \text{RA} = -19,471...^\circ$

$\therefore x = -19.5^\circ + 360^\circ k$

or

$x = 180^\circ + 19,471...^\circ + 360^\circ k$

$\therefore x = 199,5^\circ + 360^\circ k \; ; k \in \mathbb{Z}$

g) $\cos^2\theta - \sin\theta - 1 = 0$

$\therefore 1 - \sin^2\theta - \sin\theta - 1 = 0$

$\therefore \sin^2\theta + \sin\theta = 0$

$\therefore \sin\theta(\sin\theta + 1) = 0$

$\therefore \sin\theta = 0 \quad \text{or} \quad \sin\theta = -1$

$\therefore \text{RA} = 0^\circ \quad \text{or} \quad \text{RA} = -90^\circ$

$\therefore \theta = 360^\circ k \quad \text{or} \quad \theta = 180^\circ + 360^\circ k \; ; k \in \mathbb{Z}$

or

$\theta = -90^\circ + 360^\circ k \quad \text{or} \quad \theta = 270^\circ + 360^\circ k$

h) $3\sin^2\theta - \sin\theta\cos\theta - 1 = 0$

$\therefore 3\sin^2\theta - \sin\theta\cos\theta - (\sin^2\theta + \cos^2\theta) = 0$

$\therefore 2\sin^2\theta - \sin\theta\cos\theta - \cos^2\theta = 0$

$\therefore (\sin\theta - \cos\theta)(2\sin\theta + \cos\theta) = 0$

$\therefore \sin\theta = \cos\theta \quad \text{or} \quad 2\sin\theta = -\cos\theta$

$\therefore \tan\theta = 1 \quad \text{or} \quad \tan\theta = -1/2$

$\therefore \text{RA} = 45^\circ \quad \text{or} \quad \text{RA} = -26,565...^\circ$

$\therefore \theta = 45^\circ + 180^\circ k$

or $\theta = -26,6^\circ + 180^\circ k \; ; k \in \mathbb{Z}$

i) $\sin^2\theta - 7\sin\theta\cos\theta + 2 = 0$

$\therefore \sin^2\theta - 7\sin\theta\cos\theta + 2(\sin^2\theta + \cos^2\theta) = 0$

$\therefore 3\sin^2\theta - 7\sin\theta\cos\theta + 2\cos^2\theta = 0$

$\therefore (3\sin\theta - \cos\theta)(\sin\theta - 2\cos\theta) = 0$

$\therefore 3\sin\theta = \cos\theta \quad \text{or} \quad \sin\theta = 2\cos\theta$

$\therefore \tan\theta = 1/3 \quad \text{or} \quad \tan\theta = 2$

$\therefore \text{RA} = 18,434...^\circ \quad \text{or} \quad \text{RA} = 63,434...^\circ$

$\therefore \theta = 18,4^\circ + 180^\circ k$

or $\theta = 63,4^\circ + 180^\circ k \; ; k \in \mathbb{Z}$

j) $-\cos\theta = \sin(\theta - 10^\circ)$

$\therefore \cos\theta = -\sin(\theta - 10^\circ)$

$\therefore \cos\theta = \sin(-\theta + 10^\circ)$

$\therefore \cos\theta = \cos(90^\circ - [-\theta + 10^\circ])$

$\therefore \cos\theta = \cos(80^\circ + \theta)$

$\therefore \text{RA} = 80^\circ + \theta$

$\therefore \theta = \pm(80^\circ + \theta) + 360^\circ k \; ; k \in \mathbb{Z}$

$\therefore \theta = -80^\circ - \theta + 360^\circ k \quad \text{or} \quad \theta = 80^\circ + \theta + 360^\circ k$

$\therefore 2\theta = -80^\circ + 360^\circ k \quad \text{or} \quad 0 = 80^\circ + 360^\circ k$

$\therefore \theta = -40^\circ + 180^\circ k \quad \text{or no solution}$

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k) $\cos 2x = -\cos x$

$\therefore \cos 2x = \cos(180° - x)$

$\therefore \text{RA} = 180° - x$

$\therefore 2x = \pm(180° - x) + 360°k \; ; k \in \mathbb{Z}$

$\therefore 2x = 180° - x + 360°k \text{ or } 2x = -180° + x + 360°k$

$\therefore 3x = 180° + 360°k \text{ or } x = -180° + 360°k$

$\therefore x = 60° + 120°k \quad \text{or} \quad x = -180° + 360°k$

l) $\cos x = \tan(-x)$

$\therefore \cos x = -\tan x$

$\therefore \cos x = -\dfrac{\sin x}{\cos x}$

$\therefore \cos^2 x = -\sin x$

$\therefore 1 - \sin^2 x = -\sin x$

$\therefore \sin^2 x - \sin x - 1 = 0$

$\therefore \sin x = \dfrac{1-\sqrt{5}}{2} \text{ or } \sin x = \dfrac{1+\sqrt{5}}{2} \rightarrow \text{no solution}$

$\therefore \text{RA} = -38,172...$

$\therefore x = -38,2° + 360°k \text{ or } x = 180° + 38,17...° + 360°k \; ; k \in \mathbb{Z}$

$= 218,2° + 360°k$

3)

a) $x \in \{-135° ; 45° ; 180°\}$

b) $x \in \{0° ; 180° ; 30° ; 150°\}$

c) $\theta \in \{30° ; 60° ; 120°\}$

d) $x = 90°$

e) $\theta \in \{-305° ; -125° ; 55°\}$

f) $\theta \in \{-19,5° ; 199,5°\}$

g) $x \in \{0° ; 180° ; 270 ; 360°\}$

h) $x \in \{-135° ; -26,5° ; 45°\}$

i) $\theta \in \{-161,6° ; -116,6° ; 18,4° ; 63,4°\}$

j) $x \in \{-220° ; -40° ; 140°\}$

k) $\theta \in \{-300° ; -180° ; -60°\}$

l) $\theta \in \{218,2° ; 321,8°\}$

Restrictions

Exercise 6:

1)

a) $\dfrac{2\sin^3\theta + 2\sin\theta\cos^2\theta}{\cos\theta} = 2\tan\theta$

is undefined when $\cos\theta = 0$.

$\therefore \text{RA} = 90°$

$\therefore \theta = \pm 90° + 360°k \; ; \; k \in \mathbb{Z}$

b) $\left(\tan^2\theta - \sin^2\theta\right)\left(1 + \dfrac{\cos^2\theta}{\sin^2\theta}\right) = \tan^2\theta$

is undefined when $\sin^2\theta = 0$ or when $\cos^2\theta = 0$.

$\therefore \sin\theta = 0 \text{ or } \cos\theta = 0$

$\therefore \text{RA} = 0° \text{ or } \text{RA} = 90°$

$\therefore \theta = 0° + 360°k \text{ or } \theta = 180° + 360°k ; k \in \mathbb{Z}$

or

$\therefore \theta = \pm 90° + 360°k$

Mixed Exercise:

1)

a) $\sqrt{(1-\cos x)(1+\cos x)}$

$= \sqrt{1 - \cos^2 x}$

$= \sqrt{\sin^2 x}$

$= \sin x$

b) $\dfrac{\cos(360° - x)\sin(180° + x)\cos(90° - x)}{\sin(180° - x)\tan(180° - x)}$

$= \dfrac{(\cos x)(-\sin x)(\cancel{\sin x})}{(\cancel{\sin x})(-\tan x)}$

$= (\cos x)(\sin x) \div \tan x$

$= (\cos x)(\cancel{\sin x}) \times \dfrac{\cos x}{\cancel{\sin x}}$

$= \cos^2 x$

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c) $\dfrac{\tan 150^\circ \cos 135^\circ \cos 100^\circ}{\sin 585^\circ \cos 150^\circ \sin 170^\circ}$

$$= \frac{(-\tan 30^\circ)(-\cos 45^\circ)(\cos 100^\circ)}{(\sin 225^\circ)(-\cos 30^\circ)(\sin 10^\circ)}$$

$$= \frac{\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{-\sqrt{2}}{2}\right)(\cos 100^\circ)}{(-\sin 45^\circ)\left(\frac{-\sqrt{3}}{2}\right)(\sin 10^\circ)}$$

$$= \frac{\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{-\sqrt{2}}{2}\right)(-\sin 10^\circ)}{\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right)(\sin 10^\circ)} = -\frac{2}{3}$$

d) $\tan(x-180^\circ)\cos(900^\circ+x)\left[\sin(-x) - \dfrac{\cos^2(180^\circ - x)}{\cos(x-90^\circ)}\right]$

$$= (\tan x)(-\cos x)\left[(-\sin x) - \frac{\cos^2 x}{\cos\left[-(90^\circ - x)\right]}\right]$$

$$= (\tan x)(-\cos x)\left[(-\sin x) - \frac{\cos^2 x}{\sin x}\right]$$

$$= \left(\frac{\sin x}{\cos x}\right)(-\cos x)\left(\frac{-\sin^2 x - \cos^2 x}{\sin x}\right)$$

$$= \sin^2 x + \cos^2 x = 1$$

e) $3\sin^2 150^\circ + 2\sin^2 225^\circ - \cos 360^\circ - 2\cos^2 330^\circ$

$$= 3\sin^2 30^\circ + 2\sin^2 45^\circ - 1 - 2\cos^2 30^\circ$$

$$= 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{\sqrt{2}}{2}\right)^2 - 1 - 2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{3}{4} + 1 - 1 - \frac{3}{2} = -\frac{3}{4}$$

f) $\dfrac{\tan(-311^\circ)\sin(-143^\circ)}{\cos 307^\circ \tan 229^\circ}$

$$= \frac{(-\tan 311^\circ)(-\sin 143^\circ)}{\cos 53^\circ \tan 49^\circ}$$

$$= \frac{\tan 49^\circ (-\sin 37^\circ)}{\cos 53^\circ \tan 49^\circ}$$

$$= \frac{-\cos 53^\circ}{\cos 53^\circ} = -1$$

g) $\dfrac{\sin(-\theta)\sin(360^\circ + \theta) + \cos(90^\circ + \theta)}{-\sin(360^\circ - \theta) - \tan(315^\circ)}$

$$= \frac{(-\sin\theta)(\sin\theta) - \sin\theta}{\sin\theta + \tan 45^\circ}$$

$$= \frac{-\sin^2\theta - \sin\theta}{\sin\theta + 1}$$

$$= \frac{-\sin\theta(\sin\theta + 1)}{\sin\theta + 1}$$

$$= -\sin\theta$$

2)

a) $\text{LHS} = \dfrac{1 - 2\cos x + \cos^2 x}{\sin^2 x}$

$$= \frac{(\cos x - 1)^2}{(1 - \cos x)(1 + \cos x)}$$

$$= \frac{(\cos x - 1)^2}{-(\cos x - 1)(1 + \cos x)}$$

$$= \frac{\cos x - 1}{-(1 + \cos x)} = \frac{1 - \cos x}{1 + \cos x} = \text{RHS}$$

b) $\text{LHS} = \dfrac{\sin x(1 + \cos x) - \sin x(1 - \cos x)}{(1 - \cos x)(1 + \cos x)}$

$$= \frac{\sin x + \sin x\cos x - \sin x + \sin x\cos x}{1 - \cos^2 x}$$

$$= \frac{2\sin x\cos x}{\sin^2 x}$$

$$= \frac{2\cos x}{\sin x} = \frac{2}{\tan x} = \text{RHS}$$

3)

a) $\text{LHS} = 1 + \dfrac{\sin^2\theta}{\cos^2\theta}$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \text{RHS}$$

b) $\sin^2 240^\circ + \cos^2 240^\circ + \tan^2 240^\circ$

$$= \frac{1}{\cos^2 240^\circ}$$

$$= \frac{1}{\cos^2 60^\circ} = 4$$

c) It is not valid as $\dfrac{1}{\cos 240^\circ} = \dfrac{1}{-\cos 60^\circ} = -2$ and the root of any number cannot be negative in the real number system.

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4)

a) $\text{LHS} = \left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)$

$= \frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} = 1 = \text{RHS}$

b) $\left(\frac{1}{\cos\theta}+\tan\theta\right)\left(\frac{1}{\cos\theta}-\tan\theta\right)=1$

$\therefore \left(\frac{1}{\cos\theta}+\tan\theta\right)2=1$

$\therefore \frac{1}{\cos\theta}+\tan\theta=\frac{1}{2}$

5)

a) $\cos\left(\frac{x}{2}+15^\circ\right)=\tan 12^\circ \text{ and } x\in\left[-270^\circ;540^\circ\right]$

$\therefore \text{RA} = 77,727...^\circ$

$\therefore \frac{x}{2}+15^\circ = \pm\left(77,727...^\circ\right)+360^\circ k; k\in\mathbb{Z}$

$\therefore \frac{x}{2} = 62,727...^\circ+360^\circ k \text{ or } \frac{x}{2} = -92,727...^\circ+360^\circ k$

$\therefore x = 125,5^\circ+720^\circ k \text{ or } x=-185,5^\circ+720^\circ k$

$\therefore x\in\left\{-185,5^\circ;125,5^\circ;534,5^\circ\right\}$

b) $7\cos^2\theta+\sin\theta\cos\theta-6=0$

$\therefore 7\cos^2\theta+\sin\theta\cos\theta-6\left(\sin^2\theta+\cos^2\theta\right)=0$

$\therefore \cos^2\theta+\sin\theta\cos\theta-6\sin^2\theta=0$

$\therefore (\cos\theta+3\sin\theta)(\cos\theta-2\sin\theta)=0$

$\therefore \tan\theta=-\frac{1}{3} \text{ or } \tan\theta=\frac{1}{2}$

$\therefore \text{RA} = -18,434...^\circ \text{ or } \text{RA} = 26,565...^\circ$

$\therefore \theta = -18,4^\circ+180^\circ k \text{ or } \theta = 26,6^\circ+180^\circ k; k\in\mathbb{Z}$

6)

a) $\frac{\sin\theta\cos\theta\tan\theta}{1-\cos^2\theta}$

$= \frac{\cancel{\sin\theta}\,\cancel{\cos\theta}\left(\frac{\cancel{\sin\theta}}{\cancel{\cos\theta}}\right)}{\cancel{\sin^2\theta}}$

$=1$

b) $\tan A = 1$

$\therefore \text{RA} = 45^\circ$

$\therefore A = 45^\circ+180^\circ k; k\in\mathbb{Z}$

$\therefore A\in\left\{45^\circ;225^\circ\right\}$

7) $\cos^2 x-2\cos x=0$

$\therefore \cos x(\cos x-2)=0$

$\therefore \cos x=0 \text{ or } \cos x=2 \rightarrow \text{N/A}$

$\therefore \text{RA} = 90^\circ$

$\therefore x = \pm 90^\circ+360^\circ k \text{ ; } k\in\mathbb{Z}$

8) $\tan 5\theta = \tan\theta$

$\therefore \text{RA} = \theta$

$\therefore 5\theta = \theta+180^\circ k; k\in\mathbb{Z}$

$\therefore 4\theta = 180^\circ k$

$\therefore \theta = 45^\circ k$

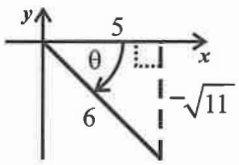
$\therefore \theta\in\left\{-180^\circ;-135^\circ;-45^\circ;0^\circ;45^\circ;135^\circ;180^\circ\right\}$

Note that: $x\neq -90^\circ$ and $x\neq 90^\circ$ as these are asymptotes

9) $y^2 = r^2-x^2$

$\therefore y^2 = 6^2-5^2$

$\therefore y = -\sqrt{11}$



a) $\sin\theta = \frac{-\sqrt{11}}{6}$

b) $\tan^2\theta\cos\theta = \left(\frac{-\sqrt{11}}{5}\right)^2\left(\frac{5}{6}\right)$

$= \frac{11}{30}$

c) $\cos\alpha = \cos\left(90^\circ-\theta\right)$

$= \sin\theta$

$= \frac{-\sqrt{11}}{6}$

10) $\cos\theta = -\frac{k}{3} \text{ and } k<0$

$\therefore \cos\theta>0 \text{ and } \sin\left(180^\circ+\theta\right)>0$

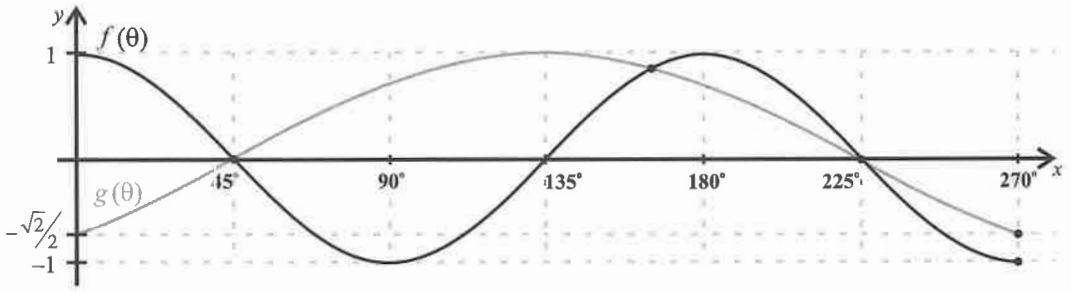
$\therefore \cos\theta>0 \text{ and } -\sin\theta>0$

$\therefore \cos\theta>0 \text{ and } \sin\theta<0$

$\therefore \theta$ lies in quadrant 4.

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11)



a) $f(\theta)=g(\theta)$

$\therefore \cos 2\theta = \sin\left(\theta - 45^\circ\right)$

$\therefore \cos 2\theta = \cos\left[90^\circ - \left(\theta - 45^\circ\right)\right]$

$\therefore \cos 2\theta = \cos\left(135^\circ - \theta\right)$

$\text{RA} = 135^\circ - \theta$

$\therefore 2\theta = \pm\left(135^\circ - \theta\right) + 360^\circ k\,;k \in \mathbb{Z}$

$\therefore 3\theta = 135^\circ + 360^\circ k \text{ or } \theta = -135^\circ + 360^\circ k$

$\therefore \theta = 45^\circ + 120^\circ k \text{ or } \theta = -135^\circ + 360^\circ k$

$\therefore \theta = 45^\circ \text{ or } \theta = 165^\circ \text{ or } \theta = 225^\circ$

b) You are looking for where $g(\theta)$ is above $f(\theta)$

$\therefore 45^\circ \le \theta \le 165^\circ$

This is also written as $\theta \in \left[45^\circ\,;165^\circ\right]$

c) You are looking for where $g(\theta)$ is positive and $f(\theta)$ is negative or where $g(\theta)$ is negative and $f(\theta)$ is positive.

Note that $f(\theta) \neq 0$

$\therefore 0^\circ \le \theta < 135^\circ \text{ and } \theta \neq 45^\circ$

This is also written as $\theta \in \left[0^\circ\,;135^\circ\right) \text{ and } \theta \neq 45^\circ$

d) $\theta = 135^\circ$

12)

a) $\cos 0^\circ + \cos 1^\circ + \cos 2^\circ + \ldots + \cos 178^\circ + \cos 179^\circ + \cos 180^\circ$

$= \cos 0^\circ + \cos 1^\circ + \cos 2^\circ + \ldots + \cos\left(180^\circ - 2^\circ\right) + \cos\left(180^\circ - 1^\circ\right) + \cos 180^\circ$

$= 1 + \cos 1^\circ + \cos 2^\circ + \ldots - \cos 2^\circ - \cos 1^\circ - 1$

$= 0$

b) $\sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \sin^2 88^\circ + \sin^2 89^\circ + \sin^2 90^\circ$

$= \sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \sin^2\left(90^\circ - 2^\circ\right) + \sin^2\left(90^\circ - 1^\circ\right) + \sin^2 90^\circ$

$= \sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \cos^2 2^\circ + \cos^2 1^\circ + 1$

$= \sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \sin^2 44^\circ + \sin^2 45^\circ + \cos^2 44^\circ + \ldots + \cos^2 2^\circ + \cos^2 1^\circ + 1$

$= 44 + 1 + \sin^2 45^\circ$

$= 45 + \left(\frac{\sqrt{2}}{2}\right)^2$

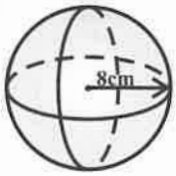
$= 45{,}5$

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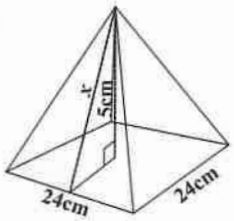
Mixed Exercise:

1) **Determine the surface area of the shapes shown below correct to 1 decimal place:**

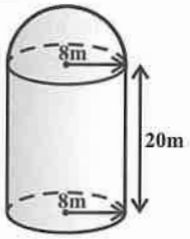
a)



b)

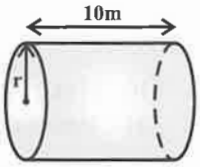


c)



2) **Determine the volume of all the shapes in Question 1 above.**

3) **The closed cylinder alongside has a volume of $282,74\,m^3$ and a height of 10m.**

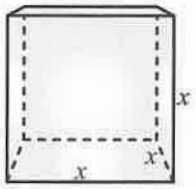


a) Determine the radius of the cylinder correct to 1 decimal place.

b) Hence, determine the surface area of the cylinder correct to 1 decimal place.

c) By what factor must the radius and height of the cylinder be multiplied by, in order to increase the volume to $35342,92\,m^3$?

4) **Given the square right prism (cube) alongside:**



a) Determine the volume in terms of x.

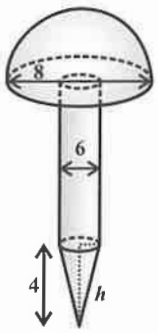
b) Determine the volume of the prism if $x = 2\,cm$.

c) If the prism has a volume of $512\,m^3$ determine the value of x.

d) If the prism has a volume of $100\,m^3$, by what factor must the sides of the cube be multiplied by to increase the volume to $900\,m^3$?

e) If the surface area of the cube is $600\,m^2$, by what factor must the sides of the cube be multiplied by to increase the surface area to $1200\,m^2$?

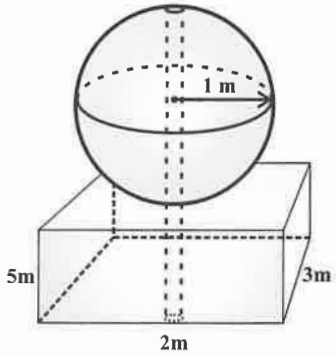
5) **The diagram alongside represents a 30cm long solar powered light designed to illuminate pathways. The head of the light is a hemisphere with a diameter of 8cm. The body of the light is a cylinder with a diameter of 6cm. The peg section, which is designed to be inserted into the ground, is a cone with a height of 4cm. Determine the surface area of the light. Assume the figure is made from 1 piece and there is no overlapping of any of the surfaces.**



6) **A garden water feature is constructed from a rectangular prism and sphere as shown in the diagram below. A cylindrical hole is bored through the centre of the feature to allow water to flow through it.**

a) If the water pump can maintain a volume of $0,22\,m^3$, determine the diameter of the hole through the feature.

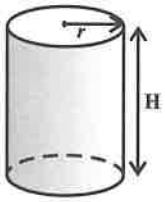
b) Hence, correct to 1 decimal place, determine the surface area of the water feature. Remember that a hole is bored through both sides of the sphere and prism.

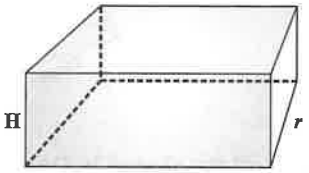


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7) **Star Cups Coffee sells its coffee beans in a cylindrical container. The open container (without a lid) has a surface area of 400π cm^2. It launches a new coffee powder in a rectangular shaped container with the following dimensions:**

- **The height is kept the same as their current cylindrical containers**
- **The breadth of the new container is equal to the radius of the cylindrical container**
- **The length of the rectangular base is twice its breadth**





a) Show that the height of the container can be written as: $H = \frac{200}{r} - \frac{r}{2}$.

b) Hence, write an expression for the volume of the new container in terms of its breadth (r).

c) Hence, determine the volume of the new container if the radius of the original container is 8cm.

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Mixed Exercise:

1)

a) $SA = 4\pi r^2$

$= 4\pi(8)^2$

$= 804,2\,\text{cm}^2$

b) $SA = A_{faces} + A_{base}$

$A_{face} = \frac{1}{2}\text{base} \times x$

$x^2 = 12^2 + 5^2 \rightarrow$ Pythag

$\therefore x = 13\,\text{cm}$

$\therefore A_{face} = \frac{1}{2}(24)(13) = 156\,\text{cm}^2$

$SA = 4(156) + (24)(24)$

$= 1200\,\text{cm}^2$

c) $SA = A_{base} + A_{cyl} + A_{open\ HS}$

$= \pi r^2 + 2\pi r\text{H} + 2\pi r^2$

$= \pi(8)^2 + 2\pi(8)(20) + 2\pi(8)^2$

$= 1608,5\,\text{m}^2$

2)

a) $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi(8)^3$

$= 2144,7\,\text{cm}^3$

b) $V = \frac{1}{3}(A_{base} \times \text{H})$

$= \frac{1}{3}(24)(24)(5)$

$= 960\,\text{cm}^3$

c) $V = V_{HS} + V_{cyl}$

$= \frac{2}{3}\pi r^3 + \pi r^2\text{H}$

$= \frac{2}{3}\pi(8)^3 + \pi(8)^2(20)$

$= 5093,6\,\text{m}^3$

3)

a) $V = \pi r^2\text{H}$

$\therefore 282,74 = \pi r^2(10)$

$\therefore r^2 = \dfrac{282,74}{10\pi}$

$\therefore r = \sqrt{\dfrac{282,74}{10\pi}} = 3\,\text{m}$

b) $SA = 2\pi r^2 + 2\pi r\text{H}$

$= 2\pi(3)^2 + 2\pi(3)(10)$

$= 245,0\,\text{m}^2$

c) If all lengths are multiplied by a factor of k the volume increases by a factor of k^3.

$k^3 = \dfrac{V_{new}}{V_{old}}$

$= \dfrac{35342,92}{282,74}$

$= 125$

$\therefore k^3 = 125$

$\therefore k = 5$

4)

a) $V = l \times B \times H$

$= x \times x \times x$

$= x^3$

b) $V = x^3$

$= 2^3$

$= 8\,\text{cm}^3$

c) $V = x^3$

$\therefore 512 = x^3$

$\therefore x = 8\,\text{m}$

d) If all lengths are multiplied by a factor of k the volume increases by a factor of k^3.

$k^3 = \dfrac{V_{new}}{V_{old}}$

$= \dfrac{900}{100}$

$= 9$

$\therefore k^3 = 9$

$\therefore k = \sqrt[3]{9}$

e) If all lengths are multiplied by a factor of k the area increases by a factor of k^2.

$k^2 = \dfrac{A_{new}}{A_{old}}$

$= \dfrac{1200}{600}$

$= 2$

$\therefore k^2 = 2$

$\therefore k = \sqrt{2}$

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5) Note that the cylinder and cone are both open and that the hemisphere is closed except where it joins the cylinder.

$SA_{total} = A_{HS} + A_{cyl} + A_{cone}$

$A_{HS} = \overset{HS}{3\pi r^2} - \overset{cyl}{\pi r^2}$

$= 3\pi(4)^2 - \pi(3)^2$

$= 39\pi\,cm^2$

$A_{cyl} = 2\pi rH$

$= 2\pi rH$

$= 2\pi(3)(30-4-4)$

$= 132\pi\,cm^2$

$A_{cone} = \pi r h_s$

$h_s = 4^2 + 3^2$

$= 25$

$\therefore h_s = 5\,cm$

$\therefore A_{cone} = \pi(3)(5)$

$= 15\pi$

$\therefore SA_{total} = 39\pi + 132\pi + 15\pi$

$= 186\pi$

$= 584,34\,cm^2$

6)

a) $V = \pi r^2 H$

$\therefore 0,22 = \pi r^2(5+2)$

$\therefore r^2 = \dfrac{0,22}{7\pi}$

$\therefore r = 0,1\,m$

$\therefore d = 2(0,1) = 0,2\,m$

b) Note that the area of the top and base of the cylinder has to be removed twice from the sphere and twice from the prism.

$SA_{sphere} = \overset{Sphere}{4\pi r^2} - \overset{cyl}{\left(2\pi r^2\right)} = 4\pi(1)^2 - 2\pi(0,1)^2 = \dfrac{199\pi}{50}\,m^2$

$SA_{prism} = \overset{Sides}{2(2\times5)} + \overset{Sides}{2(3\times5)} + \overset{Bases}{2(2\times3)} - 2\pi(0,1)^2 \approx 61,937...\,m^2$

$\therefore SA_{total} = \dfrac{199\pi}{50} + 61,937...$

$= 74,4\,m^2$

7)

a) $SA_{cyl} = \pi r^2 + 2\pi rH$

$\therefore 400\pi - \pi r^2 = 2\pi rH$

$\therefore H = \dfrac{400\pi}{2\pi r} - \dfrac{\pi r^2}{2\pi r}$

$\therefore H = \dfrac{200}{r} - \dfrac{r}{2}$

b) $V_{prism} = L \times B \times H$

$= 2B \times B \times H$

$= 2r \times r \times H$

$= 2r^2 H$

$= 2r^2\left(\dfrac{200}{r} - \dfrac{r}{2}\right)$

$= 400r - r^3$

c) $V = 400(8) - (8)^3$

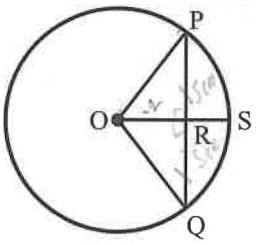
$= 2688\,cm^3$

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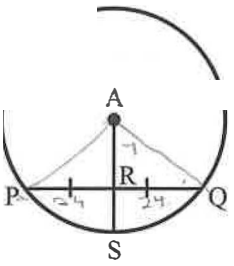
Theorems 1 and 2

Exercise 1:

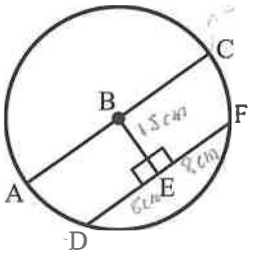
1) **In the diagram alongside, O is the centre of the circle. OR = 12 cm and chord PQ = 10 cm. If OS ⊥ PQ, determine the length of OS.**



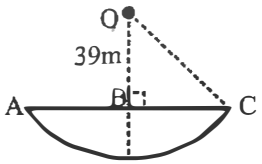
2) **In the diagram alongside, A is the centre of the circle. AS bisects chord PQ. Determine the length of the radius if PQ = 48 cm and AR = 7 cm.**



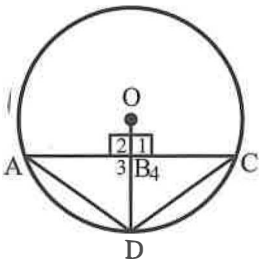
3) **In the circle alongside, AC is a diameter of the circle and BE ⊥ DF. If B is the centre of the circle, determine the length of the diameter if DF = 16 cm and BE = 15 cm.**



4) **In the diagram alongside, O is the centre of a circle and OB is 39m in length. If AC is 160m in length, determine the length of BD.**



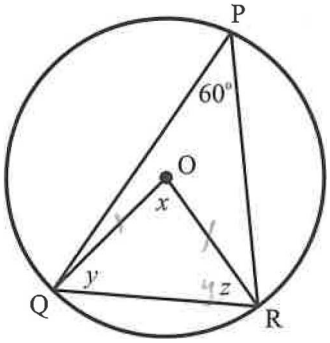
5) **In the diagram alongside, AC ⊥ OB. Show that AD = DC**



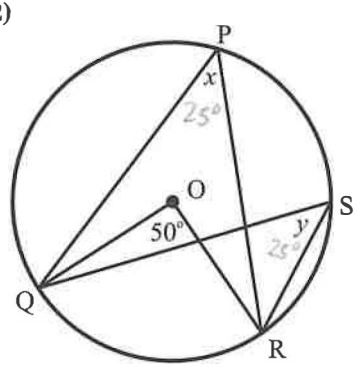
Theorem 3

Exercise 2: In each diagram below, O is the circle centre. Determine the value of x, y and z.

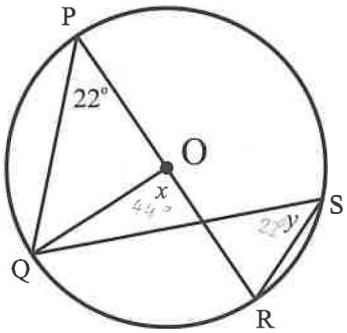
1)



2)



3)



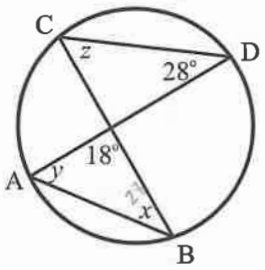
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Theorem 4

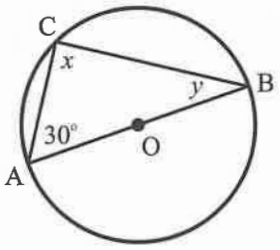
Exercise 3:

1) Determine the value of the unknown variables in each of the diagrams below:

a)

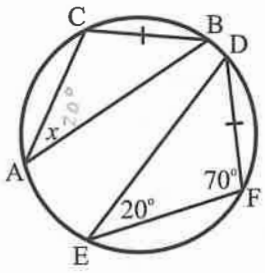


b)

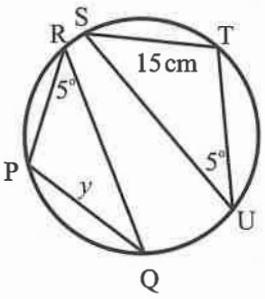


2) Determine the value of the unknown angles or sides labelled as x or y in each of the diagrams below:

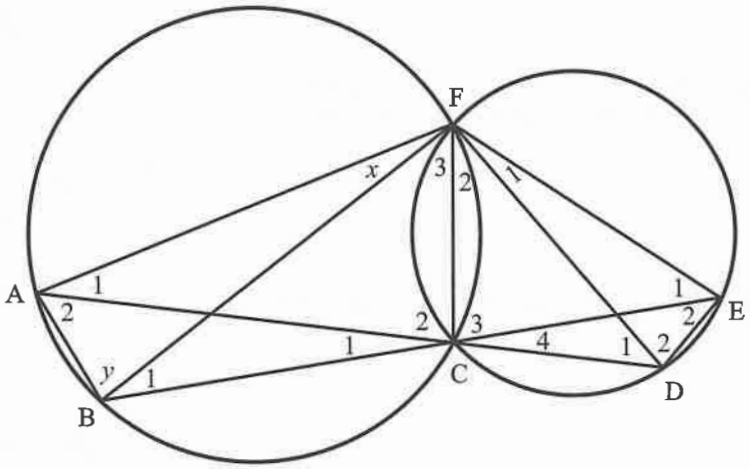
a)



b)



3) In the diagram below, $\hat{D}_2 = 60°$ and $\hat{F}_1 = 30°$. Determine the value of x and y.

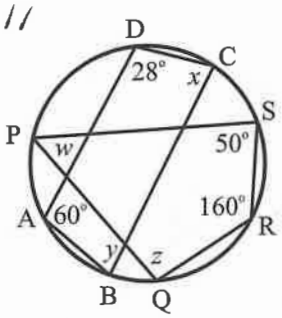


Theorem 5

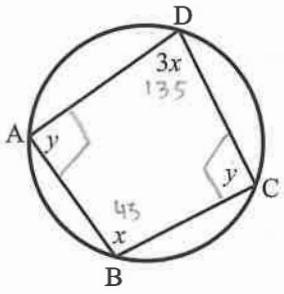
Exercise 4:

1) The vertices of the quadrilaterals below lie on the circumference of the circle. Determine the values of the unknown variables.

a)

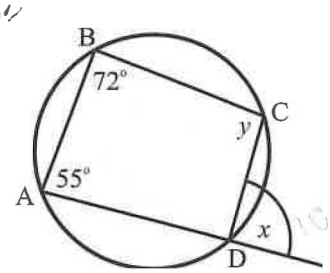


b)

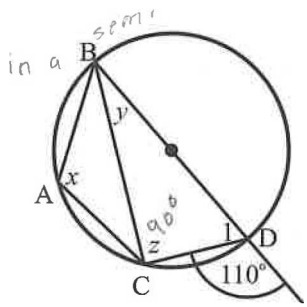


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c)

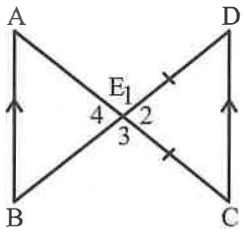


d)

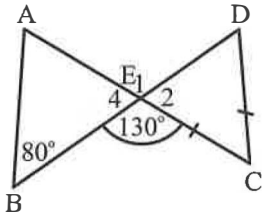


2) Prove that ABCD is a cyclic quadrilateral in each of the figures below.

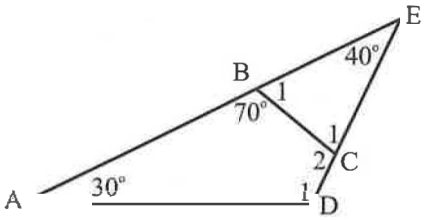
a)



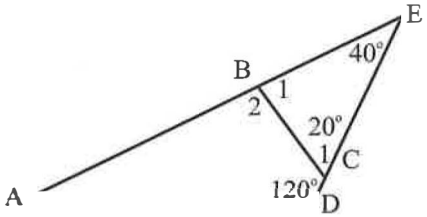
b)



c)



d)

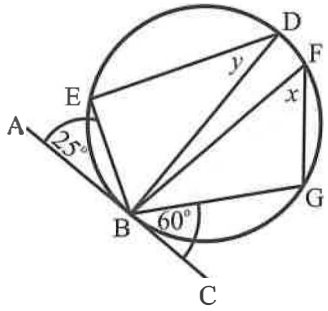


Theorems 6 and 7

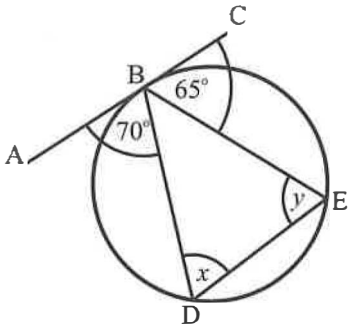
Exercise 5:

1) Determine the value of the unknown variables in each of the following diagrams. In diagrams *a* and *b*, AC is a tangent at B. In diagram *c*, AB and AC are tangents at B and C respectively. In diagram *d*, the sides of triangle PQR are all tangents to the circle at A, B and C.

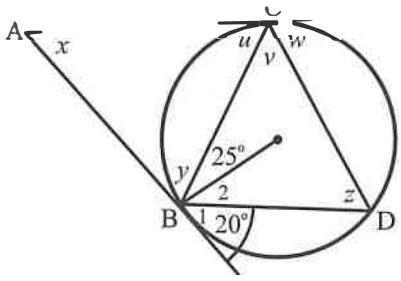
a)



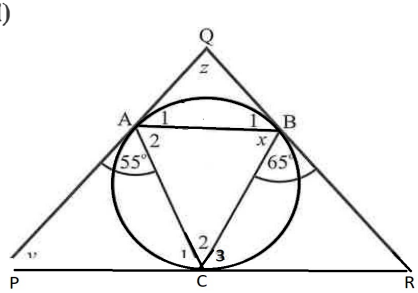
b)



c)



d)



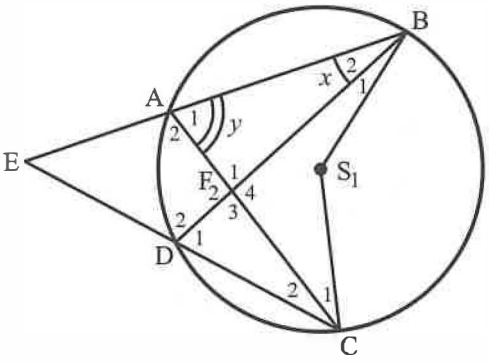
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Mixed Exercise: (Remember that diagrams are not drawn to scale.)

1) In the diagram alongside, A, B and C are on the circle with centre S. Chords BA and CD are produced to meet at E. AC and BD intersect at F. $\hat{B}_2 = x$ and $A_1 = y$.

a) Express $\hat{F}_2$ in terms of x and y.

b) Prove that $\hat{S}_1 = \hat{F}_2 + \hat{E}$

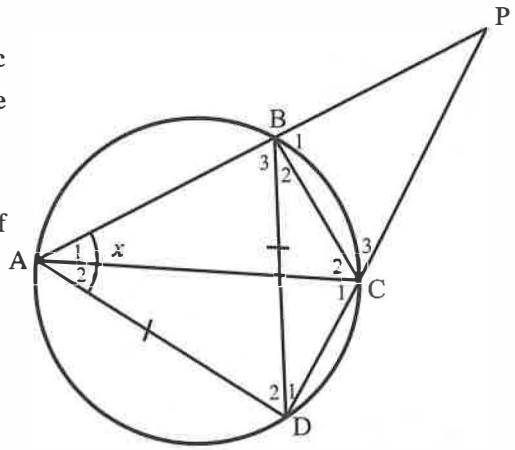


2) In the diagram alongside, ABCD is a cyclic quadrilateral. AD = DB and $B\hat{A}D = x$. AB and DC are produced to intersect at P.

a) Give 3 other angles that are equal to x.

b) If $C\hat{D}A = 65°$ and $x = 85°$, determine the value of angles $\hat{P}$, $\hat{B}_1$ and $\hat{A}_2$.

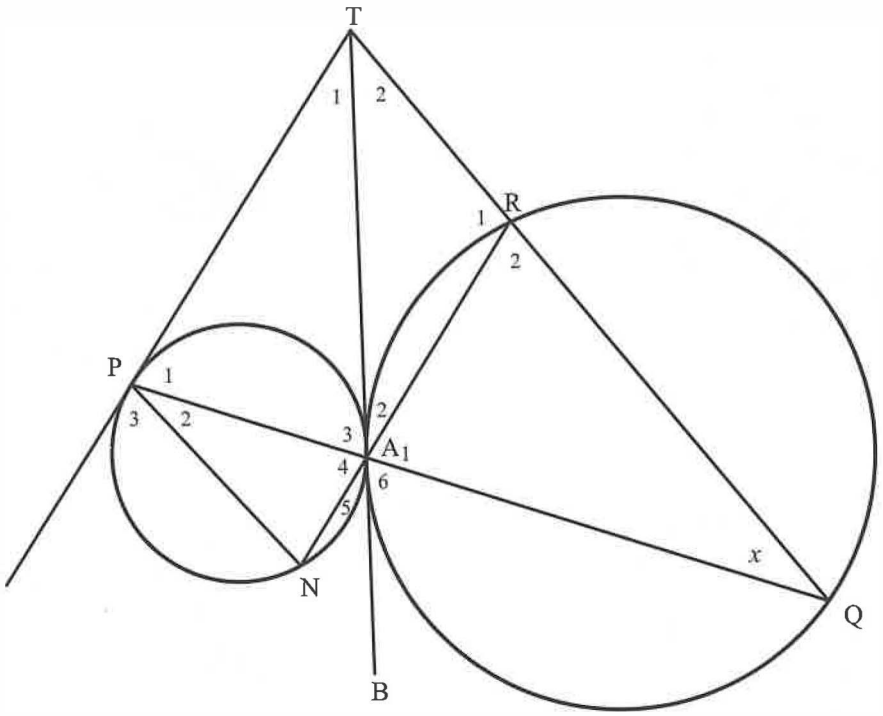
c) Prove that AD is a tangent to circle ACP.



3) In the diagram below, two circles have a common tangent TAB. PT is a tangent to the smaller circle at P. PAQ, QRT and NAR are straight lines and $\hat{Q} = x$.

a) Give 3 other angles that are equal to x.

b) Prove that APTR is a cyclic quadrilateral.



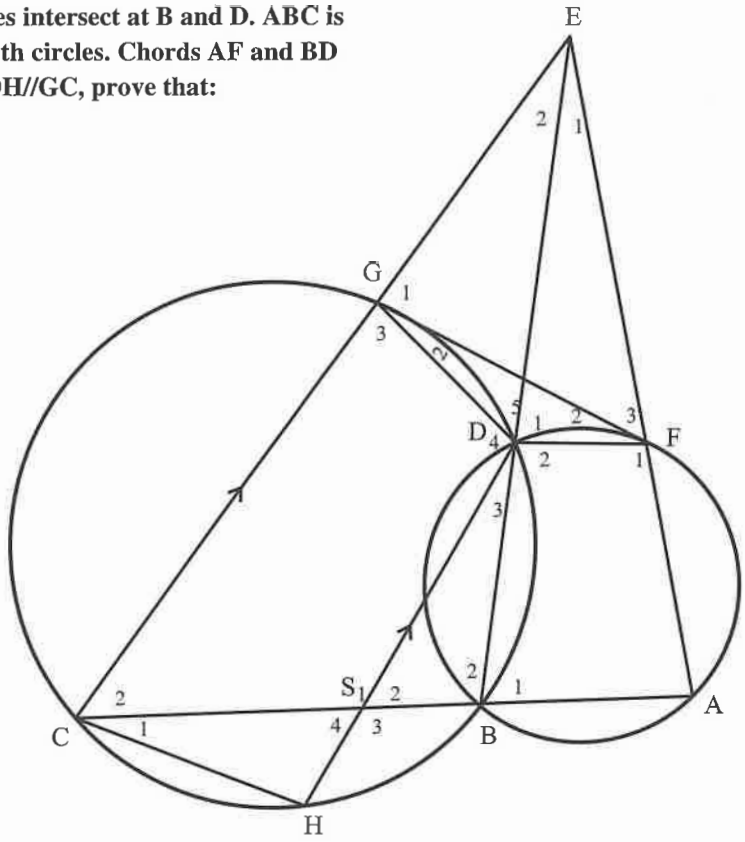
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4) **In the diagram alongside, two circles intersect at B and D. ABC is a straight line forming chords in both circles. Chords AF and BD are produced to meet CE at E. If DH//GC, prove that:**

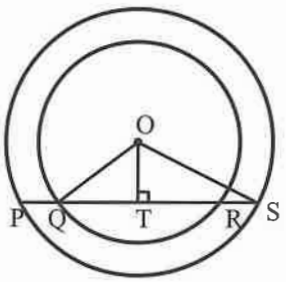
a) FDGE is a cyclic quadrilateral

b) HC is a tangent to circle BCE

c) $\hat{F}_3 = \hat{S}_2$



5) **Two concentric circles centred at O have radii of 10cm and 17cm. QR = 12cm . If OT⊥QR, determine the length of PS.**

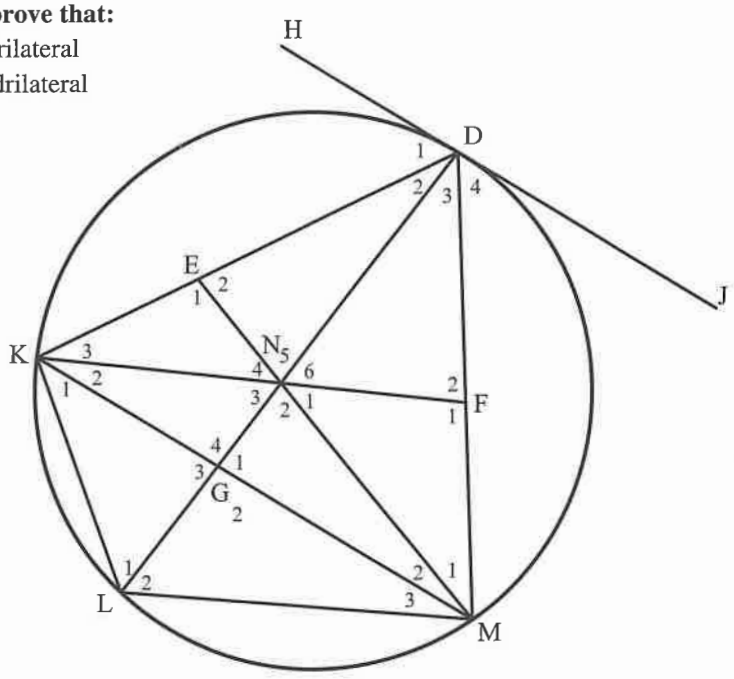


6) **In the diagram below, KF//LM and K, L, M and D are points on the circle. ME⊥KD and HJ is a tangent at D.**
If EM//KL and HJ//KM prove that:

a) DENF is a cyclic quadrilateral

b) MGED is a cyclic quadrilateral

c) DK = DM



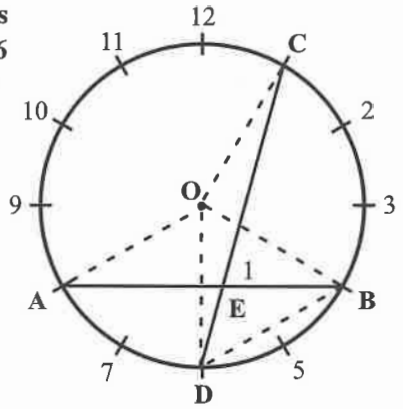
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7) The diagram alongside shows a clock face. Straight line AB joins the 8 and 4 together and straight line CD joins the 1 and 6 together. O is the centre of the clock face. Determine the value of:

a) $A\hat{O}D$ (no reason is required)

b) $C\hat{O}B$ (no reason is required)

c) $\hat{E}_1$.

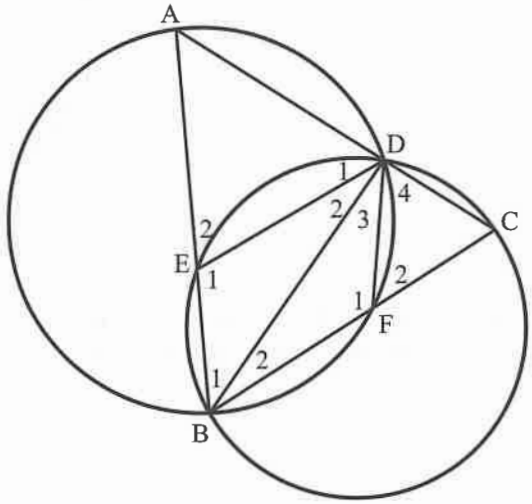


8) In the diagram below two circles intersect one another at D and B. AB is a straight line such that it intersects circle BCD at point E. BC is a straight line such that it intersects the circle ABD at F. DE, DB and DF are joined. It is given that FC = FD and $\hat{D}_4 = y$.

a) Determine the value of $\hat{E}_1$ in terms of y.

b) Determine the value of $\hat{A}$ in terms of y.

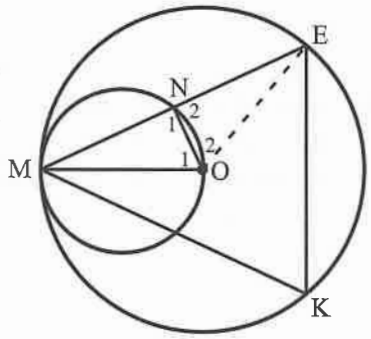
c) Hence, prove that ED//BC.



9) In the diagram alongside, O is the centre of circle MEK such that OM is the diameter of circle MNO. Chord EM of the larger circle cuts the smaller circle at N. EM has a length of $4m^2 - 4$ units and ON has a length of $4m$ units.

a) Determine the length of the radius of the larger circle in terms of m.

b) If $O\hat{M}N = 30°$, determine the value of $\hat{K}$.

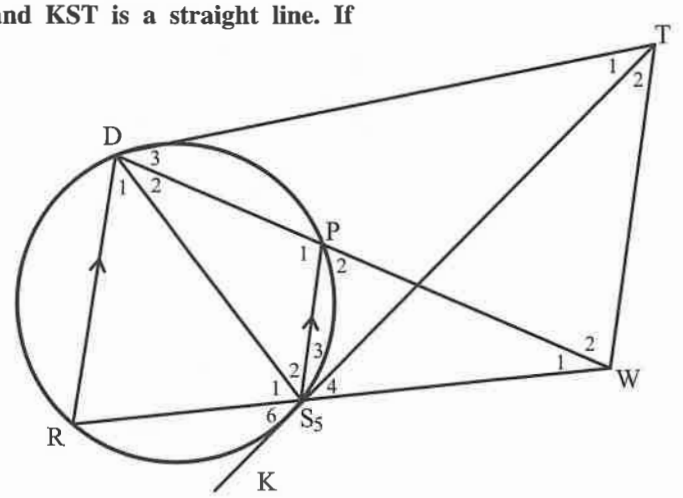


10) In the diagram alongside, TD is a tangent to the circle at D. RS and DP are produced to meet at W and KST is a straight line. If $\hat{S}_4 = \hat{S}_2$ and DR//PS, prove that:

a) SWTD is a cyclic quadrilateral.

b) TK is a tangent to the circle at S.

c) TW//PS



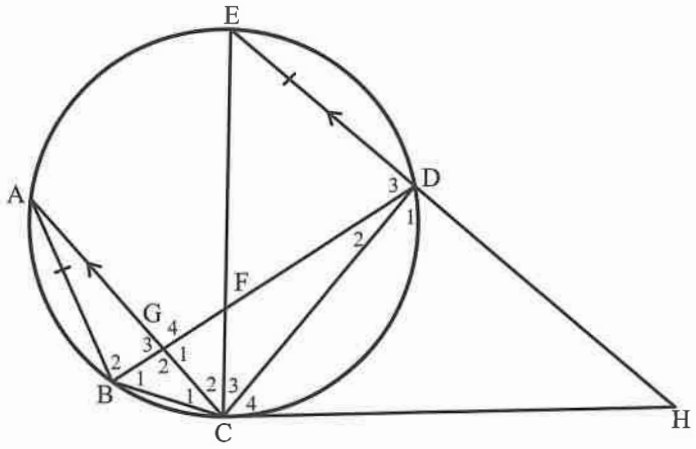
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11) In the diagram alongside, HC is a tangent to the circle at C and HDE is a straight line. DB cuts EC at F and AC at G. ED//AC and AB = ED. Prove that:

a) FC is a tangent to circle GCB.

b) $\hat{D}_1 = \hat{G}_3$.

c) HDFC is a cyclic quadrilateral.

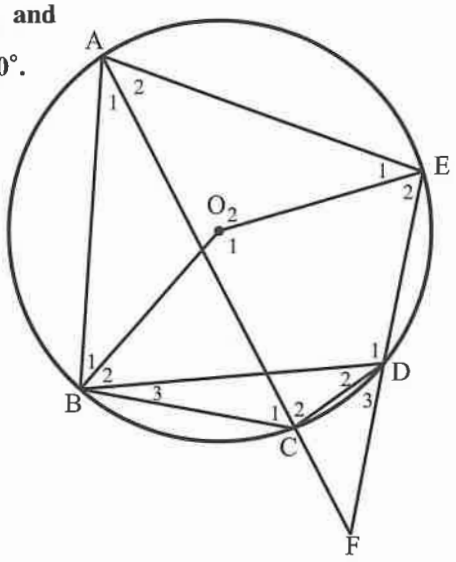


12) In the diagram alongside O is the centre of the circle. $\hat{A}_1 = \hat{A}_2$ and CD = CF. AC and ED are produced to meet at F and $\hat{O}_1 = 120°$.

a) Determine the value of $\hat{D}_3$.

b) Determine the value of $A\hat{B}D$.

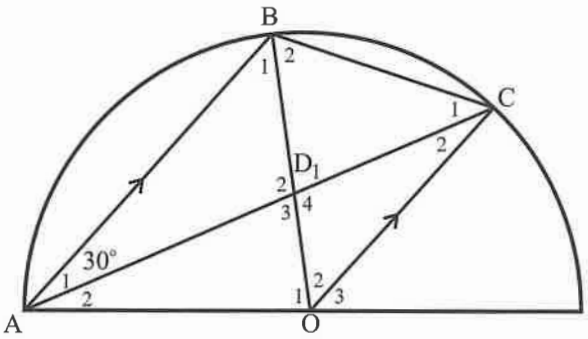
c) Hence, with reasons, state whether or not OBFD is a cyclic quadrilateral.



13) In the diagram below, O is the centre of the semi-circle. OC//AB and OB and AC intersect at D. $\hat{B}_2 = \hat{O}_2$.

a) If $A_1 = 30°$ determine the values of $\hat{D}_1, \hat{B}_2, \hat{C}_1$.

b) Prove that ABCO is a rhombus.



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Theorem 1 and 2

Exercise 1:

1)

Statement	Reason
$RQ = RP = 5\,cm$	Line from centre $\perp$ chord
$OP^2 = OR^2 + RP^2$	Pythag.
$\therefore OP^2 = 12^2 + 5^2 = 169$ $\therefore OP = 13\,cm$	
$\therefore OP = OS = 13\,cm$	Radii

2)

Statement	Reason
$AS \perp PQ$	Line from centre to midpoint of chord
$AQ^2 = AR^2 + RQ^2$	Pythag.
$\therefore AQ^2 = 7^2 + 24^2 = 625$ $\therefore AQ = 25\,cm$	
Note that AQ and AP are radii of the circle $\therefore AP = 25\,cm$	

3)

Statement	Reason
$DE = EF = 8\,cm$	Line from centre $\perp$ chord
$BF^2 = BE^2 + EF^2$	Pythag.
$\therefore BF^2 = 15^2 + 8^2 = 289$ $\therefore BF = 17\,cm$	
$\therefore AC = 2BC = 34\,cm$	Diameter = 2 x radius

4)

Statement	Reason
$AB = BC = 80\,m$	Line from centre $\perp$ chord
$OC^2 = OB^2 + BC^2$	Pythag.
$\therefore OC^2 = 39^2 + 80^2 = 7921$ $\therefore OC = 89\,m$	
$OD = OC = 89\,m$ $\therefore BD = 89 - 39 = 50\,m$	Radii

5)

Statement	Reason
In $\triangle ADB$ and $\triangle CDB$:	
$AB = BC$	Line from centre $\perp$ chord.
$DB = DB$	Common side.
$\therefore \hat{B}_3 = \hat{B}_4 = 90^\circ$	Given
$\therefore \triangle ADB \equiv \triangle CDB$	SAS
$\therefore AD = DC$	

Theorem 3

Exercise 2:

1)

Statement	Reason
$x = 2\hat{P} = 120^\circ$	$\angle$ at centre $= 2\angle$ at circumference
$OQ = OR$	Radii
$\therefore y = z$	$\angle$'s opp. = sides
$\therefore y = z = 30^\circ$	$\angle$'s in a $\triangle$

2)

Statement	Reason
$x = y = \frac{1}{2}(50^\circ) = 25^\circ$	$\angle$ at centre $= 2\angle$ at circumference

3)

Statement	Reason
$x = 2\hat{P} = 44^\circ$	$\angle$ at centre $= 2\angle$ at circumference
$y = \frac{1}{2}x = 22^\circ$	$\angle$ at centre $= 2\angle$ at circumference

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Theorem 4

Exercise 3:

1)

a)

Statement	Reason
$x = 28^\circ$	$\angle$'s in same segment
$y = 180^\circ - 28^\circ - 18^\circ = 134^\circ$	$\angle$'s in a Δ
$y = z = 134^\circ$	$\angle$'s in same segment

b)

Statement	Reason
$x = 90^\circ$	$\angle$ in a semi-circle
$y = 180^\circ - 30^\circ - 90^\circ$	$\angle$'s in a Δ
$\therefore y = 60^\circ$	

2)

a)

Statement	Reason
$x = 20^\circ$	= chords subtend = $\angle$'s

b)

Statement	Reason
ST = PQ = $y = 15$ cm	chords subtended by = $\angle$'s

3)

Statement	Reason
$\hat{F}_1 = \hat{C}_4 = 30^\circ$	$\angle$'s in same segment
$\hat{C}_4 = \hat{C}_1 = 30^\circ$	Vert. Opp. $\angle$'s
$\hat{C}_1 = x = 30^\circ$	$\angle$'s in same segment
$\hat{D}_2 = \hat{C}_3 = 60^\circ$	$\angle$'s in same segment
$\hat{C}_2 = 180^\circ - \hat{C}_1 - \hat{C}_3$ $= 180^\circ - 30^\circ - 60^\circ = 90^\circ$	Adj. $\angle$'s on a str. line
$\hat{C}_2 = y = 90^\circ$	$\angle$'s in same segment

Theorem 5

Exercise 4:

1)

a)

Statement	Reason
$w = 180^\circ - 160^\circ = 20^\circ$	Opp. $\angle$'s of a cyclic quad.
$x = 180^\circ - 60^\circ = 120^\circ$	Opp. $\angle$'s of a cyclic quad.
$y = 180^\circ - 28^\circ = 152^\circ$	Opp. $\angle$'s of a cyclic quad.
$z = 180^\circ - 50^\circ = 130^\circ$	Opp. $\angle$'s of a cyclic quad.

b)

Statement	Reason
$x + 3x = 180^\circ$	Opp. $\angle$'s of a cyclic quad.
$\therefore x = 45^\circ$	
$y + y = 180^\circ$	Opp. $\angle$'s of a cyclic quad.
$\therefore y = 90^\circ$	

c)

Statement	Reason
$x = \hat{B} = 72^\circ$	Ext. $\angle$ of a cyclic quad.
$y = 180^\circ - 55^\circ = 125^\circ$	Opp. $\angle$'s of a cyclic quad.
$\therefore y = 125^\circ$	

d)

Statement	Reason
$x = 110^\circ$	Ext. $\angle$ of a cyclic quad.
$z = 90^\circ$	$\angle$ in a semi-circle
$\hat{D}_1 = 180 - 110^\circ = 70^\circ$	Opp. $\angle$'s of a cyclic quad.
$y = 180^\circ - \hat{D}_1 - z$ $\therefore y = 180^\circ - 70^\circ - 90^\circ$ $\therefore y = 20^\circ$	$\angle$'s in a Δ

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2)

a)

Statement	Reason
$\hat{A} = \hat{C}$	Alt. $\angle$'s $(AB//CD)$
$\hat{B} = \hat{D}$	Alt. $\angle$'s $(AB//CD)$
$\hat{D} = \hat{C}$	$\angle$'s opp. = sides
$\therefore \hat{A} = \hat{B}$	
$\therefore \hat{A} = \hat{D}$	
$\therefore$ ABCD is a cyclic quad.	line seg. subt. = $\angle$'s on the same side

b)

Statement	Reason
$\hat{E}_2 = 180° - 130° = 50°$	Adj. $\angle$'s on a str. line
$\hat{E}_2 = \hat{D} = 50°$	$\angle$'s opp. = sides
$\hat{C} = 180° - \hat{E}_2 - \hat{D}$	$\angle$'s in a Δ
$\therefore \hat{C} = 180° - 100° = 80°$	
$\therefore \hat{C} = \hat{B}$	
$\therefore$ ABCD is a cyclic quad.	line seg. subt. = $\angle$'s on the same side

c)

Statement	Reason
$\hat{B}_1 = 180° - 70° = 110°$	Adj. $\angle$'s on a str. line
$\hat{C}_1 = 180° - \hat{B}_1 - \hat{E}$	$\angle$'s in a Δ
$\therefore \hat{C}_1 = 180° - 110° - 40°$	
$\therefore \hat{C}_1 = 30° = \hat{A}$	
$\therefore$ ABCD is a cyclic quad.	Ext. $\angle$ = int. opp. $\angle$

d)

Statement	Reason
$\hat{A} = 180° - \hat{D} - \hat{E}$	$\angle$'s in a Δ
$\therefore \hat{A} = 180° - 120° - 40°$	
$\therefore \hat{A} = 20°$	
$\therefore \hat{A} = C_1$	
$\therefore$ ABCD is a cyclic quad.	Ext. $\angle$ = int. opp. $\angle$

Theorems 6 and 7

Exercise 5:

1)

a)

Statement	Reason
$x = 60°$	$\angle$ between tan. and chord
$y = 25°$	$\angle$ between tan. and chord

b)

Statement	Reason
$x = 65°$	$\angle$ between tan. and chord
$y = 70°$	$\angle$ between tan. and chord

c)

Statement	Reason
$y + 25° = 90°$	Tan $\perp$ radius
$\therefore y = 65°$	
AB = AC	Tangents from same pt.
$\therefore y = u = 65°$	$\angle$'s opp. = sides
$x = 180° - 2(65°)$	$\angle$'s in a Δ
$\therefore x = 50°$	
$z = u = y = 65°$	$\angle$ between tan. and chord
$\hat{B}_1 = 20°$	Given
$v = 20°$	$\angle$ between tan. and chord
$w = 180° - u - v$	Adj. $\angle$'s on a str. line
$\therefore w = 180° - 65° - 20°$	
$\therefore w = 95°$	

d)

Statement	Reason
$x = 55°$	$\angle$ between tan. and chord
AP = CP	Tangents from same pt.
$\therefore \hat{C}_1 = 55°$	$\angle$'s opp. = sides
$y = 180° - 2(55°)$	$\angle$'s in a Δ
$\therefore y = 70°$	
$\hat{B}_1 = 180° - x - 65°$	Adj. $\angle$'s on a str. line
$\therefore \hat{B}_1 = 180° - 55° - 65°$	
$\therefore \hat{B}_1 = 60°$	
QA = QB	Tangents from same pt.
$\therefore \hat{B}_1 = \hat{A}_1 = 60°$	$\angle$'s opp. = sides
$z = 180° - 2(60°)$	$\angle$'s in a Δ
$\therefore z = 60°$	

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Mixed Exercise:

1)

a)

Statement	Reason
$\hat{F}_2 = \hat{A}_1 + \hat{B}_2$	Ext. $\angle$ of Δ
$\therefore \hat{F}_2 = x + y$	

b)

Statement	Reason
$\hat{A}_1 = \hat{C}_2 + \hat{E}$	Ext. $\angle$ of Δ
$\hat{C}_2 = \hat{B}_2 = x$	$\angle$'s in same segment
$\therefore \hat{A}_1 = x + \hat{E}$	
$\hat{S}_1 = 2\hat{A}_1$	$\angle$ at centre $= 2\angle$ at circumference
$\hat{S}_1 = \hat{A}_1 + x + \hat{E}$	
$\hat{A}_1 + x = \hat{F}_2$	Ext. $\angle$ of Δ
$\therefore \hat{S}_1 = \hat{F}_2 + \hat{E}$	

2)

a)

Statement	Reason
$\hat{B}_3 = x$	$\angle$'s opp. = sides
$\hat{C}_1 = x$	$\angle$'s in same segment
$\hat{C}_3 = x$	Ext. $\angle$ of a cyclic quad.

b)

Statement	Reason
$\hat{B}_1 = C\hat{D}A = 65°$	Ext. $\angle$ of a cyclic quad.
$\hat{P} = 180° - P\hat{A}D - C\hat{D}A$	$\angle$'s in a Δ
$\therefore \hat{P} = 180° - 85° - 65°$	
$\therefore \hat{P} = 30°$	
$\hat{B}_2 = 180° - \hat{B}_3 - \hat{B}_1$	Adj. $\angle$'s on a str. line
$\therefore \hat{B}_2 = 180° - 85° - 65°$	
$\therefore \hat{B}_2 = 30°$	
$\therefore \hat{B}_2 = \hat{A}_2 = 30°$	$\angle$'s in same segment

c)

Statement	Reason
$\hat{P} = \hat{A}_2 = 30°$	Proved above
$\therefore$ AD is a tangent	$\angle$ between line and chord

3)

a)

Statement	Reason
$\hat{A}_2 = \hat{Q} = x$	$\angle$ between tan and chord
$\hat{A}_5 = \hat{A}_2 = x$	Vert. opp. $\angle$'s
$\hat{P}_2 = \hat{A}_5 = x$	$\angle$ between tan and chord

b)

Statement	Reason
PT = TA	Tangents from same pt.
$\therefore \hat{P}_1 = \hat{A}_3$	$\angle$'s opp. = sides
$\hat{A}_3 = \hat{A}_6$	Vert. opp. $\angle$'s
$\hat{A}_6 = \hat{R}_2$	$\angle$ between tan. and chord
$\therefore \hat{P}_1 = \hat{R}_2$	
$\therefore$ APTR is a cyclic quad.	Ext $\angle$ = int. opp $\angle$

4)

a)

Statement	Reason
$\hat{F}_2 + \hat{F}_3 = \hat{B}_1$	Ext. $\angle$ of a cyclic quad.
$\hat{B}_1 = \hat{G}_3$	Ext. $\angle$ of a cyclic quad.
$\therefore \hat{F}_2 + \hat{F}_3 = \hat{G}_3$	
$\therefore$ FDGE is a cyclic quad	Ext $\angle$ = int. opp $\angle$

b)

Statement	Reason
$\hat{C}_1 = \hat{D}_3$	$\angle$'s in same segment
$\hat{D}_3 = \hat{E}_2$	Corr. $\angle$'s (DH // CG)
$\therefore \hat{C}_1 = \hat{E}_2$	
$\therefore$ HC is a tangent	$\angle$ between line and chord

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c)

Statement	Reason
$\hat{F}_3 = \hat{D}_5$	$\angle$'s in same segment
$\hat{D}_5 = \hat{C}_2$	Ext. $\angle$ of a cyclic quad.
$\hat{C}_2 = \hat{S}_2$	Corr. $\angle$'s (DH // CG)
$\therefore \hat{S}_2 = \hat{F}_3$	

5)

Statement	Reason
$QT = TR = 6\,cm$	Line from centre $\perp$ chord
$OT^2 = OQ^2 - QT^2 = 10^2 - 6^2 = 64$	Pythag.
$TS^2 = OS^2 - OT^2 = 17^2 - 64 = 225$	Pythag.
$\therefore TS = 15\,cm$	
$TS = TP = 15\,cm$	Line from centre $\perp$ chord
$\therefore PS = 30\,cm$	

6)

a)

Statement	Reason
$\hat{F}_2 = F\hat{M}L$	Corr. $\angle$'s (KF // LM)
$\hat{E}_2 = E\hat{K}L$	Corr. $\angle$'s (EM // KL)
$F\hat{M}L + E\hat{K}L = 180°$	Opp. $\angle$'s of a cyclic quad
$\hat{F}_2 + \hat{E}_2 = 180°$	
$\therefore$ DENF is a cyclic quad	Opp. $\angle$'s supplementary

b)

Statement	Reason
$\hat{E}_2 = 90°$	Given ME $\perp$ KD
$\hat{F}_2 + \hat{E}_2 = 180°$	Opp. $\angle$'s of a cyclic quad.
$\therefore \hat{F}_2 = \hat{E}_2 = 90°$	
$\hat{F}_2 = F\hat{M}L$	Corr. $\angle$'s (KF // LM)
$F\hat{M}L = \hat{D}_1 + \hat{D}_2$	$\angle$ between tan. and chord
$\hat{D}_1 + \hat{D}_2 = \hat{G}_1 = 90°$	Alt. $\angle$'s (HJ // KM)
$\therefore \hat{E}_2 = \hat{G}_1 = 90°$	
$\therefore$ MGED is a cyclic quad	$\angle$'s in same segment

c)

Statement	Reason
$\hat{M}_1 + \hat{M}_2 = \hat{D}_4$	Alt. $\angle$'s (HJ // KM)
$\hat{D}_4 = \hat{K}_2 + \hat{K}_3$	$\angle$ between tan. and chord
$\therefore \hat{M}_1 + \hat{M}_2 = \hat{K}_2 + \hat{K}_3$	
$\therefore MD = KD$	$\angle$'s opp. = sides

7)

a)

Statement	Reason
$360° \div 12 = 30°$	
$A\hat{O}D = 2 \times 30° = 60°$	

b)

Statement	Reason
$C\hat{O}B = 3 \times 30° = 90°$	

c)

Statement	Reason
$C\hat{D}B = \frac{1}{2} C\hat{O}B = 45°$	$\angle$ at centre $= 2\angle$ at circumference
$A\hat{B}D = \frac{1}{2} A\hat{O}B = 30°$	$\angle$ at centre $= 2\angle$ at circumference
$\hat{E}_1 = A\hat{B}D + C\hat{D}B$	Ext. $\angle$ of Δ
$\hat{E}_1 = 30° + 45°$	
$\hat{E}_1 = 75°$	

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8)

a)

Statement	Reason
$DF = FC$	Given
$\therefore \hat{D}_4 = \hat{C} = y$	$\angle$'s opp. = sides
$\hat{E}_1 = 180° - \hat{C} = 180° - y$	Opp. $\angle$'s of a cyclic quad.

b)

Statement	Reason
$\hat{A} = \hat{F}_2$	Ext. $\angle$ of a cyclic quad.
$\hat{F}_2 = 180° - D_4 - \hat{C}$	$\angle$'s in a Δ
$\therefore \hat{A} = 180° - 2y$	

c)

Statement	Reason
$\hat{E}_1 = \hat{D}_1 + \hat{A}$	Ext. $\angle$ of Δ
$\hat{D}_1 = \hat{E}_1 - \hat{A}$	
$\therefore \hat{D}_1 = 180° - y - (180° - 2y) = y$	
$\hat{C} = y$	Proved above
$\therefore \hat{D}_1 = \hat{C}$	
$\therefore DE \,//\, CB$	Corr. $\angle$'s are =

9)

a)

Statement	Reason
$\hat{N}_1 = 90°$	$\angle$ in a semi-circle
$MN = NE = 2m^2 - 2$	Line from centre $\perp$ chord
$\therefore OM^2 = ON^2 + MN^2$	Pythag.
$\therefore OM^2 = (4m)^2 + (2m^2 - 2)^2$	
$\therefore OM^2 = 16m^2 + 4m^4 - 8m^2 + 4$	
$\therefore OM^2 = 4m^4 + 8m^2 + 4$	
$\therefore OM^2 = (2m^2 + 2)^2$	
$\therefore OM = 2m^2 + 2$ units	

b)

Statement	Reason
$M\hat{O}N = 180° - 90° - 30°$	$\angle$'s in a Δ
$\therefore M\hat{O}N = 60°$	
In ΔMON and ΔEON:	
$\hat{N}_1 = \hat{N}_2 = 90°$	Proved above
$OM = OE$	Radii
$ON = ON$	Common side
$\therefore \Delta MON \equiv \Delta EON$	RHS
$\therefore \hat{O}_1 = \hat{O}_2 = 60°$	$\Delta MON \equiv \Delta EON$
$\therefore M\hat{O}E = 120°$	
$\therefore \hat{K} = 60°$	$\angle$ at centre $= 2\angle$ at circumference

10)

a)

Statement	Reason
$\hat{S}_4 = \hat{S}_2$	Given
$\hat{D}_3 = \hat{S}_2$	$\angle$ between tan and chord
$\therefore \hat{D}_3 = \hat{S}_4$	
$\therefore$ SWTD is a cyclic quad.	line seg. subt. = $\angle$'s on the same side

b)

Statement	Reason
$\hat{D}_1 = \hat{S}_2$	Alt. $\angle$'s $(DR \,//\, PS)$
$\hat{S}_4 = \hat{S}_2$	Given
$\hat{S}_4 = \hat{S}_6$	Vert. opp $\angle$'s
$\therefore \hat{D}_1 = \hat{S}_6$	
$\therefore$ TK is a tangent	$\angle$ between line and chord

c)

Statement	Reason
$\hat{S}_3 = \hat{D}_2$	$\angle$ between tan and chord
$\hat{T}_2 = \hat{D}_2$	$\angle$'s in same segment
$\therefore \hat{S}_3 = \hat{T}_2$	
$\therefore PS \,//\, TW$	Alt. $\angle$'s are =

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11)

a)

Statement	Reason
$\hat{C}_2 = \hat{E}$	Alt. $\angle$'s $(ED // AC)$
$\hat{E} = \hat{B}_1$	$\angle$'s in same segment
$\therefore \hat{C}_2 = \hat{B}_1$	
$\therefore$ FC is a tangent	$\angle$ between line and chord

b)

Statement	Reason
$\hat{D}_1 = \hat{E} + \hat{C}_3$	Ext. $\angle$ of Δ
$\hat{G}_3 = \hat{B}_1 + \hat{C}_1$	Ext. $\angle$ of Δ
$\hat{E} = \hat{B}_1$	Proved above
$\hat{C}_1 = \hat{C}_3$	$\angle$'s subtended by = chords
$\therefore \hat{D}_1 = \hat{G}_3$	

c)

Statement	Reason
$\hat{B}_1 = \hat{C}_4$	$\angle$ between tan and chord
$\hat{C}_1 = \hat{C}_3$	Proved above
$\therefore \hat{C}_4 + \hat{C}_3 = \hat{B}_1 + \hat{C}_1$	
$\hat{G}_3 = \hat{B}_1 + \hat{C}_1$	Proved above
$\hat{D}_3 = \hat{G}_3$	Corr. $\angle$'s $(AC // ED)$
$\therefore \hat{D}_3 = \hat{C}_4 + \hat{C}_3$	
$\therefore$ HDFC is a cyclic quad.	Ext. $\angle$ = int. opp. $\angle$

12)

a)

Statement	Reason
$\hat{A}_1 + \hat{A}_2 = 60°$	$\angle$ at centre $= 2\angle$ at circumference
$\hat{A}_1 = \hat{A}_2$	Given
$\therefore \hat{A}_1 = \hat{A}_2 = 30°$	
$\hat{D}_3 = \hat{A}_2 = 30°$	Ext. $\angle$ of a cyclic quad.

b)

Statement	Reason
In ΔCDF:	
$\hat{F} = \hat{D}_3 = 30°$	$\angle$'s opp. = sides
In ΔAEF:	
$A\hat{E}F = 180° - \hat{A}_2 - \hat{F}$	$\angle$'s in a Δ
$A\hat{E}F = 180° - 30° - 30° = 120°$	
$\therefore A\hat{B}D = 180° - 120° = 60°$	Opp. $\angle$'s of a cyclic quad.

c)

Statement	Reason
$\hat{B}_1 \neq \hat{A}_1$	AC is not a diameter
$\therefore \hat{B}_2 \neq 30° \neq \hat{F}$	
$\therefore$ OBFD is not a cyclic quad.	line seg. does not subt. = $\angle$'s on the same side

13)

a)

Statement	Reason
$\hat{C}_2 = \hat{A}_1 = 30°$	Alt. $\angle$'s $(AB // OC)$
$\hat{O}_2 = 60°$	$\angle$ at centre $= 2\angle$ at circumference
$\therefore \hat{B}_2 = \hat{O}_2 = 60°$	Given
$\therefore \hat{C}_1 = 30°$	$\angle$'s in a Δ
$\therefore \hat{D}_1 = 90°$	$\angle$'s in a Δ

b)

Statement	Reason
$\hat{O}_1 = 2\hat{C}_1$	$\angle$ at centre $= 2\angle$ at circumference
$\therefore \hat{O}_1 = 60°$	
$\therefore \hat{O}_3 = 60°$	Adj. $\angle$'s on a str. line
$\hat{O}_3 = 2\hat{A}_2$	$\angle$ at centre $= 2\angle$ at circumference
$\therefore \hat{A}_2 = 30°$	
$\therefore \hat{A}_2 = \hat{C}_1$	
$\therefore BC // AO$	Alt. $\angle$'s are =
$\hat{D}_1 = \hat{D}_2 = \hat{D}_3 = \hat{D}_4 = 90°$	$\angle$'s in a Δ
$\therefore$ ABCO is a rhombus	Diagonals intersect at right angles and opp. sides are parallel

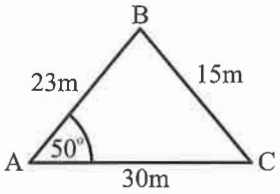
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Area Rule

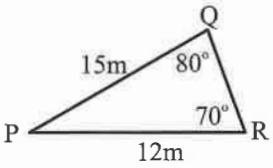
Exercise 1:

1) Determine the area of the triangles shown below:

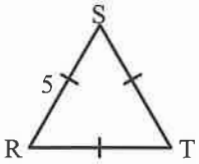
a)



b)

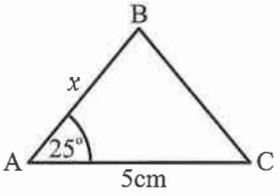


c)

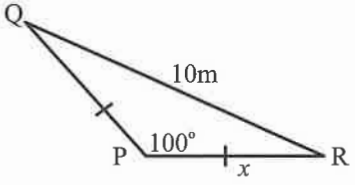


2) Given the area of each of the triangles below, determine the value of x.

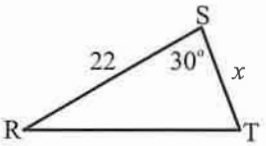
a) Area $= 8,45\,\text{cm}^2$



b) Area $= 20,96\,\text{m}^2$



c) Area $= 99\,\text{units}^2$



3) The area of $\triangle ABC$ is $75\,\text{m}^2$. $a = 10\,\text{m}$ and $c = 30\,\text{m}$. Determine the size/s of $\hat{B}$.

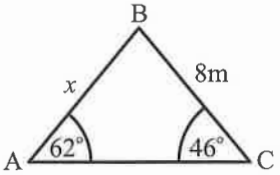
4) The area of $\triangle PQR$ is $46,67\,\text{cm}^2$. $r = 6\,\text{cm}$ and $q = 22\,\text{cm}$. Determine the size of $\hat{P}$ if $\hat{P}$ is obtuse.

Sine and Area Rule

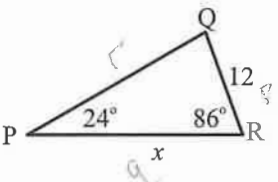
Exercise 2:

1) Determine the possible value/s of x in each of the triangles below:

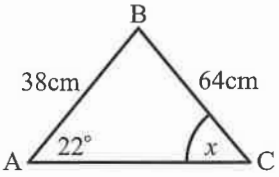
a)



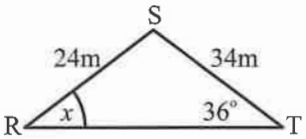
b)



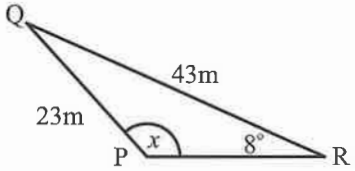
c)



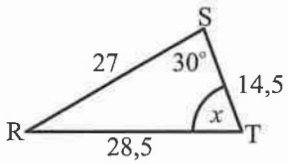
d)



e)



f)

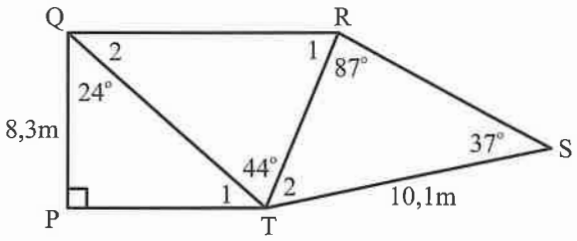


2) In the diagram alongside, determine:

a) The length of TR.

b) The length of QT.

c) The area of triangle QTR.

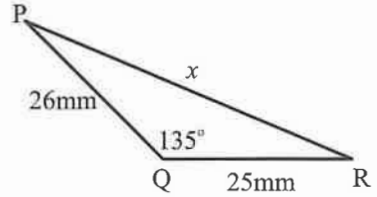


Cos Rule

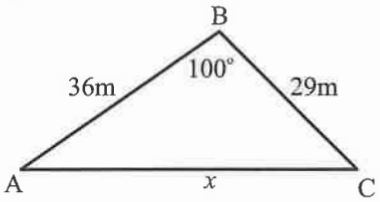
Exercise 3:

1) Determine the value of x in each of the triangles below:

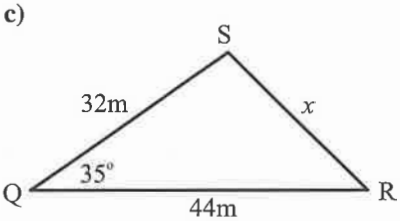
a)



b)



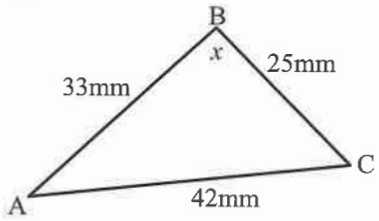
c)



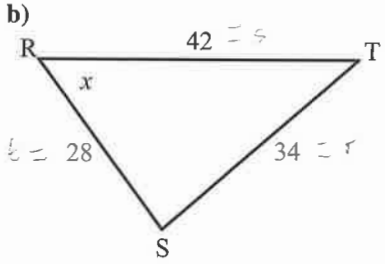
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2) Solve for x in each of the triangles below:

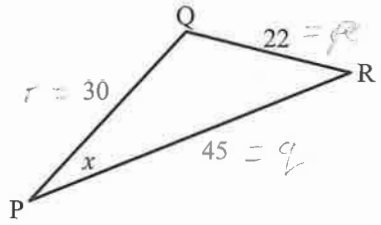
a)



b)

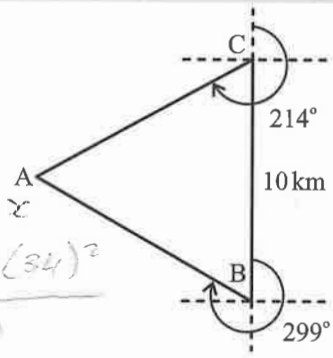


c)



Mixed Exercise:

1) Two ships are sailing towards a harbour at point A, from points C and B respectively. B is 10km due south of C. From B, the harbour at A is on a bearing of 299° and from C the harbour is on a bearing of 214°. How far is ship B from the harbour?



$(34)^2 = (28)^2 + (42)^2 - 2(28)(42)\cos x$

$\frac{2(28)(44)\cos x}{2(28)(44)} = \frac{(28)^2 + (42)^2 - (34)^2}{2(28)(44)}$

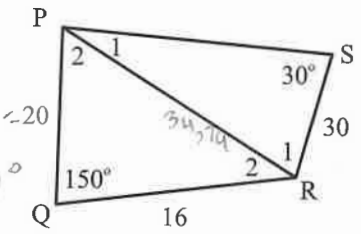
2) PQRS is a quadrilateral. PQ = 20m, QR = 16m and RS = 30m. If Q = 150° and S = 30°, determine:

a) The length of PR.

b) The size of $\hat{R}_1$.

$\cos x = \frac{20}{49}$ (-20)

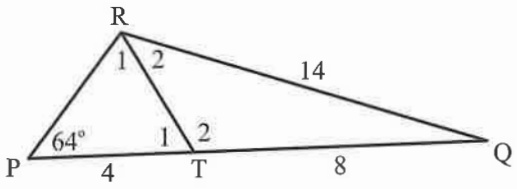
$x = 53,71^\circ$



3) In the diagram alongside, $\hat{P} = 64°$, QR = 14cm, PT = 4cm and QT = 8cm. Determine:

a) The size of $\hat{Q}$.

b) The length of RT.

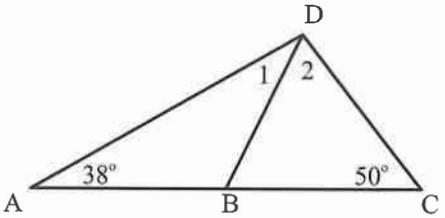


4) In the diagram alongside, ABC is a straight line. The distance between A and C is 114 units. $\hat{A}$ and $\hat{C}$ are 38° and 50° respectively. Determine:

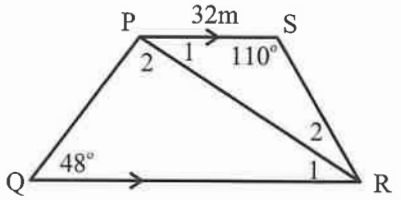
a) The distance from A to D.

b) The distance from B to D if B and C are 49 units apart.

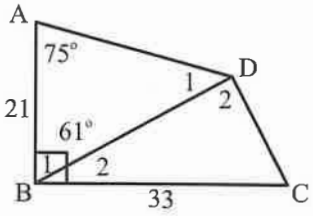
c) The area of triangle DAC.



5) In the diagram alongside, PS//QR and PS = 32m. $\hat{Q} = 48°$, $\hat{S} = 110°$ and $\hat{R}_1 = 35°$. Determine the length QR.

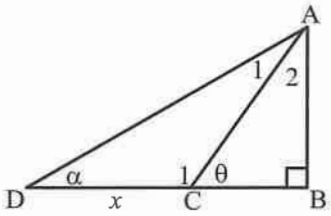


6) In the diagram alongside, $\hat{A} = 75°$, $\hat{B}_1 = 61°$, AB = 21m and BC = 33m. Determine the area of triangle BCD.



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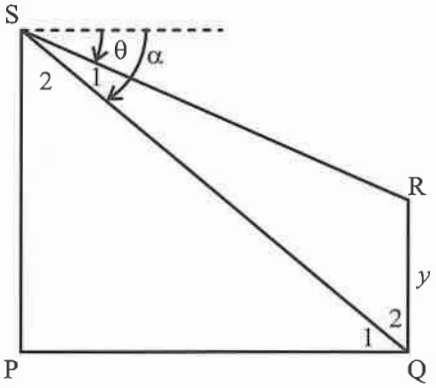
7) **In the diagram alongside, show that** $\text{BC} = \dfrac{x\sin\alpha\cos\theta}{\sin(\theta-\alpha)}$.



8) **In the diagram alongside, P and Q are two points in the same horizontal plane. R is a point y metres vertically above Q and S is a point vertically above P. From S, the angles of depression of points Q and R are α and θ respectively.**

a) Show that: $\text{PS} = \dfrac{y\cos\theta\sin\alpha}{\sin(\alpha-\theta)}$.

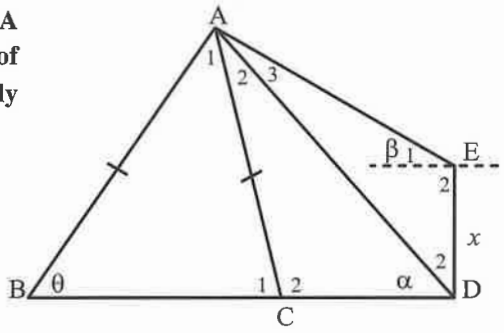
b) Hence, determine the length of PS if $y = 2000\,\text{m}$, $\theta = 23^\circ$ and $\alpha = 40^\circ$.



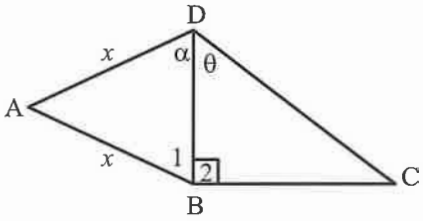
9) **In the diagram alongside, the angle of elevation of A from B is θ and AB = AC . From D, the angle of elevation of A is α and from E, a point vertically above D, the angle of elevation of A is β.**

a) Determine the values of: $\hat{C}_2, \hat{E}$ and $\hat{A}_3$.

b) Prove that $\text{AC} = \dfrac{x\cos\beta\sin\alpha}{\sin(\alpha-\beta)\sin\theta}$.



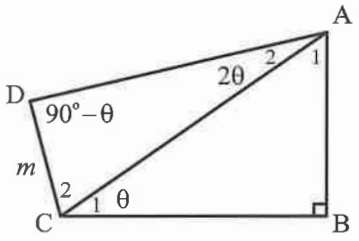
10) **In the diagram alongside, AB = AD = x. $\hat{\text{ADB}} = \alpha$ and $\hat{\text{BDC}} = \theta$. If BD⊥BC, show that:** $\text{BC} = x\tan\theta\sqrt{2(1+\cos 2\alpha)}$.



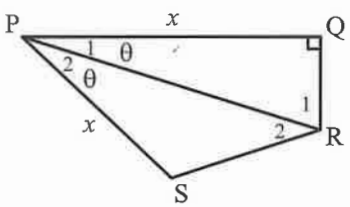
11) **A lake is divided into 2 sections. ABC is reserved for motorised water sports whilst ADC is used for swimming and non-motorised water sports, as shown in the diagram alongside. $\hat{C}_1 = \theta$, $\hat{A}_2 = 2\theta$ and $\hat{D} = 90^\circ - \theta$. If CD = m units, show that:**

a) $\text{AB} = \dfrac{m\cos\theta\sin\theta}{\sin 2\theta}$.

b) The area of the swimming section is: $\dfrac{m^2\cos^2\theta}{2\sin 2\theta}$.



12) **In the diagram alongside, $\hat{Q} = 90^\circ$, PQ = PS = x and $\hat{P}_1 = \hat{P}_2 = \theta$. Show that SR = $x\tan\theta$.**



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Chapter 9– Area, Sine and Cosine Rules (Memo)

Area Rule

Exercise 1:

1)

a) $A=\frac{1}{2}(23)(30)\sin 50°$

$=264,29\,m^2$

b) $\hat{P}=180°-80°-70°=30°$

$A=\frac{1}{2}(15)(12)\sin 30°$

$=45\,m^2$

c) $A=\frac{1}{2}(5)(5)\sin 60°$

$=10,83\,units^2$

2)

a) $8,45=\frac{1}{2}(5)(x)\sin 25°$

$\therefore x=8\,cm$

b) $20,96=\frac{1}{2}(x)(x)\sin 100°$

$\therefore x^2=21,52$

$\therefore x=6,52\,m$

c) $99=\frac{1}{2}(x)(22)\sin 30°$

$\therefore x=18\,units$

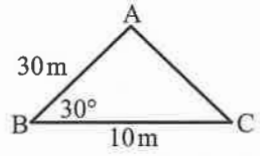
3)

$75=\frac{1}{2}(30)(10)\sin\hat{B}$

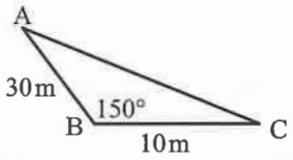
$\therefore \sin\hat{B}=\frac{1}{2}$

$\therefore \hat{B}=30°$ or $\hat{B}=180°-30°$

$=150°$



or



4)

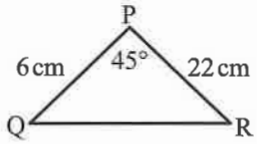
$46,67=\frac{1}{2}(6)(22)\sin\hat{P}$

$\therefore \sin\hat{P}=0,707...$

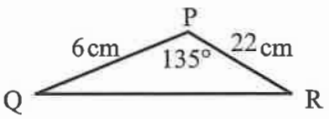
$\therefore \hat{P}=45°$ or $\hat{P}=180°-45°=135°$

But $\hat{P}$ is obtuse $\left(\text{greater than } 90°\right)$

$\therefore \hat{P}=135°$



or



Sine and Area Rule

Exercise 2:

1)

a) $\frac{x}{\sin 46°}=\frac{8}{\sin 62°}$

$\therefore x=\frac{8\sin 46°}{\sin 62°}$

$\therefore x=6,52\,m$

b) $\frac{x}{\sin\left(180°-24°-86°\right)}=\frac{12}{\sin 24°}$

$\therefore x=\frac{12\sin\left(70°\right)}{\sin 24°}$

$\therefore x=27,72\,units$

c) $\frac{\sin x}{38}=\frac{\sin 22°}{64}$

$\therefore \sin x=0,222...$

$\therefore x=12,85°$

d) $\frac{\sin x}{34}=\frac{\sin 36°}{24}$

$\therefore \sin x=0,832...$

$\therefore x=56,38°$ or $x=180°-56,38°$

$=123,62°$

e) $\frac{\sin x}{43}=\frac{\sin 8°}{23}$

$\therefore \sin x=0,260...$

$\therefore x=15,08°$ or $x=180°-15,08°$

$=164,92°$

f) $\frac{\sin x}{27}=\frac{\sin 30°}{28,5}$

$\therefore \sin x=0,473...$

$\therefore x=28,27°$

2)

a) $\frac{TR}{\sin 37°}=\frac{10,1}{\sin 87°}$

$\therefore TR=6,09\,m$

b) $\cos 24°=\frac{8,3}{QT}$

$\therefore QT=\frac{8,3}{\cos 24°}$

$=9,09\,m$

c) $A=\frac{1}{2}(9,09)(6,09)\sin 44°$

$=19,23\,m^2$

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Exercise 3:

1)

a) $x^2 = 26^2 + 25^2 - 2(26)(25)\cos 135°$
$= 2\,220,23...$
$\therefore x = 47,12\,\text{mm}$

b) $x^2 = 36^2 + 29^2 - 2(36)(29)\cos 100°$
$= 2\,499,57...$
$\therefore x = 50\,\text{m}$

c) $x^2 = 32^2 + 44^2 - 2(32)(44)\cos 35°$
$= 653,267...$
$\therefore x = 25,56\,\text{m}$

2)

a) $42^2 = 33^2 + 25^2 - 2(33)(25)\cos x$
$\therefore \cos x = -\frac{1}{33}$
$\therefore x = 91,74°$

b) $34^2 = 28^2 + 42^2 - 2(28)(42)\cos x$
$\therefore \cos x = \frac{29}{49}$
$\therefore x = 53,71°$

c) $22^2 = 30^2 + 45^2 - 2(30)(45)\cos x$
$\therefore \cos x = \frac{2441}{2700}$
$\therefore x = 25,30°$

Mixed Exercise:

1) $\hat{B} = 360° - 299° = 61°$
$\hat{C} = 214° - 180° = 34°$
$\hat{A} = 180° - 61° - 34° = 85°$
$\frac{AB}{\sin 34°} = \frac{10}{\sin 85°}$
$\therefore AB = \frac{10\sin 34°}{\sin 85°} = 5,6\,\text{km}$

2)

a) $PR^2 = 20^2 + 16^2 - 2(20)(16)\cos 150°$
$= 1210,256...$
$\therefore PR = 34,79\,\text{m}$

b) $\frac{\sin \hat{P}_1}{SR} = \frac{\sin 30°}{PR}$
$\therefore \sin \hat{P}_1 = \frac{30\sin 30°}{34,79}$
$\therefore \hat{P}_1 = 25,541...°$
$\therefore \hat{R}_1 = 180° - 25,541...° - 30° = 124,46°$

3)

a) $\frac{\sin \hat{R}}{12} = \frac{\sin 64°}{14}$
$\therefore \sin \hat{R} = 0,770...$
$\therefore \hat{R} = 50,389...°$
$\therefore \hat{Q} = 180° - 50,389...° - 64° = 65,61°$

b) $RT^2 = 8^2 + 14^2 - 2(8)(14)\cos(65,61°)$
$= 167,5$
$\therefore RT = 12,94\,\text{cm}$

4)

a) $\hat{D} = 180° - 38° - 50° = 92°$
$\frac{AD}{\sin 50°} = \frac{114}{\sin 92°}$
$\therefore AD = 87,38\,\text{units}$

b) $AB = 114 - 49 = 65\,\text{units}$
$BD^2 = 87,38^2 + 65^2 - 2(87,38)(65)\cos 38°$
$= 2908,935...$
$\therefore BD = 53,93\,\text{units}$

c) $A = \frac{1}{2}(87,38)(114)\sin 38°$
$= 3066,4\,\text{units}^2$

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5) $\hat{R}_1 + \hat{R}_2 = 70^\circ \rightarrow$ co-int $\angle$'s, PS//PR

$\therefore \hat{R}_2 = 70^\circ - 35^\circ = 35^\circ$

$$\frac{PR}{\sin 110^\circ} = \frac{32}{\sin 35^\circ}$$

$$\therefore PR = \frac{32\sin 110^\circ}{\sin 35^\circ} = 52,425...$$

$\hat{P}_2 = 180^\circ - 48^\circ - 35^\circ = 97^\circ$

$$\frac{QR}{\sin 97^\circ} = \frac{52,425...}{\sin 48^\circ}$$

$\therefore QR = 70,02\,m$

6) $$\frac{BD}{\sin 75^\circ} = \frac{21}{\sin\left(180^\circ - 75^\circ - 61^\circ\right)}$$

$$\therefore BD = \frac{21\sin 75^\circ}{\sin\left(44^\circ\right)} = 29,20...$$

$$A = \frac{1}{2}(29,20...)(33)\sin\left(90^\circ - 61^\circ\right)$$

$\therefore A = 233,59\,m^2$

7) $$\cos\theta = \frac{BC}{AC}$$

$BC = AC\cos\theta$

$\hat{C}_1 = 180^\circ - \theta$

$\therefore \hat{A}_1 = 180^\circ - \left(180^\circ - \theta\right) - \alpha$

$\therefore \hat{A}_1 = \theta - \alpha$

$$\frac{AC}{\sin\alpha} = \frac{x}{\sin(\theta - \alpha)}$$

$$\therefore AC = \frac{x\sin\alpha}{\sin(\theta - \alpha)}$$

$$\therefore BC = \frac{x\sin\alpha\cos\theta}{\sin(\theta - \alpha)}$$

8)

a) $\hat{Q}_1 = \alpha \rightarrow$ alt $\angle$'s

$$\sin\alpha = \frac{PS}{SQ}$$

$\therefore PS = SQ\sin\alpha$

$\hat{Q}_2 = 90^\circ - \alpha$

$\hat{S}_1 = \alpha - \theta$

$\therefore \hat{R} = 180^\circ - (90 - \alpha) - (\alpha - \theta)$

$= 90^\circ + \theta \rightarrow$ alt. $\angle$'s can also be used

$$\frac{QS}{\sin\left(90^\circ + \theta\right)} = \frac{y}{\sin(\alpha - \theta)}$$

$$\therefore QS = \frac{y\sin\left(90^\circ + \theta\right)}{\sin(\alpha - \theta)} = \frac{y\cos\theta}{\sin(\alpha - \theta)}$$

$$\therefore PS = \frac{y\cos\theta\sin\alpha}{\sin(\alpha - \theta)}$$

b) $$PS = \frac{y\cos\theta\sin\alpha}{\sin(\alpha - \theta)}$$

$$= \frac{2000\cos 23^\circ \sin 40^\circ}{\sin\left(40^\circ - 23^\circ\right)}$$

$= 4047,51\,m$

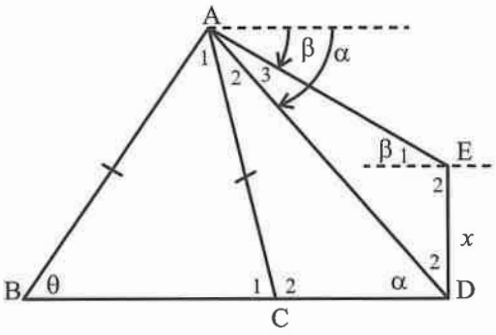
9)

a) $\hat{C}_1 = \theta$

$\therefore \hat{C}_2 = 180^\circ - \theta$

$\hat{E} = 90^\circ + \beta$

$\hat{A}_3 = \alpha - \beta$



b) $$\frac{AC}{\sin\alpha} = \frac{AD}{\sin\left(180^\circ - \theta\right)}$$

$$\therefore AC = \frac{AD\sin\alpha}{\sin\theta}$$

$$\frac{AD}{\sin\left(90 + \beta\right)} = \frac{ED}{\sin(\alpha - \beta)}$$

$$\therefore AD = \frac{x\cos\beta}{\sin(\alpha - \beta)}$$

$$\therefore AC = \frac{x\cos\beta\sin\alpha}{\sin(\alpha - \beta)\sin\theta}$$

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10) $\tan\theta = \dfrac{BC}{BD}$

$\therefore BC = BD\tan\theta$

$\hat{A} = 180° - 2\alpha \rightarrow \hat{B}_1 = \alpha$

$BD^2 = x^2 + x^2 - 2(x)(x)\cos(180° - 2\alpha)$

$= 2x^2 + 2x^2\cos 2\alpha$

$\therefore BD = \sqrt{2x^2(1+\cos 2\alpha)}$

$\therefore BD = x\sqrt{2(1+\cos 2\alpha)}$

$\therefore BC = x\tan\theta\sqrt{2(1+\cos 2\alpha)}$

11)

a) $\sin\theta = \dfrac{AB}{AC}$

$\therefore AB = AC\sin\theta$

$\dfrac{AC}{\sin(90° - \theta)} = \dfrac{m}{\sin 2\theta}$

$\therefore AC = \dfrac{m\sin(90° - \theta)}{\sin 2\theta} = \dfrac{m\cos\theta}{\sin 2\theta}$

$\therefore AB = \dfrac{m\cos\theta\sin\theta}{\sin 2\theta}$

b) $\hat{C}_2 = 180° - (90° - \theta) - 2\theta = 90 - \theta$

$A = \dfrac{1}{2}(DC)(AC)\sin\hat{C}_2$

$= \dfrac{1}{2}(m)\left(\dfrac{m\cos\theta}{\sin 2\theta}\right)\sin(90° - \theta)$

$= \dfrac{m^2\cos^2\theta}{2\sin 2\theta}$ units^2

12) $\cos\theta = \dfrac{x}{PR}$

$\therefore PR = \dfrac{x}{\cos\theta}$

$SR^2 = x^2 + \left(\dfrac{x}{\cos\theta}\right)^2 - 2(x)\left(\dfrac{x}{\cos\theta}\right)(\cos\theta)$

$= x^2 + \dfrac{x^2}{\cos^2\theta} - 2x^2$

$= \dfrac{x^2}{\cos^2\theta} - x^2$

$= x^2\left(\dfrac{1}{\cos^2\theta} - 1\right)$

$= x^2\left(\dfrac{1-\cos^2\theta}{\cos^2\theta}\right)$

$= x^2\left(\dfrac{\sin^2\theta}{\cos^2\theta}\right)$

$= x^2\tan^2\theta$

$\therefore SR = \sqrt{x^2\tan^2\theta} = x\tan\theta$

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Chapter 10– Financial Mathematics

Simple and Compound Interest and Decay

Exercise 1:

1) If an investment of R8000 earns 12% simple interest per annum, determine the value of the investment after 5 years.

2) If an investment of R75000 earns 8,5% compounded annually, determine the value of the investment after 9 years.

3) Determine the simple interest rate if an investment of R20000 is worth R30800 after 3 years.

4) How long would it take an investment to halve in value if it depreciates at a rate of 12,5% per annum on a straight line method?

5) If an investment of R95000 depreciates at 6,5% per annum using the reducing balance method, determine the value of the investment after 3 years.

6) Determine how long it would take an investment of R13250 to depreciate to R6625 if the rate of depreciation is 5% per annum and the straight line method is used.

7) Determine the rate of depreciation that would be used if a car depreciates from R120 000 to R63 327,80 in a period of 5 years on a reducing balance.

The Effects of Different Compounding Periods

Exercise 2:

1) If Jane wishes to have R120000 in 5 years time, how much must she invest if she is guaranteed a return of 8,5% p.a. compounded monthly on her investment?

2) Thabo purchases a car for R250000. How much will the car be worth in 5 years, if it depreciates at a rate 12,5% p.a. compounded quarterly?

3) If Tazneen invests R20000 into a bank account, what interest rate would be required in order for her money to double in 3 years if interest is compounded monthly?

4) Tebogo wishes to have R25000 in his bank account in 5 years. If interest is compounded monthly and he invests R13760, what interest rate will he require to fulfil his wish?

5) Thandi invests R65000 and sees that her investment doubled over a period of 5 years. Calculate the annual interest rate (correct to one decimal place) if interest is compounded quarterly.

6) If R5000 is invested for 10 years at 7,25% p.a. compounded quarterly, what simple interest rate would have to be used to obtain the same result (give your answer correct to two decimal places)?

7) If a car costing R85000 depreciates at rate of 12% p.a. for 3 years using the straight line method, what would the depreciation rate be using a reducing balance method?

Time Lines

Exercise 3: Draw a timeline for each question.

1) Naledi invests R90000 into a savings account that offers an interest rate of 6,5% p.a. compounded monthly. After 3 years the interest rate changes to 7% p.a. compounded quarterly. Calculate how much she will have in her account after 7 years.

2) Peter takes out a loan of R1000000 to start a new business. After 4 years his business grows and he borrows a further R200000 to build a storage warehouse. The bank offers him an interest rate of 11,2% p.a. compounded monthly and it is agreed that he will pay back R1000000 after 7 years and settle the remainder of the loan 3 years later. What will his final payment be to settle the loan?

3) For tax purposes Sipho depreciates his car using a reducing balance at a rate of 8,5% p.a. compounded quarterly. After 9 months he realises the rate he used was incorrect and changes it to 10% p.a. compounded monthly in order to account for his mistake. What will the book value of his car be at the end of the tax year if it was worth R80000 at the start of the tax year?.

$A = P(1-i)^n$

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4) Sally invested R10000 into a savings account. The interest rate for the first two years was 8% p.a. compounded monthly. Thereafter, it changed to 11,5% p.a. compounded semi-annually. After the first year she also deposited an additional R500 into the account and another R1000 after 5 years. How much did she have in her account after 6 years?

5) During school holidays Nadine worked as a waitress. She deposited all her earnings into unit trusts that earned a return of 10% per annum compounded monthly. After 6 years she makes an additional deposit of R5000. If Nadine has R20000 after 10 years, determine the amount that she originally invested.

6) Joseph picked 4 correct numbers in the National Lottery and deposited his winnings into a savings account offering an interest rate of 12% p.a. compounded monthly. After 2 years he deposited an additional R8000 into the account. Joseph then withdrew R1000 two years later for a family holiday. If Joseph had R30000 eight years after opening the account, how much did he win in the Lottery?

Nominal and Effective interest rates

Exercise 4:

1) Harry takes out a loan and the bank quotes him an interest rate of 20% p.a. compounded monthly. Calculate the effective interest rate that he will pay.

2) If a bank offers an interest rate of 10,2% p.a. compounded monthly determine the effective interest rate correct to one decimal place.

3) Calculate the effective interest rate if a savings account offers an interest rate of 12,5% p.a. compounded quarterly.

4) Jamie has the choice of investing his money into one of two savings accounts. One offers an interest rate of 7,79% p.a. compounded daily and the other offers a rate of 8,1% p.a. compounded annually. Which account should Jamie choose?

5) Convert an effective interest rate of 8% per annum to a nominal rate, compounded quarterly.

6) Convert a nominal interest rate of 25,5% p.a. compounded daily to a nominal rate that is compounded quarterly.

Mixed Exercise:

1) **Albert deposits R18000 for 7 years at 14,5% per annum compounded monthly for the first 3 years and 16% compounded quarterly for the next 4 years. How much will Albert receive in total after 7 years?**

2) **Justin invests R8500 into a savings account and is quoted a nominal interest rate of 12,29% p.a. compounded monthly.**
   a) Determine the effective annual interest rate.
   b) Use the effective rate (without rounding) to calculate the value of Justin's investment after 4 years.

3) **Steven makes a bad investment in the stock market. He purchases shares and finds that after 3 years his shares are worth one third of their original value.**
   a) Determine the rate of depreciation if depreciation is calculated on a straight line basis.
   b) Determine the rate of depreciation if depreciation is calculated on a reducing balance basis.

4) **Mike invests his money into a savings account for 7 years. For the first three years he receives an interest rate of 11% p.a. compounded monthly and 12,5% p.a. compounded semi-annually for the remainder of the investment. After 7 years he has a total of R90000 in his account.**
   a) Determine the effective rate that Mike received on his investment. Do not round off your answer.
   b) Using the effective rate, determine Mike's initial investment?

5) **After a rainstorm a puddle in the sun begins to evaporate. If 2% of the water evaporates every minute how long to the nearest minute will it take for half of the water in the puddle to evaporate?**
   (Hint: Use the formula for reducing balance depreciation and trial and error to solve for the exponent)

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6) **How long will it take for an investment to quadruple in value at a simple interest rate of 22% p.a. Give your answer correct to the nearest year and month.**

7) **On the day that Bob turned 8 his father invested a sum of money into a fixed deposit savings account to buy a car for him on his 18th birthday. The account promised an interest rate of 13% per annum compounded monthly. On Bob's 13th birthday his father deposited twice the initial amount into the account. If Bob had R50 000 on his 18th birthday, how much money was initially invested?**

8) **In order for an economy to be considered to be in a stage of hyperinflation the inflation rate of the country has to be greater than or equal to 50% per month. In February 2007 the inflation rate in Zimbabwe exceeded 50% per month. Determine what the effective yearly inflation rate was in Zimbabwe during February 2007.**

9) **One of the primary causes of ozone depletion is the release of Chlorofluorocarbons (CFC) into the atmosphere by human beings. Since the late 1970's it has been observed that the rate of decline in the volume of the earth's ozone has been steady at a rate of approximately 4% per decade from the time of initial observation. At this rate of decline how many years and months will it take for the ozone layer to deplete to a third of its original size?**

10) **Mr Deeb invested R30 000 into unit trusts for 10 years. During the first 3 years the stock market crashed and his investment depreciated at a rate of 12% p.a. using straight line depreciation. Thereafter, his investment increased at a rate of 9,8% p.a. compounded quarterly. How much will Mr Deeb's investment be worth after 10 years?**

11) **On Jimmy's 8th birthday his grandfather set up a fund for him. The rules of the fund stated that Jimmy will be allowed to withdraw R20 000 on his 18th birthday and a further R25 000 on his 21st birthday. If the fund offers an interest rate of 10% per annum compounded quarterly for the first 4 years and thereafter 8,5% per annum compounded monthly, how much did Jimmy's grandfather initially invest?**

12) **Lindiwe purchases a car for R160 000 which depreciates at 20% p.a.**
   a) Determine what the car will be worth after 5 years if it depreciates on a reducing balance.
   b) If a new car of the same model costs R280 000 after 5 years, determine the rate of inflation.
   c) Lindiwe starts saving to buy the new model in 5 years time. She deposits x into a savings account that offers an interest rate of 9,2% p.a. compounded semi-annually. Two years later she deposits an additional $4x$ into the account. Determine the value of x.

13) **R16 725 is deposited into a savings account at interest rate of 11,25% p.a. compounded monthly for the first 2 and a half years and 12% p.a. compounded quarterly thereafter. After 51 months R5 700 was withdrawn from the account. How much will be in the account after 7 years?**

14) **A bank account offers an interest rate of 7,9% p.a. compounded every 2 months.**
   a) Determine the effective interest rate correct to 9 decimal places.
   b) Use your effective interest to determine how much must be invested in order to receive R421 180,15 after 6 years.
   c) Verify that your answer in b, is correct using the nominal interest rate.

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Simple and Compound Interest and Decay

Exercise 1:

1) $A = P(1+in)$
$= 8000[1+0,12(5)]$
$= R12800$

2) $A = P(1+i)^n$
$= 75000(1+0,085)^9$
$= R156289,18$

3) $A = P(1+in)$
$\therefore 30800 = 20000(1+3i)$
$\therefore 1,54 = 1+3i$
$\therefore i = 0,18$
$= 18\%$

4) $A = P(1-in)$
$\therefore \frac{x}{2} = x(1-0,125n)$
$\therefore \frac{1}{2} = 1-0,125n$
$\therefore -\frac{1}{2} = -0,125n$
$\therefore n = 4$ years

5) $A = P(1-i)^n$
$= 95000(1-0,065)^3$
$= R77653,04$

6) $A = P(1-in)$
$\therefore 6625 = 13250(1-0,05n)$
$\therefore \frac{1}{2} = 1-0,05n$
$\therefore -\frac{1}{2} = -0,05n$
$\therefore n = 10$ years

7) $A = P(1-i)^n$
$\therefore 63327,8 = 120000(1-i)^5$
$\therefore 0,5277... = (1-i)^5$
$\therefore \sqrt[5]{0,5277} = 1-i$
$\therefore i = 0,12$
$= 12\%$

The Effects of Different Compounding Periods

Exercise 2:

1) $A = P(1+i)^n$
$\therefore 120000 = P\left(1+\frac{0,085}{12}\right)^{5\times12}$
$\therefore P = \frac{120000}{\left(1+\frac{0,085}{12}\right)^{5\times12}} = R78569,99$

2) $A = P(1-i)^n$
$= 250000\left(1-\frac{0,125}{4}\right)^{4\times5}$
$= R132487,32$

3) $A = P(1+i)^n$
$\therefore 40000 = 20000\left(1+\frac{i}{12}\right)^{3\times12}$
$\therefore 2 = \left(1+\frac{i}{12}\right)^{36}$
$\therefore \sqrt[36]{2} = 1+\frac{i}{12}$
$\therefore \frac{i}{12} = \sqrt[36]{2}-1$
$\therefore i = 0,2333$
$= 23,33\%$

4) $A = P(1+i)^n$
$\therefore 25000 = 13760\left(1+\frac{i}{12}\right)^{5\times12}$
$\therefore \frac{625}{344} = \left(1+\frac{i}{12}\right)^{60}$
$\therefore \sqrt[60]{\frac{625}{344}} = 1+\frac{i}{12}$
$\therefore \frac{i}{12} = \sqrt[60]{\frac{625}{344}}-1$
$\therefore i = 0,12$
$= 12\%$

5) $A = P(1+i)^n$
$\therefore 130000 = 65000\left(1+\frac{i}{4}\right)^{4\times5}$
$\therefore 2 = \left(1+\frac{i}{4}\right)^{20}$
$\therefore \sqrt[20]{2} = 1+\frac{i}{4}$
$\therefore \frac{i}{4} = \sqrt[20]{2}-1$
$\therefore i = 0,141$
$= 14,1\%$

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6) $A = P(1+i)^n$

$= 5000\left(1+\frac{0,0725}{4}\right)^{10\times4}$

$= 10256,85...$

$A = P(1+in)$

$\therefore 10256,85... = 5000(1+10i)$

$\therefore 2,051... = 1+10i$

$\therefore i = 0,1051$

$= 10,51\%$

7) $A = P(1-in)$

$= 85000[1-(0,12)(3)]$

$= R54400$

$A = P(1+i)^n$

$\therefore 54400 = 85000(1-i)^3$

$\therefore 0,64 = (1-i)^3$

$\therefore \sqrt[3]{0,64} = 1-i$

$\therefore i = 1-\sqrt[3]{0,64}$

$= 0,1382$

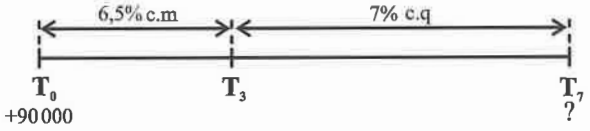
$= 13,82\%$

Timelines

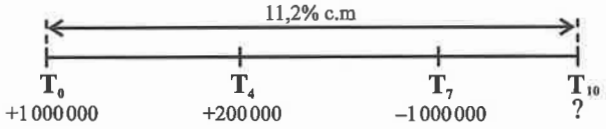
Exercise 3:

1) $A = 90000\left(1+\frac{0,065}{12}\right)^{3\times12}\left(1+\frac{0,07}{4}\right)^{4\times4}$

$= R144295,27$



2)

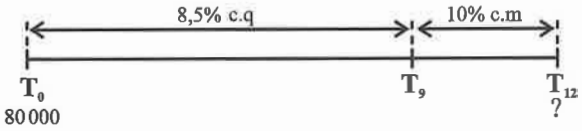


$A = 1000000\left(1+\frac{0,112}{12}\right)^{10\times12} + 200000\left(1+\frac{0,112}{12}\right)^{6\times12} - 1000000\left(1+\frac{0,112}{12}\right)^{3\times12}$

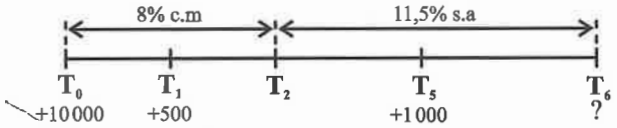
$= R2042226,65$

3) $A = 80000\left(1-\frac{0,085}{4}\right)^{\frac{9}{3}}\left(1-\frac{0,1}{12}\right)^{3}$

$= R73148,00$



4)



$A = 10000\left(1+\frac{0,08}{12}\right)^{2\times12}\left(1+\frac{0,115}{2}\right)^{4\times2} + 500\left(1+\frac{0,08}{12}\right)^{1\times12}\left(1+\frac{0,115}{2}\right)^{4\times2} + 1000\left(1+\frac{0,115}{2}\right)^{2\times1}$

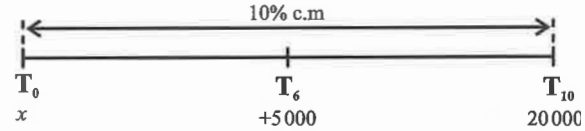
$= R20309,46$

5) $20000 = x\left(1+\frac{0,1}{12}\right)^{10\times12} + 5000\left(1+\frac{0,1}{12}\right)^{4\times12}$

$\therefore x\left(1+\frac{0,1}{12}\right)^{10\times12} = 20000 - 5000\left(1+\frac{0,1}{12}\right)^{4\times12}$

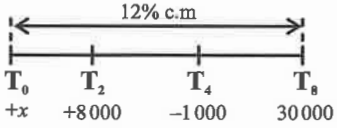
$\therefore x = \frac{20000 - 5000\left(1+\frac{0,1}{12}\right)^{4\times12}}{\left(1+\frac{0,1}{12}\right)^{10\times12}}$

$\therefore x = R4637,25$



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6) $30000 = x\left(1+\frac{0,12}{12}\right)^{8\times12} + 8000\left(1+\frac{0,12}{12}\right)^{6\times12} - 1000\left(1+\frac{0,12}{12}\right)^{4\times12}$



$$\therefore x\left(1+\frac{0,12}{12}\right)^{8\times12} = 30000 - 8000\left(1+\frac{0,12}{12}\right)^{6\times12} + 1000\left(1+\frac{0,12}{12}\right)^{4\times12}$$

$$\therefore x = \frac{30000 - 8000\left(1+\frac{0,12}{12}\right)^{6\times12} + 1000\left(1+\frac{0,12}{12}\right)^{4\times12}}{\left(1+\frac{0,12}{12}\right)^{8\times12}}$$

$$\therefore x = \text{R}5861,42$$

Nominal and Effective interest rates

Exercise 4:

1) $1+i_{eff} = \left(1+\frac{0,2}{12}\right)^{12}$

$\therefore i_{eff} = 0,2194$

$\therefore i_{eff} = 21,94\%$

2) $1+i_{eff} = \left(1+\frac{0,102}{12}\right)^{12}$

$\therefore i_{eff} = 0,1069$

$\therefore i_{eff} = 10,7\%$

3) $1+i_{eff} = \left(1+\frac{0,125}{4}\right)^{4}$

$\therefore i_{eff} = 0,1310$

$\therefore i_{eff} = 13,10\%$

4) $1+i_{eff} = \left(1+\frac{0,0779}{365}\right)^{365}$

$\therefore i_{eff} = 0,0810$

$\therefore i_{eff} = 8,10\%$

$\therefore$ They are the same rate.

5) $1+0,08 = \left(1+\frac{i_{nom}}{4}\right)^{4}$

$\therefore \sqrt[4]{1+0,08} = 1+\frac{i_{nom}}{4}$

$\therefore \frac{i_{nom}}{4} = \sqrt[4]{1+0,08} - 1$

$\therefore i_{nom} = 0,0777$

$\therefore i_{nom} = 7,77\%$

6) $1+i_{eff} = \left(1+\frac{0,255}{365}\right)^{365}$

$\therefore i_{eff} = 0,29...$

$1+i_{eff} = \left(1+\frac{i_{nom}}{4}\right)^{4}$

$\therefore \sqrt[4]{1+0,29..} = 1+\frac{i_{nom}}{4}$

$\therefore \frac{i_{nom}}{4} = \sqrt[4]{1+0,29..} - 1$

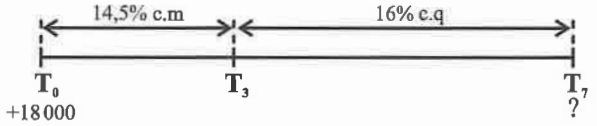
$\therefore i_{nom} = 0,2632$

$\therefore i_{nom} = 26,32\%$

Mixed Exercise:

1) $\text{A} = 18000\left(1+\frac{0,145}{12}\right)^{3\times12}\left(1+\frac{0,16}{4}\right)^{4\times4}$

$= \text{R}51950,74$



2)

a) $1+i_{eff} = \left(1+\frac{0,1229}{12}\right)^{12}$

$\therefore i_{eff} = 0,1300647296$

$\therefore i_{eff} = 13,01\%$

b) $\text{A} = \text{P}(1+i)^{n}$

$= 8500(1+0,1300647296)^{4}$

$= \text{R}13862,20$

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3)

a) $A = P(1-in)$

$\therefore \frac{x}{3} = x(1-3i)$

$\therefore \frac{1}{3} = 1-3i$

$\therefore 3i = \frac{2}{3}$

$\therefore i = 0,2222$

$\therefore i = 22,22\%$

b) $A = P(1-i)^n$

$\therefore \frac{x}{3} = x(1-i)^3$

$\therefore \frac{1}{3} = (1-i)^3$

$\therefore \sqrt[3]{\frac{1}{3}} = 1-i$

$\therefore i = 0,3066$

$\therefore i = 30,66\%$

4)

a) $\left(1+i_{eff}\right)^7 = \left(1+\frac{0,11}{12}\right)^{3\times12}\left(1+\frac{0,125}{2}\right)^{4\times2}$

$\therefore 1+i_{eff} = \sqrt[7]{\left(1+\frac{0,11}{12}\right)^{3\times12}\left(1+\frac{0,125}{2}\right)^{4\times2}}$

$\therefore i_{eff} = 0,1232355217$

$\therefore i_{eff} = 12,32355217\%$

b) $A = P(1+i)^n$

$\therefore 90000 = P(1+0,1232355217)^7$

$\therefore P = \frac{90000}{(1+0,1232355217)^7}$

$= R39897,59$

5) $A = P(1-i)^n$

$\therefore \frac{x}{2} = x(1-0,02)^n$

$\therefore \frac{1}{2} = (1-0,02)^n$

$\therefore n = 34$ minutes

6) $A = P(1+in)$

$\therefore 4x = x(1+0,22n)$

$\therefore 4 = 1+0,22n$

$\therefore n = 13,64$

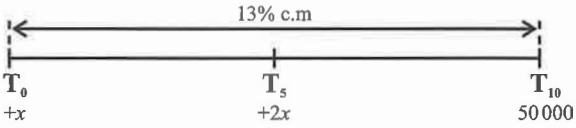
$\therefore n = 13$ years and 8 months

7) $50000 = x\left(1+\frac{0,13}{12}\right)^{10\times12} + 2x\left(1+\frac{0,13}{12}\right)^{5\times12}$

$\therefore 50000 = x\left[\left(1+\frac{0,13}{12}\right)^{10\times12} + 2\left(1+\frac{0,13}{12}\right)^{5\times12}\right]$

$\therefore x = \frac{50000}{\left(1+\frac{0,13}{12}\right)^{10\times12} + 2\left(1+\frac{0,13}{12}\right)^{5\times12}}$

$\therefore x = R6701,11$



8) $1+i_{eff} = \left(1+\frac{0,5}{12}\right)^{12}$

$\therefore i_{eff} = 0,6321$

$\therefore i_{eff} = 63,21\%$

9) $A = P(1-in)$

$\therefore \frac{x}{3} = x(1-0,04n)$

$\therefore \frac{1}{3} = 1-0,04n$

$\therefore n = 16,66..$ decades

$\therefore n = 166$ years and 8 months

10) $A = P(1-in)$

$= 30000[1-0,12(3)]$

$= R19200$

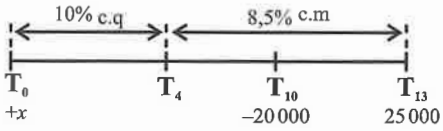
$A = P(1+i)^n$

$= 19200\left(1+\frac{0,098}{4}\right)^{7\times4}$

$= R37812,57$

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11) $25\,000 = x\left(1+\frac{0,1}{4}\right)^{4\times4}\left(1+\frac{0,085}{12}\right)^{9\times12} - 20\,000\left(1+\frac{0,085}{12}\right)^{3\times12}$



$$\therefore x\left(1+\frac{0,1}{4}\right)^{4\times4}\left(1+\frac{0,085}{12}\right)^{9\times12} = 25\,000 + 20\,000\left(1+\frac{0,085}{12}\right)^{3\times12}$$

$$\therefore x = \frac{25\,000 + 20\,000\left(1+\frac{0,085}{12}\right)^{3\times12}}{\left(1+\frac{0,1}{4}\right)^{4\times4}\left(1+\frac{0,085}{12}\right)^{9\times12}}$$

$$\therefore x = \text{R}15\,962,41$$

12)

a) $A = P(1-i)^n$

$= 160\,000(1-0,2)^5$

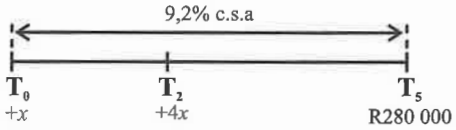
$= \text{R}52\,428,80$

b) $280\,000 = 160\,000(1+i)^n$

$\therefore 1,75 = (1+i)^5$

$\therefore i = 0,1184 = 11,84\%$

c) $280\,000 = x\left(1+\frac{0,092}{2}\right)^{5\times2} + 4x\left(1+\frac{0,092}{2}\right)^{3\times2}$

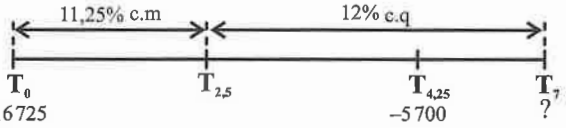


$$\therefore 280\,000 = x\left[\left(1+\frac{0,092}{2}\right)^{5\times2} + 4\left(1+\frac{0,092}{2}\right)^{3\times2}\right]$$

$$\therefore x = \frac{280\,000}{\left(1+\frac{0,092}{2}\right)^{5\times2} + 4\left(1+\frac{0,092}{2}\right)^{3\times2}}$$

$$\therefore x = \text{R}41134,64$$

13)



$$A = 16725\left(1+\frac{0,1125}{12}\right)^{2,5\times12}\left(1+\frac{0,12}{4}\right)^{4,5\times4} - 5700\left(1+\frac{0,12}{4}\right)^{2,75\times4}$$

$= \text{R}29\,781,33$

14)

a) $1+i_{eff} = \left(1+\frac{0,079}{6}\right)^6$

$\therefore i_{eff} = 0,08164652162$

$\therefore i_{eff} = 8,164652162\%$

b) $A = P(1+i)^n$

$\therefore 421180,15 = P(1+0,08164652162)^6$

$\therefore P = \text{R}263\,000$

c) $A = P(1+i)^n$

$\therefore 421180,15 = P\left(1+\frac{0,079}{6}\right)^{6\times6}$

$\therefore P = \text{R}263\,000$

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Grade 10 Revision and Dependent and Independent Events

Exercise 1:

1) **If a fair coin is tossed and a fair die is rolled, determine the probability of obtaining:**
   - **a)** A head and a 3.
   - **b)** A tail and a prime number.
   - **c)** A tail and not a 5.
   - **d)** A head and not a 6.

2) **Roulette is a popular gambling game played in casinos worldwide. The game consists of a wheel with 36 slots numbered 1 to 36. Each number is randomly placed on the wheel, with an equal amount of red and black numbers. The 37th slot on the wheel is a green zero. In the game the wheel is spun and a white ball is dropped on the spinning wheel. Determine the probability that the ball will land on:**
   - **a)** A black number
   - **b)** An even number
   - **c)** A number less than 13
   - **d)** The number 0

3) **What is the probability of rolling at least one 6 in four rolls of a fair die?**

4) **With reason, say whether or not the following events are complementary:**
   - **a)** Obtaining a head and obtaining a tail when a coin is tossed.
   - **b)** Drawing a 2 and drawing a red card from a deck of cards.

5) **With reason, say whether or not the following events are independent:**
   - **a)** Drawing 2 cards from a deck with replacement.
   - **b)** Drawing a card from a deck and rolling a die.

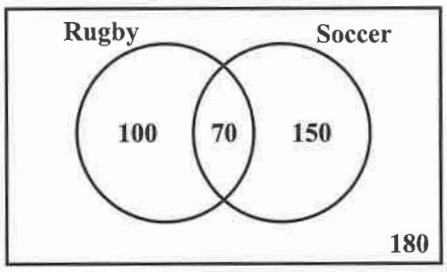
6) **The following probabilities are given: $P(A) = 0,45$; $P(B) = 0,3$ and $P(A \text{ or } B) = 0,615$:**
   - **a)** Determine if events A and B are mutually exclusive.
   - **b)** Hence, determine $P(A \text{ and } B)$.
   - **c)** Are events A and B independent?

7) **Thabo likes red sweets. In a jar there are 5 red sweets and 8 yellow sweets. If Thabo randomly selects a sweet and eats it and then selects another sweet and eats it, determine the probability that:**
   - **a)** He selects 2 sweets that he likes.
   - **b)** He likes the first sweet but not the second.
   - **c)** He likes the first or second sweet.
   - **d)** He likes the second sweet but not the first.

Venn Diagrams

Exercise 2:

1) **In a survey, a group of Grade 11 learners were asked if they play rugby or soccer. The Venn diagram below shows the results of the survey.**



Determine the probability that a learner chosen at random from the group:
- **a)** Plays rugby, but not soccer.
- **b)** Does not play either sport.
- **c)** Plays soccer.
- **d)** Plays both sports.

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2) **In a survey a group of 100 Grade 11 learners were asked if they listen to Rock or Classical music.**

- **48 listened to Rock**
- **52 listened to Classical**
- **10 listened to neither**

a) Draw a Venn diagram to represent the above information and determine the number of learners that listen to both rock and classical music.
b) Determine the probability of selecting a learner that listens to classical music.
c) Determine the probability of selecting a learner that listens to rock or classical music.

3) **At a school, 500 learners were asked how they travel to school. The options were by car, bus or taxi.**

- **280 travel by bus**
- **205 travel by car**
- **185 travel by taxi**
- **35 don't travel by car, bus or taxi**
- **100 travel by car and bus**
- **45 travel by car and taxi**
- **80 travel by taxi and bus**
- ***x* travel by car, bus and taxi**

a) Draw a Venn diagram to represent the above information and determine the number of learners that use all 3 modes of transport.
b) Determine the probability that a learner travels by car only.
c) Determine the probability that a learner travels by car and bus.
d) Determine the probability that a learner does not travel by car.

4) **At a school, 110 learners were asked if they take Geography, History or Science.**

- **43 take History**
- **46 take Geography**
- **67 take Science**
- **7 don't take any of the subjects**
- **12 take History and Science but not Geography**
- **5 take all 3 subjects**
- **20 take Geography and Science but not History**
- ***x* take Geography and History but not Science**

a) Draw a Venn diagram to represent the above information and determine the value of x.
b) Determine the probability that a learner doesn't take Geography or Science.
c) Determine the probability that a learner takes History or Geography but not Science.
d) Determine the probability that a learner taking Science and Geography does not take History.
e) Determine the probability that a learner taking History does not take Geography.

Tree Diagrams

Exercise 3:

1) **The table below represents the probability of a soccer team's outcome for two matches.**

First Match	Probability	Second Match	Probability
Win (W)	50%	**Win**	80%
		Draw	5%
		Lose	15%
Draw (D)	10%	**Win**	40%
		Draw	30%
		Lose	30%
Lose (L)	40%	**Win**	20%
		Draw	10%
		Lose	70%

a) Draw a Tree Diagram to represent the information in the above table.
b) Determine the probability that the team will lose their first match and win their second.
c) Determine the probability that the team will win 1 of their matches.
d) Determine the probability that the team will lose at least 1 match.
e) Determine the probability that the team will draw 1 match.

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2) **A bag contains 12 blue balls, 10 red balls and 18 green balls. 2 balls are then drawn without replacement. Draw a Tree Diagram to represent the above information and use it to determine the probability that:**

a) Both balls are green.

b) One blue and one red ball is chosen.

c) A green ball is chosen given that the 1st ball was blue.

d) A green ball is chosen second but not first.

3) **At a company dinner there are 2 choices for starters, 3 choices for mains and 2 choices for desserts, as shown in the table below.**

Starters	Mains	Desserts
• Salad (S) • Soup (R)	• Meat (M) • Chicken (C) • Fish (F)	• Ice-cream and chocolate sauce (I) • Cake (K)

If all choices are made randomly and have an equal chance of being chosen, determine the probability of:

a) Having salad and cake.

b) Having soup or chicken.

c) Not having cake or fish.

d) Not having cake and fish.

Contingency Tables

Exercise 4:

1) **In a survey, 800 Grade 11 and 12 learners were polled to see which brands of soft drink they prefer. The results are tabulated in the contingency table below. Fill in the missing values in the table and answer the questions that follow:**

	Grade 11	Grade 12	Total
Likes brand A	200		364
Likes brand B		196	
Total	440		

a) Determine the probability of selecting a grade 12.

b) Determine the probability of selecting a learner that likes brand A.

c) Determine the probability of selecting a learner in Grade 12 that likes brand A.

d) Is a learner's preference for brand independent of Grade? Show all relevant calculations.

2) **Orlando Towers is a power station in Soweto which was constructed in 1939 and was in operation until 1998. Since then it has been converted into a tourist attraction offering extreme sports. Among other sports, it boasts the world's first bungee jump between 2 cooling towers and the first suspended catch air device (SCAD) freefall inside a cooling tower. A group of people visiting the towers was surveyed on whether they have done a SCAD freefall or bungee jump. The contingency table below shows the probabilities of these events. Fill in the missing values in the table and answer the questions that follow:**

	Performed a SCAD freefall	Not performed a SCAD freefall	Total
Performed a bungee jump			0,6
Not performed a bungee jump	0,26		
Total		0,35	

a) Determine the probability of selecting a person who has performed a SCAD freefall and bungee jump.

b) Determine the probability of selecting a person who has performed a SCAD freefall.

c) Determine the probability of selecting a person who has performed a bungee jump.

d) Are performing a SCAD freefall and bungee jump independent events? Show all relevant calculations.

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Mixed Exercise:

1) **State whether the following events are mutually exclusive, complementary or neither:**
   a) Rolling an even number on a die and rolling a prime number.
   b) Rolling a number less than 6 and rolling a 6 on a die.
   c) Rolling an odd number on a die and rolling a 4.
   d) Rolling an even number and rolling an odd number.

2) **Two cards are drawn from a fair pack of playing cards without replacement. Draw a Tree Diagram to show the probabilities of drawing a specific suit in the first draw and a red or black card in the second draw. Giving answers as a percentage, use your diagram to determine the probability of:**
   a) Drawing 2 black cards.
   b) Drawing a spade and a red card.
   c) Drawing a spade and a black card.
   d) Not drawing a spade or a red card.

3) **Given the following probabilities say whether the following events are mutually exclusive, independent or neither.**
   a) $P(A) = 0,2$; $P(B) = 0,35$; $P(A \text{ or } B) = 0,55$
   b) $P(A) = 0,235$; $P(B) = 0,565$; $P(A \text{ or } B) = 0,7$
   c) $P(A) = 0,2$; $P(B) = 0,5$; $P(A \text{ or } B) = 0,6$
   d) $P(A) = 0,35$; $P(B) = 0,55$; $P(A \text{ or } B) = 0,7075$

4) **In a recent survey, 125 people were asked if they went on holiday in the summer, winter or spring in the past year. The results of the survey are shown below. Draw a Venn diagram and use it to answer the questions that follow:**
   - **39 went on holiday in summer**
   - **43 went on holiday in spring**
   - **65 went on holiday in winter**
   - **105 went on holiday in one of the seasons**
   - **5 went on holiday in all 3 seasons**
   - **17 went on summer and winter holidays**
   - **23 went on holiday in winter and spring**
   - ***x* went on holiday in summer and spring but not in winter**

   a) Determine the probability that a person selected went on holiday in one of the seasons?
   b) Determine the probability that a person selected went on holiday in summer **or** winter but not spring.
   c) Determine the probability that a person selected went on holiday in winter **and** summer.
   d) Determine the probability that a person selected went on holiday in spring **and** winter but not summer.

5) **A study was performed on 800 patients over a 2 year period to determine the number of patients that have side effects from 3 different drugs. All patients were given Drugs A, B and C throughout the study at separate times to ensure that there was no interaction between the drugs. Draw a Venn diagram to represent the information below and use it to answer the questions which follow.**
   - **400 reported side effects from drug A**
   - **180 reported side effects from drugs B and C**
   - **350 reported side effects from drug B**
   - **680 reported side effects from at least 1 of the drugs**
   - **400 reported side effects from drug C**
   - **70 reported side effects from all 3 drugs**
   - **210 reported side effects from drugs A and C**
   - ***x* reported side effects from drugs A and B but not C**

   a) Determine the probability that a person had no side effects.
   b) Determine the probability that a person in the trial had side effects.
   c) Determine the probability that a person had side effects from drug A.
   d) Determine the probability that a person had side effects from Drug B or C.

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6) **In order to determine if side effects on drugs A are related to gender, another study was conducted. The results are tabulated in the contingency table below. Complete the table below and use it to answer the questions below:**

	Male	Female	Total
Had side effects	1500		3000
No side effects		1000	
Total			5000

a) Determine the probability that a person chosen at random will not have side effects and be male.

b) Determine the probability that a person that had side effects will be female.

c) Show that being female and not having side effects are independent events.

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Chapter 11– Probability (Memo)

Grade 10 Revision and Dependent and Independent Events

Exercise 1:

1)

a) $P(\text{H and 3}) = \frac{1}{2} \times \frac{1}{6}$
$= \frac{1}{12}$
$= 0,0833$

b) $P(\text{T and P}) = \frac{1}{2} \times \frac{3}{6}$
$= \frac{1}{4}$
$= 0,25$

c) $P(\text{T and 5'}) = \frac{1}{2} \times \frac{5}{6}$
$= \frac{5}{12}$
$= 0,4167$

d) $P(\text{H and 6'}) = \frac{1}{2} \times \frac{5}{6}$
$= \frac{5}{12}$
$= 0,4167$

2)

a) $P(\text{B}) = \frac{18}{37}$
$= 0,4865$

b) $P(\text{E}) = \frac{18}{37}$
$= 0,4865$

c) $P(\text{Less than 13}) = \frac{13}{37}$
$= 0,3514$

d) $P(0) = \frac{1}{37}$
$= 0,027$

3) $P(\text{at least one 6}) = 1 - P(\text{no 6's})$
$= 1 - \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right)$
$= 1 - \left(\frac{5}{6}\right)^4$
$= \frac{671}{1296}$
$= 0,5177$

4)

a) $P(\text{H}) + P(\text{T}) = 1$
Events are mutually exclusive.
$\therefore$ Events are complementary.

b) $P(\text{2 and R}) = \frac{2}{52} \neq 0$
$\therefore$ Events are not mutually exclusive.
$\therefore$ Events are not complementary.

5)

a) 1st event doesn't affect the outcome of the 2nd. Therefore, events are independent.

b) 1st event doesn't affect the outcome of the 2nd. Therefore, events are independent.

6)

a) $P(\text{A}) + P(\text{B}) = 0,45 + 0,3$
$= 0,75$
$P(\text{A or B}) = 0,615$
$\therefore P(\text{A or B}) \neq P(\text{A}) + P(\text{B})$
$\therefore$ Events are not mutually exclusive.

b) $\therefore P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$
$\therefore 0,615 = 0,45 + 0,3 - P(\text{A and B})$
$\therefore P(\text{A and B}) = 0,135$

c) $P(\text{A}) \times P(\text{B}) = 0,45 \times 0,3$
$= 0,135$
$P(\text{A and B}) = P(\text{A}) \times P(\text{B})$
$\therefore$ Events are independent.

7)

a) $P(\text{R and R}) = \frac{5}{13} \times \frac{4}{12}$
$= \frac{5}{39}$
$= 0,1282$

b) $P(\text{R and Y}) = \frac{5}{13} \times \frac{8}{12}$
$= \frac{10}{39}$
$= 0,2564$

c) $P[(\text{R and Y}) \text{ or } (\text{Y and R})] = \left(\frac{5}{13} \times \frac{8}{12}\right) + \left(\frac{8}{13} \times \frac{5}{12}\right)$
$= \frac{20}{39}$
$= 0,5128$

d) $P(\text{Y and R}) = \frac{8}{13} \times \frac{5}{12}$
$= \frac{10}{39}$
$= 0,2564$

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Exercise 2:

1)

a) $P(\text{R and S}') = \frac{100}{500} = \frac{1}{5} = 0,2$

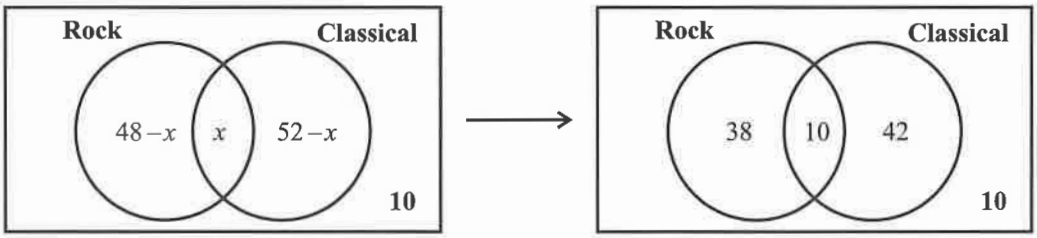
b) $P(\text{No Sport}) = \frac{180}{500} = \frac{9}{25} = 0,36$

c) $P(\text{S}) = \frac{150+70}{500} = \frac{11}{25} = 0,44$

d) $P(\text{S and R}) = \frac{70}{500} = \frac{7}{50} = 0,14$

2)

a)



$48 - x + x + 52 - x + 10 = 100$

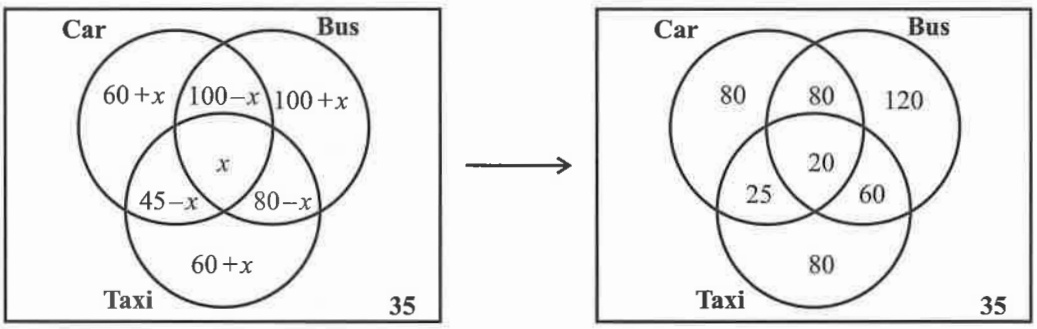
$\therefore x = 10$

b) $P(\text{C}) = {}^{52}/_{100} = 0,52$

c) $P(\text{R or C}) = {}^{90}/_{100} = 0,9$

3)

a)



Car and Bus: $100 - x$

Car and Taxi: $45 - x$

Taxi and Bus: $80 - x$

Car: $205 - (100 - x + x + 45 - x) = 60 + x$

Bus: $280 - (100 - x + x + 80 - x) = 100 + x$

Taxi: $185 - (45 - x + x + 80 - x) = 60 + x$

$\therefore 500 = 60 + x + 100 + x + 60 + x + 100 - x + 45 - x + 80 - x + x + 35$

$\therefore x = 20$

b) $P(\text{C only}) = \frac{80}{500}$

$= 0,16$

c) $P(\text{C and B}) = \frac{80+20}{500}$

$= \frac{1}{5}$

$= 0,2$

d) $P(\text{C}') = \frac{120+60+80+35}{500}$

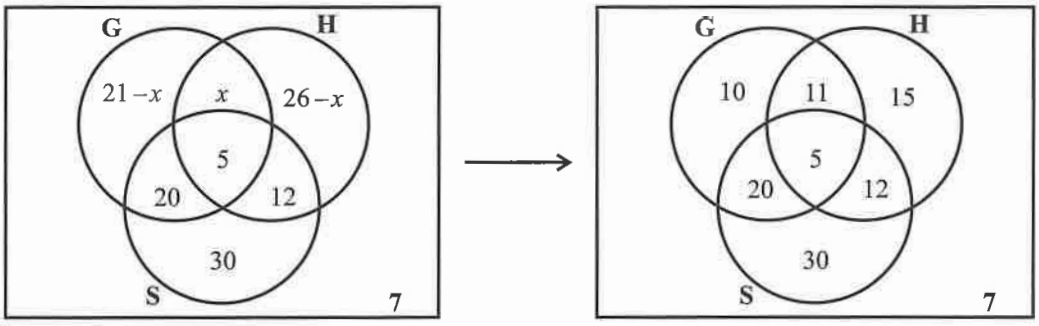
$= \frac{59}{100}$

$= 0,59$

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4)

a)



S: $67-(20+5+12)=30$

G: $46-(20+5+x)=21-x$

H: $43-(x+5+12)=26-x$

$\therefore 110=21-x+x+26-x+20+5+12+30+7$

$\therefore x=11$

b) $P(\text{G or S})'=\dfrac{15+7}{110}$

$=0,2$

c) $P(\text{H or G and S'})=\dfrac{10+11+15}{110}$

$=0,32$

d) $P(\text{S and G and H'})=\dfrac{20}{110}$

$=0,18$

e) $P(\text{H and G'})=\dfrac{15+12}{110}$

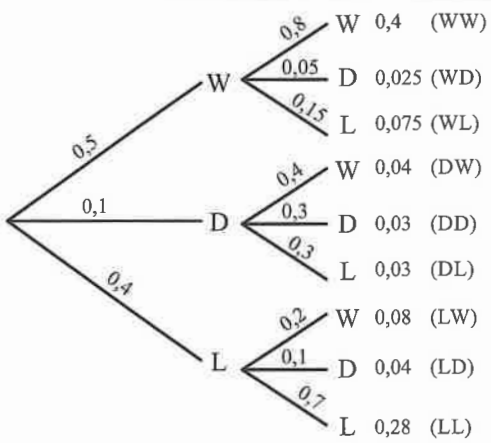
$=0,25$

Tree Diagrams

Exercise 3:

1)

a)



b) $P(\text{Loss and Win})=0,08$

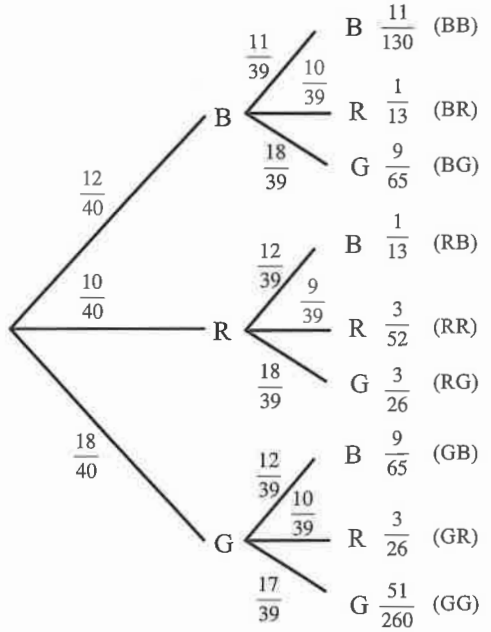
c) $P(\text{one win})=0,025+0,075+0,04+0,08=0,22$

d) $P(\text{at least 1 Loss})=0,075+0,03+0,08+0,04+0,28$

$=0,505$

e) $P(\text{1 Draw})=0,025+0,04+0,03+0,04=0,135$

2)



a) $P(\text{G and G})=\dfrac{51}{260}=0,1962$

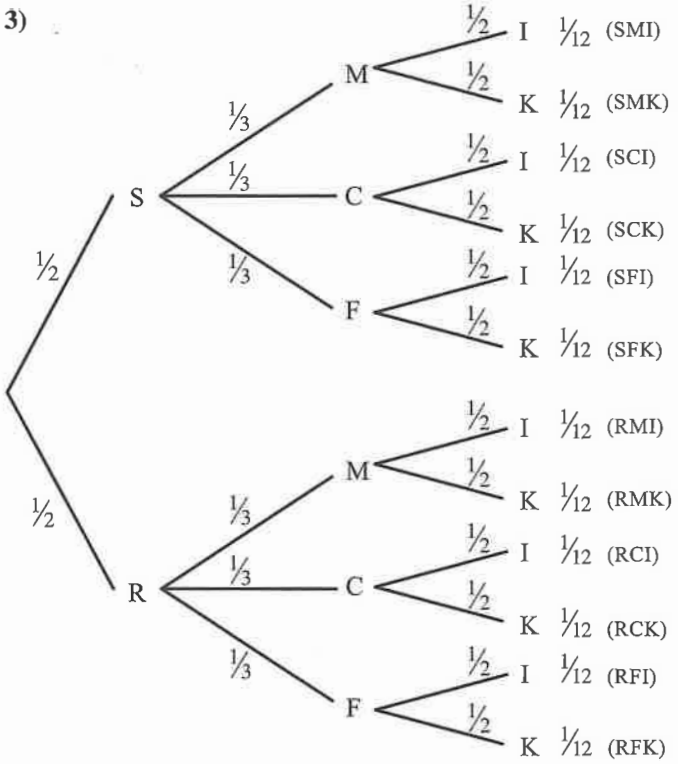
b) $P(\text{B and R})=\dfrac{1}{13}+\dfrac{1}{13}=0,1538$

c) $P(\text{Blue and then Green})=\dfrac{9}{65}=0,1385$

d) $P(\text{G' and then G})=\dfrac{9}{65}+\dfrac{3}{26}=0,2538$

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3)



a) $P(\text{S and K}) = 3\left(\frac{1}{12}\right) = 0,25$

b) $P(\text{R or C}) = 8\left(\frac{1}{12}\right) = 0,6667$

c) $P(\text{K}' \text{ or } \text{F}') = 4\left(\frac{1}{12}\right) = 0,3333$

d) $P(\text{K and F})' = 10\left(\frac{1}{12}\right) = 0,8333$

Contingency Tables

Exercise 4:

1)

	Grade 11	Grade 12	Total
Likes brand A	200	**164**	364
Likes brand B	**240**	196	**436**
Total	440	**360**	**800**

a) $P(\text{GR12}) = \frac{360}{800} = 0,45$

b) $P(\text{A}) = \frac{364}{800} = 0,455$

c) $P(\text{GR12 and A}) = \frac{164}{800} = 0,205$

d) $P(\text{GR12}) \times P(\text{A}) = \frac{360}{800} \times \frac{364}{800} = 0,20475... \approx 0,205$

$\therefore P(\text{GR12 and A}) = P(\text{GR12}) \times P(\text{A})$

$\therefore$ Events are independent.

2)

	Performed a SCAD freefall	Not Performed a SCAD freefall	Total
Performed a bungee jump	**0,39**	**0,21**	0,6
Not performed a bungee jump	0,26	**0,14**	**0,4**
Total	**0,65**	0,35	**1**

a) $P(\text{S and B}) = 0,39$

b) $P(\text{S}) = 0,65$

c) $P(\text{B}) = 0,6$

d) $P(\text{S}) \times P(\text{B}) = 0,65 \times 0,6 = 0,39$

$\therefore P(\text{S and B}) = P(\text{S}) \times P(\text{B})$

$\therefore$ Events are independent.

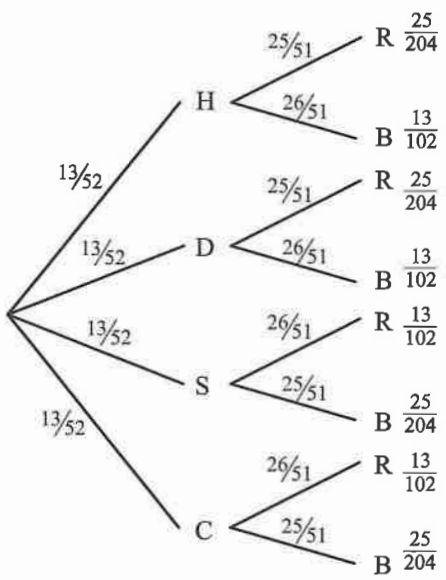
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Mixed Exercise:

1)

a) Neither **b)** Complementary **c)** Mutually Exclusive **d)** Complementary

2)



a) $P(2\text{ Black}) = 2\left(\frac{25}{204}\right) = 24,51\%$

b) $P(\text{Spade and Red}) = \frac{13}{102} = 12,75\%$

c) $P(\text{Spade and Black}) = \frac{25}{204} = 12,25\%$

d) $P(\text{S' or R'}) = \frac{25}{204} = 12,25\%$

3)

a) $P(A)+P(B) = 0,2+0,35 = 0,55$

$P(A \text{ or } B) = 0,55$

$\therefore P(A \text{ or } B) = P(A)+P(B)$

$\therefore$ A and B are mutually exclusive.

b) $P(A)+P(B) = 0,235+0,565 = 0,8$

$P(A \text{ or } B) = 0,7$

$\therefore P(A \text{ or } B) \neq P(A)+P(B)$

$\therefore$ A and B are not mutually exclusive.

$P(A \text{ or } B) = P(A)+P(B)-P(A \text{ and } B)$

$\therefore 0,7 = 0,235+0,565-P(A \text{ and } B)$

$\therefore P(A \text{ and } B) = 0,1$

$P(A)\times P(B) = 0,235\times 0,565 = 0,132$

$\therefore$ A and B are not mutually exclusive or independent.

c) $P(A)+P(B) = 0,2+0,5 = 0,7$

$P(A \text{ or } B) = 0,6$

$\therefore P(A \text{ or } B) \neq P(A)+P(B)$

$\therefore$ A and B are not mutually exclusive.

$P(A \text{ or } B) = P(A)+P(B)-P(A \text{ and } B)$

$\therefore 0,6 = 0,2+0,5-P(A \text{ and } B)$

$\therefore P(A \text{ and } B) = 0,1$

$P(A)\times P(B) = 0,2\times 0,5 = 0,1$

$\therefore$ A and B are independent.

d) $P(A)+P(B) = 0,35+0,55 = 0,9$

$P(A \text{ or } B) = 0,7075$

$\therefore P(A \text{ or } B) \neq P(A)+P(B)$

$\therefore$ A and B are not mutually exclusive.

$P(A \text{ or } B) = P(A)+P(B)-P(A \text{ and } B)$

$\therefore 0,7075 = 0,35+0,55-P(A \text{ and } B)$

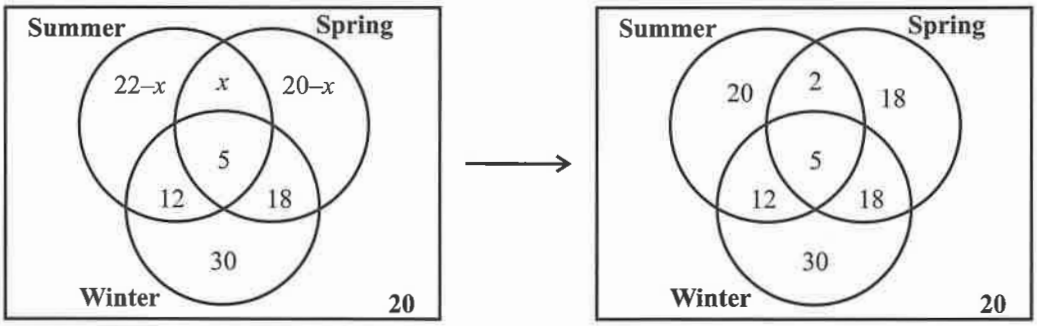
$\therefore P(A \text{ and } B) = 0,1925$

$P(A)\times P(B) = 0,35\times 0,55 = 0,1925$

$\therefore$ A and B are independent.

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4)



$\therefore 22-x+x+20-x+5+12+18+30+20=125$

$\therefore x=2$

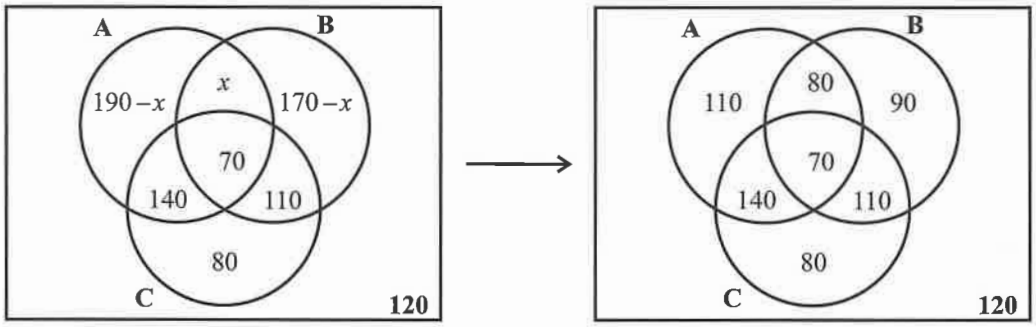
a) $P(\text{at least 1})=1-P(\text{none})=1-\frac{20}{125}=0,84$

b) $P(\text{Su or Wi and Sp'})=\frac{20+12+30}{125}=0,496$

c) $P(\text{Wi and Su})=\frac{5+12}{125}=0,136$

d) $P(\text{Sp and W but Su'})=\frac{18}{125}=0,144$

5)



B and C: $180-70=110$ A and C: $210-70=140$ A and B: x

C: $400-140-70-110=80$ A: $400-140-70-x=190-x$ B: $350-70-110-x=170-x$

$\therefore 800=190-x+x+170-x+140+70+110+80+120$

$\therefore x=80$

a) $P(\text{A or B or C})'=\frac{120}{800}=0,15$

b) $P(\text{A or B or C})=1-P(\text{A or B or C})'=1-0,15=0,85$

c) $P(\text{A})=\frac{110+80+70+140}{800}=0,5$

d) $P(\text{B or C})=\frac{80+90+70+110+140+80}{800}=0,7125$

6)

	Male	Female	Total
Had side effects	1500	**1500**	3000
No side effects	**1000**	1000	**2000**
Total	**2500**	**2500**	5000

a) $P(\text{NS and M})=\frac{1000}{5000}=\frac{1}{5}=0,2$

b) $P(\text{S}|\text{F})=\frac{1500}{3000}=\frac{1}{2}=0,5$

c) $P(\text{F and NS})=\frac{1000}{5000}=\frac{1}{5}=0,2$

$P(\text{F})=\frac{2500}{5000}=\frac{1}{2}=0,5$

$P(\text{NS})=\frac{2000}{5000}=\frac{2}{5}=0,4$

$\therefore P(\text{F})\times P(\text{NS})=0,5\times 0,4=0.2$

$\therefore P(\text{F and NS})=P(\text{F})\times P(\text{NS})$

$\therefore$ Side effects are independent of gender.

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Histogram and Frequency Polygons

Exercise 1:

1) A survey was conducted on the amount of time that a group of Grade 11 learners spend on their homework. The results of the survey are show below.

Time (t)	$10 \leq t < 20$	$20 \leq t < 30$	$30 \leq t < 40$	$40 \leq t < 50$	$50 \leq t < 60$	$60 \leq t < 70$
Frequency	4	6	10	4	4	2

a) Draw a histogram for the above data.

b) Draw a frequency polygon on the histogram.

c) Use the histogram to determine the modal class.

2) The weights (in kg) of 39 newborn babies in a hospital are recorded in the frequency table below:

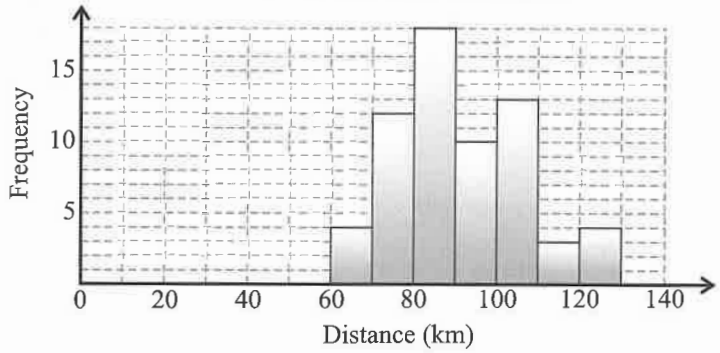
Weight	Frequency	Cumulative frequency
$0 < w \leq 1$	0	0
$1 < w \leq 2$	3	3
$2 < w \leq 3$	28	31
$3 < w \leq 4$	8	39
$4 < w \leq 5$	0	39

a) Draw a histogram for the above data.

b) Draw a frequency polygon on the histogram.

c) Determine the value of the median.

3) The histogram below represents the distances that a salesman travels in 1 month on each trip. Draw a frequency table of this data and use it to draw a frequency polygon.



Ogives

Exercise 2:

1) Use the following data sets to construct an ogive. In addition, label the median, upper and lower quartiles on the ogive with Q_1, Q_2 and Q_3.

a)

Interval	Frequency
$40 \leq x < 50$	15
$50 \leq x < 60$	27
$60 \leq x < 70$	18
$70 \leq x < 80$	10

b)

Interval	Frequency
$20 \leq x < 40$	5
$40 \leq x < 60$	10
$60 \leq x < 80$	20
$80 \leq x < 100$	5

2) A traffic officer records the following speeds of cars on a highway while manning a speed trap:

Speed (s)	Frequency	Cumulative frequency
$60 < s \leq 70$	5	
$70 < s \leq 80$	10	
$80 < s \leq 90$	15	
$90 < s \leq 100$	30	
$100 < s \leq 110$	40	
$110 < s \leq 120$	32	
$120 < s \leq 130$	11	
$130 < s \leq 140$	7	
$140 < s \leq 150$	5	
$150 < s \leq 160$	2	

a) Complete the table and draw an ogive for the above data.

b) Using the ogive, determine the inter-quartile range.

c) Construct a new table and use it to determine the estimated mean speed.

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Standard Deviation and Variance

Exercise 3:

1) Given the following data set:

15	23	45	28	32	35	52	25	70

a) Determine the variance and standard deviation for the above data.

b) If 15 were added to each value in the data set, what would be the new mean?

c) If 15 were added to each value in the data set, what would be the new standard deviation?

2) The table below shows the amount of time that 8 Grade 11 learners took to complete a race.

Time	$(x-\bar{x})$	$(x-\bar{x})^2$
12		
19		
40		
33		
25		
24		
15		
38		

a) Complete the table and determine the standard deviation.

b) How many learners lie within one standard deviation of the mean?

3) The table below shows the monthly salaries of employees at a manufacturing company.

Salary	R3 500	R5 000	R7 500	R12 200	R22 000	R40 000
Frequency	4	6	30	15	4	1

a) Determine the standard deviation

b) Determine the modal salary.

c) Determine the median salary.

d) What percentage of employees earn amounts within 1 standard deviation of the mean?

4) If the standard deviation of a data set is: $\sigma = \frac{\sqrt{385}}{5}$

a) Determine the variance.

b) Determine the number of elements in the data set.

Symmetrical and Skewed Data

Exercise 4:

1) The consumer price index is a measure of the inflation rate in South Africa. The table below shows the CPI inflation rate according to STATS SA from 2001 through to 2009.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009
CPI	5,7	9,2	5,8	1,4	3,4	4,7	7,1	11,5	7,1

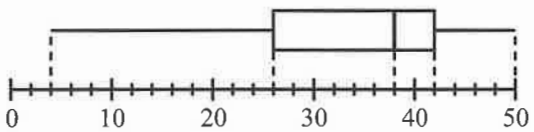
a) Determine the five number summary for the above data.

b) Draw a box and whisker plot for the above data.

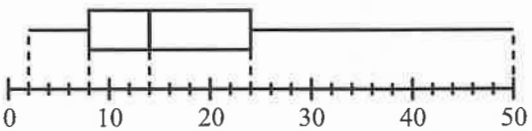
c) Determine if there are any outliers in the above data.

2) Given the box and whisker diagrams below, comment on the symmetry of the data. In addition, determine if there are any outliers in the data set.

a)

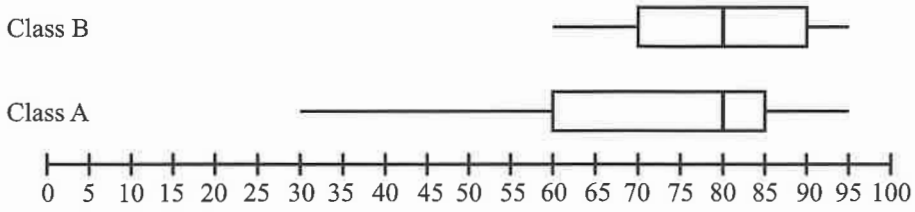


b)



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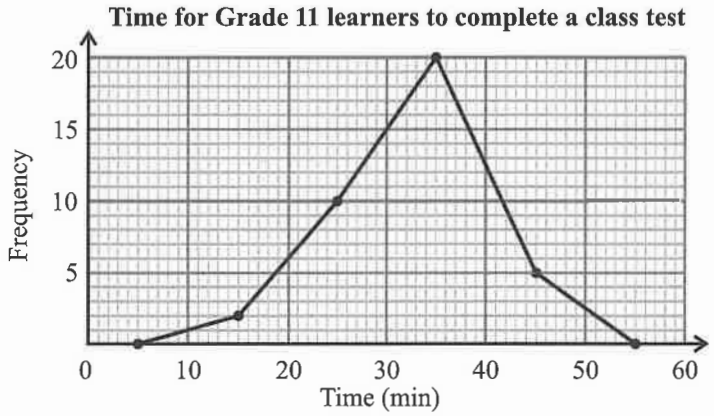
3) The following box and whisker diagrams summarise the final end of year exams for two Grade 11 Mathematics classes.



a) What is the median mark for class A and B?
b) What is the minimum mark for class A?
c) What percentage of the class marks lie between the maximum and median marks?
d) Are there any outliers in class A?

Mixed Exercise:

1) The frequency polygon below shows the time that it took the learners in a Grade 11 class to complete a test.



a) How many learners wrote the test?
b) Draw an ogive for this information and show where the median, upper and lower quartiles lie.

2) The table below shows the distribution of the heights of 200 plants.

Height (cm)	Frequency	Cumulative frequency
$0 < x \leq 10$	20	
$10 < x \leq 20$	40	
$20 < x \leq 30$	60	
$30 < x \leq 40$	50	
$40 < x \leq 50$	20	
$50 < x \leq 60$	10	

a) Complete the table and use it to draw an ogive of the data.
b) Show the median height on the ogive (Q_2).
c) The tallest 20% of the plants are to be sold. Determine the cut off height of the plants to be sold.

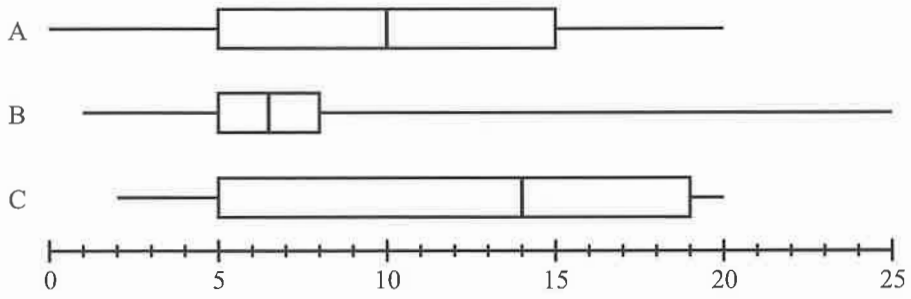
3) The ages of 24 people who watched a play are shown below:

78	69	75	49	89	54	85	82	42	64	90	72
77	70	65	72	60	68	97	53	35	67	71	25

a) Determine the 5 number summary for the data.
b) Represent the above data on a box and whisker diagram.
c) Describe the distribution of data and determine if there are any outliers in the data.

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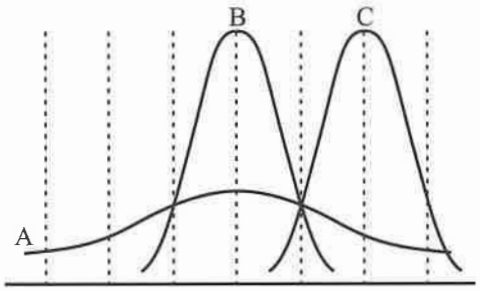
4) The box and whisker diagrams below represent the waiting times (in minutes) for 3 different call centres.



a) Which of the 3 call centres has the largest inter-quartile range.
b) Which of the 3 call centres has the largest range?
c) Call centre B has very long whiskers, what does this tell you about the data?
d) What percentage of the waiting times lies between the maximum waiting time and the median?

5) Three different data sets are shown by the normal distributions below:

a) Which data set has the greatest mean?
b) Which two data sets have the same mean?
c) Which data sets have the same variance?
d) Data set A has a mean of 12 and a standard deviation of 6. Determine the interval of scores that lie within 1 standard deviation.



6) Given the data in the table below:

Data point	16	14	20	26	32
Frequency	5	21	8	18	4

a) Determine the mean of the data.
b) Determine the standard deviation of the data.
c) If x was added to every data point, what would the new standard deviation be?
d) If x was added to every data point, what would the new mean be in terms of x?

7) The standard deviation of the 5 data sets is given below:

Data Set	Standard Deviation
1 ; 2 ; 3 ; 4 ; 5	$\sqrt{2}$
3 ; 5 ; 7 ; 9 ; 11	$2\sqrt{2}$
2 ; 5 ; 8 ; 11 ; 14	$3\sqrt{2}$
10 ; 14 ; 18 ; 22 ; 26	$4\sqrt{2}$
5 ; 10 ; 15 ; 20 ; 25	$5\sqrt{2}$

a) Using the table above determine the standard deviation of: $a\,;\,a+d\,;\,a+2d\,;\,a+3d\,;\,a+4d$.
b) Prove that your answer in a above is correct.

8) The data set 9; 13 ; 15 ; a ; b has a standard deviation of 4 and a mean of 15. Determine the value/s of a and b.

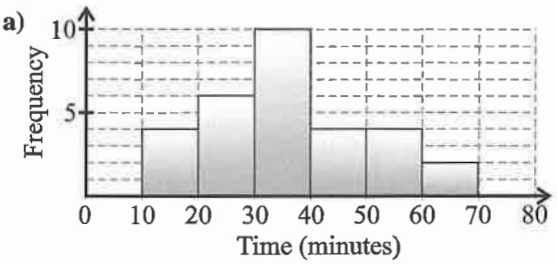
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Histogram and Frequency Polygons

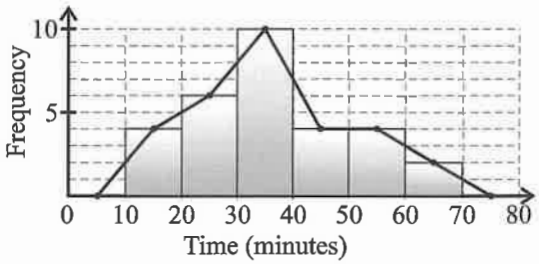
Exercise 1:

1)

a)



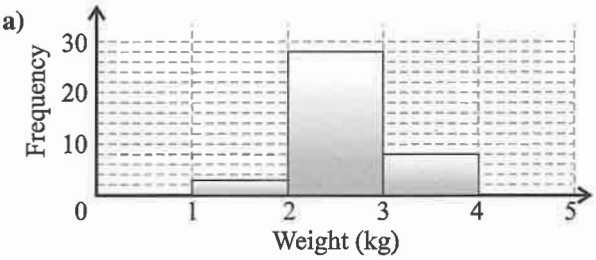
b)



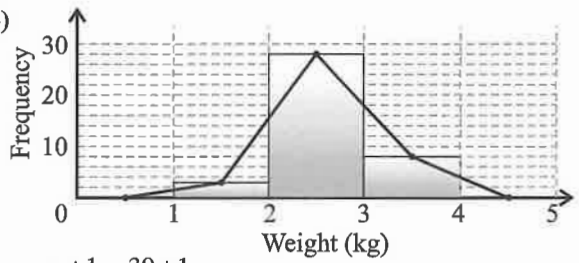
c) It can be seen from the histogram that the modal class is $30 \le t < 40$.

2)

a)



b)

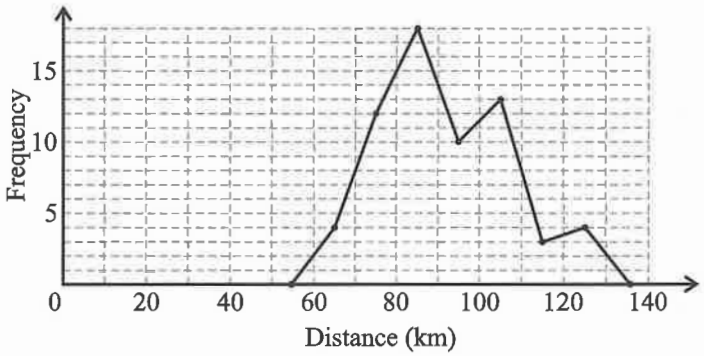


c) There are 39 elements in the data set. $\therefore$ median position $= \frac{n+1}{2} = \frac{39+1}{2} = 20.$

Therefore, the median lies in the interval $2 < w \le 3$ and the median weight is 2,5 kg.

3)

Interval	Frequency	Midpoint
$60 \le D < 70$	4	**65**
$70 \le D < 80$	12	**75**
$80 \le D < 90$	18	**85**
$90 \le D < 100$	10	**95**
$100 \le D < 110$	13	**105**
$110 \le D < 120$	3	**115**
$120 \le D < 130$	4	**125**

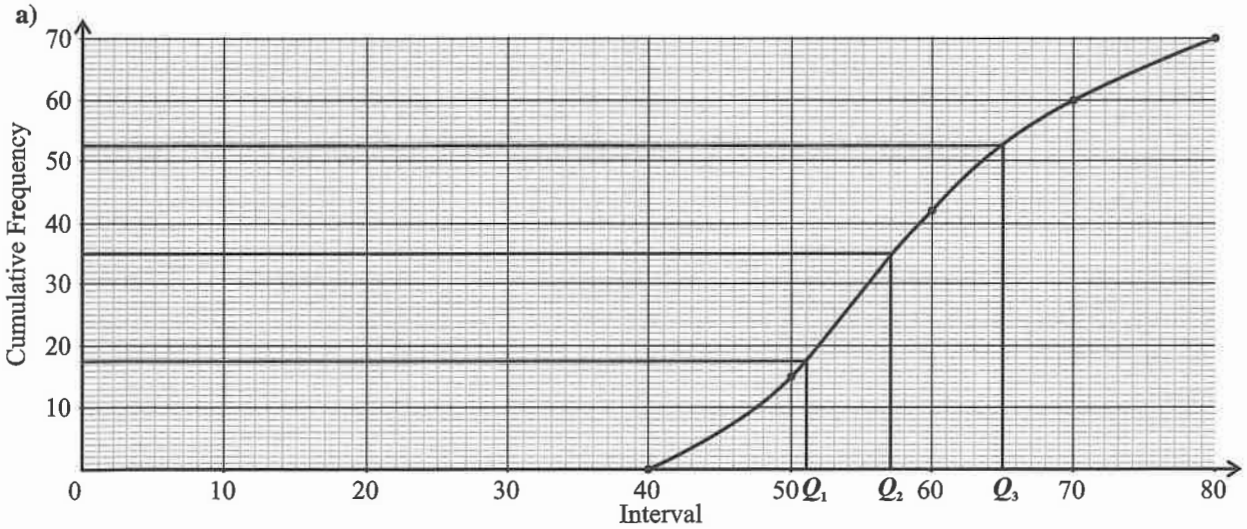


Ogives

Exercise 2:

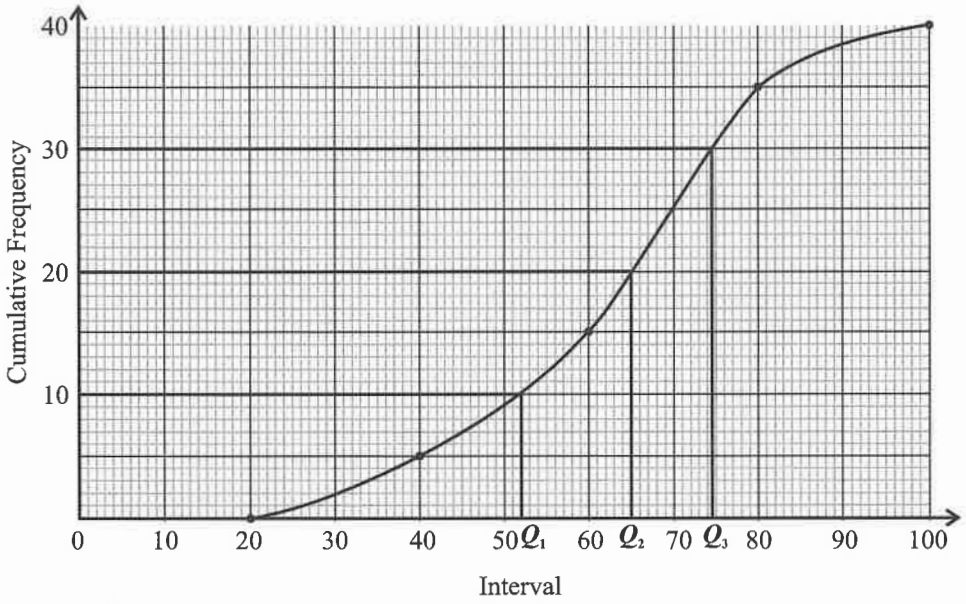
1)

a)



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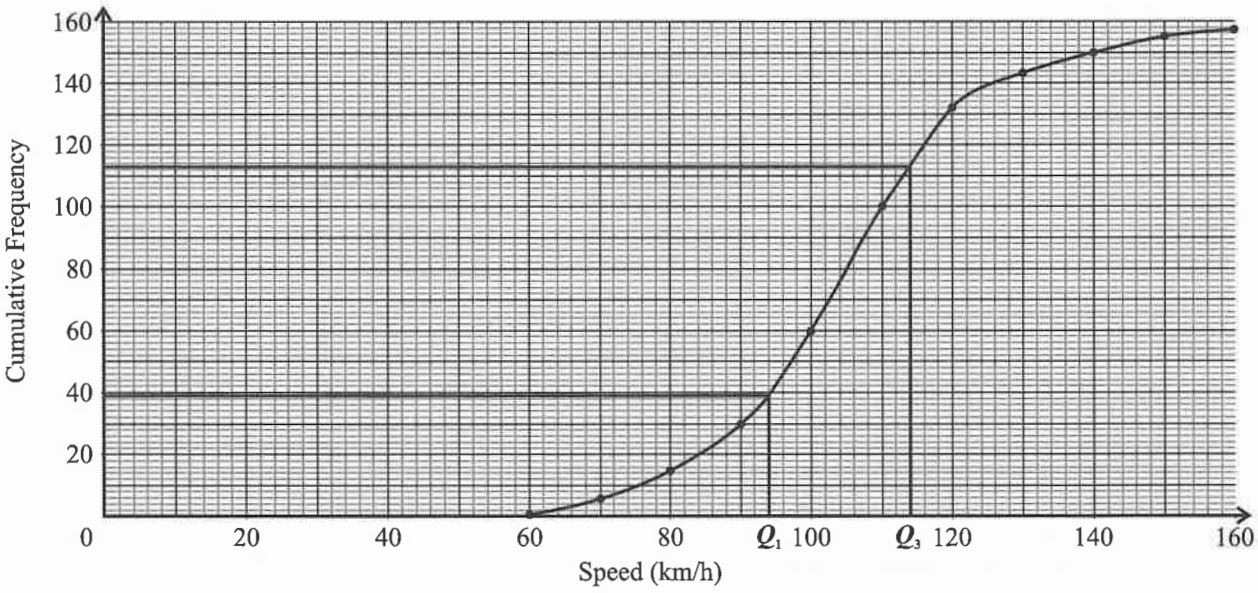
b)



2)

a)

Speed (s)	Frequency	Cumulative frequency
$60 < s \leq 70$	5	5
$70 < s \leq 80$	10	15
$80 < s \leq 90$	15	30
$90 < s \leq 100$	30	60
$100 < s \leq 110$	40	100
$110 < s \leq 120$	32	132
$120 < s \leq 130$	11	143
$130 < s \leq 140$	7	150
$140 < s \leq 150$	5	155
$150 < s \leq 160$	2	157



b) As shown on the ogive, $Q_1 = 94$ and $Q_3 = 114$

$\therefore$ IQR $= 114 - 94 = 20$ km/h

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c)

Speed (s)	Frequency	Midpoint	Midpoint x Frequency
$60 < s \leq 70$	5	65	325
$70 < s \leq 80$	10	75	750
$80 < s \leq 90$	15	85	1 275
$90 < s \leq 100$	30	95	2 850
$100 < s \leq 110$	40	105	4 200
$110 < s \leq 120$	32	115	3 680
$120 < s \leq 130$	11	125	1 375
$130 < s \leq 140$	7	135	945
$140 < s \leq 150$	5	145	725
$150 < s \leq 160$	2	155	310
Sum	**157**		**16 435**

$\therefore$ Estimated mean $= \frac{16435}{157} = 104,68$ km/h

Standard Deviation and Variance

Exercise 3:

1)

a) $\bar{x} = \frac{15+23+45+28+32+35+52+25+70}{9} = 36,11$ minutes

x	$x - \bar{x}$	$(x-\bar{x})^2$
15	–21,11	445,63
23	–13,11	171,87
45	8,89	79,03
28	–8,11	65,77
32	–4,11	16,89
35	–1,11	1,23
52	15,89	252,49
25	–11,11	123,43
70	–33,89	1 148,53
	Sum	**2 304,87**

$\therefore \sigma^2 = \frac{2304,87}{9} = 256,10$

$\therefore \sigma = \sqrt{256,10} = 16$

b) 51,11

c) 16 (standard deviation remains the same)

2)

a) $\bar{x} = \frac{12+19+40+33+25+24+15+38}{8} = 25,75$

Time	$(x-\bar{x})$	$(x-\bar{x})^2$
12	–13,75	189,06
19	–6,75	45,56
40	14,25	203,06
33	7,25	52,56
25	–0,75	0,56
24	–1,75	3,06
15	–10,75	115,56
38	12,25	150,06
Sum		759,48

$\therefore \sigma = \sqrt{\frac{759,48}{8}} = 9,74$

b) One standard deviation lies between 16,01 and 35,49. Therefore, there are 4 learners that lie within 1 standard deviation of the mean.

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3)

a) Using a calculator: $\therefore \sigma = \text{R}5842{,}99$.

b) R7 500.

c) There are 60 elements in the data set.

$$\text{median position} = \frac{60+1}{2} = 30{,}5$$

$\therefore$ median = R7 500

d) One standard deviation lies between 3 823,68 and 15 509,66.

$\therefore$ 51 salaries lie within 1 standard deviation.

$\therefore$ 85% lie within 1 standard deviation.

4)

a) $\sigma^2 = \left(\frac{\sqrt{385}}{5}\right)^2 = \frac{385}{25} = 15{,}4$

b) 25

Symmetrical and Skewed Data

Exercise 4:

1) Write the data in order:

1,4	3,4	4,7	5,7	5,8	7,1	7,1	9,2	11,5

a) Minimum value = 1,4

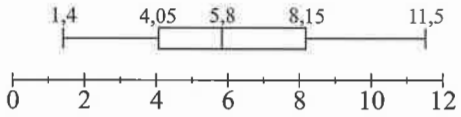
$$\text{Lower quartile} = \frac{3{,}4+4{,}7}{2} = 4{,}05$$

Median = 5,8

$$\text{Upper quartile} = \frac{7{,}1+9{,}2}{2} = 8{,}15$$

Maximum value = 11,5

b)



c) $IQR = 8{,}15 - 4{,}05 = 4{,}1$

$4{,}05 - (1{,}5 \times 4{,}1) = -2{,}1 \rightarrow Q_1 - 1{,}5 \times IQR$

$8{,}15 + (1{,}5 \times 4{,}1) = 14{,}3 \rightarrow Q_3 + 1{,}5 \times IQR$

$\therefore$ There are no outliers in the data set

2)

a) It can be seen from the diagram that the data is skewed to the left.

$IQR = 42 - 26 = 16$

$26 - (1{,}5 \times 16) = 2 \rightarrow Q_1 - 1{,}5 \times IQR$

$42 + (1{,}5 \times 16) = 66 \rightarrow Q_3 + 1{,}5 \times IQR$

$\therefore$ There are no outliers in the data set.

b) It can be seen from the diagram that the data is skewed to the right.

$IQR = 24 - 8 = 16$

$8 - (1{,}5 \times 16) = -16 \rightarrow Q_1 - 1{,}5 \times IQR$

$24 + (1{,}5 \times 16) = 48 \rightarrow Q_3 + 1{,}5 \times IQR$

$\therefore$ There are outliers in the data set.

3)

a) 80

b) 30

c) 50%

d) $IQR = 85 - 60 = 25$

$60 - (1{,}5 \times 25) = 22{,}5 \rightarrow Q_1 - 1{,}5 \times IQR$

$85 + (1{,}5 \times 25) = 122{,}5 \rightarrow Q_3 + 1{,}5 \times IQR$

$\therefore$ There are no outliers in class A.

Mixed Exercise:

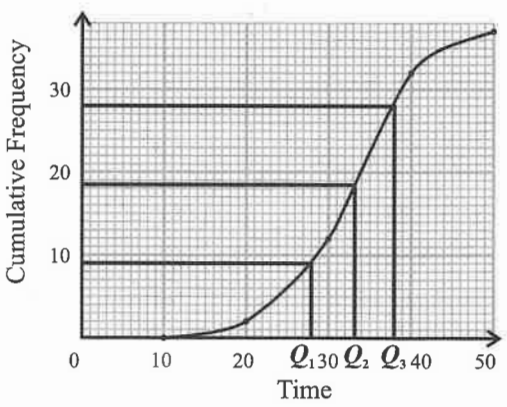
1)

a) Determine the sum of the frequencies on the frequency polygon:

$0 + 2 + 10 + 20 + 5 + 0 = 37$

$\therefore$ 37 learners wrote the test.

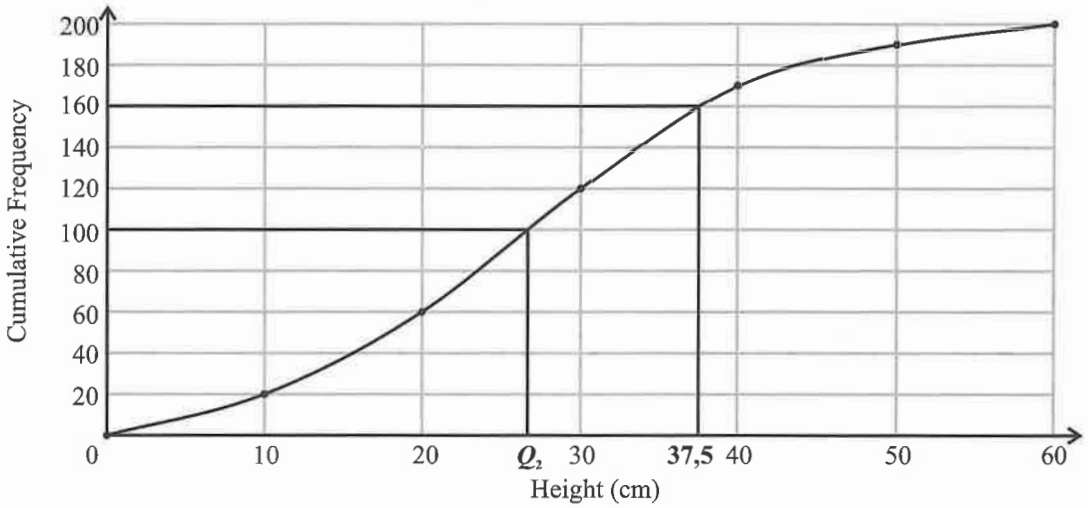
b)



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2a)

Height (cm)	Frequency	Cumulative frequency
$0 < x \leq 10$	20	20
$10 < x \leq 20$	40	60
$20 < x \leq 30$	60	120
$30 < x \leq 40$	50	170
$40 < x \leq 50$	20	190
$50 < x \leq 60$	10	200



b) Shown on the diagram above.

c) You need to calculate the 80th percentile:
$200 \times 0,8 = 160$

$\therefore$ The cut off height lies at the 160th position.

$\therefore$ Plants with a height above 37,5 cm will be sold.

3) Write the data in order:

25	35	42	49	53	54	60	64	65	67	68	69	70	71	72	72	75	77	78	82	85	89	90	97

a) Minimum value = **25**

Q_2 position $= \frac{24+1}{2} = 12,5$

$\therefore Q_2 = \frac{69+70}{2} = \mathbf{69,5}$

Q_1 position $= \frac{12+1}{2} = 6,5$

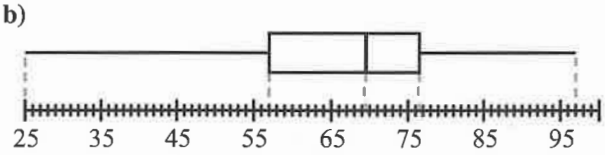
$\therefore Q_1 = \frac{54+60}{2} = \mathbf{57}$

Q_3 position $= 12 + 6,5 = 18,5$

$\therefore Q_3 = \frac{77+78}{2} = \mathbf{77,5}$

Maximum value = **97**

b)



c) From the diagram it can be seen the data is skewed to the left.

$IQR = 77,5 - 57 = 20,5$

$57 - (1,5 \times 20,5) = 26,25 \rightarrow Q_1 - 1,5 \times IQR$

$77,5 + (1,5 \times 20,5) = 108,25 \rightarrow Q_3 + 1,5 \times IQR$

$\therefore$ 25 is an outlier as it is below 26,25.

4)

a) C

b) B

c) It most likely has an outlier in the data set

d) 50%

5)

a) C

b) A and B

c) B and C

d) 6 to 18

6) Using a calculator:

a) 20,18

b) 6,06

c) There would be no change.

d) new mean $= 20,18 + x$

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7)

a) $d\sqrt{2}$

b) $\bar{x} = \dfrac{a+a+d+a+2d+a+3d+a+4d}{5} = \dfrac{5a+10d}{5} = \dfrac{5(a+2d)}{5} = a+2d$

x	$x-\bar{x}$	$(x-\bar{x})^2$
a	$-2d$	$4d^2$
$a+d$	$-d$	d^2
$a+2d$	0	0
$a+3d$	d	d^2
$a+4d$	$2d$	$4d^2$
	Sum	$\mathbf{10d^2}$

$\therefore \sigma = \sqrt{\dfrac{10d^2}{5}} = \sqrt{2d^2} = d\sqrt{2}$

8) $\bar{x} = 15$

$\therefore \dfrac{9+13+15+a+b}{5} = 15$

$\therefore 37+a+b = 75$

$\therefore a = 38-b \rightarrow \boxed{1}$

Now determine the standard deviation in terms of a and b:

x	$x-\bar{x}$	$(x-\bar{x})^2$
9	-6	36
13	-2	4
15	0	0
a	$a-15$	$(a-15)^2$
b	$b-15$	$(b-15)^2$
	Sum	$\mathbf{40+(a-15)^2+(b-15)^2}$

$\sigma = \sqrt{\dfrac{40+(a-15)^2+(b-15)^2}{5}}$

$\therefore 4 = \sqrt{\dfrac{40+(a-15)^2+(b-15)^2}{5}}$

$\therefore 16 = \dfrac{40+(a-15)^2+(b-15)^2}{5}$

$\therefore 40 = (a-15)^2+(b-15)^2 \rightarrow \boxed{2}$

Sub $\boxed{1}$ into $\boxed{2}$:

$40 = (38-b-15)^2+(b-15)^2$

$\therefore 40 = 529-46b+b^2+b^2-30b+225$

$\therefore 2b^2-76b+714 = 0$

$\therefore b^2-38b+357 = 0$

$\therefore (b-21)(b-17) = 0$

$\therefore b = 21$ or $b = 17 \rightarrow \boxed{3}$

Sub $\boxed{3}$ into $\boxed{1}$:

$\therefore a = 17$ or $a = 21$

$\therefore a = 17$ and $b = 21$ or $a = 21$ and $b = 17$

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CAPS - The tables below show the curriculum as prescribed by the National Department of Education for examination purposes.

1. Functions	Chapter 5
Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	
Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and / or reflection about the y axis.	
Problem solving and graph work involving the prescribed functions. Average gradient between two points.	

2. Number Patterns, Sequences and Series	Chapter 3
Investigate number patterns leading to those where there is a constant second difference between consecutive terms, and the general term is therefore quadratic.	

3. Finance, Growth and Decay	Chapter 10
Use simple and compound decay formulae $A = P(1 + in)$ and $A = P(1- i)^n$ to solve problems (including straight line depreciation and depreciation on a reducing balance). Link to work on functions.	
The effect of different periods of compounding growth and decay (including effective and nominal interest rates).	

4. Algebra	Chapters 1 and 2
(a) Apply the laws of exponents to expressions involving rational exponents. (b) Add, subtract, multiply and divide simple surds.	
Take note that there exist numbers other than those on the real number line, the so-called nonreal numbers. It is possible to square certain nonreal numbers and obtain negative real numbers as answers. Nature of roots.	
Revise factorisation.	
Solve: • quadratic equations; • quadratic inequalities in one variable and interpret the solution graphically; and • equations in two unknowns, one of which is linear the other quadratic, algebraically or graphically.	

5. Differential Calculus	
None.	

6. Probability	Chapter 11
(a) Work with dependent and independent events. (b) Use Venn diagrams or contingency tables and tree diagrams as aids to solving probability problems (where events are not necessarily independent).	

7. Euclidean Geometry Measurement	Chapter 8
(a) Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles. (b) Solve circle geometry problems, providing reasons for statements when required. (c) Prove riders.	
Revise Grade 10 work.	

8. Trigonometry	Chapters 6 and 9
(a) Derive and use the identities: $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\sin^2\theta + \cos^2\theta = 1$ (b) Derive the reduction formulae. (c) Determine the general solution and/or specific solutions of trigonometric equations. (d) Establish the sine, cosine and area rules.	
Solve problems in 2-dimensions.	

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CAPS - The tables below show the curriculum as prescribed by the National Department of Education for examination purposes.

9. Analytical Geometry	Chapter 4
Use a Cartesian co-ordinate system to derive and apply : • the equation of a line through two given points; • the equation of a line through one point and parallel or perpendicular to a given line; and • the inclination of a line.	

10. Statistics	Chapter 12
(a) Represent measures of central tendency and dispersion in univariate numerical data by: • using ogives; and • calculating the variance and standard deviation of sets of data manually (for small sets of data) and using calculators (for larger sets of data) and representing results graphically. (b) Represent Skewed data in box and whisker diagrams, and frequency polygons. Identify outliers.	

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Recommended Teaching Plan by the National Department of Education

1	TERM 1										
Weeks	**Week 1**	**Week 2**	**Week 3**	**Week 4**	**Week 5**	**Week 6**	**Week 7**	**Week 8**	**Week 9**	**Week 10**	**Week 11**
Topics	Exponents and Surds			Equations and Inequalities			Number Patterns		Analytical Geometry		
Assessment	Investigation or Project						Test				
Date Completed											

2	TERM 2										
Weeks	**Week 1**	**Week 2**	**Week 3**	**Week 4**	**Week 5**	**Week 6**	**Week 7**	**Week 8**	**Week 9**	**Week 10**	**Week 11**
Topics	Functions				Trigonometry (reduction formulae, graphs, equations)						
Assessment	Assignment / Test								MID-YEAR EXAMINATION		
Date Completed											

3	TERM 3									
Weeks	**Week 1**	**Week 2**	**Week 3**	**Week 4**	**Week 5**	**Week 6**	**Week 7**	**Week 8**	**Week 9**	**Week 10**
Topics	Measurement	Euclidean Geometry			Trigonometry (sine, cosine and area rules)		Finance, Growth and Decay		Probability	
Assessment	Test					Test				
Date Completed										

4	TERM 4									
Weeks	**Week 1**	**Week 2**	**Week 3**	**Week 4**	**Week 5**	**Week 6**	**Week 7**	**Week 8**	**Week 9**	**Week 10**
Topics	Statistics			Revision			FINAL EXAMINATION			Admin
Assessment	Test									
Date Completed										

PAPER 1: 3 HOURS		PAPER 2: 3 HOURS	
Algebraic Expressions and Equations (and Inequalities)	45	Euclidean Geometry and Measurement	40
Number Patterns	25	Analytical Geometry	30
Functions and Graphs	45	Trigonometry	60
Finance, Growth and Decay	15	Statistics	20
Probability	20		
Total Marks	150	Total Marks	150

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ANNUAL TEACHING PLAN
Summary - Grade 11

	Subject	Time	Chapter/s	Paper
Term 1	Exponents and Surds	3 Weeks	1	1
	Equations and inequalities	3 Weeks	2	1
	Number Patterns	2 Weeks	3	1
	Analytical geometry	3 Weeks	4	2
Term 2	Functions	4 Weeks	5	1
	Trigonometric (reduction formulas, graphs and equations)	4 Week	5 and 6	1/2
	Mid-year examinations	3 Weeks	–	–
Term 3	Measurement	1 Weeks	7	2
	Euclidean geometry	3 Weeks	8	2
	Trigonometry (area, sine and cosine rules)	2 Weeks	9	2
	Financial mathematics	2 Weeks	10	1
	Probability	2 Weeks	11	1
Term 4	Statistics	3 Weeks	12	2
	Revision	3 Weeks	–	–
	End of year examinations	3 Weeks	–	–

The teaching plan above is recommended by the National Department of Education from the CURRICULUM AND ASSESSMENT POLICY STATEMENT (CAPS)

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