THE CHEAT SHEET

Grade **11**

CAPS

MATHEMATICS PAPER 2



Miss Angler

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THE CHEAT SHEET:
GR. 11 MATHEMATICS PAPER 2

Miss Angler

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HOW TO USE THIS STUDY GUIDE

PLEASE REMEMBER: This study guide does not replace your textbook or the instruction of your teacher. Its purpose is to help make learning and revision EASY!

1. The Cheat Sheet is laid out in the same order that your final exam will be laid out.
   - Keep this Cheat Sheet next to you when working through past papers for guided examples and explanations.

2. The pro-tips point out common mistakes you should be careful of when answering questions.

PRO-TIP

- Use these pro-tips for exam success!
- These tips are often the reason why my students get 80%+ in their exams and finals.

3. **QUESTION TYPES**

Each question is labelled with its difficulty level (Look out for either **K, R, C or P** next to each question!). This will help you identify the types of questions you need to familiarise yourself with to achieve your goals and to help you prepare for your assessments.

See **the next page** for a detailed explanation of the type of questions that each difficulty level could entail.

Each section is broken up into the QUESTION TYPES that most frequently show up in final examinations.

REMEMBER: ALWAYS USE THE EXAM GUIDELINE FROM YOUR SCHOOL WHEN PREPARING FOR AN EXAM OR TEST!

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PAPER 2: TOPICS & WEIGHTINGS

In your November examinations you should write two 3-hour papers for Mathematics, each out of 150 marks.

The 150 marks for Paper 1 should be split as follows:

Statistics	±20 marks
Analytical Geometry	±30 marks
Trigonometry	±50 marks
Euclidean Geometry & Measurement	±50 marks

EXAM TIPS

It's a good idea to write past papers under exam conditions to get used to the time allocation. Use the mark allocations as a guide for how long you should spend on each question. If a question is taking too long, leave it and rather focus on completing other, easier questions first. There are quick and easy questions and more time-consuming, complicated questions in each section. Make sure you "bank" the easier marks and then see how much time you have left to spend on the challenging questions. The more you practice, the quicker you will become and the more questions you will be able to answer.

It is important to be aware that these guidelines are based off the CAPS document as governmental papers are written at the end of your Grade 11 year. However, some schools write their own papers and therefore their breakdown could differ slightly.

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EXPLAINING THE DIFFICULTY LEVELS OF QUESTIONS

Any assessment you write for Mathematics should be divided into four levels of questions according to the percentages below. Each category of question requires a different level of understanding and approach to the question. In this guide you will see the level next to each type of question in order to help you achieve your goals.



Knowledge (20%) - K: reading information from a diagram, identifying and substituting into the correct formulae from the formula sheet, using mathematical facts and terminology and reciting definitions.

Everyone should feel comfortable with this level of question. They are meant to be your **guaranteed marks** so if there are any questions in this guide indicated as K (knowledge) that you are unsure about, you are encouraged to familiarise yourself with the concepts required to feel confident in answering all the questions of this level type.



Routine procedure (35%) - R: performing familiar procedures and simple calculations that could involve a few steps, answering questions that are similar to those **taught in class** and should be **well practiced** from homework and past papers.

If you are achieving less than **50%** in an assessment, you need to focus on the questions marked as K (knowledge) and R (routine) in this guide. Once you are confident in both these levels, you can start to focus on the next level, complex procedure.



Complex procedure (30%) - C: multiple step calculations that require **higher-order** thinking to solve. Being able to make connections between different representations of information and integrating different topics.

If you are achieving more than 50% but want to achieve a mark closer to **70-80%** you need to be able to **easily** answer all the K (knowledge) and R (routine) questions quite quickly so you have time to spend on the complex procedure questions. Practice all the C (complex) questions in this guide to familiarise yourself with this type of question and write as many past papers (under exam conditions) as you can leading up to any assessments.



Problem-solving (15%) - P: problems that are **unfamiliar** (unseen) but not necessarily difficult. Questions that are asked differently from how they were in class, but can be solved using existing knowledge on the topic.

These questions are aimed at the students achieving **80%** or more and are for those trying to achieve top marks in their assessments. You should only spend time on these types of questions if you have **confidently** answered **ALL** the K, R and C questions in your assessment and have the spare time to spend on these more out-of-the box style questions. If you aren't achieving 80%, don't let these types of questions throw you off. They only count a small percentage of each assessment.

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STATISTICS

<u>In order to achieve maximum marks for this section, you need to be able to:</u>

- Be able to **calculate** the measures of central tendency (**mean, median and mode**) of grouped and ungrouped data.
- Be able to **calculate** the measure of dispersion (**range and interquartile range**) of grouped and ungrouped data.
- Find the values for the **five-number summary.**
- Draw and interpret **histograms, frequency polygons** and **ogives.**
- Calculate and interpret **standard deviation** of grouped data.
- Understand **skewness** and identify **outliers.**

There are three types of data groupings:

1. **Ungrouped data** - individual data values.
2. **Grouped data** - data values that are arranged into intervals (usually of equal width).
3. **Bivariate data** - data values that compare two variables (regression lines are covered in Grade 12).



IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

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Question Type 1: Ungrouped Data

In **ungrouped data** we are given the individual raw data values as a list. The list can be unordered or ordered from smallest to largest.

You need to be able to calculate the following measures of central tendency and dispersion:

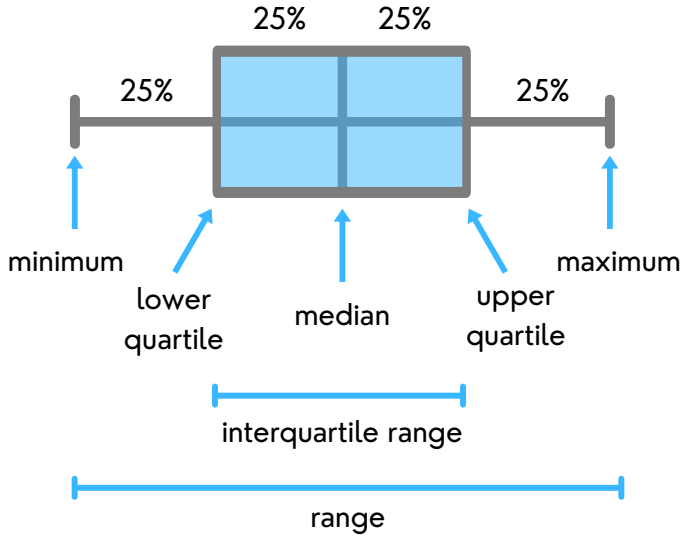
1. **Mean** - the average of all of the data values.
2. **Mode** - the data value that occurs the most (the value with the highest frequency).
3. **Median** - the number that is exactly in the middle of all of the values, when the values are placed in order (can be a data value or between data values)
4. **Lower quartile** - the value that is a quarter of the way from the minimum, when the values are placed in order
5. **Upper quartile** - the value that is a three-quarters of the way from the minimum, when the values are placed in order
6. **Range** - the difference between the maximum and minimum values.
7. **Interquartile range** - the difference between the upper and lower quartiles.
8. **Standard deviation** - the measure of how spread out the data values are in relation to the mean.

You need to be able to answer the following types of interpretation questions:

1. How many values are **within/outside** of one/two/three **standard deviations** from the mean.
2. **Skewness** - data can either be symmetrical, skewed left (negatively skewed) or skewed right (positively skewed).
3. **Outliers** - an outlier is a data value that is significantly different to the rest of the data set.

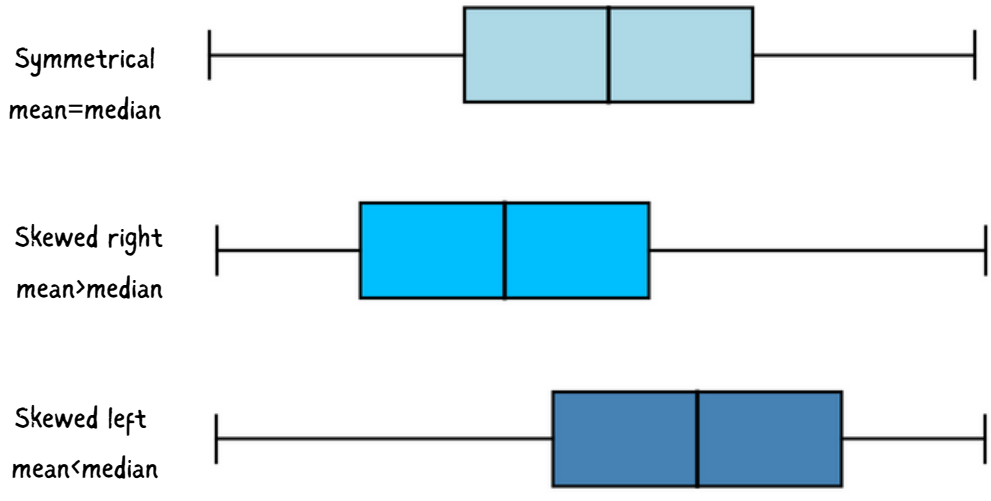
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A **box-and-whisker plot** is a way to visually represent the **five number summary** (minimum, lower quartile, median, upper quartile, maximum) and each section represents 25% of the data (one quarter).



A note on **notation**:
Lower quartile = LQ or Q_1
Upper quartile = UQ or Q_3
Interquartile range = IQR

If data values are evenly spread, the box-and-whisker plot will be **symmetrical** but more likely the data will be **skewed**. If the mean is greater than the median, the data is skewed to the right. You can also compare the length of the box-and-whisker plot on the left of the median and the right. When the data is skewed right, the right tail of the plot is longer from the median to the maximum compared to the left tail (from the median to the minimum). If the mean is less than the median, the data is skewed to the left and the left side of the box-and-whisker plot is longer than the right side.



IQR = interquartile range
LQ = lower quartile
UQ = upper quartile

To calculate an **outlier** we use the following formula: $(LQ - 1{,}5 \times IQR; UQ + 1{,}5 \times IQR)$
If a data value is within these boundaries, it is NOT an outlier. If the data values is smaller than the minimum boundary or larger than the maximum boundary, it is an outlier.

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Worked example

An ice-cream truck was interested in the age of its customers so they decided to record every customer's ages over a weekend. The results are recorded below:

32	34	33	18	28	24	28	21	26	25	23	28	14	16	21

Determine the following values:

K Mean

$$\frac{32 + 34 + 33 + 18 + 28 + 24 + 28 + 21 + 26 + 25 + 23 + 28 + 14 + 16 + 21}{15}$$

$$= \frac{371}{15}$$

$$= 24{,}73$$

- To calculate the mean, add all the numbers together and divide by how many data values there are.
- Always show the total of the sum before your final answer to get method marks if you make a mistake.
- Round off decimal values to decimal places.

K Median

n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11	n=12	n=13	n=14	n=15
14	16	18	21	21	23	24	25	26	28	28	28	32	33	34

$$n = \frac{15 + 1}{2}$$

$$n = 8$$

Median = 25

- To find the median, first you need to order to data by rewriting the values from smallest to largest.
- Next, you need to find WHERE the median is (n).
- Add 1 to how many numbers are in the data set (15+1), then divide this by 2. This is NOT the median, it is the location of the median in the ordered data.
- Count until you have the 8th number, this is the median.

K Mode

28



- The mode is the value that appears most often in the data set.
- In this data set, all the numbers appear once except for 21, which appears twice, and 28, which appears three times. Therefore, 28 appears the most and is the mode.

K Range

$$34 - 14$$

$$= 20$$

- The range is the difference between the largest and smallest values, therefore RANGE = MAX-MIN.
- Show the calculation to earn method marks in case you make a mistake.

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Worked example

A class of Grade 11 students are discussing how many pets they each have. The results are summarised in the frequency table below:

Pets	0	1	2	3	4	5	16
Freq	3	5	6	2	3	1	1

First, we need to understand how the frequency table works. The top row shows how many pets a student could have and the bottom row show how many students have that number of pet. For example, there are 3 people in the class who have 0 pets and 5 people in the class who have 1 pet. If we started to write all the data values we would write a 0 three times, and a 1 five times: 0 0 0 1 1 1 1 1 ...
Since the same number is repeated multiple times, it is more efficient to represent these values in a frequency table to summarise the values but you need to understand how to convert the values into a list if needed.



How many students are in the class?



$3 + 5 + 6 + 2 + 3 + 1 + 1$

$= 21$ students

- To calculate the total number of students, we add the frequencies together.



What is the modal number of pets?



2 pets

- The highest frequency is 6, therefore most students in the class have 2 pets.



Calculate the mean.



$$\frac{(0 \times 3) + (1 \times 5) + (2 \times 6) + (3 \times 2) + (4 \times 3) + (5 \times 1) + (16 \times 1)}{21}$$

$$= \frac{56}{21}$$

$$= 2{,}67$$

- To calculate the total, multiply each value by the frequency.
- Then, to calculate the mean, divide the total by the total frequency
- Round off your answer to 2 decimal places.

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R Calculate the interquartile range.

Interquartile range = upper quartile – lower quartile

$= 3{,}5 - 1$

$= 2{,}5$

- To calculate the interquartile range, first we need to find the upper and lower quartiles.
- For this is example, let's write out all the numbers to visualise the data set but if you are comfortable with the frequency table, it will save a lot of time to skip this step.
- First, let's find the median.
- There are 21 data values, therefore the median is in position (21+1)/2 = 11
- There are 10 values on either side of the median. To find the upper and lower quartiles, we need to find the middle of these 10 vlues.
- Position = (10+1)/2 = 5.5
- To find the value in position 5.5 we average the values in position 5 and 6 for the lower quartile and position 16 and 17 for the upper quartile (5.5 from the median or 5.5 from the maximum is between values 16 and 17)

0, 0, 0, 1, 1, 1, 1, 1, 2, 2, (2), 2, 2, 2, 3, 3, 4, 4, 4, 5, 16

median

lower quartile = 1

upper quartile =3,5

- For the lower quartile, halfway between 1 (n=5) and 1 (n=6) is 1. For the upper quartile, halfway between 3 (n=16) and 4 (n=17) is 3.5.
- Now, we can calculate the interquartile range by subtracting the lower quartile from the upper quartile.

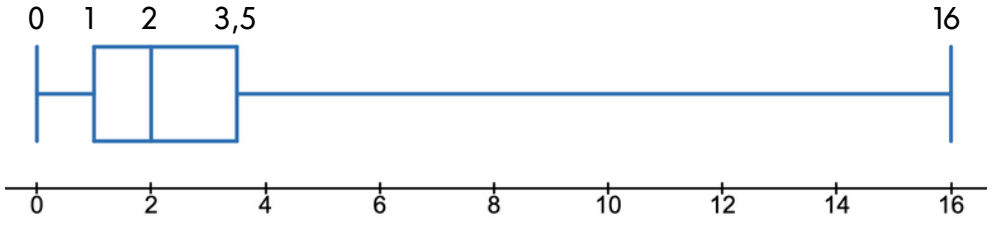
K Comment on the skewness of the data.

The data is skewed to the right.

- We need to compare the mean and the median to determine skewness. Thankfully, we have already calculated the mean and the median.
- Mean>median ∴ skewed right

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Represent the data set on a box-and-whisker plot.



- To summarise the data in a box-and-whisker plot we need the five number summary: minimum, lower quartile, median, upper quartile and maximum
- We already have the lower quartile, median, and upper quartile from calculating the interquartile range so we just need the minimum (smallest value = 0) and the maximum (largest value = 16)
- Usually, in an assessment, you will be given a number line as a scale to use but if you are not given one, you will need to draw your own.
- Draw vertical lines at each of the five numbers. Connect the lower and upper quartiles to create a box and connect the box to the minimum and maximum.
- You must label the box-and-whisker with the five values above the plot.

Determine whether there is an outlier in the data set. Justify your answer with a suitable calculation.

Outlier: $\{LQ - 1{,}5 \times IQR\,;\, UQ + 1{,}5 \times IQR\}$

$\{1 - 1{,}5 \times 2{,}5;\, 3{,}5 + 1{,}5 \times 2{,}5\}$

$\{-2{,}75\,;\, 7{,}25\}$

∴ Yes, the student with 16 pets is an outlier

- This is the formula to determine the bounds of an outlier. If it is outside these bounds, it is an outlier.
- LQ = lower quartile
  IQR = interquartile range
  UQ = upper quartile
- In our data set there is only one value not within these values, which means we have one outlier of 16 pets.

What percentage of values fall within 2 standard deviations of the mean?

Standard deviation = 3,29

Two standard deviations from the mean = $\{\bar{x} - 2\sigma;\, \bar{x} + 2\sigma\}$

∴ $\{2{,}67 - 2 \times 3{,}29\,;\, 2{,}67 + 2 \times 3{,}29\}$

∴ $\{-3{,}91\,;\, 9{,}25\}$

∴ $\frac{22}{23} \times 100 = 95{,}65\%$

- See the next page for steps on how to calculate the standard deviation (and an alternate method to calculate the mean).
- Now, take the mean we calculated earlier and subtract two standard deviations and then separately, add two standard deviations to get the boundary values
- Count how many values are within these bounds.
- To calculate the percentage, divide the number of values by the total frequency and100.
- Round off your answer to 2 decimal places

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If every student were to adopt 2 cats, without calculating the new values, how would this affect the mean and the standard deviation?

The mean would increase and the standard deviation would stay the same.

- Since the total number of pets has increased, but the number of students has remained the same, this will increase the mean.
- Standard deviation is a measure of how spread out the data values are from each other. If all the data values increase by the same amount together, the data isn't any more or less spread out than before, so the standard deviation will stay the same.

If the students who has 16 pets were to give away 2 fish to a friend in another class, without calculating the new values, how would this affect the mean and the standard deviation?

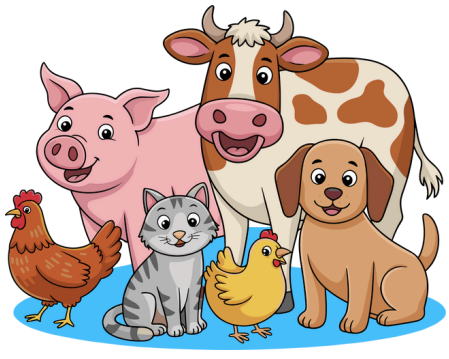
The mean would decrease and the standard deviation would decrease.

- Since the total number of pets has decreased, but the number of students has remained the same, this will decrease the mean.
- Since the value furthest from the mean is now closer to the mean (the student with the most pets now has fewer pets, which is more similar to the mean), this will make the data less spread out and decrease the standard deviation.

If one of the students who has 5 pets were to give all their pets to the student who already has 16 pets, without calculating the new values, how would this affect the mean and the standard deviation?

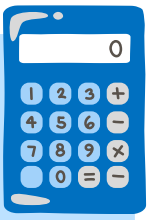
The mean would stay the same and the standard deviation would increase.

- Since the total number of pets has stayed the same, and the number of students has remained the same, this mean will stay the same.
- Since one of the values that is closer to the mean (the student with 5 pets) is moving further away from the mean as they will now have zero pets, and the largest value (16 pets) is getting even bigger and therefore even further away from the mean, this will make the data more spread out and increase the standard deviation.



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How to use your CASIO scientific calculator to calculate the MEAN and the STANDARD DEVIATION:



- Press [MODE] and then select the [STAT] option.
- Select [1: 1 - VAR].
- Now type the data values in the "X" column. Press [=] after each value.
- If you have a frequency table, type the frequencies into the "FREQ" column.
- Press [AC].
- Press [SHIFT] then [1] and select [4: Var].
- To calculate the mean, select [2].
- To calculate the standard deviation, select [3].

In a frequency questions like this one, you can save yourself time by having the **frequency column** turned on in your calculator settings.

- Press [SHIFT] and then select the [SETUP] option.
- Press the down arrow to get more options.
- Select [4: STAT].
- Select [1: ON].

PRO-TIP

Double check you have typed in your data values correctly. The answers from the calculator don't have any method marks.

Worked example



Seven numbers, in ascending order, are given as:

7	x	10	13	15	y	24

If the mean is 14 and the interquartile range is 9, determine the values of x and y.



$$\frac{7+x+10+13+15+y+24}{7}=14$$

$$x+y+69=98$$
$$\therefore x+y=29$$

$$y-x=9$$
$$\therefore y=9+x$$

$$x+(9+x)=29$$
$$2x=20$$
$$x=10 \text{ and } y=19$$

- There are two variables in the list of data values, therefore we are given two pieces of information in order to make two separate equations and solve simultaneously.
- Since the mean is 14, we can create an equation with all the values added together and divided by the total number of data values (7) and equate it to 14.
- By multiplying both sides by 7, we can simplify to get rid of the fraction and add the values on the LHS.
- Simplifying even further, we have the variables on one side (LHS) and the numbers on the other (RHS)
- At this point, we can't do anything further with this equation so we pivot to our other piece of information: the IQR, which is 9.
- The IQR is the difference between the UQ and LQ. Since there are 7 values, the median is in the 4th position. The middle of the remaining 3 values below the median is the LQ (x) and the middle of the 3 values above the median is the UQ (y). Now we have a second equation in terms of x and y.
- Choose a variable to isolate, here we have chosen to isolate y in the second equation.
- By substituting the expression for y into the other equation, we now have an equation with only one type of variable and we can solve for x.
- Once you have the value for x, don't forget to go back to the expression for y to find its value.

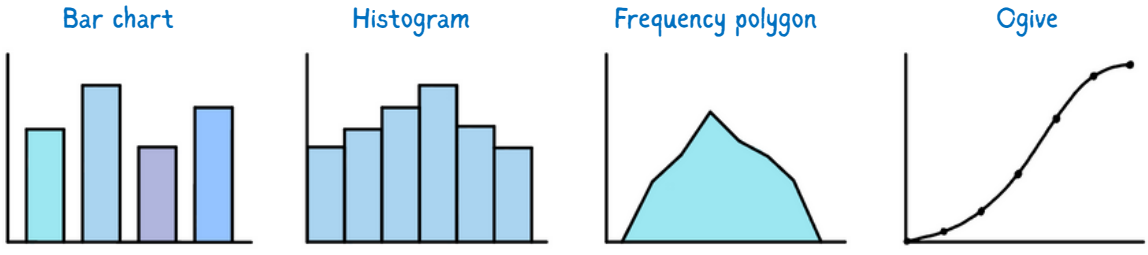
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Question Type 2: Grouped Data

In **grouped data** we do not have the raw data values as they have been summarised into intervals. Usually these intervals are of equal width and the frequency of the interval is given. Since we do not have the individual data values, most calculations are an estimate.

There are two types of data:
Discrete data is counted and recorded as individual units e.g. number of people or marks in a test.
Continuous data is measured and recorded as a range as the values are most often decimals eg. distance or time.

Visualising grouped data:
Grouped data can either be displayed as a bar graph (discrete data), a histogram or frequency polygon (continuous data) or an ogive (both discrete and continuous data).



Estimated mean:
To calculate the **estimated mean**, we use the midpoint of each interval (the midpoint acts as the average of the interval):

- If the data is **discrete**, you only include **one** of the **endpoints** of the interval.
- If the data is **continuous**, you include **both** of the **endpoints** of the interval.

For example, the class interval is $0 < x \leqslant 10$:

If the data is **discrete**

$$\text{Midpoint} = \frac{1+10}{2}$$
$$= 5,5$$

If the data is **continuous**

$$\text{Midpoint} = \frac{0+10}{2}$$
$$= 5$$

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Worked example

The table below shows the heights, in metres, of students in a Grade 11 class.

Height (h)	Frequency (f)
$150 \leq h < 156$	2
$156 \leq h < 162$	4
$162 \leq h < 168$	15
$168 \leq h < 174$	7
$174 \leq h < 180$	3

K How many students are in the class?

$$\text{Total} = 2 + 4 + 15 + 7 + 3$$
$$= 31$$

- Add the frequencies of each interval to get the total number of students in the class

K What is the modal interval?

$$162 \leq h < 168$$

- The interval with the highest frequency is the modal interval (modal is the adjective of mode)

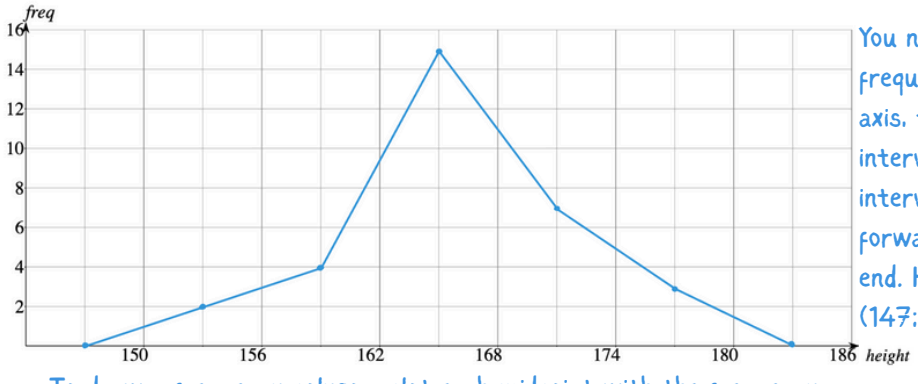
R Complete the table below and calculate the approximate mean.

Height (h)	Frequency (f)	Midpoint (m)	$f \times m$
$150 \leq h < 156$	2	153	2x153=306
$156 \leq h < 162$	4	159	4x159=636
$162 \leq h < 168$	15	165	15x165=2475
$168 \leq h < 174$	7	171	7x171=1197
$174 \leq h < 180$	3	177	3x177=531

$$\text{Approx mean} = \frac{306+636+2475+1197+531}{31}$$
$$= \frac{5145}{31}$$
$$= 165{,}97\ m$$

- First, complete the table. Since this is continuous data (distance) we use both endpoints to determine the midpoints e.g. (150+156)/2 = 153
- Multiply the frequency of each interval by its midpoint to calculate the total of each interval
- Add the totals together and divide by the total frequency to calculate the approximate (estimated) mean.

R Draw a frequency polygon displaying the above data on the grid below.



You need to ground the frequency polygon to the x-axis, therefore using the same interval width, go back one interval from the start and forward one interval from the end. Here, we ground at (147;0) and (183;0).

- To draw a frequency polygon, plot each midpoint with the frequency from that interval e.g. (153;2) and (159;4)

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Worked example

The table below shows the marks of 28 students for a recent Maths class test out of 80.

Mark (m)	Frequency (f)
$0 \leq m < 16$	1
$16 \leq m < 32$	3
$32 \leq m < 48$	9
$48 \leq m < 64$	13
$64 \leq m < 80$	2

K How many students achieved an A (above 80%) in the test ?

2

- 80% of 80 = 64
- There are two students in the 64<m<80 interval

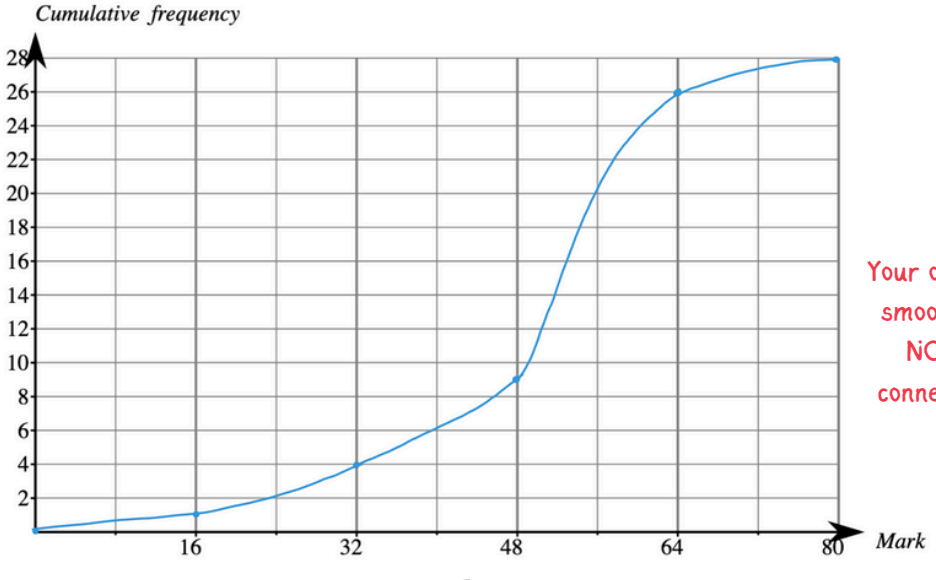
K Complete the cumulative frequency column in the table below.

Mark (m)	Frequency (f)	Cumulative Frequency
$0 \leq m < 16$	1	1
$16 \leq m < 32$	3	4
$32 \leq m < 48$	9	13
$48 \leq m < 64$	13	26
$64 \leq m < 80$	2	28

- The cumulative frequency is the running total of all the intervals up to that point.
- The first cumulative frequency is always the same as the frequency, the next cumulative frequency is the total frequency of both intervals, 1+3=4.
- For the next cumulative frequency, take the current total (4) and add the next frequency, 4+9=13.
- Continue until you have the total for all the intervals.

R Draw an ogive for the data on the grid provided below.

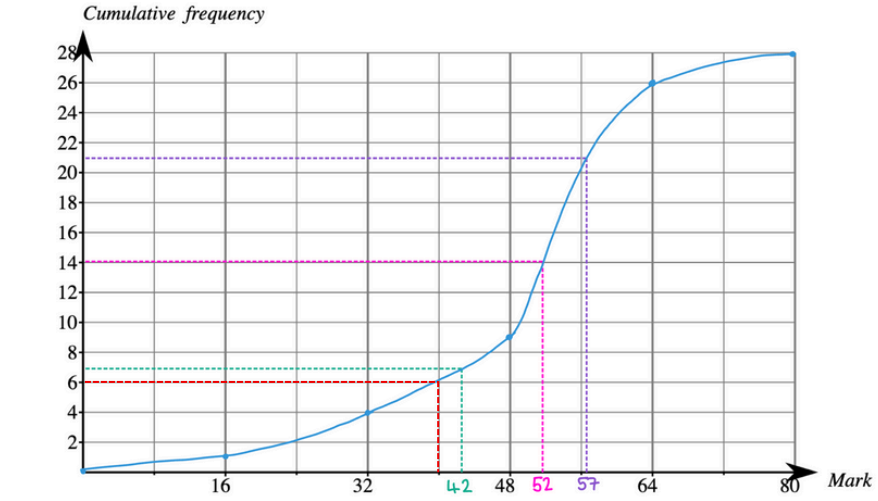
- To get the co-ordinates to draw an ogive, use the upper boundary of each interval with the cumulative frequency up to that point, therefore list of co-ordinates is (16;1), (32;4), (48; 13), (64;26) and (80;28).
- The ogive must be grounded to the x-axis at the lower boundary of the first interval with zero, therefore (0;0).



Your ogive must be a nice, smooth curve. You MAY NOT use a ruler to connect the coordinates.

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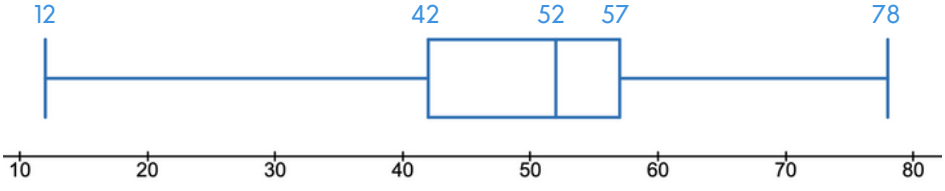
R Use the ogive to determine the lower quartile, median and upper quartile.



Lower quartile = 42 Median = 52 Upper quartile = 57

- The median is halfway and since there are 28 students, halfway is at a cumulative frequency of 14. Draw a dotted line across from the y-axis at 14 and once you connect to the ogive, draw a vertical line down to the x-axis. Read off the value from the x-axis (in this case, 52).
- The upper and lower quartiles are a quarter of the way from the top and from the bottom of the ogive. A quarter of 28 is 7, therefore draw a dotted horizontal line from the y-axis at 7 (lower quartile) and at 21 (upper quartile). Where these lines meet the ogive, draw a vertical line down to the x-axis. Read off these values (LQ=42; UQ=57).
- Most likely, there will be a range of values accepted in any assessment as the values can vary slightly depending on the curve of your ogive. This is why it is VITAL to draw the dotted lines so your teacher and see how you've read off these values.

R If the lowest mark was 12 and the highest was 78, draw a box and whisker diagram for the data.



R Comment on the distribution of the heights of the learners.

Skewed left

- We don't have the value for the mean so we need to rely on the box-and-whisker plot to determine the skewness of the data.
- The left side is longer than the right (from median to minimum is longer than from median to maximum).

C If the pass mark was 50%, how many students passed this test?

$50\% \times 80 = 40$
6 students scored less than 40 marks.
$\therefore 28 - 6 = 22$ students passed.

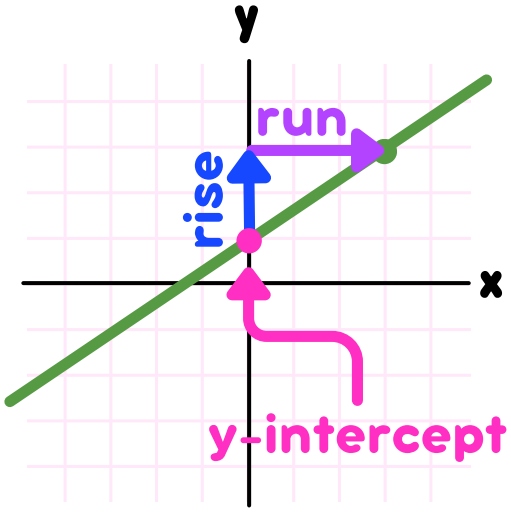
- Calculate the pass mark by multiplying the total (80) by 50%
- To read off the cumulative frequency, drawing a dotted line vertically from the x-axis at 40 until the line reaches the ogive. Now, draw a horizontal line to the y-axis to read off how many students achieved 40 marks or less (6 students).
- Subtract the students who didn't pass from the total to determine the number of students who passed.

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ANALYTICAL GEOMETRY

<u>In order to achieve maximum marks for this section, you need to be able to:</u>

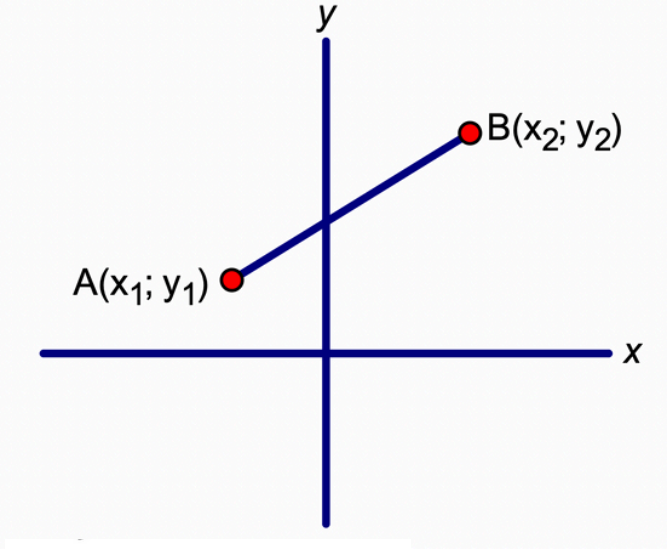
- calculate the **gradient, distance** and **midpoint** between two points.
- determine the **equation of a line** though two given points
- determine the equation of a line through one point and **parallel or perpendicular** to a given line
- calculate the equation of a **median** and **altitude** of a triangle, and **perpendicular bisector** of a line.
- determine the coordinates of the **fourth point** of a **parallelogram**
- calculate the **angle of inclination** of a line to the x-axis
- use the angle of inclination to **determine angles** in various shapes
- calculate the **area** of various shapes



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Question Type 1:
Determining the gradient, midpoint and distance

There are some basic formulae that you need to be able to apply. These formulae were all taught in Grade 9/10 and are given on the formula sheet.



- Gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

- Midpoint:

$$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

- Distance:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Equation of a line:

$y = mx + c$ OR $y - y_1 = m(x - x_1)$

If you have the gradient (m) and the y-intercept (c)

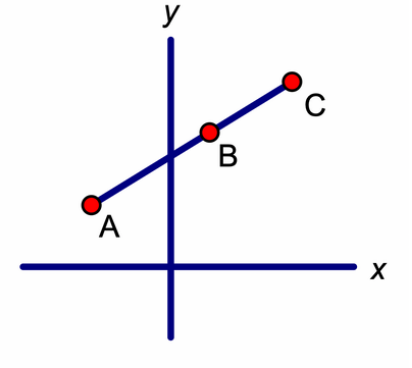
If you have the gradient (m) and a point $(x_1; y_1)$.

- We call points **collinear** when they exist on the same line. The gradients between the points must be equal, and since they share a point, they will then lie on the same line.

- **Parallel lines** stay the same distance from each other and will never meet, therefore they have the same gradient.

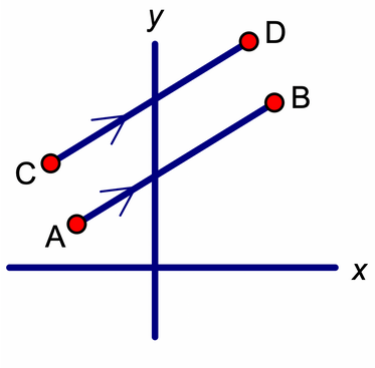
- **Perpendicular lines** meet at a right-angle, therefore they have opposite gradients (called negative reciprocals). This means the product of the gradients will be -1.

Collinear points



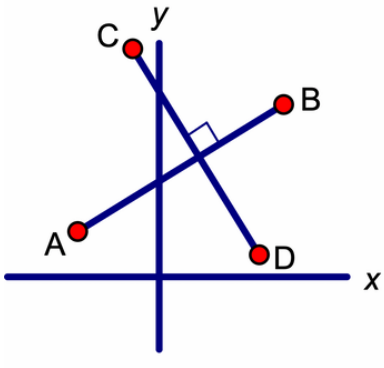
$m_{AB} = m_{BC} = m_{AC}$

Parallel lines



$m_{AB} = m_{CD}$

Perpendicular lines



$m_{AB} \times m_{CD} = -1$

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Worked example

Given the points A(4;-1) and B(-2;2):

K Calculate the gradient of AB.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{AB} = \frac{2-(-1)}{-2-4}$$

$$m_{AB} = \frac{3}{-6}$$

$$m_{AB} = -\frac{1}{2}$$

- To calculate the gradient of a line we substitute the points into the gradient formula.
- Here we have used A as coordinate 1 and B as coordinate 2 but you would get the same answer if you switched them around.
- Be careful not to switch halfway through your calculation. if you have chosen A is coordinate 1 when you substituted in the y-values. you can't now switch and use B as coordinate 1 for the x-values.
- Always simplify your answer fully and write the negative in front of the fraction.

K Calculate the length of AB.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{\left(4 - (-2)\right)^2 + (-2 - 1)^2}$$

$$AB = 3\sqrt{5}$$

$$AB = 6{,}71$$

- To calculate length we use the distance formula.
- Substitute the coordinates of A and B into the formula. Here we have used A as coordinate 1 and B as coordinate 2 but just as with gradient. you would get the same answer if you switched them around.
- If the question does not specify. round off your answer to two decimal places.

K Determine the coordinates of M, the midpoint of AB.

$$M\left(\frac{x_1+x_2}{2}; \frac{y_1+x_2}{2}\right)$$

$$M\left(\frac{4+(-2)}{2}; \frac{-2+1}{2}\right)$$

$$M\left(1; -\frac{1}{2}\right)$$

- The formula for midpoint finds the average x-value and the average y-value using two points.
- As with gradient and distance. here we have used A as coordinate 1 and B as coordinate 2 but you would get the same answer if you switched them around.
- Gradient = m (lower case) and Midpoint = M (uppercase).

K **Worked example**

Prove $A(-2;1), B(1;10)$ and $C(5;22)$ are collinear.

$$m_{AB} = \frac{10-1}{1-(-2)} = \frac{9}{3} = 3 \qquad m_{BC} = \frac{22-10}{5-1} = \frac{12}{4} = 3 \qquad m_{AC} = \frac{22-1}{5-(-2)} = \frac{21}{7} = 3$$

- For points to be collinear. they need to have the same gradient and share a point.
- You do not need to find all three gradients to prove the points are collinear. any two of the three is sufficient.

A, B and C are collinear since the gradients are equal and they share a point.

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Worked example

If $D(-6;11), E\left(-1;\frac{17}{2}\right)$ and $F(8;k)$ are collinear, find the value of k.

$$m_{DE} = \frac{\frac{17}{2}-11}{-1-(-6)}$$

$$= \frac{-\frac{5}{2}}{5}$$

$$= -\frac{1}{2}$$

$$m_{DF} = \frac{k-11}{8-(-6)}$$

$$-\frac{1}{2} = \frac{k-11}{14}$$

$$-7 = k - 11$$

$$\therefore k = 4$$

- Here we are told the points are collinear, therefore we can use that the gradients are equal to solve for k.
- First, we find the gradient between the two known points (D and E).
- Now, you can find an expression for either DF or EF, in terms of k (since E has a fraction it is easier to use D).
- Set this expression equal to the gradient found for DE.
- To solve for k, we multiply by the denominator (x14).
- Now we can get the value for k by adding 11 to both sides of the equation.

Worked example

Given a triangle with vertices $C(1;1), A(4;3)$ and $T(3;-2)$ prove that ΔCAT is right-angled.

$$m_{CA} = \frac{3-1}{4-1} \qquad m_{AT} = \frac{-2-3}{3-4} \qquad m_{Tc} = \frac{-2-1}{3-1}$$

$$m_{CA} = \frac{2}{3} \qquad m_{AT} = 5 \qquad m_{TC} = \frac{-3}{2}$$

$$\therefore m_{CA} \times m_{TC} = -1$$

$\therefore$ The triangle is right-angled at C.

- To prove a triangle is right-angled we need two sides to be perpendicular.
- Either you can draw yourself a small sketch to try and guess which two sides are more likely to be the ones you need, but it's safer (and potentially quicker) to just calculate all three gradients.
- If two lines are perpendicular, the product of their gradients will equal -1, therefore CA and TC have opposite gradients and will be perpendicular to each other at the point they share, C.
- Since the question said PROVE the triangle IS right-angled, you cannot say this is false. If your calculations do not prove two of the sides are perpendicular, you have made a mistake.

Worked example

Given the points $A(4;6), B(-2;-2), C(x;-10)$ and $D(-4;2)$, determine the value of x if:

B is the midpoint of AC.

$$\frac{4+x}{2} = -2$$

$$4 + x = -4$$

$$\therefore x = -8$$

- Since we know B is the midpoint of AC, we can create an equation to solve for x using only the x-values in the midpoint formula.
- By adding the x-values of A and C (and dividing by 2), we know the answer must be -2 (the x-coordinate of B).
- Now we can solve for x by multiplying by 2 and then subtracting 4.

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R AB is parallel to CD.

$$m_{AB} = \frac{-2-6}{-2-4}$$

$$m_{AB} = \frac{4}{3}$$

$$m_{CD} = \frac{-10-2}{x-(-4)}$$

$$\frac{4}{3} = \frac{-12}{x+4}$$

$$4(x+4) = -36$$

$$4x + 16 = -36$$

$$4x = -52$$

$$\therefore x = -13$$

- Since AB is parallel to CD they will have the same gradient.
- First, we calculate the gradient of the known coordinates, AB.
- Then, we find an expression for CD, in terms of x.
- Since the gradients are equal, we can set this expression equal to the value we found for AB's gradient.
- Now we can solve for x. The easiest way to get x out from the denominator is to cross multiply. You multiply the left numerator by the right denominator and that will equal the right numerator multiplied by the left denominator.

$$\frac{4}{3} \times \frac{-12}{x+4}$$

Only cross multiply when you have FRACTION = FRACTION

- Now, multiply the 4 into the bracket by distributing.
- Lastly to get x on its own, subtract 16 and divide by 4.

R AB is perpendicular to CD.

$$m_{AB} = \frac{4}{3}$$

$$\therefore m_{CD} = -\frac{3}{4}$$

$$-\frac{3}{4} = \frac{-12}{x+4}$$

$$-3(x+4) = -48$$

$$-3x - 12 = -48$$

$$-3x = -36$$

$$\therefore x = 12$$

- Since AB is perpendicular to CD, their gradients will be opposites (and the product of their gradients must equal -1).
- We already have the gradient for AB and an expression for the gradient of CD from the previous question so there is no need to calculate them again
- Now we can set the expression for CD equal to the opposite gradient of AB.
- To solve for x, we cross multiply again, as we did in the previous question.
- Finally, to get x on its own, we add 12 and divide by -3.

C $AC = 2\sqrt{65}$

$$AC = \sqrt{(x-4)^2 + (-10-6)^2}$$

$$2\sqrt{65} = \sqrt{x^2 - 8x + 16 + 256}$$

$$\left(2\sqrt{65}\right)^2 = \left(\sqrt{x^2 - 8x + 16 + 256}\right)^2$$

$$260 = x^2 - 8x + 272$$

$$0 = x^2 - 8x + 12$$

$$0 = (x-2)(x-6)$$

$$\therefore x = 2 \text{ OR } x = 6$$

- We have been given the length of AC, therefore we will use the distance formula and substitute in the coordinates from A and C.
- To get rid of the square roots, we square both sides. Be careful to square the ENTIRE left hand side.
- Since the equation is quadratic, we move everything to the same side, in this case the RHS as x^2 will be positive, and the other side (the LHS) is zero.
- Factorise the trinomial to get the two factors to solve for x.
- Without any further information, there is no way to eliminate either of the values of x.

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Question Type 2:
Determining the equations of lines between two points

Worked example

Given the points A(-3;3) and B(1;6):

K Determine the equation of the line AB.

$$m_{AB} = \frac{6-(3)}{1-(-3)}$$

$$= \frac{3}{4}$$

$$y = \frac{3}{4}x + c$$

$$6 = \frac{3}{4}(1) + c$$

$$c = \frac{21}{4}$$

$$y = \frac{3}{4}x + \frac{21}{4}$$

OR

$$y - y_1 = m(x - x_1)$$

$$y - 6 = \frac{3}{4}(x - 1)$$

$$y - 6 = \frac{3}{4}x - \frac{3}{4}$$

$$y = \frac{3}{4}x + \frac{21}{4}$$

- To determine the equation, we need the calculate the gradient (m) and the y-intercept (c).
- First, we substitute the coordinates of A and B into the gradient formula.
- To calculate the y-intercept, either we can substitute a point into the standard form (y=mx+c) or into the point-slope form ($y - y_1 = m(x - x_1)$).
- Make sure to write your answer in standard form, this means you need y on its own as the subject.

K Determine whether AB is parallel to the line $3y + 6 = 4x$.

$$3y = 4x - 6$$

$$y = \frac{4}{3}x - 2$$

No, the lines are not parallel since their gradients are not equal.

- To identify whether the lines are parallel, we need to compare their gradients.
- To read the gradient off this line, we need to rewrite it in standard form.
- First, get the "y" term on its own by subtracting 6.
- Now we need to coefficient of y to be 1, not 3, therefore we divide everything by 3.

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K Find the equation of a the line, which is parallel to AB, and passes through the point $\left(2;\frac{1}{2}\right)$.

$$y = \frac{3}{4}x + c$$
$$\frac{1}{2} = \frac{3}{4}(2) + c$$
$$\frac{1}{2} = \frac{3}{2} + c$$
$$c = -1$$
$$\therefore y = \frac{3}{4}x - 1$$

- Since the line we want is parallel to AB, they will have the same gradient.
- We already know the gradient of AB=¾, now we need to find the y-intercept (c).
- To calculate c, we substitute a point for x and y, leaving only c as a variable.
- Don't forget to write the full equation as your final answer.

K Find the equation of a the line, which is perpendicular to AB, and passes through the point (6;-5).

$$y = -\frac{4}{3}x + c$$
$$-5 = -\frac{4}{3}(6) + c$$
$$-5 = -8 + c$$
$$c = 3$$
$$\therefore y = -\frac{4}{3}x + 3$$

- Since the lines are perpendicular, their gradients will be opposite (opposite in sign and opposite fraction). You can also use that the product of the gradients will be -1 to find the perpendicular gradient.
- Next, we need to find c so we substitute in a point for x and y so have c as the only variable.
- Don't forget to write the full equation as your final answer.

IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

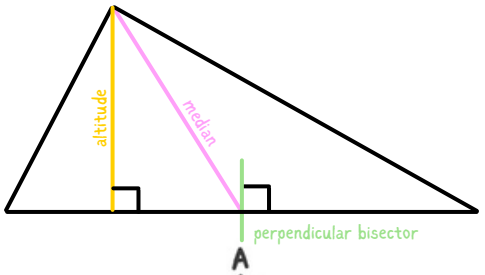
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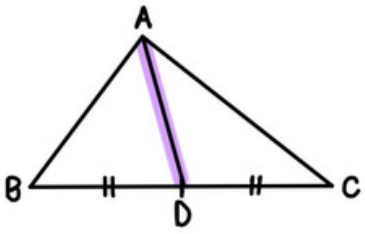
Question Type 3:
Determining the equations of lines in triangles

There are three special lines that can be found in a triangle:

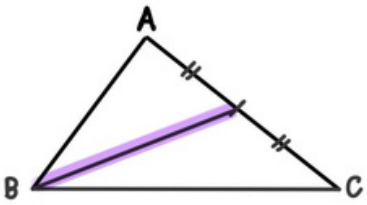
- The **altitude** is perpendicular to the base, and passes through the opposite vertex (it does not pass through the midpoint of the base).
- The **perpendicular bisector** is perpendicular to the base, and passes through the midpoint of the base (it does not pass through the opposite vertex).
- The **median** passes through the midpoint of the base, and passes through the opposite vertex (it is not perpendicular to the base).

	perpendicular	midpoint	opposite vertex
altitude	✓	✗	✓
perpendicular bisector	✓	✓	✗
median	✗	✓	✓

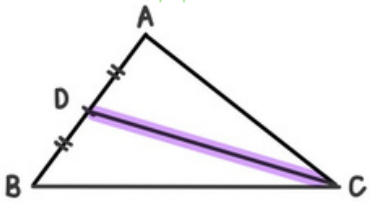




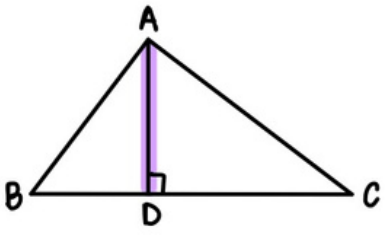
The median from vertex A which bisects the opposite side BC.



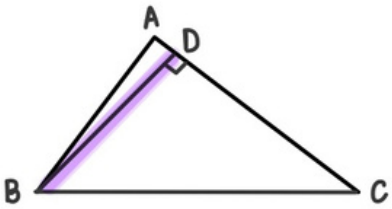
The median from vertex B which bisects the opposite side AC.



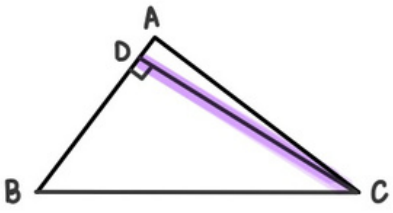
The median from vertex C which bisects the opposite side AB.



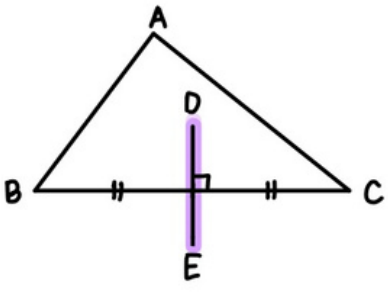
The altitude from vertex A, which is perpendicular to BC.



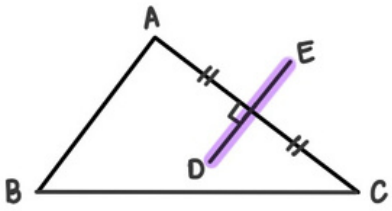
The altitude from vertex B, which is perpendicular to AC.



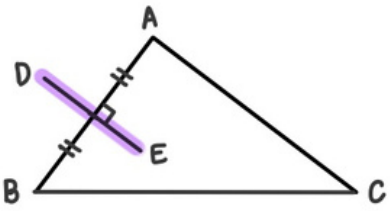
The altitude from vertex C, which is perpendicular to AB.



The perpendicular bisector of BC, which passes through the midpoint and is perpendicular to BC.



The perpendicular bisector of AC, which passes through the midpoint and is perpendicular to AC.

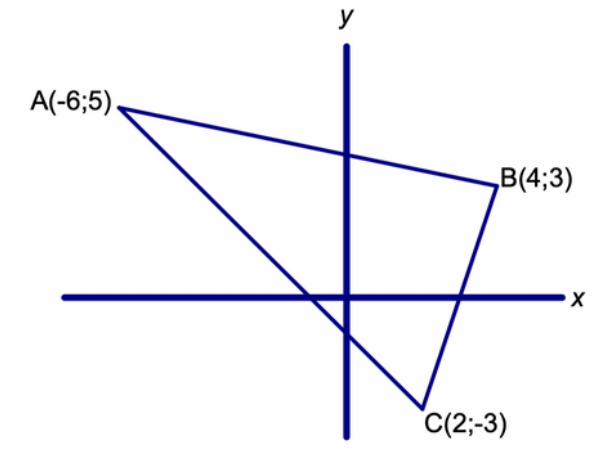


The perpendicular bisector of AB, which passes through the midpoint and is perpendicular to AB.

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Worked example

Given ΔABC with $A(-6;5)$, $B(4;3)$ and $C(2;-3)$, determine the following:



K The midpoint of AC.

$$M\left(\frac{-6+2}{2};\frac{5+(-3)}{2}\right)$$

$$M(-2;1)$$

- To find the midpoint of A and C, we substitute those coordinates into the midpoint formula.
- We aren't told what to call the coordinate of the midpoint so we can call it M since there isn't already an "M" in the diagram.

K The gradient of AC.

$$m_{AC} = \frac{5-(-3)}{-6-2}$$

$$m_{AC} = -1$$

- To calculate the gradient of the line passing through points A and C, we substitute both coordinates into the gradient formula.
- Don't forget to simplify your answer as much as possible.

R The perpendicular bisector of AC.

$$y = 1x + c$$

$$5 = 1(-6) + c$$

$$c = 11$$

$$\therefore y = x + 11$$

- To find the equation of a perpendicular bisector you need to gradient of the line AC to then find the opposite gradient (perpendicular gradient) and you need the midpoint of the line.
- We have already calculated the gradient of AC, so we can go straight to using the opposite gradient (opposite gradients are formally called negative reciprocals and will have a product of -1).
- We have also already calculated the midpoint of AC as M(-6;5) so we can substitute this coordinate into our straight line equation to solve for c.
- Don't forget to write the full equation in standard form as your final answer.

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R The altitude of AC.

$y = 1x + c$

$3 = 1(4) + c$

$c = -1$

$\therefore y = x - 1$

- The altitude of a line will be perpendicular and pass through the opposite vertex (in this case B is the opposite vertex).
- Using the gradient we already calculated for AC we can say the perpendicular gradient is 1 (as we did in the previous question).
- Now we substitute the opposite vertex, B(4;3), as x and y to solve for c.
- Don't forget to write the full equation in standard form as your final answer.
- The altitude of a triangle can also be called the perpendicular height, which is useful for calculating the area of the triangle.



The vertical height from the base of a mountain to its peak is often called its altitude. It is the perpendicular height.

R The median of AC.

$m_{MB} = \frac{3-1}{4-(-2)}$

$m_{MB} = \frac{1}{3}$

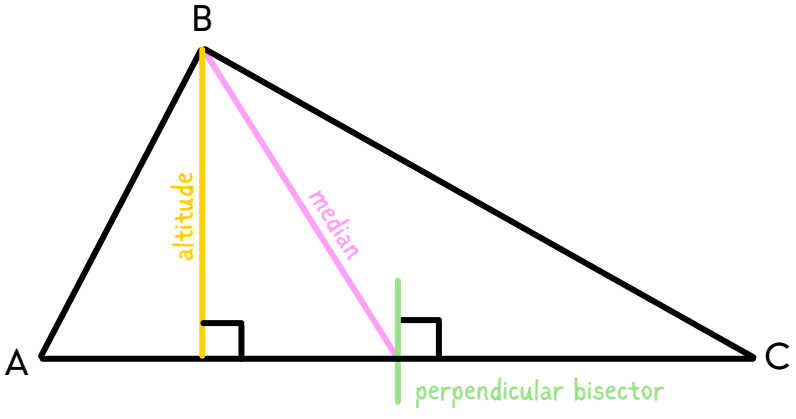
$y = \frac{1}{3}x + c$

$3 = \frac{1}{3}(4) + c$

$c = \frac{5}{3}$

$\therefore y = \frac{1}{3}x + \frac{5}{3}$

- To find the median we need to find the equation of the line through the midpoint of AC and the opposite vertex, B.
- This means we need to calculate a new gradient, MB, by substituting the coordinates of M and B into the gradient formula.
- Now, to find c, you can either substitute M or B as the values for x and y. Either way, you will get the same answer for c.
- Don't forget to write the full equation in standard form as your final answer.

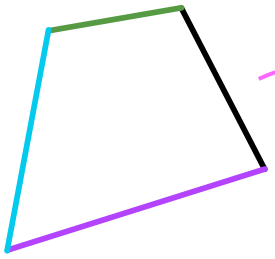


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Question Type 4: Properties of quadrilaterals

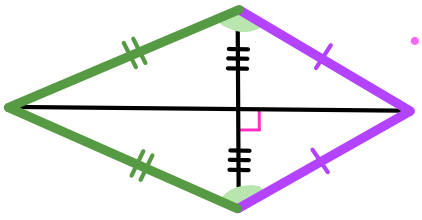
Quadrilaterals Family Tree

Quadrilateral



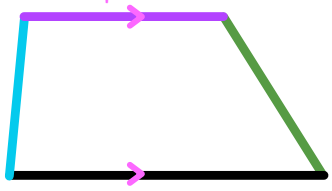
- Interior angles add up to 360°

Kite



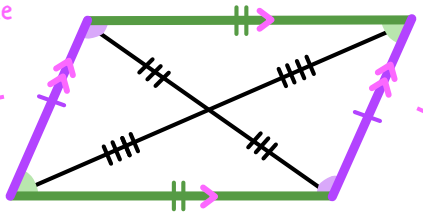
- 2 pairs adjacent sides equal
- Diagonals perpendicular, with one diagonal bisected

Trapezium



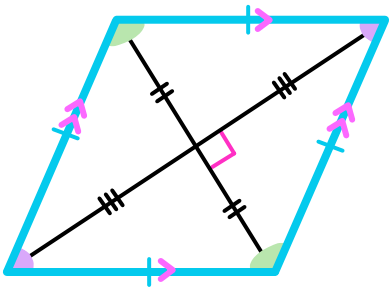
- 1 pair opposite sides parallel

Parallelogram



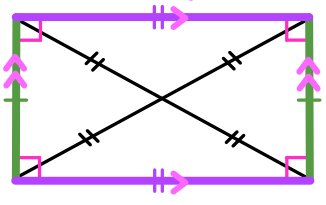
- 2 pairs opposite sides parallel
- 2 pairs opposite sides equal
- Diagonals bisect each other
- 2 pairs opposite angles equal

Rhombus



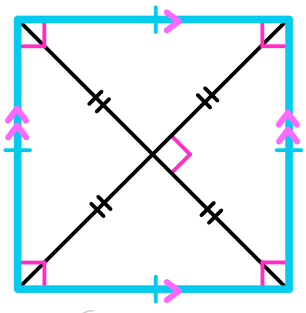
- All sides equal
- Diagonals bisect each other at right-angles
- Diagonals bisect interior angles

Rectangle



- Interior angles are right-angles
- Diagonals are equal in length

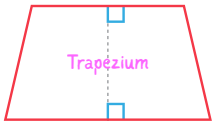
Square



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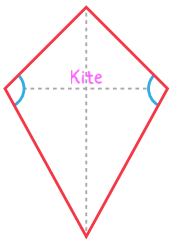
Proving the different quadrilaterals:

To prove lines are **parallel** → equal gradients
To prove lines are **perpendicular** → product of gradients = -1
To prove lines **bisect** → prove same midpoint
To prove **equal lengths** (sides/diagonals) → prove same distance



To prove a quadrilateral is a **trapezium**:

- The quadrilateral has one pair of opposite sides parallel



To prove a quadrilateral is a **kite**:

- The quadrilateral has two pairs of adjacent sides equal
- One diagonal bisects the other diagonal at right angles



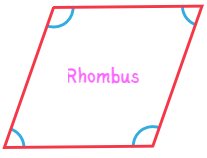
There are five ways to prove a quadrilateral is a **parallelogram**:

- The diagonals bisect each other
- Both pairs opposite sides are parallel
- Both pairs of opposite sides are equal in length
- One pair of opposite sides are equal and parallel
- Both pairs of opposite angles are equal



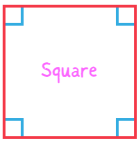
To prove a quadrilateral is a **rectangle**:

- A parallelogram with one interior right angle.
- The diagonals of the quadrilateral are equal in length and bisect each other.



To prove a quadrilateral is a **rhombus**:

- A parallelogram with a pair of adjacent sides equal
- All four sides are equal in length
- The diagonals bisect at right angles



To prove a quadrilateral is a **square**:

- The quadrilateral is a parallelogram with one interior right angle and a pair of adjacent sides equal.
- The quadrilateral is a rectangle with one pair of adjacent sides equal
- The quadrilateral is a rectangle with diagonals bisecting at right angles.
- The quadrilateral is a rhombus with an interior right angle.
- The quadrilateral is a rhombus with equal diagonals.

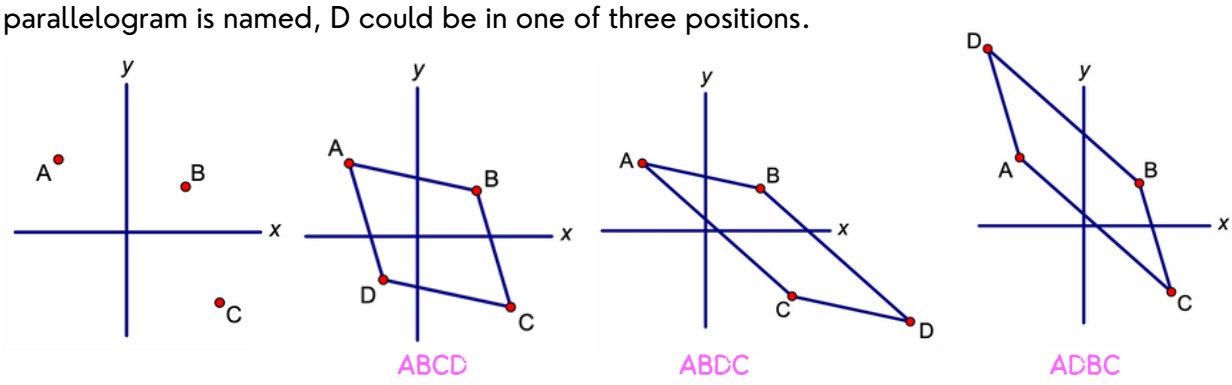
Property pairs - if your quadrilateral has one of these properties, it will also have the pair property.

Adjacent sides equal ⟷ diagonals perpendicular
Sides perpendicular ⟷ diagonals equal in length

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To find the fourth vertex in a parallelogram we use the method of inspection. If you have a parallelogram called ABCD, to get from point A to B you will follow the same path as from D to C (or from B to C is the same path as from A to D) ∴ we can use the known path to find the missing vertex.

Take note of the order of the letters in naming the parallelogram. Consider points A, B and C drawn below. If you need to add point D to make a parallelogram, depending on how the parallelogram is named, D could be in one of three positions.



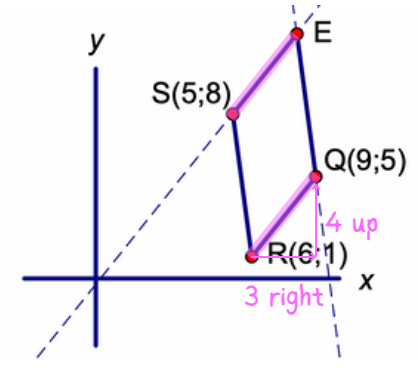
These questions are usually marked as answer only, which means there are no marks awarded to the working out. If you accidentally place the vertex in the wrong position, you will get zero for the question so double check the name of the parallelogram before you commit to your answer.

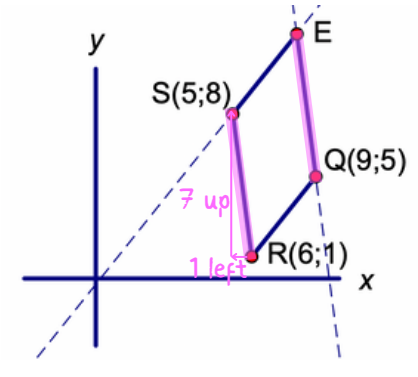
Worked example

The points S(5;8), Q(9;5), R(6;1) and E(a;b) are the vertices of a parallelogram.

R Find the coordinates of E if SEQR is a parallelogram.

E(8;12)





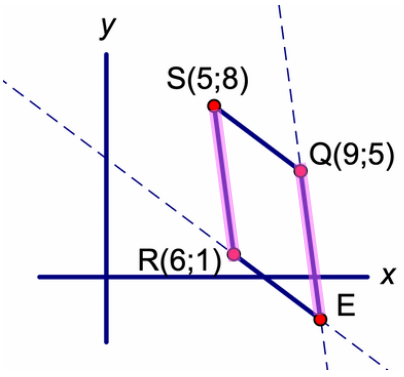
- Pay very close attention to the order of the letters. since the parallelogram is called SEQR. we are moving clockwise with E between S and Q.
- It helps to draw roughly where the vertices will be to give you an idea of a realistic answer.
- Since SEQR is a parallelogram. we have two pairs of opposite sides that are parallel and equal.
- Either we can say the path from R → Q is the same as S → E or we can use R → S = Q → E.
- Let's us R → Q first. To move from R to Q. we move 3 units right and 4 units up ∴ we add 3 to the x-coordinate and add 4 to the y-coordinate of S.
- E(5+3;8+4) ∴E(8;12)

- OR we can use R → S: move left 1 unit and up 7 units ∴ we apply the same movement to Q to get E(9-1;5+7) ∴E(8;12)

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Find the coordinates of E if SREQ is a parallelogram.

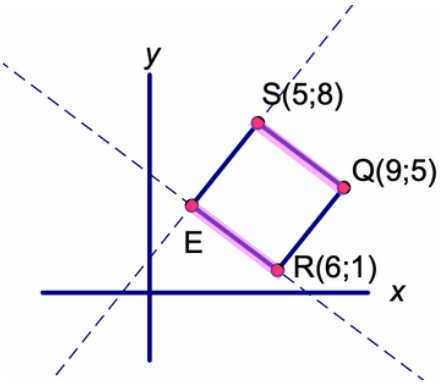
E(10:-2)



- Now, the coordinates have been named in anti-clockwise order with E between R and Q.
- To find E, we will copy the same path as S → R = Q → E.
- You always want to head towards the unknown coordinate (in this case, E).
- From S to R we move 1 right and 7 down
- Therefore, take Q's coordinate and do the same
- E(9+1:5-7) ∴ E(10:-2)
- You could also use S → Q = R → E

Find the coordinates of E if SERQ is a parallelogram

E(2:4)



- These coordinates are also named in anticlockwise order with E between S and R.
- Here we are using Q → S = R → E
- The path from Q to S is move left 4 and up 3
- Therefore from R, we also move left 4 and up 3
- E(6-4:1+3) ∴E(2:4)
- You could also use Q → R = S → E

Is SERQ a parallelogram, rectangle, rhombus or square?

- We already know SERQ is a parallelogram, so we don't need to prove this but we do need to decide if it's a more specific parallelogram (a rectangle, rhombus or square) or just a basic parallelogram.
- To prove whether or not it is a rectangle or a square, we can check whether the sides meet at a right angle.
- To determine if the sides are perpendicular, we find the gradients of any two adjacent sides.
- Now, check whether the product of their gradients is -1, since it is, the parallelogram is now either a rectangle or a square and definitely not a rhombus.
- To check whether SERQ is a rectangle or a square we can check if the sides are equal in length. The opposite sides will already be equal (since it is a parallelogram) so we need to find the lengths of any two adjacent sides.
- Since the adjacent sides are equal in length (have the same distance), the quadrilateral is a square.
- You could also have proved the diagonals were perpendicular (which would make the quadrilateral either a rhombus or a square) and you could check whether the diagonals are equal in length (which would make the quadrilateral either a rectangle or a square).

$$m_{SQ} = \frac{5-8}{9-5}$$

$$m_{SQ} = -\frac{3}{4}$$

$$m_{QR} = \frac{1-5}{6-9}$$

$$m_{QR} = \frac{4}{3}$$

$$\therefore m_{SQ} \times m_{QR} = -1 \ \therefore SQ \perp QR$$

$$SQ = \sqrt{(5-9)^2 + (8-5)^2}$$

$$SQ = 5$$

$$QR = \sqrt{(9-6)^2 + (5-1)^2}$$

$$QR = 5$$

$$\therefore SQ = QR$$

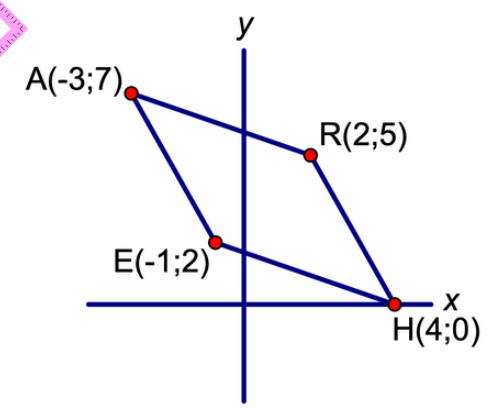
SERQ is a square (parm with adj sides perp and adj sides =)

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Worked example

Given H(4;0), E(-1;2), A(-3;7) and R(2;5) are the vertices of quadrilateral HEAR, prove that the quadrilateral is a:

R parallelogram



$M_{AH}\left(\frac{-3+4}{2};\frac{7+0}{2}\right)$

$M_{AH}\left(\frac{1}{2};\frac{7}{2}\right)$

$M_{RE}\left(\frac{-1+2}{2};\frac{2+5}{2}\right)$

$M_{RE}\left(\frac{1}{2};\frac{7}{2}\right)$

∴ HEAR is a parallelogram (diags bisect each other)

- The quickest and easiest method to prove a quadrilateral is to prove whether the diagonals bisect.
- To do this, first draw a rough sketch to determine which vertices are opposite each other.
- Next, find the midpoints of the two diagonals.
- If the midpoints are the same, the diagonals bisect and the quadrilateral is a parallelogram.

R rhombus

$AR = \sqrt{(-3-2)^2+(7-5)^2}$

$AR = \sqrt{29}$

$RH = \sqrt{(2-4)^2+(5-0)^2}$

$RH = \sqrt{29}$

∴ HEAR is a rhombus (parm with adj sides =)

- To prove this parallelogram is a rhombus, either you need the adjacent sides to be equal (since the opposite sides of a parallelogram are already equal, if you prove one pair of adjacent sides are equal, this will prove all the sides are equal).

OR

$m_{AH} = \frac{7-0}{-3-4}$

$m_{AH} = -1$

$m_{RE} = \frac{5-2}{2-(-1)}$

$m_{RE} = 1$

$\therefore m_{AH} \times m_{RE} = -1$

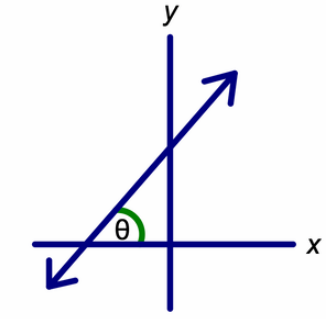
∴ HEAR is a rhombus (parm with diags perpendicular)

- Or, you can prove the diagonals are perpendicular by calculating the gradients of the diagonals and checking whether the product of the diagonals is -1.

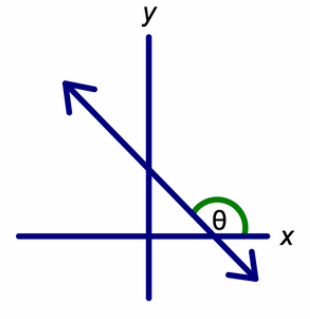
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Question Type 5:
Finding the angle of inclination

The angle of inclination of a line is the angle between that line and the positive x-axis. See the images below for examples of angles of inclination.



$m > 0$ ∴ θ is acute



$m < 0$ ∴ θ is obtuse

We use the following formula to calculate the angle inclination of a line:

$\tan\theta = m$, where m is the gradient and θ is the angle of inclination.

If $m > 0$, $\theta = \tan^{-1}(m)$

If $m < 0$, $\theta = 180° - \tan^{-1}(m)$

To find an angle that is not on the x-axis (usually an angle in a triangle):

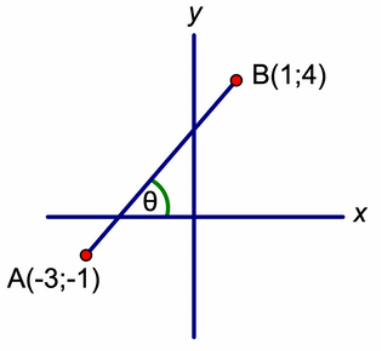
- First find the gradients of both the lines that meet at that angle
- Then find the angle of inclination of each of the lines (usually we label these as α and β).
- Lastly, you need to combine these two angles to find the angle you want. Sometimes you need to subtract both the angles (α and β) from 180, or subtract them from each other.

Remember that in order to solve for the angle, you have to type [SHIFT] and then [tan] into the calculator.

Worked example

Determine the size of θ in the diagrams below:

R A(-3;-1) and B(1;4)



$$m_{AB} = \frac{4-(-1)}{1-(-3)}$$

$$m_{AB} = \frac{5}{4}$$

$$\tan\theta = \frac{5}{4}$$

$$\theta = \tan^{-1}\left(\frac{5}{4}\right)$$

$$\theta = 51{,}34°$$

- To calculate the angle of inclination between the x-axis and a line, we need the gradient of that line.
- Since the gradient is positive, the angle will be an acute value.
- Now, we set the gradient equal to $\tan\theta$
- To calculate θ, use "SHIFT" and "tan" on the calculator and type the gradient in the brackets.
- Round your answer to two decimal places, unless told otherwise.

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R A(-2;3) and B(4;-3)

$$m_{AB} = \frac{3-(-3)}{-2-(4)}$$

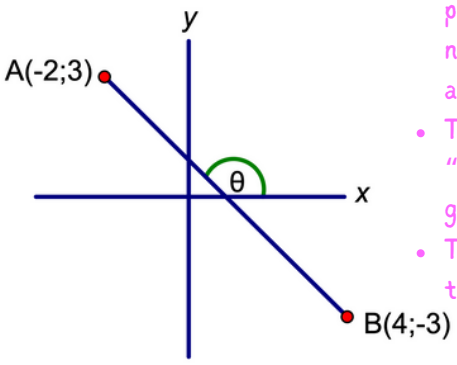
$$m_{AB} = -1$$

$$\tan\theta = -1$$

$$\theta = 180° - \tan^{-1}(1)$$

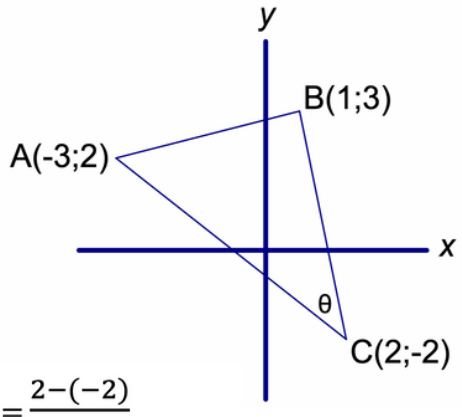
$$\theta = 180° - 45°$$

$$\theta = 135°$$



- Here, we do the same method as the previous question except this gradient is negative, therefore θ will be an obtuse angle.
- To calculate θ, we still use "SHIFT" and "tan" of the gradient, but we make the gradient positive to get an acute value.
- Then, since tan is negative in Q2 we use the Q2 reduction of 180-(acute angle).

R A(-3;2), B(1;3) and C(2;-2)



$$m_{AC} = \frac{2-(-2)}{-3-2}$$

$$m_{AC} = -\frac{4}{5}$$

$$\tan\alpha = -\frac{4}{5}$$

$$\alpha = 180° - \tan^{-1}\left(\frac{4}{5}\right)$$

$$\alpha = 180° - 38{,}65\ldots°$$

$$\alpha = 141{,}34\ldots°$$

$$m_{BC} = \frac{3-(-2)}{1-2}$$

$$m_{BC} = -5$$

$$\tan\beta = -5$$

$$\beta = 180° - \tan^{-1}(5)$$

$$\beta = 180° - 78{,}69\ldots°$$

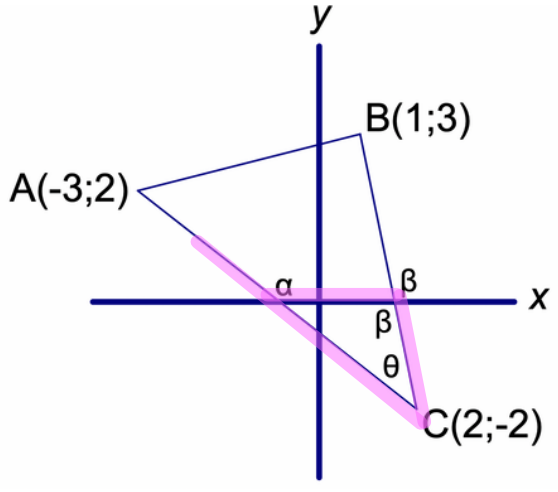
$$\beta = 101{,}30\ldots°$$

$$\theta = \alpha - \beta \qquad (ext\angle of\ \Delta)$$

$$\theta = 141{,}34\ldots° - 101{,}30\ldots°$$

$$\theta = 40{,}03°$$

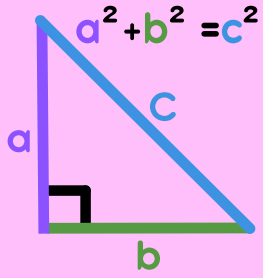
- The angle of inclination only determines angles between the x-axis and the line, therefore we need to use these angles to calculate θ.
- First, you need to find the gradients of both the lines that meet at θ (at C).
- Once you have the gradients of AC and BC, you need to find each of their angles of inclination. It is helpful to assign some new values in the sketch. Here, we have used α as the angle for AC and β as the angle for BC.
- Now, calculate the values of both α and β using "SHIFT" and "tan" of each gradient. Keep in mind both gradients are negative, therefore you need to use 180-(acute angle) having used the gradient as positive.
- In the diagram below the exterior angle of the triangle has been highlighted in pink to help you visualise the relationship more clearly.
- We can place β inside the triangle using vertically opposite angles.
- Since α=θ+β then θ=α-β



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Question Type 6:
Calculate the area of a figure

You can be asked to find the areas of various shapes, most often triangles and quadrilaterals. To find the area of a triangle, you first need to establish whether it is right-angled or not as this determines which of the two formulae you can use. If you need to calculate the area of a quadrilateral you are not familiar with, you can rather break it down into two triangles and add the two triangles' areas together.

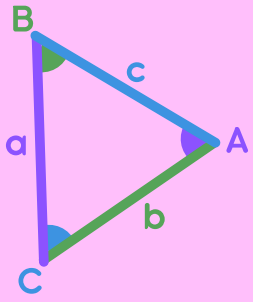


$$Area = \frac{1}{2}(base)\,(height)$$

$$Area = \frac{1}{2}ba$$

$$Area = \frac{1}{2}ab$$

To find the area of a right-angled triangle, use the lengths of **two sides** that **meet at the right-angle** (do not use the hypotenuse). It does not matter which is considered the "base" and which side is the "perpendicular height" as long as those two sides meet at $90°$.



$$Area = \frac{1}{2}ab\sin C$$

$$Area = \frac{1}{2}bc\sin A$$

$$Area = \frac{1}{2}ac\sin B$$

To find the area of a non-right-angled triangle, you need the lengths of **two sides** and the angle between those sides. It must be the **included angle**.

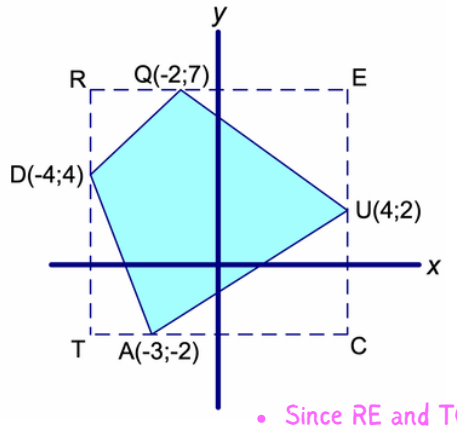
PRO-TIP

Most often, you will have already calculated some of the lengths or angles needed before being asked to find the area. Do not recalculate them. You may use information you already have and only calculate any missing information if you don't already have everything you need.

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Worked example

RECT is a rectangle with RE||TC||x-axis and RT||EC||y-axis. Points Q, U, A and D lie on the sides of the rectangle.



- Since RE and TC are horizontal lines, each pair will have the same y-value as any other coordinate on that line, therefore the y-value of R and E is the same as the y-value of Q (7) and the y-value of T and C is the same as the y-value of A (-2).
- Similarly, the x-values of R, D and T are all -4 and the x-values of E, U and C are all 4.
- The length of RT (or EC) = 7-(-2)=9
- The length of RE (or TC) = 4-(-4)=8

C Calculate the area of rectangle RECT.

$R(-4;7) \quad E(4;7) \quad C(4;-2) \quad T(-4;-2)$

$\therefore Area\ RECT = 8 \times 9$

$= 72 \text{ units}^2$

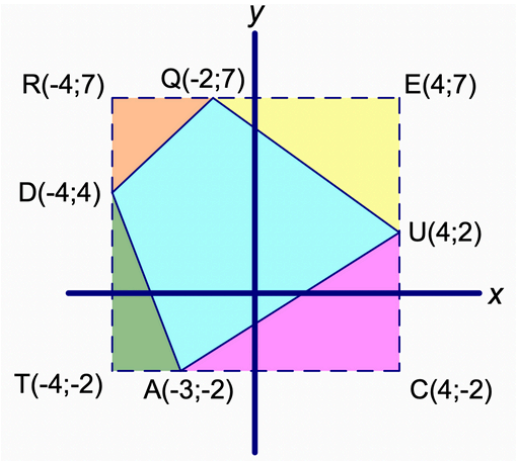
P Calculate the area of the shaded quadrilateral, QUAD.

$Area\ QUAD = 72 - Area\ \Delta DRQ - Area\ \Delta QEU - Area\ \Delta UCA - Area\ \Delta ATD$

$= 72 - \left(\frac{1}{2} \times 3 \times 2\right) - \left(\frac{1}{2} \times 6 \times 5\right) - \left(\frac{1}{2} \times 7 \times 4\right) - \left(\frac{1}{2} \times 1 \times 6\right)$

$= 72 - 3 - 15 - 14 - 3$

$= 37 \text{ units}^2$



- To find the area of the shaded quadrilateral, we need to subtract the areas of the four triangles from the area of RECT.
- The calculation for each triangle's area has been highlighted to match the shading in the diagram.
- All the triangles are right-angled (since they share an angle with the rectangle) and you can use either perpendicular side as the base or the height.

P If A is reflected in the line RE, determine the coordinates of A'.

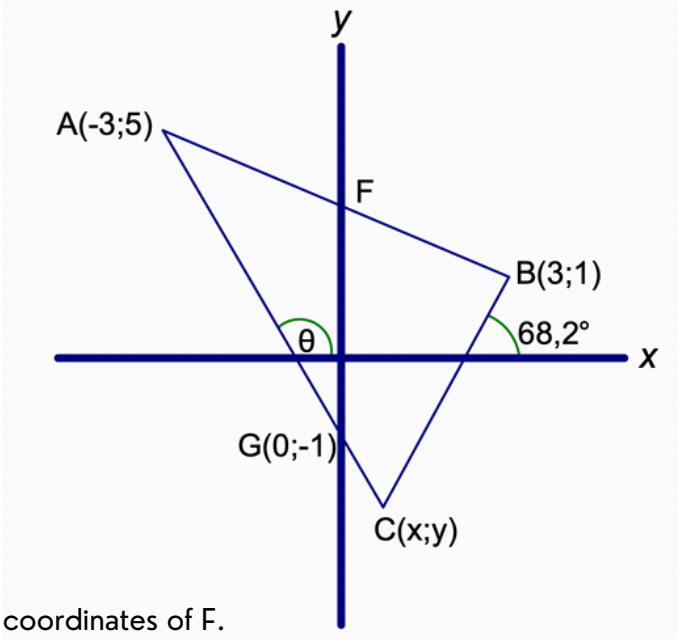
A'(-3;16)

- When you reflect a point in a horizontal line, the x-value will stay the same.
- Currently, A is 9 units below the line RE, so after being reflected it will now be 9 units above the line ∴A'(-3;7+9)=A'(-3;16)

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Worked example

△ABC is drawn with vertices A(-4;10), B(3;1) and C(x;y). F is midpoint of AB and G(0;-1) is the y-intercept of AC. The angle between BC and the axis is $68,2°$ and the angle between AC and the x-axis is θ.



K Determine the coordinates of F.

$$F\left(\frac{-3+3}{2};\frac{5+1}{2}\right)$$

$$F(0;3)$$

- Since F is the midpoint of AB, we substitute the coordinates of A and B into the midpoint formula.
- Make sure to call the coordinate F, not M.

R Determine the magnitude of θ, correct to one decimal place.

$$m_{AG} = \frac{5-(-1)}{-3-0}$$

$$m_{AG} = -2$$

$$\tan\theta = -2$$

$$\theta = 180° - \tan^{-1}(2)$$

$$\theta = 180° - 63,43°$$

$$\theta = 116,6°$$

- To calculate θ we first need to gradient of AG (or AC since they are the same line but we don't know C yet so we will use AG here).
- The gradient of AG is negative, therefore we use "SHIFT" and "tan" of the positive gradient and use the Q2 reduction of 180- (acute angle).
- The question specified to round off to one decimal place so be sure to read the question and any instructions carefully to avoid losing unnecessary rounding marks.

R Determine the equation of BC.

$$m_{AG} = \tan(68,2°)$$

$$m_{AG} = 2,50$$

$$y = 2,5x + c$$

$$1 = 2,5(3) + c$$

$$c = -6,5$$

$$\therefore y = 2,5x - 6,5$$

- To find the equation of BC we need to gradient and the y-intercept.
- Usually, to find the gradient we would use two points on the line and substitute in to the gradient formula but here, we only have one coordinate.
- Instead, we have been given the angle of inclination so we can use that to find the gradient, and then substitute the only coordinate we have on the line, B, to calculate c (the y-intercept).

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C Hence, given BC= $\sqrt{29}$, show the coordinates of C are (1;-4).

$$BC = \sqrt{(x-3)^2 + (y-1)^2}$$
$$\sqrt{29} = \sqrt{(x-3)^2 + (2{,}5x - 6{,}5 - 1)^2}$$
$$29 = (x-3)^2 + (2{,}5x - 7{,}5)^2$$
$$29 = x^2 - 6x + 9 + 6{,}25x^2 - 37{,}5x + 56{,}25$$
$$0 = 7{,}25x^2 - 43{,}5x + 36{,}25$$
$$x = \frac{-(-43{,}5) \pm \sqrt{(-43{,}5)^2 - 4(7{,}25)(36{,}25)}}{2(7{,}25)}$$
$$x = 1 \text{ or } x = 5$$
$$y = 2{,}5(1) - 6{,}5$$
$$y = -4$$
$$C(1;-4)$$

- Given the length (distance) of BC we can substitute the coordinates of B and C into the distance formula (with C's coordinates being x and y since we don't know their values yet).
- We can set the value of the distance to $\sqrt{29}$ since we were given the value in the question.
- We can also replace the y-value with the equation we found in the previous question as this was the equation for the line BC therefore the y-value of the coordinate of C and the y-value of the equation are the same y.
- We are now solving this equation simultaneously using the substitute method.
- Simplify the bracket and square both sides to get rid of the square roots.
- Simplify the equation by moving all terms to the LHS and collecting like terms.
- Use the quadratic formula to solve for x. Since C was left of B in the diagram, the x value should be less than 3, therefore we choose x=1.
- Now substitute x=1 into the equation of BC to solve for y.
- Double check you have the same values as you were told to show in the question.

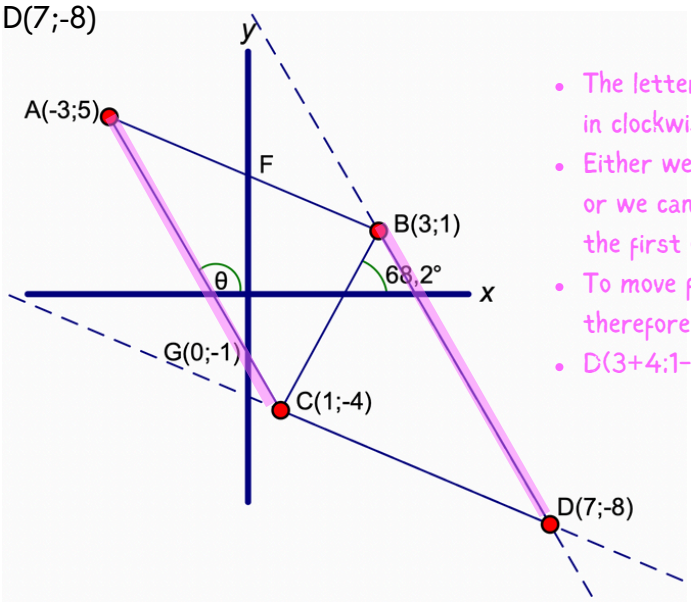
K Determine the length of AC. Leave your answer in surd form.

$$AC = \sqrt{(-3-1)^2 + \left(5-(-4)\right)^2}$$
$$AC = \sqrt{97}$$

- Using the coordinate of C we found (or given in the previous question) and of A, we substitute these values into the distance formula to find the length of AC.
- The previous question was far more complicated than this question. This is a reminder that if you find a question challenging, don't give up on the question that follow. They might be much easier!

R Given ABDC is a parallelogram, write down the coordinates of D.

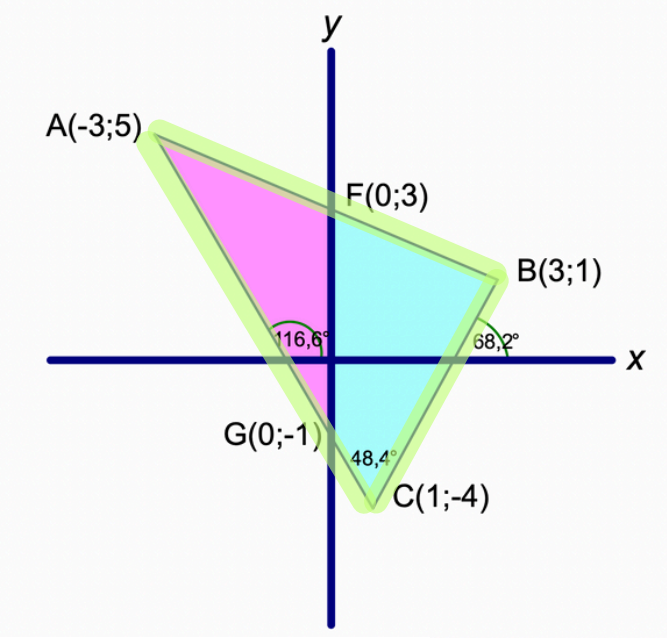
D(7;-8)



- The letters of this parallelogram have been labeled in clockwise order with D between B and C.
- Either we can use the path from A → C = B → D or we can use A → B = C → D (here we are using the first option).
- To move from A to C, we move 4 right and 9 down therefore from B we do the same to get to D.
- D(3+4;1-9) ∴D(7;-8)

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P Determine the area of quadrilateral FBCG.



$A\hat{C}B = 116{,}6° - 68{,}2° \qquad (ext\angle of\ \Delta)$
$A\hat{C}B = 48{,}4°$

$Area\ FBCG = Area\ \Delta ABC - Area\ \Delta AFG$

$Area\ \Delta AFG = \frac{1}{2}(4)(3)$
$= 6\ \text{units}^2$

$Area\ \Delta ABC = \frac{1}{2}(\sqrt{97})(\sqrt{29})\sin(48{,}4°)$
$= 19{,}83\ \text{units}^2$

$Area\ FBCG = 19{,}83 - 6$
$= 13{,}83\ \text{units}^2$

- To find the area of the quadrilateral we need to find the area of the larger triangle ABC and subtract the smaller triangle AFG.
- To find the area of triangle AFG we can use FG as the base (3--1=4) and the perpendicular height from FG to A is 3.
- To find the area of triangle ABC, we need an angle (since the triangle is not right-angled).
- Using the information we already have, it will be easiest to find angle C, therefore we need the lengths of side AC (already calculated) and BC (given previously).
- Substitute these value into the Area rule formula.
- Now, subtract the area of the smaller triangle from the larger triangle to find the area of the quadrilateral.

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TRIGONOMETRY

<u>In order to achieve full marks for this section, you need to be able to:</u>

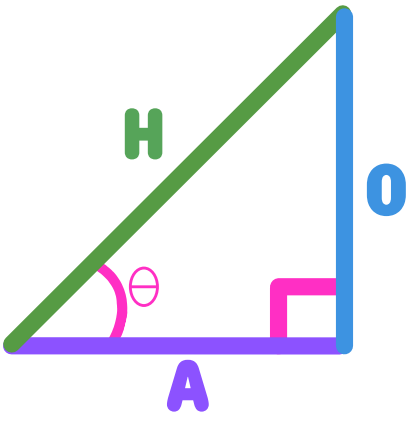
- prove trigonometric **identities**
- simplify a trigonometric expression using **reduction formulae**
- **draw a right-angled triangle** using given information
- use a right-angled triangle to write **expressions for trigonometric ratios**
- solve trigonometric **equations**
- work with **special triangles** to simplify expressions without using a calculator

There are 6 typical questions to expect in any assessment on Trig:

1. Simplifying a trigonometric expression
2. Constructing a triangle
3. Proving a trigonometric identity
4. Solving a trigonometric equation
5. 2D Trig
6. Trig Graphs

We will look at each of these types of questions separately, but let's first look at some revision of basic information that you need to know.

<u>The definitions of the trigonometric ratios</u>



In Grade 10 we defined three ratios in a right-angled triangle using an angle, θ, and the sides (either the **H**ypotenuse, **O**pposite side or **A**djacent side). If you are struggling to label the sides correctly, remember the hypotenuse is always opposite the right angle and θ is always between the hypotenuse and the adjacent side.

SOHCAHTOA

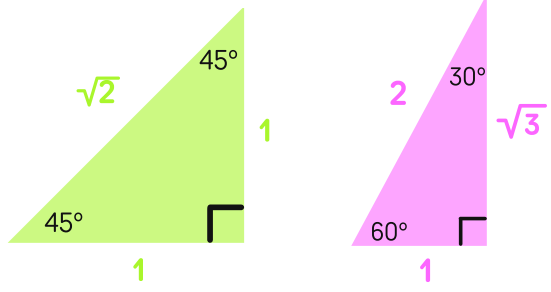
$\sin\theta = \frac{O}{H}$ $\cos\theta = \frac{A}{H}$ $\tan\theta = \frac{O}{A}$

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Question Type 1:
Simplifying trigonometric expression

Special angles:

There are 3 **special angles** that you can be used with one of the Trig ratios and you need to be able to find these values without the use of a calculator. These 3 special angles are shown in the special triangles below and if you can remember the special triangles, you can use them to write all the Trig ratios.



θ	30^o	45^o	60^o
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

- Take note of the three fractions that have been highlighted. These fractions all have roots (surds) in their denominator, which means when you type them into your calculator, your calculator will simplify these fractions and give you an answer that looks different, but has the same value (it has been rationalised to make the denominator a rational number).
- All the other fractions you can get from your calculator without using the special triangles, but these three are different. Some schools might not award full marks if you use the values from the calculator for these three, so please check with your teacher if you need to memorise them.
- Some questions are also made easier by using the values in the table, rather than the values on the calculator, so it is useful to know them.

Worked example

R Simplify the following, without the use of a calculator:

$$\frac{\sin 60°.\tan 45°}{\tan 30°}$$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)(1)}{\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{1}$$

$$= \frac{3}{2}$$

- First, we use our special triangles to determine the numerical value of each Trig ratio.
- Next, in order to divide two fractions we keep the first fraction the same, swop the divide to multiply, and flip the second fraction over.
- When multiply two square roots together, either we can say that a root and a square cancel each other out, $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$ or we can say $\sqrt{3} \times \sqrt{3} = \sqrt{3 \times 3} = \sqrt{9} = 3$

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Reduction formulae:

Quadrant II

$\sin\theta = \frac{y}{r}$ ⊕

$\cos\theta = \frac{-x}{r}$ ⊖

$\tan\theta = \frac{y}{-x}$ ⊖

$\therefore$ **sin θ**

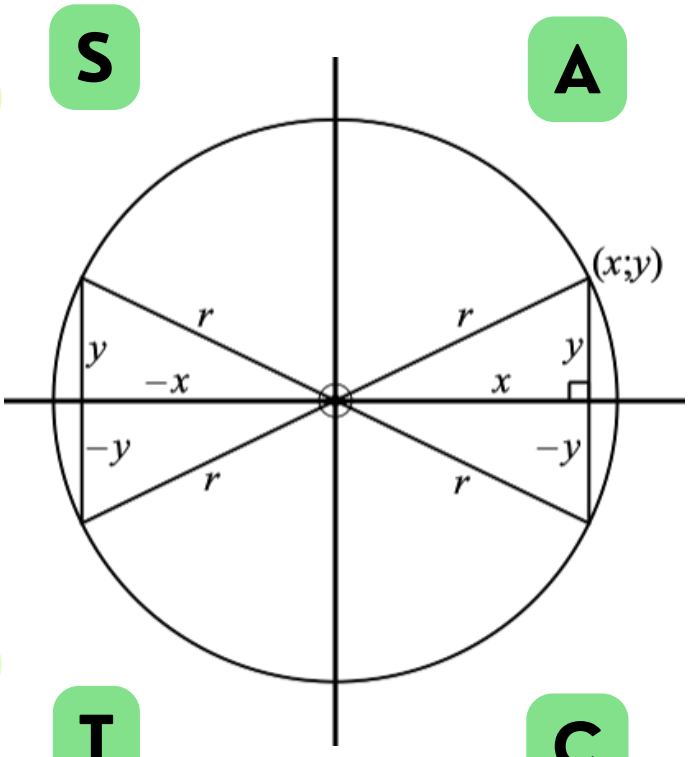
Quadrant I

$\sin\theta = \frac{y}{r}$ ⊕

$\cos\theta = \frac{x}{r}$ ⊕

$\tan\theta = \frac{y}{x}$ ⊕

$\therefore$ ***ALL***



Quadrant III

$\sin\theta = \frac{-y}{r}$ ⊖

$\cos\theta = \frac{-x}{r}$ ⊖

$\tan\theta = \frac{-y}{-x}$ ⊕

$\therefore$ **tan θ**

Quadrant IV

$\sin\theta = \frac{-y}{r}$ ⊖

$\cos\theta = \frac{x}{r}$ ⊕

$\tan\theta = \frac{-y}{r}$ ⊖

$\therefore$ **cos θ**

The Cartesian Plane is divided into four quadrants. In all the quadrants, a right-angled triangle is drawn from a co-ordinate that lies on a circle. The radius of the circle (r) is positive in all quadrants, the adjacent side (x) is positive on the right of the y-axis and the opposite side (y) is positive above the x-axis. The diagram is often referred to as the CAST diagram as it shows which trigonometric ratio is positive in each quadrant: Q1 = All, Q2 = Sinx, Q3 = Tanx, Q4 = Cosx.

The purpose of the CAST diagram is to reduce any trigonometric ratio with an angle greater than 90° to an acute angle in the first quadrant since they have the same value, once taking into account whether the ratio should be positive or negative.

Worked example

K Simplify the following:

Q2

$\cos(180^\circ - \theta)$

$= -\cos\theta$

Q1

$\tan(360^o + \theta)$

$= \tan\theta$

- First, look at the angle and decide which quadrant the ratio is in.
- Use the CAST diagram to decide whether the ratio is positive or negative in that quadrant.
- Reduce the angle to θ and write the correct sign with the ratio.

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Worked example

R Simplify the following:

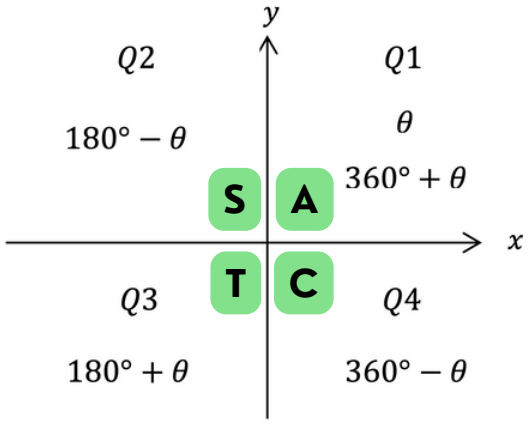
$$\frac{\cos(180°+\theta)\sin(180°-\theta)}{\cos(360°-\theta)}$$

(Q3, Q2)

$$= \frac{(-\cos\theta)(\sin\theta)}{(\cos\theta)}$$

(Q4)

$$= -\sin\theta$$



- First we use the reduction rules for each quadrant to determine whether each ratio will become positive or negative when we simplify each angle to θ.
- Once all the angles are simplified, we can identify ratios that are identical in the numerator and denominator and cancel them.
- The goal is to simplify as much as possible, hopefully to a single Trig ratio if we can.

R

$$\frac{\cos(180°+x)\tan(360°-x)}{\sin(360°+x)\cos(180°-x)}$$

(Q3, Q4, Q1, Q2)

$$= \frac{(-\cos x)(-\tan x)}{(\sin x)(-\cos x)}$$

$$= -\frac{\frac{\sin x}{\cos x}}{\frac{\sin x}{1}}$$

$$= -\frac{\sin x}{\cos x} \times \frac{1}{\sin x}$$

$$= -\frac{1}{\cos x}$$

- Take note of how many negatives there are in the question. If there is an even number, the answer will be positive and if there is an odd number of negatives, the answer will be negative.
- Often identities need to be used to further simplify the answer. Here, the tan identity was used and rather than having a fraction in a fraction, we write the denominator separately.
- Now, we can tip and times the denominator. Remember you can always write a fraction over 1 in order to flip it.

IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

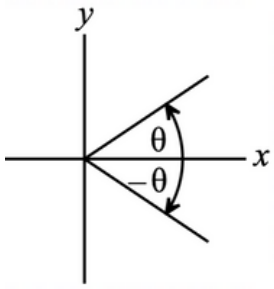
C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

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Negative angles:

When an angle is negative, it means it is rotating around the Cartesian plane in the opposite direction (i.e. a positive angle rotates anti-clockwise whereas a negative angle is clockwise). You can also add 360° to a negative angle to make it positive. We'll look at both options in the examples below.



Worked example

Simplify the following:

K

Q4

$\cos(-x)$

$= \cos x$

OR

$= \cos(-x + 360°)$
$= \cos(360° - x)$

Q4

$= \cos x$

- Shown alongside each answer in black is the alternative in green by adding 360° to get positive angles instead.
- x is an acute angle, since it is negative, we move clockwise, therefore it's in Q4, where cos is positive.
- OR, you can add 360° to -x, 360°-x is the reduction for Q4, where cos is positive.

K

Q3

$\sin(x - 180°)$

$= -\sin x$

OR

$= \sin(x - 180° + 360°)$
$= \sin(x + 180°)$
$= \sin(180° + x)$

Q3

$= -\sin x$

- x-180° first starts with a positive acute angle (Q1) but then goes backwards (clockwise) by 180° so it lands in Q3, where sin is negative.
- OR, you can add 360°, which results in 180°+x, which is the normal reduction for Q3, where sin is negative.

PRO-TIP

You can add or subtract 360° as many times as you like, it won't change the value of the expression.

K

Q1

$\tan(x - 360°)$

$= \tan x$

OR

Q1

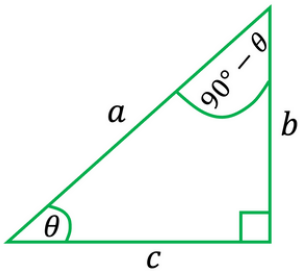
$= \tan(x - 360° + 360°)$
$= \tan x$

- Here, if we add 360° it cancels with the -360° leaving only x, which is Q1 where everything is positive.

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Co-ratios:

If we have one of the angles in a triangle is θ, we can find the other angle by using the fact that angles in a triangle add up to $180°$. This makes the other angle $90° - \theta$. Using the triangle shown below, we can see there is a **special** relationship between **sinθ** and **cosθ** when using a complementary angle. This relationship is called **co-ratios**.



$$\sin\theta = \frac{b}{a} \quad \text{and} \quad \cos(90° - \theta) = \frac{b}{a}$$
$$\cos\theta = \frac{c}{a} \quad \text{and} \quad \sin(90° - \theta) = \frac{c}{a}$$

$\cos(90° - \theta) = \sin\theta$	$\sin(90° - \theta) = \cos\theta$	$\cos(270° - \theta) = -\sin\theta$	$\sin(270° - \theta) = -\cos\theta$
$\cos(90° + \theta) = -\sin\theta$	$\sin(90° + \theta) = \cos\theta$	$\cos(270° + \theta) = \sin\theta$	$\sin(270° + \theta) = -\cos\theta$

PRO-TIP

Rather than trying to memorise all the options for co-ratios, rather decide what sign the original ratio would have in that quadrant, then switch sin↔cos

Worked example

Simplify the following:

R

$$\frac{\overset{Q1}{\cos(90° - x)}}{\sin(90° + x)}$$

$$= \frac{\sin x}{\cos x} \quad (Q2)$$
$$= \tan x$$

- Both of these angles are co-ratios so we switch cos↔sin
- $90°-x$ is in Q1, so the ratio is positive (since ALL Trig ratios are positive in Q1)
- $90°+x$ is in Q2, where sin is positive so even though the sin switches to cos, it stays positive.

R

$$\frac{\overset{Q1}{\cos(90°-x)} - \overset{Q4}{\sin(360°-x)}}{\underset{Q1}{\sin(90°-x)} - \underset{Q3}{\cos(180°+x)}}$$

$$= \frac{\sin x - (-\sin x)}{\cos x - (-\cos x)}$$
$$= \frac{2\sin x}{2\cos x}$$
$$= \tan x$$

- First, decide which quadrant each of the angles lies within
- Next, determine whether the angle is a reduction (ratio stays the same) or a co-ratio (sin ↔ cos)
- Simplify by collecting like terms
- Now that there is only one term in the numerator, and one term in the denominator, cancel the 2s.
- Use the tan identity to simplify to one Trig ratio.

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R

$$\frac{\sin 120° . \tan 300°}{\cos(-60°) . \tan 315°}$$

$$= \frac{(\sin 60°).(-\tan 60°)}{(\cos 60°).(-\tan 45°)}$$

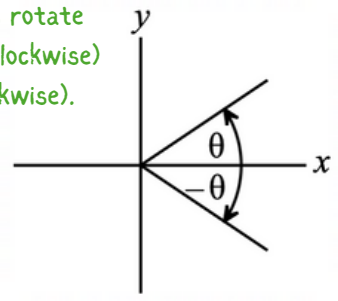
$$= \frac{\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{1}\right)}{\left(\frac{1}{2}\right)(-1)}$$

$$= \frac{-\frac{3}{2}}{-\frac{1}{2}}$$

$$= -\frac{3}{2} \times -\frac{2}{1}$$

$$= 3$$

- Remember, negative angles rotate in the opposite direction (clockwise) to positive angles (anticlockwise).



- Reduce each angle to an acute angle, keeping in mind whether the original value would be positive or negative in that quadrant.
- In order to determine the value of a Trig ratio without a calculator, either the angles must be special angles or they must cancel out.
- In order to divide by a fraction, we convert to multiply and flip the fraction over (tips and times!).

Question source: EC November 2023 Grade 11 Paper 2

R

$$\frac{\overset{Q3}{\tan 240°}.\overset{Q1}{\cos 73°}.\overset{Q4}{\sin 300°}}{\underset{Q2}{\cos 120°}.\underset{Q1}{\sin 17°}}$$

$$= \frac{\tan 60° . \cos 73° . (-\sin 60°)}{(-\cos 60) . \sin 17°}$$

$$= \frac{(\sqrt{3}).\cancel{\cos 73°}.\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right).\cancel{\cos 73°}}$$

$$= -\frac{3}{2} \times -\frac{2}{1}$$

$$= 3$$

- As with the previous question, we reduce each angle to an acute angle, keeping in mind whether the original value would be positive or negative in that quadrant.
- Some of these angles aren't special angles so they must cancel out.
- The angles $73°$ and $17°$ aren't special angles but they are complementary angles, which means we can use co-ratios to convert them to matching ratios that will cancel out.
- Either we convert $\cos 73°$ to $\sin 17°$ or we keep $\cos 73°$ in the numerator and swop $\sin 17°$ to $\cos 73°$
- You must show converting to the co-ratio before you cancel them.



IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

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Worked example

C Simplify the following, to a single trigonometric ratio:

$$\frac{\cos(540° - x)\tan(x - 180°)}{1 - \sin^2(270° + x)}$$

$$\frac{\cos(180° - x).\tan(180° + x)}{1 - (-\cos x)^2}$$

$$\frac{(-\cos x)(\tan x)}{1 - \cos^2 x}$$

$$\frac{-\cos x \times \frac{\sin x}{\cos x}}{\sin^2 x}$$

$$\frac{-\sin x}{\sin^2 x}$$

$$-\frac{1}{\sin x}$$

- First, we can simplify these angles into angles which are less than 360° we are more familiar with. This isn't necessary, but it does make the question easier.
- The first ratio has an angle more than 360° therefore we can subtract 360° to get an answer more easily recognisable in one of the four quadrants. This angle is in Q2, where cos is negative.
- The second ratio has a negative angle. We can add 360° to make the angle positive (or start moving anticlockwise by an angle of x, then move clockwise by 180°). Either way, you end up in Q3, where tan is positive.
- The third ratio has a square, which means regardless of whether the ratio ends up being positive or negative, once it is squared, it will be positive. You MUST SHOW whether the ratio is positive or negative BEFORE it is squared. Here, 270°+x is in Q3, where sin would be negative but it is also a co-ratio, so sin switches to cos.
- Now, we can use our identities to simplify the answer. In the numerator we use the tan identity and in the denominator we use the square identity.
- We can cancel ratios that appear in the numerator and denominator.
- Since have have one ratio is negative, the final answer will be negative.

PRO-TIP

If you have a squared ratio, you MUST show whether the reduced ratio is positive or negative BEFORE you square it.

Worked example

C Simplify the following, without the use of a calculator:

$$\frac{\tan 225° - \sin(180° + \theta)\cos(\theta + 90°)}{\sin\theta.\sin(360° + \theta) + \cos\theta.\cos(-\theta)}$$

$$= \frac{(\tan 45°) - (-\sin\theta).(-\sin\theta)}{\sin\theta.\sin\theta + \cos\theta.\cos\theta}$$

$$= \frac{1 - \sin^2\theta}{\sin^2\theta + \cos^2\theta}$$

$$= \frac{\cos^2\theta}{1}$$

$$= \cos^2\theta$$

- Reduce each Trig ratio to an acute angle, using brackets to show what each reduction becomes.
- You must show each reduction and not simplify in this first step, otherwise you will likely lose marks.
- Simplify each term in the numerator and denominator. You cannot cancel while you have separate terms.
- Use special angles to replace $\tan 45° = 1$
- Use the square identity to replace the numerator with a single term and simplify the denominator to 1.

Question source: GP June 2024 Grade 11 Paper 2

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Question Type 2: Constructing a triangle

When a question states that it should be done without using a calculator, that often points to one of three methods:

1. Working with **special triangles** to determine the values of special angles (see page 42)
2. Constructing an acute-angled right triangle where the **ratio is a variable**
3. Drawing a right-angled triangle on the **Cartesian plane**

When you are required to draw a triangle, you will be given a trigonometric ratio and information about the angle to determine which quadrant to draw the triangle. The triangle will ALWAYS be **right-angled** and the **hypotenuse** is ALWAYS **positive.**

Either the angle will be a variable with a value as the ratio, or the ratio will be a variable and the angle is a known value.

You will be given enough information to find all three sides of the triangle and all three of the angles, either as a value or in terms of the given variable.

Remember the definitions of the Trig ratios:

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x}$$

When given the value of a Trig ratio, we can equate the numerator of the ratio to the side of the triangle that corresponds to that relationship.
For example, if $\sin\theta = \frac{1}{2}$, the side opposite to θ would be 1 and the hypotenuse would be 2.

If the ratio doesn't have a denominator, write the ratio over 1.
For example, if $\cos\theta = m$, rewrite the ratio as $\cos\theta = \frac{m}{1}$ then the side adjacent to θ would be m and the hypotenuse would be 1.

PRO-TIP

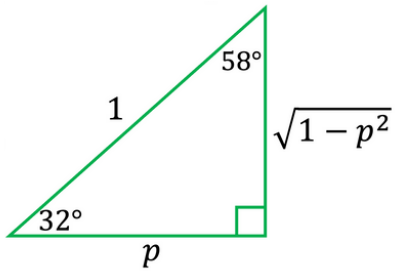
If you have two sides of a right-angled triangle, you can find the third side using Pythagoras (don't forget to write the reason **Pythag**) and if you have one other angle in the right-angled triangle, you can calculate the third angle since all the angles must add to 180° (don't forget to write **sum of ∠s in △**).

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Worked example

Given $\cos 32 = p$:

R Represent this information in a sketch.



- We need the given ratio to be written as a fraction in order to match the definition of the ratio for cos.
- $\cos 32 = \frac{p}{1} = \frac{adjacent}{hypotenuse}$
- Now we can assign the given sides to the triangle and use Pythagoras to find the missing side.
- Lastly, we complete the triangle by finding the missing angle = $180° - 90° - 32° = 58°$

R Determine the value of the following:

$$\sin\theta = \frac{opp}{hyp} \quad \cos\theta = \frac{adj}{hyp} \quad \tan\theta = \frac{opp}{adj}$$

i) $\sin 32$

$$= \frac{\sqrt{1-p^2}}{1}$$
$$= \sqrt{1-p^2}$$

- Using the constructed triangle, we can read off the sides needed to find these ratios.
- Remember, if your answer is a fraction over 1, we simplify by only writing the numerator (all numbers are over 1 unless told otherwise so keeping the 1 in the denominator is not necessary and is considered unsimplified so you can lose marks).

ii) $\cos 58$

$$= \frac{\sqrt{1-p^2}}{1}$$
$$= \sqrt{1-p^2}$$

- When applying the square to the fraction, you need to square the numerator (which cancels out the square root) and square the denominator.

iii) $1 - \tan^2 32$

$$= 1 - \left(\frac{\sqrt{1-p^2}}{p}\right)^2$$
$$= 1 - \frac{1-p^2}{p^2}$$
$$= \frac{p^2 - (1-p^2)}{p^2}$$
$$= \frac{p^2 - 1 + p^2}{p^2}$$
$$= \frac{2p^2 - 1}{p^2}$$

- Then, we combine these two separate terms by using their LCD of p^2.
- Don't forget to write your numerator in a bracket to apply the minus throughout the expression.
- Collect the like terms
- You cannot cancel the p^2 since you have more than one term in the numerator (you can only cancel in fractions when both the numerator and denominator are only one term or when everything is being multiplied together).

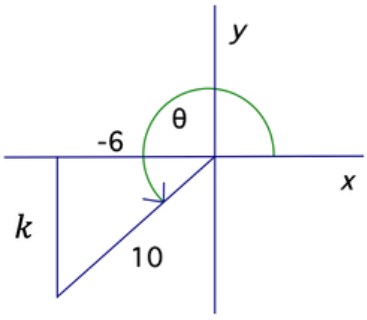
PRO-TIP

These questions only have triangles in Quadrant 1, therefore all the sides are **always positive**.

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Worked example

Use the sketch below to determine the values of the following, without the use of a calculator:



R k

$k^2 = (10)^2 - (-6)^2 \quad (pythag)$

$\sqrt{k^2} = \sqrt{64}$

$k = \pm 8$

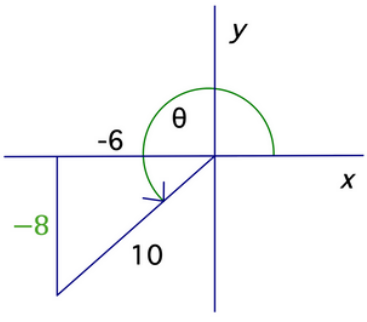
$k = -8$

- Using the sides in the given triangle, we use Pythagoras to find the missing side.
- Since we are finding one of the shorter sides (not the hypotenuse) we subtract the squares of the other sides.
- Remember, when you root a value, it could be a positive or negative number.
- Keeping in mind our triangle is in Q3, both the opposite (y) and adjacent sides (x) will be negative so we choose k=-8.

R $\sin\theta$

$= -\frac{8}{10}$

$= -\frac{4}{5}$



- Replacing k with -8 on the diagram, we can read off the opposite side and the hypotenuse to write the value of $\sin\theta$.

R $\cos\theta \times \tan\theta$

$= \frac{\cancel{-6}}{10} \times \frac{-8}{\cancel{-6}}$

$= -\frac{8}{10}$

$= -\frac{4}{5}$

- First we read off the sides for $\cos\theta = \frac{Adjacent}{Hypotenuse}$ and $\tan\theta = \frac{Opposite}{Adjacent}$
- Since there is a -6 in both the numerator and denominator, they cancel to simplify
- Always simplify your answer as much as possible

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Worked example

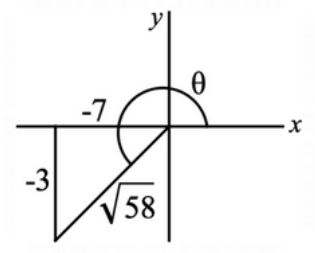
If $7\tan\theta = 3$ and $\cos\theta < 0$, use a sketch in the correct quadrant and without the use of a calculator, determine the value of: $\dfrac{\sin\theta + \cos\theta}{2\sin\theta}$

R

$$\tan\theta = \frac{3}{7} = \frac{-3}{-7}$$

$$r^2 = (-7)^2 + (-3)^2 \quad (pythag)$$
$$r^2 = 58$$
$$r = \sqrt{58}$$

- First, get the Trig ratio isolated.
- The only quadrant where tan is positive and cos is negative is Q3
- In Q3, both the opposite (or y value) and the adjacent sides (or x value) are negative therefore we write the fraction with both a negative numerator and negative denominator.
- Using pythag we calculate the length of the hypotenuse (called r, which is ALWAYS positive).



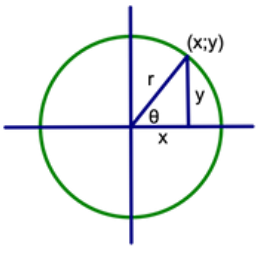
$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r} \qquad \cos\theta = \frac{adj}{hyp} = \frac{x}{r} \qquad \tan\theta = \frac{opp}{adj} = \frac{y}{x}$$

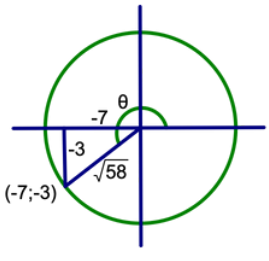
$$\frac{\sin\theta + \cos\theta}{2\sin\theta}$$
$$= \frac{\frac{-3}{\sqrt{58}} + \frac{-7}{\sqrt{58}}}{2\left(-\frac{3}{\sqrt{58}}\right)}$$
$$= \frac{-\frac{10}{\sqrt{58}}}{-\frac{6}{\sqrt{58}}}$$
$$= -\frac{10}{\sqrt{58}} \times -\frac{\sqrt{58}}{6}$$
$$= \frac{5}{3}$$

With a calculator, skip straight from here to the final answer.

- Using the sides of the triangle, we can write down the values of the Trig ratios.
- Simplify the numerator into a single fraction (the denominators are already the same).
- Simplify the denominator by multiplying the fraction by the coefficient of 2.
- Divide by the fraction in the denominator by converting to multiplication and flipping the fraction over.
- Cancel the roots and simplify your answer.
- If you are allowed to use a calculator, skip from the substitution straight to the answer.

Question source: EC November 2023 Grade 11 Paper 2





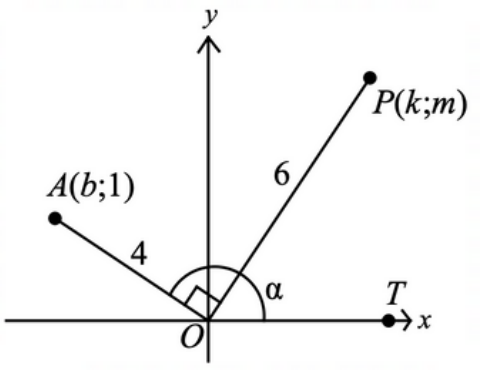
PRO-TIP

The hypotenuse is often called r for **r**adius. It is the radius of a circle that would pass through the point with the coordinates (x;y) to correspond with the ratio. Since the radius is a length, The hypotenuse is ALWAYS POSITIVE.

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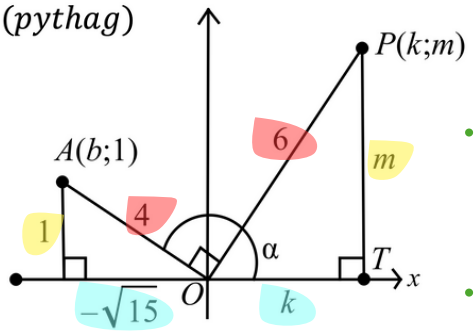
Worked example

In the diagram alongside, A is a point on the Cartesian plane such that $A\hat{O}T = \alpha$ with OA = 4. P(k;m) is a point such that OP = 6 and AO ⊥ OP.



C Determine the value of b.

$$b^2 = 4^2 - 1^2 \quad (pythag)$$
$$b^2 = 15$$
$$b = -\sqrt{15}$$



- By constructing two right angled triangles we can use the coordinates of A and P to fill in the values of the sides.
- Using pythag, we can calculate the value of b. Don't forget to say "pythag" and to root your calculation.

C Determine, without the use of a calculator, the value of:

$$\sin\alpha = \frac{opp}{hyp} \quad \cos\alpha = \frac{adj}{hyp} \quad \tan\alpha = \frac{opp}{adj}$$

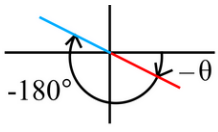
i) $\tan\alpha$

$$\tan\alpha = -\frac{1}{\sqrt{15}}$$

ii) $\sin(-\alpha - 180°) + \cos(-\alpha)$ (Q2, Q4)

$$= \sin\alpha + \cos\alpha$$
$$= \left(\frac{1}{4}\right) + \left(\frac{-\sqrt{15}}{4}\right)$$
$$= \frac{1-\sqrt{15}}{4}$$

- Reduce both the Trig ratios using negative angles.
- Use the sides in the triangle to write both the values for the Trig ratios.
- Simplify your answer by combining the fractions into a single fraction - they already have the same denominator.
- Either you can visualise the angle $-\alpha - 180°$ as going anti-clockwise as shown in the image, or you can add 360° to get a positive $180° - \alpha$



P Without the use of a calculator, determine the values of k and m.

$$\cos(\alpha - 90°) = \frac{k}{6}$$
$$\sin\alpha = \frac{k}{6}$$
$$\frac{1}{4} = \frac{k}{6}$$
$$k = \frac{6}{4}$$
$$k = \frac{3}{2}$$

$$\sin(\alpha - 90°) = \frac{m}{6}$$
$$-\cos\alpha = \frac{m}{6}$$
$$-\left(\frac{-\sqrt{15}}{4}\right) = \frac{m}{6}$$
$$\frac{6\sqrt{15}}{4} = m$$
$$m = \frac{3\sqrt{15}}{2}$$

- The angle in the triangle on the right (Q1) is $\alpha - 90°$
- Using co-ratios we can reduce these ratios and find their values using the sides of the triangle in Q1.
- Equate the values found previously using the triangle in Q2 to the values found using the triangle in Q1.
- Isolate the variable in each equation.

Question source: GP November 2022 Grade 11 Paper 2

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Question Type 3:
Proving trigonometric identities

The Tan Identity

$$\tan\theta = \frac{opp}{adj}$$

$$\begin{aligned}\frac{\sin\theta}{\cos\theta} &= \frac{opp}{hyp} \div \frac{adj}{hyp} \\ &= \frac{opp}{hyp} \times \frac{hyp}{adj} \\ &= \frac{opp}{adj} \\ &= \tan\theta\end{aligned}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

The Square Identity

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{opp}{hyp}\right)^2 + \left(\frac{adj}{hyp}\right)^2 \\ &= \frac{opp^2}{hyp^2} + \frac{adj^2}{hyp^2} \\ &= \frac{opp^2 + adj^2}{hyp^2} \\ &= \frac{hyp^2}{hyp^2} \\ &= 1\end{aligned}$$

$$\therefore \sin^2\theta + \cos^2\theta = 1$$

Take note of the following tips when proving an identity:

- **You have to use the correct layout.** Choose one side of the identity to work on and label your first line with that chosen side. Use "LHS" for the left-hand side of the identity, and "RHS" for the right-hand side of the identity. **You MAY NOT work on both sides of the identity at the same time.** You may, however, work with each side separately, and show that both sides simplify to the same expression.

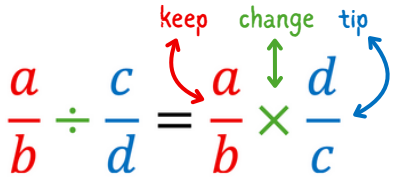
- The point is to show that the expressions on each side of the given identity are equal to each other, and then conclude at the end **"LHS = RHS"**.

- Look out for opportunities to use the **square and tan identities**. Simply applying these identities should earn you a mark.

- Look out for opportunities to **factorise** expressions, either by taking out a common factor, factorising a difference of two squares or factorising trinomials.

- If an expression consists of multiple separate fractions, and the other side of the identity consists of **only one fraction**, make sure to write the separate fractions as one by finding the lowest common denominator of the separate fractions. When dividing by a fraction, remember to **tip** and **times**.

- There are often multiple ways of proving an identity, so just **try something!** Even if you are unable to prove the identity, you could get some marks for applying any of these tips.



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Worked example

R Prove the following identity: $\frac{\tan x \,.\, \cos x}{\sin x} = 1$

$$
\begin{aligned}
LHS &= \frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1} \\
&= \frac{\cancel{\sin x}}{\cancel{\sin x}} \\
&= 1 \\
&= RHS
\end{aligned}
$$

- First we chose the LHS as it is the more complicated side
- Using the tan identity we change tanx to $\frac{\sin x}{\cos x}$
- Since cosx is being multiplied by a fraction, it is easier to write it over 1 so they are both fractions
- Looking at the two fractions in the numerator, the two cosx's cancel leaving just sinx
- Now the sinx is in the numerator and the sinx in denominator cancel out.
- Remember, anything divided by itself is 1.

Reminder:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

PRO-TIP

It is easier to start working with the more complicated side of the identity and simplifying it to match the other side.

You don't need to work with one side the whole way. You can simplify each side as much as needed to get them to match each other.

Worked example

R Prove the following identity: $\cos\theta\,(1 + \tan^2\theta) = \frac{1}{\cos\theta}$

$$
\begin{aligned}
LHS &= \cos\theta\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right) \\
&= \cos\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right) \\
&= \frac{\cos\theta}{1}\left(\frac{1}{\cos^2\theta}\right) \\
&= \frac{1}{\cos\theta} \\
&= RHS
\end{aligned}
$$

- Starting with the LHS first convert $\tan^2\theta = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{\sin^2\theta}{\cos^2\theta}$
- Combine the two fractions using an LCD
- Use the square identity to simplify the numerator
- Write the cosθ in front of the bracket over 1 to multiply the fractions together more easily.
- Multiply the fractions by cancelling out cosθ
- Don't forget to conclude LHS=RHS



You MUST work with the LHS and RHS as **two separate expressions** and prove that these expressions have the same value. You **CANNOT** treat these questions as an equation or you will not get marks for these questions.

You can work with the LHS all the way until you have the RHS, you can work with only the RHS until you have the same expression as the LHS or you can work with each for a bit (separately) until you end with the same expression on each side.

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Worked example

C Prove the following identity: $\frac{2\sin^2 x - 1}{\sin x - \cos x} = \sin x + \cos x$

$$
\begin{aligned}
LHS &= \frac{2\sin^2 x - (\sin^2 x + \cos^2 x)}{\sin x - \cos x} \\
&= \frac{\sin^2 x - \cos^2 x}{\sin x - \cos x} \\
&= \frac{(\sin x - \cos x)(\sin x + \cos x)}{(\sin x - \cos x)} \\
&= \sin x + \cos x \\
&= RHS
\end{aligned}
$$

- Here we replace 1 with the square identity since $\sin^2 x + \cos^2 x = 1$
- Don't forget to write the identity in brackets as the identity needs to be multiplied through by the negative that was with 1.
- Collecting like terms in the numerator, we now have two square terms separated by a minus. In order to cancel we need to factorise.
- In the numerator we factorise by DOTS and since the denominator cannot be factorised, we place it in a bracket to become one term.
- Cancel identical brackets in the numerator and denominator.
- Don't forget to conclude that LHS=RHS

Worked example

C Prove the following identity: $\frac{\cos^2 x - \cos x - \sin^2 x}{2\sin x \cos x + \sin x} = \frac{1}{\tan x} - \frac{1}{\sin x}$

$$
\begin{aligned}
LHS &= \frac{\cos^2 x - \cos x - (1 - \cos^2 x)}{\sin x(2\cos x + 1)} \\
&= \frac{\cos^2 x - \cos x - 1 + \cos^2 x}{\sin x(2\cos x + 1)} \\
&= \frac{2\cos^2 x - \cos x - 1}{\sin x(2\cos x + 1)} \\
&= \frac{(2\cos x + 1)(\cos x - 1)}{\sin x(2\cos x + 1)} \\
&= \frac{\cos x - 1}{\sin x}
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{1}{\frac{\sin x}{\cos x}} - \frac{1}{\sin x} \\
&= 1 \div \frac{\sin x}{\cos x} - \frac{1}{\sin x} \\
&= 1 \times \frac{\cos x}{\sin x} - \frac{1}{\sin x} \\
&= \frac{\cos x - 1}{\sin x} \\
&= LHS
\end{aligned}
$$

- Starting with the LHS we use the square identity to swop out $\sin^2 x$ in order to have the numerator only in terms of cosx. Be careful to use brackets so you don't mess up your signs.
- We also factorise the denominator by taking a common factor of sinx.
- By collecting like terms, we now have a trinomial that we can factorise. The denominator is hint of what you'd hope one of the trinomial's brackets will be as we are hoping to cancel a common bracket.
- Once we have simplified the LHS as much as possible, we can switch to the RHS to see if we can get both expressions to simplify the same.
- On the RHS we use the tan identity and then simplify the fraction within a fraction by converting divide to multiply and flipping the fraction around.
- Now we have an LCD of sinx so these two fractions can be combined resulting in a matching expression to our LHS.

Question source: EC November 2023 Grade 11 Paper 2

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C For which value(s) of x is the identity undefined if $x \in [0°; 180°]$?

$$\frac{\cos^2 x - \cos x - \sin^2 x}{2\sin x \cos x + \sin x} = \frac{1}{\tan x} - \frac{1}{\sin x}$$

$$2\sin x \cos x + \sin x = 0$$
$$\sin x\,(2\cos x + 1) = 0$$
$$\sin x = 0 \quad \text{OR} \quad \cos x = -\frac{1}{2}$$
$$x = 0°; 180° \qquad x = 120°$$

$$\tan x = 0$$
$$x = 0°; 180°$$

$$\tan x \ \textit{is undefined}$$
$$x = 90°$$

$$\sin x = 0$$
$$x = 0°; 180°; 360°$$

$$\therefore x = 0°; 90°; 120°; 180°$$

An identity could be **invalid** (undefined or non-real) in three scenarios:

- You have asymptotes (tanx) → undefined
- You have a fraction (denominator cannot equal zero) → undefined
- You have a root (you cannot root a negative) → non-real

tanx | asymptotes | $\frac{\text{anything}}{\text{zero}}$ | $\sqrt{\text{negative}}$

- We have three areas of concern for the identity potentially being undefined: each denominator could be zero, and we have tanx, which has asymptotes.
- For the denominators, make each equal to zero and solve for x.

- For the first denominator, we factorise out the common factor of sinx (as we did when proving the identity in the previous question). Now we set sinx equal to zero, and the bracket equal to zero.
- For sinx to equal zero, we need to x-intercepts of the sinx graph, which happens at x=0°and then every 180°in both directions. These are the same answers as the third denominator equals zero.
- For the bracket to equal zero, we solve for cosx and use [SHIFT] and [cos] on our calculator to find the reference angle (RA). Since cosx equals a negative ratio, we place the RA in Q2, where cosx is negative (it would also be negative in Q3 but we only want values of x between 0° and 180°).
- The second denominator has two potential issues, tanx cannot equal zero, since it's in the denominator, but tanx also has asymptotes so we need two lists of values.
- For tanx = 0, we need the x-intercepts of tanx, which starts at x=0°and reocurrs every 180°(the period of tanx).
- tanx has asymptotes at x=90°and every 180°(the period of tanx)
- We have now tackled each problem individually, and we have four separate lists of values for x that would make the identity undefined. We compile the lists into one single list of values between 0° and 180°, ignoring any duplicate values.
- The question asked for the values of x where the identity IS undefined, therefore we say x does equal these values to make the identity undefined.
- If you are struggling to grasps some of the concepts discussed here, come back after you are comfortable with the Trig graphs and Trig equations sections as these sections overlap in this question.

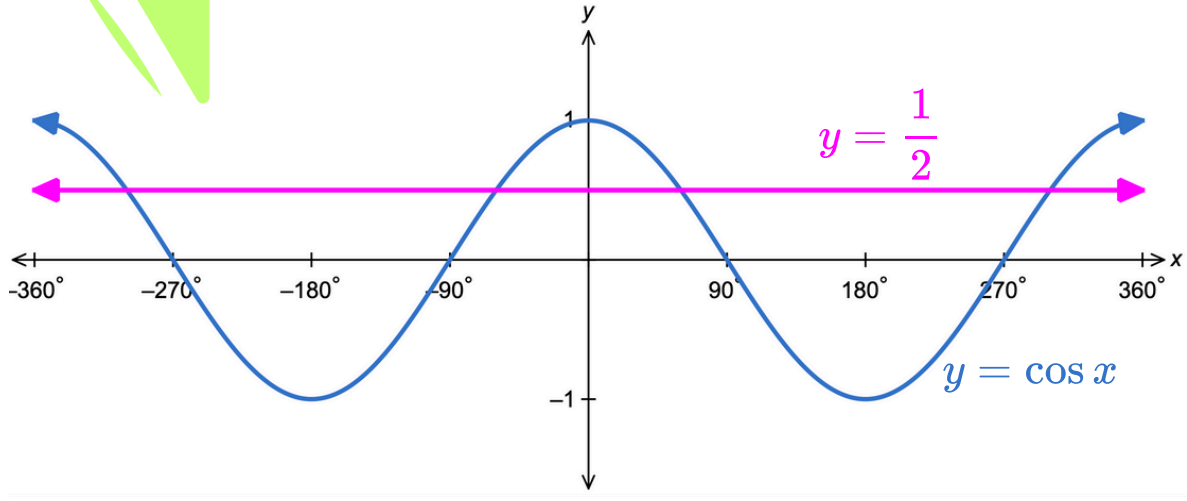
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Question Type 4:
Solving a trigonometric equation

When solving a **Trigonometric equation**, we are finding where two graphs intersect and since the Trigonometric functions repeat forever, there are infinitely many solutions to any Trigonometric equation. In order to save ourselves from writing a list of solutions that never ends, we either write our answer as a **General Solution** or we are given a region within which we can provide a list of **Specific Solutions**.

The General Solution resembles the general formula for a pattern and uses the period of the graph to provide a rule to find any number of solutions to an equation. Unlike in a pattern, where n is a natural number, in a General Solution, **n is an integer** as the graph **repeats in both directions**. We can then use this rule to find Specific Solutions within a provided region by substituting n = -1, 0, 1, etc.

Below is an example of a graphical representation of the equation **cosx = ½**. This is where the blue graph (**y=cosx**) has points of intersection with the pink graph (**y= ½**).



- Notice that there are **two solutions for each repetition** of the period.
- In this given domain, there are already four solutions and the **graphs continue** in both directions **indefinitely**.
- We therefore use the **general solution** to write a formula for each unique solution, which can be used to find all the solutions within a given domain.

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There are **five** types of general solution questions:

a) Basic
b) Factorise
c) Tan identity
d) Co-ratio
e) Square identity

We will practice at least one of each type. You need to be able to identify the type of general solution in order to solve it. Once you have the general solution, you can be asked to find specific solutions. We will tackle finding specific solutions later in the Trig Graphs section.

Worked example - basic

R Solve for x given: $2\sin(x+20°) = \sqrt{3}$

$\sin(x+20°) = \frac{\sqrt{3}}{2}$
$RA = 60°$

Q1
$x + 20° = 60° + n.360°$
$x = 40° + n.360° \quad n \in \mathbb{Z}$

Q2
$x + 20° = 180° - 60° + n.360°$
$x = 100° + n.360° \quad n \in \mathbb{Z}$

- First, isolate the trig ratio by dividing both sides by 2
- We find the reference angle (RA) by doing $RA = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ To get $\sin^{-1}$ on your calculator press [SHIFT] and [sin] to calculate the angle, although this is a special angle so a calculator is not strictly necessary.
- Since the ratio was positive, we choose the quadrants where sine is positive (Q1 and Q2).
- Do not solve for x until you have placed the RA in the correct quadrants, with the relevant reduction and period.
- The RA is an acute angle so it's the same as the Q1 angle but for Q2 we use the reduction of that quadrant ($180°$-angle).
- Simplify to solve for x in each quadrant.
- Since the period of sine is $360°$ we say $+n360°$ to get all the possible solutions.
- Don't forget to say $n \in \mathbb{Z}$ at least once in your solution (some schools use $k.360° \; k \in \mathbb{Z}$, which does the same thing).

Worked example - basic

R Solve for x given: $\cos 2x = -0{,}737$

$\cos 2x = -0{,}737$
$RA = 42{,}52°$

Q2
$2x = 180° - 42{,}52° + n.360°$
$2x = 137{,}48° + n.360°$
$x = 68{,}74° + n.180° \quad n \in \mathbb{Z}$

Q3
$2x = 180° + 42{,}52° + n.360°$
$2x = 222{,}52° + n.360°$
$x = 111{,}26° + n.180° \quad n \in \mathbb{Z}$

- To find the reference angle (RA) we always use a positive value $\therefore RA = \cos^{-1}(0{,}737) = 42{,}5235\ldots$
- Since the ratio is negative, we use the negative quadrants for cosine (Q2 and Q3)
- The reduction for Q2 is $180°$ - RA and Q3 is $180°$ + RA
- When we divide by 2 to isolate x, we divide everything by 2, including the period.
- Do not divide the RA by 2 until you have placed the RA in the correct quadrants, with the relevant reduction and period.

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Worked example -factorise

R Solve for x given: $2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$
$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x = 1$

$RA = 90°$ $\sin x = \frac{1}{2}$

$RA = 30°$

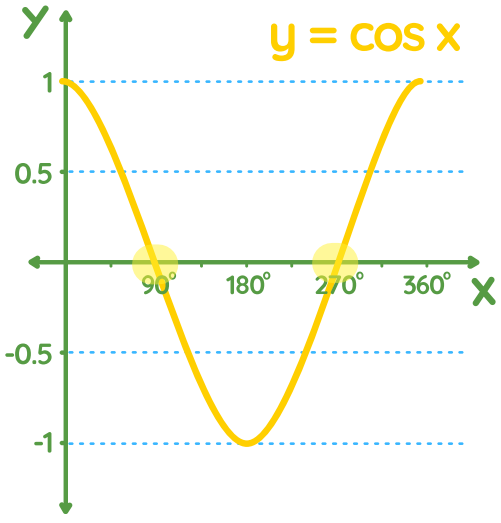
Q1:
$x = 90° + n.360° \quad n \in \mathbb{Z}$

Q4:
$x = 360° - 90° + n.360°$
$x = 270° + n.360° \quad n \in \mathbb{Z}$

$x = 90° + n.180° \quad n \in \mathbb{Z}$

Q1:
$x = 30° + n.360° \quad n \in \mathbb{Z}$

Q2:
$x = 180° - 30° + n.360°$
$x = 150° + n.360 \quad n \in \mathbb{Z}$



- In order to factorise, all terms need to be on the same side of the equation.
- cosx is a common factor, which gives two sets of answers (one for the common factor and one for the bracket).
- Looking at the solution for cosx=0, you can either consider 0 to be positive (Q1 and Q4) or negative, the final answer will be exactly the same.
- The two quadrants can also be summarised into a single general solution by visualising the x-intercepts of the cosx graph, as shown here.
- For the 2sinx=1 solution, first isolate the trig ratio before calculating the RA (here the RA is a special angle so this entire question could be done without a calculator)
- The ratio is positive so we use the positive quadrants for sine (Q1 = RA and Q2 = 180°-RA).

Question source: DBE November 2016 Grade 11 Paper 2

Worked example - Tan identity

R Solve for x given: $3\cos x - \sin x = 0$

$3\cos x - \sin x = 0$
$3\cos x = \sin x$
$3 = \tan x$
$RA = 71{,}57°$

Q1
$x = 71{,}57° + n.360° \quad n \in \mathbb{Z}$
Q3
$x = 180° + 71{,}57° + n.360°$
$x = 251{,}57° + n.360° \quad n \in \mathbb{Z}$

$x = 71{,}57° + n.180° \quad n \in \mathbb{Z}$

- You can identify this is a "tan trick" question because there are only two terms, one with "cos" and one with "sin" and their angles are identical.
- By dividing by cosx, we convert sinx to tanx.
- For tan, either you can write both the positive quadrants (Q1 and Q3) with a period of 360° OR you can do one quadrant (Q1) and use a period of 180°. The solution you get from Q3 is the first repetition you'd get from using the period of 180° (using n=1).

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Worked example - Co-ratio

R/C Solve for x given: $\sin(x + 12°) = \cos 2x$

$\sin(x + 12°) = \cos 2x$
$\sin(x + 12°) = \sin(90° - 2x)$
Q1:
$x + 12° = 90° - 2x$
$3x = 78° + n.360$
$x = 26° + n.120° \quad n \in \mathbb{Z}$

Q2:
$x + 12° = 180° - (90° - 2x)$
$x + 12° = 90° + 2x$
$-x = 78° + n.360°$
$x = -78° - n.360° \quad n \in \mathbb{Z}$

- You can identify this is a "co-ratio trick" question because there are only two terms, one with sin and one with cos but unlike the "tan trick", the angles are different.
- By using co-ratios we can get the ratios to match, which means the angles are equal. Or think of canceling out sine on both sides.
- DO NOT SOLVE FOR x before moving the angles to the correct quadrant.
- When using a reduction, don't forget to use brackets otherwise you could make mistakes with the signs.
- Don't forget to divide everything when isolating x: both the angle and the period.

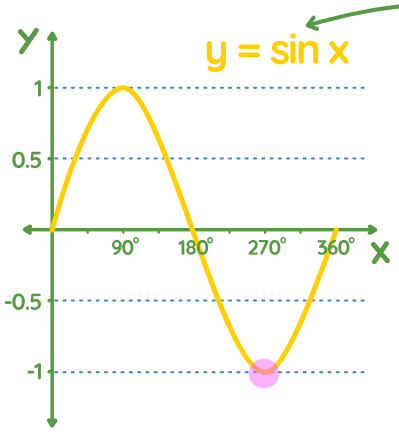
Worked example - Square identity

C Solve for x given: $3\cos^2 x - 5\sin x = 1$

$3\cos^2 x - 5\sin x = 1$
$3(1 - \sin^2 x) - 5\sin x = 1$
$3 - 3\sin^2 x - 5\sin x - 1 = 0$
$-3\sin^2 x - 5\sin x + 2 = 0$
$3\sin^2 x + 5\sin x - 2 = 0$
$(3\sin x - 1)(\sin x + 2) = 0$
$\sin x = \frac{1}{3}$ or $\sin x \neq -2$
$RA = 19{,}47°$

Q1:
$x = 19{,}47° + n.360°$
Q2:
$x = 180° - 19{,}47° + n.360°$
$x = 160{,}53° + n.360° \quad n \in \mathbb{Z}$

- You can identify this is a "square identity trick" question because there are three terms, one trig ratio is squared, one isn't and you have a constant. It has the format of a trinomial but you cannot factorise it yet because the Trig ratios are different.
- Replace the squared cos ratio using the square identity. Now the Trig ratios match.
- When solving for x in a quadratic equation we bring all the terms to one side so we have EvERYTHING = 0
- Divide everything by a negative, to get make the trinomial easier to factorise.
- Now factorise the trinomial to get two factors each = 0
- Solve for x in each factor.
- The factor (3sinx-1) has a solution in Q1 and Q2 (the positive quadrants for sin)



You can see from the sinx graph, that the minimum of sinx is -1 so there isn't a value for x that will result in sinx=-2, therefore the factor (sinx+2) doesn't have a solution.

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Worked example - Negative co-ratio

C Solve for x given: $\cos(2x + 30°) = -\sin x$

$\cos(2x + 30°) = -\cos(90° - x)$

Q2:

$2x + 30° = 180° - (90° - x)$

$2x + 30° = 90° + x$

$x = 60° + n.360° \quad n \in \mathbb{Z}$

Q3:

$2x + 30° = 180° + (90° - x)$

$2x + 30° = 270° - x$

$3x = 240° + n.360°$

$x = 80° + n.120° \quad n \in \mathbb{Z}$

- Since we have two terms with different ratios, and the angles are different, we use co-ratios to get the ratios to match.
- In this question, one of the ratios is negative, so we use the negative quadrants for cos (Q2 and Q3).
- Don't forget to use the reductions for these quadrants BEFORE you solve for x.
- When isolating x, if you divide by a value, divide the period as well.

PRO-TIP

For a general solution we need to say $n \in \mathbb{Z}$ at least once during the solution. You will likely loose marks if you leave it out.

Since n is an integer, you can add or subtract the period as many times as needed to get a list of values if asked for specific solutions (n = ...-2, -1, 0, 1, 2, ...)

Worked example

Given: $\frac{\sin\theta + 2}{\sin\theta - 1} = \frac{a}{b}$

C Prove $\sin\theta = \frac{2b + a}{a - b}$

$b\sin\theta + 2b = a\sin\theta - a$

$2b + a = a\sin\theta - b\sin\theta$

$2b + a = \sin\theta\,(a - b)$

$\frac{2b + a}{a - b} = \sin\theta$

- To isolate sinθ, we cross multiply the fractions.
- Next, we move the letters together on the left and the sinθ terms together on the right.
- On the right, we take out sinθ as a common factor.
- Now, divide by the bracket to get the same equation as given in the question.

P Hence, determine θ if $\frac{\sin\theta + 2}{\sin\theta - 1} = -\frac{15}{17}$ and $\theta \in [90°; 270°]$.

$a = -15$ and $b = 17$

$\therefore \sin\theta = \frac{2(17) + (-15)}{(-15) - (17)}$

$\sin\theta = -\frac{19}{32}$

$RA = 36{,}42°$

Q3:

$\theta = 180° + 36{,}42°$

$\theta = 216{,}42°$

- Using the previous question, we allocate the numerator as "a" and the denominator as "b".
- Now we can substitute the value of a=-15 and b=17 into the equation we proved in the previous question.
- Use your calculator to simplify the fraction.
- To solve for θ we need the reference angle (RA).
- Use [SHIFT] and [sin] of the ratio as a positive value to get an acute angle for the RA.
- Since sinθ equals a negative ratio, we are in Q3 or Q4 but the question speficied θ was between 90°and 270°(Q2 or Q3) therefore the only quadrant that is true for both is Q3.
- We use the reduction of 180° + RA to get the answer for θ in Q3.

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Question Type 5: 2D Trigonometry

<u>In order to achieve full marks for this section, you need to be able to:</u>

- Use the **trigonometric ratios** to calculate the lengths of sides or sizes of angles of a right-angled triangle.
- Use the **sine rule** to calculate the length of a side or the size of an angle of a non-right angled triangle.
- Use the **cosine rule** to calculate the length of a side or the size of an angle of a non-right angled triangle.
- Use the **area rule** to calculate the area of a non-right angled triangle.
- **Prove** the three triangle rules.

We can use **trigonometry** to solve both right-angled and non-right angled triangles.

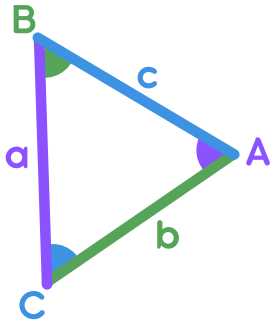
"Solving triangles" means to calculate the lengths of sides, and the sizes of angles, of triangles.

To solve **right-angled triangles**, we use the **trigonometric ratios**, as explained in the previous section.

To solve **non-right angled triangles**, we use the **triangle rules:** the sine rule, cosine rule and area rule.

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THE SINE RULE:



To calculate a side:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}}$$

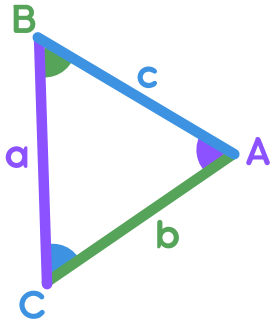
To calculate an angle:

$$\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$$

To use the Sine Rule, we need a pair of a known side opposite a known angle. We can then find an unknown side opposite a known angle, or an unknown angle opposite a known side.

To make the calculation easier: if you are looking for a side, use the formula with sides on top, and if you are calculating an angle, put sin(angle) on top.

THE COSINE RULE:



To calculate a side:

$$a^2 = b^2 + c^2 - 2bc.\cos\hat{A}$$

$$b^2 = a^2 + c^2 - 2ac.\cos\hat{B}$$

$$c^2 = a^2 + b^2 - 2ab.\cos\hat{C}$$

To calculate an angle:

$$\cos\hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\hat{B} = \frac{a^2 + c^2 - b^2}{2ac}$$

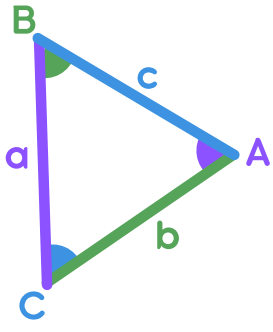
$$\cos\hat{C} = \frac{a^2 + b^2 - c^2}{2ab}$$

We use Cosine Rule if we can't use Sine Rule. This will either be because you have SAS (two sides and the included angle) or SSS (all three sides and no angles).

Start by identifying which angle you HAVE or which angle you NEED. The side opposite this angle is the subject of the Cosine Rule.

You can learn both the "side" and "angle" versions of the Cosine Rule, or you can use the "side" version and manipulate the equation to solve for the angle.

THE AREA RULE:



To calculate the area of a non-right angled triangle:

$$Area = \frac{1}{2}ab.\sin\hat{C}$$

$$Area = \frac{1}{2}bc.\sin\hat{A}$$

$$Area = \frac{1}{2}ac.\sin\hat{B}$$

To calculate the area of a non-right angled triangle you need the lengths of two sides and the INCLUDED angle.

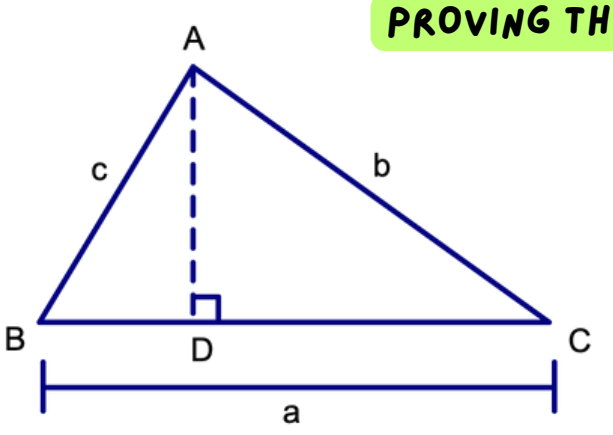
The highlighted formulae are the ones given on the formula sheet (see page 135).

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There are several proofs you need to know in preparation for your exams. There are 3 proofs in 2D Trig: the Sine Rule, the Cosine Rule and the Area Rule proofs.

The given triangle might be slightly different (labelled with different letters or in a different orientation) but the method of the proof remains the same. You must understand each proof to be able to adapt them to the given triangle. All the Trigonometric proofs start with the same construction to create two right-angled triangles. If you do not do the construction correctly, you will likely not receive any marks for your proof.

PROVING THE SINE RULE:



Required to prove:

$$\frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

Construction:

$AD \perp BC$

Proof:

In ΔABD:

$$\sin \hat{B} = \frac{AD}{c}$$
$$AD = c.\sin \hat{B}$$

In ΔACD:

$$\sin \hat{C} = \frac{AD}{b}$$
$$AD = b.\sin \hat{C}$$

Since AD = c.sinB̂ and AD = b.sinĈ, then c.sinB̂ must be equal to b.sinĈ (both = AD).

$$\therefore c.\sin \hat{B} = b.\sin \hat{C}$$
$$\frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

If we want to prove the fractions where the sides are in the numerator, we divide both sides by sinB̂ and sinĈ. On the left, the sinB̂ will cancel, leaving sinĈ in the denominator and on the right, sinĈ will cancel, leaving sinB̂ in the denominator.

$$\frac{c.\sin \hat{B}}{\sin \hat{B}.\sin \hat{C}} = \frac{b.\sin \hat{C}}{\sin \hat{B}.\sin \hat{C}}$$
$$\frac{c}{\sin \hat{C}} = \frac{b}{\sin \hat{B}}$$

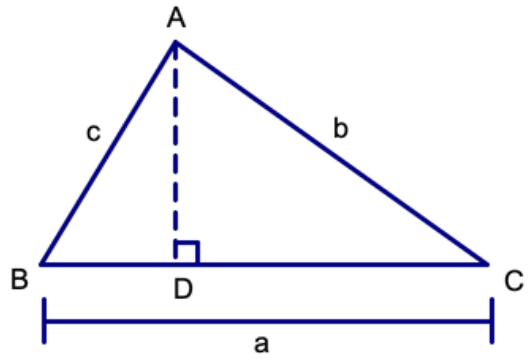
Then we switch left and right to match the order of the fractions in the "required to prove".

PRO-TIP

As shown in this example of the proof, you will only ever be asked to prove **two** fractions are equal, of the three fractions in the Sine Rule.

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PROVING THE COSINE RULE:



Required to prove:

$$c^2 = a^2 + b^2 - 2ab.\cos\hat{C}$$

Construction:

$AD \perp BC$

Proof:

In ΔABD:

$c^2 = AD^2 + BD^2 \quad (pythag)$

But $BD = a - DC$

$\therefore c^2 = AD^2 + (a - DC)^2$

$c^2 = AD^2 + a^2 - 2a.DC + DC^2$

In ΔACD:

$b^2 = AD^2 + DC^2 \quad (pythag)$

And $\cos\hat{C} = \frac{DC}{b}$

$\therefore DC = b.\cos\hat{C}$

$c^2 = a^2 + b^2 - 2a.b.\cos\hat{C}$

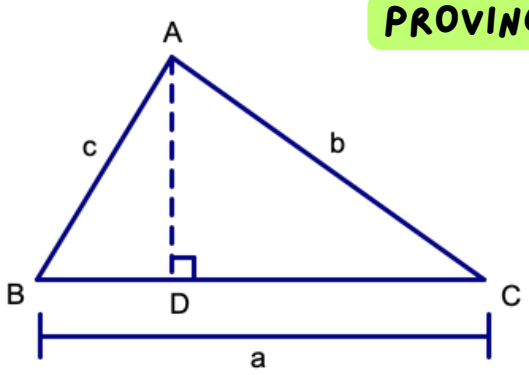
For the Cosine Rule proof, we do Pythagoras in both the right-angled triangles. We also use the ratio for cos to find a ratio in terms of the angle in the Cosine Rule, in this proof that is cosĈ.

Side a = BD + DC, therefore BD = a - DC. We then expand the square bracket (you can write the bracket and FOIL if you aren't comfortable going directly to the next step).

Using Pythagoras in the right-hand side triangle, we find an equation for b^2, in terms of AD and DC, and for DC, in terms of the angle Ĉ, which are the pieces of our current equation that are missing.

We replace $AD^2 + DC^2$ with b^2 and replace DC with b.cosĈ.

PROVING THE AREA RULE:



Required to prove:

$$Area\ \Delta ABC = \frac{1}{2}\ ab.\sin\hat{C}$$

Construction:

$AD \perp BC$

Proof:

$Area\ \Delta ABC = \frac{1}{2}\ a \times AD$

In ΔACD:

$\sin\hat{C} = \frac{AD}{b}$

$AD = b.\sin\hat{C}$

$\therefore Area\ \Delta ABC = \frac{1}{2}\ ab.\sin\hat{C}$

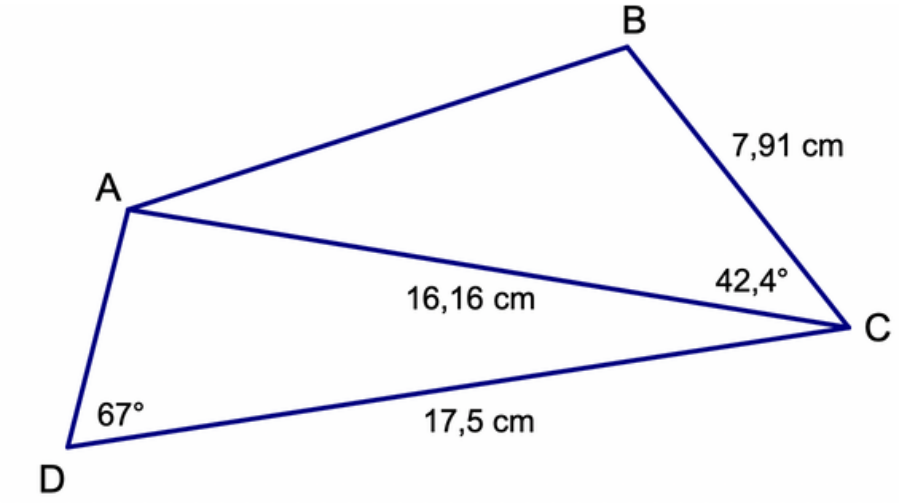
Area = ½ base x perpendicular height

We can replace AD with an expression of the same value using the ratio for sin in the right-hand side triangle.

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Worked example

Quadrilateral ABCD is drawn with BC = 7,91 cm, AC = 16,16 cm and CD = 17,5 cm. It is also given that $B\hat{C}A = 42{,}4°$ and $A\hat{D}C = 67°$.



R Determine the area of ΔABC.

$$\text{Area} = \frac{1}{2}(AC)(BC)\sin\hat{C}$$
$$= \frac{1}{2}(16{,}16)(7{,}91)\sin(42{,}4°)$$
$$= 43{,}10\ cm^2$$

- To use the Area rule, you need two sides and the angle between them. In triangle ABC we have everything we need and can substitute the values straight into the area formula
- Don't forget to write your answer rounded to 2 decimal places and use cm squared as the units.

R Determine the length of AB.

$$AB^2 = AC^2 + BC^2 - 2(AC)(BC)\cos A\hat{C}B$$
$$= (16{,}16)^2 + (7{,}91)^2 - 2(16{,}16)(7{,}91)\cos 42{,}4°$$
$$\sqrt{AB^2} = \sqrt{134{,}9267\ldots}$$
$$AB = 11{,}62\ cm$$

- Since you don't have any pairs of sides with their opposite angle, you cannot use the Sine rule, therefore we use the Cos rule.
- The length that is the subject of the formula is opposite the angle for the calculation.
- Don't forget to square root the value to obtain the length, and round to 2dp.

R Determine the size of $D\hat{A}C$.

$$\frac{\sin D\hat{A}C}{DC} = \frac{\sin A\hat{D}C}{AC}$$

$$\frac{\sin D\hat{A}C}{17{,}5} = \frac{\sin 67°}{16{,}16}$$

$$\sin D\hat{A}C = \frac{\sin 67°}{16{,}16} \times 17{,}5$$

$$\sin D\hat{A}C = 0{,}996\ldots$$

$$D\hat{A}C = \sin^{-1}(0{,}996\ldots)$$

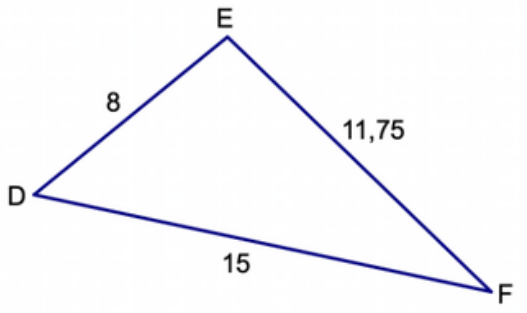
$$D\hat{A}C = 85{,}44°$$

- In triangle ACD, we have a pair with a known angle (D) opposite a known side (AC), therefore we can use the Sine rule.
- The Sine rule can either have the sides in the numerator, or the angles, depending on which you wish to calculate. Since we want an angle, we chose to have the angles in the numerator.
- Substitute the side opposite the unknown angle (DC) and the pair with both known values into the formula.
- To solve for the angle, make it the subject by multiplying by the denominator of 17.5 and then isolate the angle by using SHIFT and SIN of the ratio.

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Worked example

R ΔDEF is drawn with $DE = 8, EF = 11{,}75$ and $FD = 15$. Calculate the size of $\hat{E}$.



$$15^2 = 8^2 + 11{,}75^2 - 2(8)(11{,}75)\cos\hat{E}$$

$$15^2 - 8^2 - 11{,}75^2 = -2(8)(11{,}75)\cos\hat{E}$$

$$\frac{15^2 - 8^2 - 11{,}75^2}{-2(8)(11{,}75)} = \cos\hat{E}$$

$$-\frac{367}{3008} = \cos\hat{E}$$

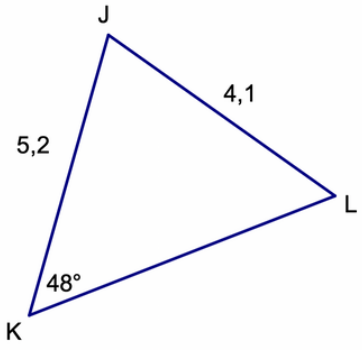
$$\hat{E} = \cos^{-1}\left(-\frac{367}{3008}\right)$$

$$\hat{E} = 97°$$

- Since we don't have any angles, we can't have a known pair to use the Sine rule, therefore we need to use the Cos rule.
- If we want angle E, we need to use the side opposite E (DF) as the subject of the Cos rule.
- To get the angle isolated, first we subtract the separate terms and then we divide by the coefficient of cosE.
- Now, we use SHIFT and COS of the ratio to calculate the angle. Since the ratio was negative, the angle will be an obtuse angle.

Worked example

C ΔJKL is drawn with $JL = 4{,}1;\ JK = 5{,}2$ and $J\hat{K}L = 48°$. Calculate the size of $\hat{L}$.



$$\frac{\sin\hat{L}}{JK} = \frac{\sin\hat{K}}{JL}$$

$$\frac{\sin\hat{L}}{5{,}2} = \frac{\sin 48°}{4{,}1}$$

$$\sin\hat{L} = \frac{\sin 48°}{4{,}1} \times 5{,}2$$

$$\sin\hat{L} = 0{,}9425\ldots$$

$$\hat{L} = \sin^{-1}(0{,}9425\ldots)$$

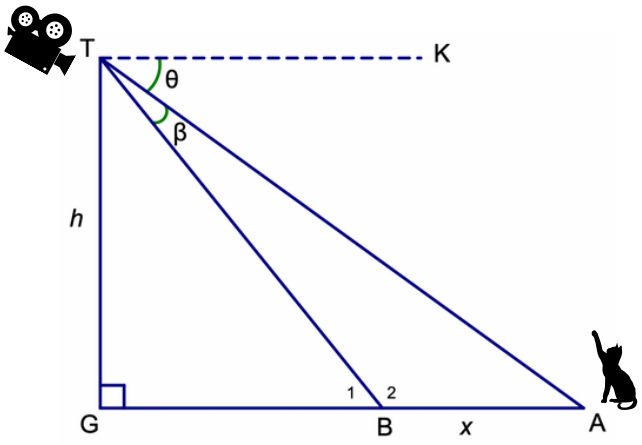
$$\hat{L} = 70{,}48° \quad \text{or} \quad \hat{L} = 180° - 70{,}48° = 109{,}52°$$

- We have a known pair (angle K and JL) therefore we can use the Sine rule.
- To calculate an angle, we choose the formula with angles in the numerator.
- Since the angle we need is opposite the larger side, it must be the larger angle, but since Sine is positive in Q1 and Q2, the angle could be acute (Q1) or obtuse (Q2). This is called the ambiguous case.
- There is not enough information in the question to choose between the two possible values for L, therefore we write both answers.

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Worked example

A cameraperson is stationed at the top of a vertical tower (TG), where TG = h. She films a cat starting at A, where the angle of depression of T to A is θ. The camera follows the cat as it walks (in the same vertical plane) a distance of x metres horizontally to B, creating angle β.



R Determine $\hat{B}_1$ in terms of θ and β, with reasons.

$\hat{A} = \theta \qquad (alt\ \angle s; TK||GA)$

$\therefore \hat{B}_1 = \beta + \theta \quad (ext\ \angle\ of\ \Delta)$

- Since TK and GA are horizontal lines, they are parallel to each other. Using alternate angles, we can say KTA=A.
- The exterior angle of a triangle is equal to the sum of the opposite interior angles.

C Show that $x = \dfrac{h \sin\beta}{\sin(\theta + \beta)\sin\theta}$

$\sin(\beta + \theta) = \dfrac{h}{TB}$

$TB = \dfrac{h}{\sin(\beta + \theta)}$

$\dfrac{AB}{\sin B\hat{T}A} = \dfrac{TB}{\sin\hat{A}}$

$\dfrac{x}{\sin\beta} = \dfrac{\frac{h}{\sin(\beta+\theta)}}{\sin\theta}$

$x = \dfrac{h}{\sin(\beta + \theta)} \times \dfrac{1}{\sin\theta} \times \dfrac{\sin\beta}{1}$

$x = \dfrac{h \sin\beta}{\sin(\beta + \theta)\sin\theta}$

- Starting in the right-angled triangle TGB, we use sin(B1) to find side TB (hypotenuse) using the given side TG (opposite).
- Now working in the non-right angled triangle TBA, we use the Sine rule to find an expression for side AB (x).
- Substitute the expression found for TB into the Sine rule equation.
- Isolate x by multiplying by its denominator of sinB.
- We now have a fraction within a fraction, so to simplify we can use the "tip and times" rule by converting the division of sinθ into multiply by $\frac{1}{\sin\theta}$
- Multiply numerators by numerators and denominators by denominators to fully simplify and match the expression given in the question.

R Determine the distance the cat calked from A to B, i.e. x, if the tower was $20\ m$ high, $\theta = 65°$ and $\beta = 37°$.

$x = \dfrac{20 \sin 37°}{\sin(37° + 65°)\sin 65°}$

$x = 13{,}58\ m$

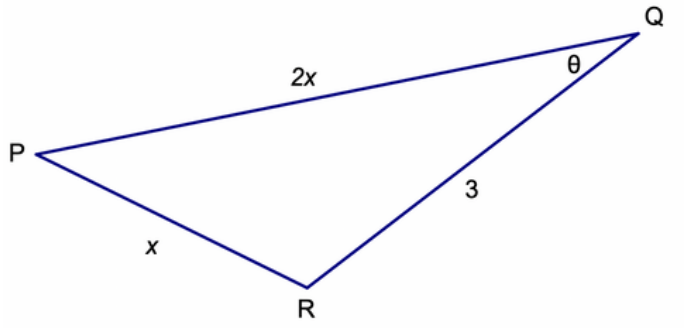
- Substitute h=20, $\theta = 65°$, and $\beta = 37°$ into the given formula for x.
- Even if you did not complete the previous question, you can still get full marks for this question using the given information.

Question source: Grade 11 GP November 2022 Paper 2

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Worked example

In ΔPQR, $QR = 3\ units$, $PR = x\ units$, $PQ = 2x$ and $P\hat{Q}R = \theta$.



C Show that $\cos\theta = \frac{x^2+3}{4x}$

$$x^2 = (2x)^2 + 3^2 - 2(2x)(3)\cos\theta$$

$$x^2 = 4x^2 + 9 - 12x\cos\theta$$

$$12x\cos\theta = 3x^2 + 9$$

$$\cos\theta = \frac{3x^2 + 9}{12x}$$

$$\cos\theta = \frac{3(x^2 + 3)}{12x}$$

$$\cos\theta = \frac{x^2 + 3}{4x}$$

- To get cosθ as the subject, we will use the Cos rule.
- The side opposite θ (x) will start as the subject.
- Substitute all the values into the Cos rule, making sure to use brackets so the terms are squared correctly.
- To make cosθ the subject, move the term to the LHS and the x^2 to the RHS and simplify the like terms. This also means everything is now positive.
- Divide both sides of the equation by 12x to get cosθ on its own
- By factorising out the common factor of 3 in the numerator, we can cancel with the denominator to get the same fraction as given in the question.

R If $x = 2{,}4$ units, calculate θ.

$$\cos\theta = \frac{(2{,}4)^2 + 3}{4(2{,}4)}$$

$$\cos\theta = \frac{73}{80}$$

$$\theta = \cos^{-1}\left(\frac{73}{80}\right)$$

$$\theta = 24{,}15°$$

- Even though the previous question was Complex (C), this question is Routine (R) since you can use the answer given in the previous question and substitute x = 2.4 into the given expression to determine the value of cosθ.
- To solve for θ we use SHIFT and COS to calculate the angle using the ratio.
- Round your answer to 2 decimal places (dp) unless told otherwise.

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R Hence, calculate the area of ΔPQR.

Area $= \frac{1}{2}(PQ)(RQ)\sin\hat{Q}$

$= \frac{1}{2}(4{,}8)(3)\sin 24{,}15°$

$= 2{,}95\ units^2$

- Using x = 2,4 we can find the length of PQ = 2(2,4) and substitute RQ = 3 (given) and the value for θ calculated in the previous question.
- The question used "units" so this answer is in $units^2$

P Calculate the values of x for which the triangle exists.

$x + 2x > 3$ AND $x + 3 > 2x$ AND $2x + 3 > x$

$x > 1$ $x < 3$ $x > -3$

$\therefore 1 < x < 3$

- For a triangle to exist, the sum of any two sides must be greater than the third side.
- We can create three inequalities using this fact to find a range of values for x.
- Since x > 1 we can ignore the last inequality (x > -3) since the values greater than 1 will be greater than -3 already.
- Usually with inequalities we say OR but in this case, all three inequalities have to be true at the same time, therefore we say AND.
- Since x > 1 AND x < 3, then x is between 1 and 3.

OR

Since $x > 0,\ 0 < \cos\theta < 1$

$$0 < \frac{x^2 + 3}{4x} < 1$$

$$x^2 + 3 < 4x$$

$$x^2 - 4x + 3 < 0$$

$$(x - 3)(x - 1) < 0$$

$$1 < x < 3$$

- x must be positive since it is the length of one of the side's of the triangle (PR).
- Looking at the equation given for $\cos\theta = \frac{x^2+3}{4x}$, the numerator of that fraction must be positive (a square number is either zero or positive and then add 3 will definitely be positive) and the denominator is positive (since x is positive) therefore cosθ is positive but cosθ has a maximum of 1 (picture the maximum turning point of a cosx graph).
- This means 0<cosθ<1 and since $\cos\theta = \frac{x^2+3}{4x}$, then $0 < \frac{x^2+3}{4x} < 1$
- Usually we cannot multiply by x, since we don't know if it is positive or negative, therefore we don't know whether to swop the direction of the inequality signs, but here we do know x is positive so we can.
- Now, move all the terms to the same side since the inequality is quadratic and factorise the trinomial.
- The critical values are 1 and 3, and the values of x that will make the inequality negative are between the critical values.

Question source: DBE Grade 11 November 2017 Paper 2

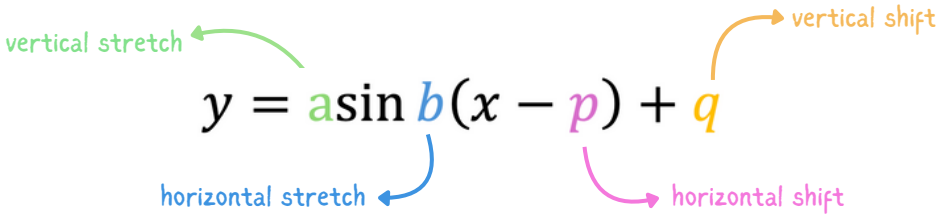
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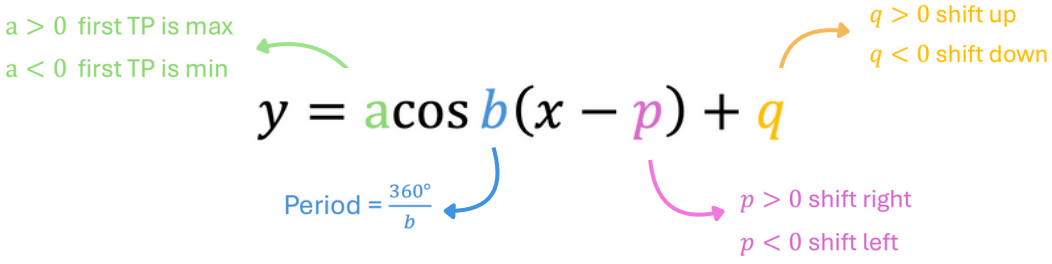
Question Type 6: Trigonometric functions

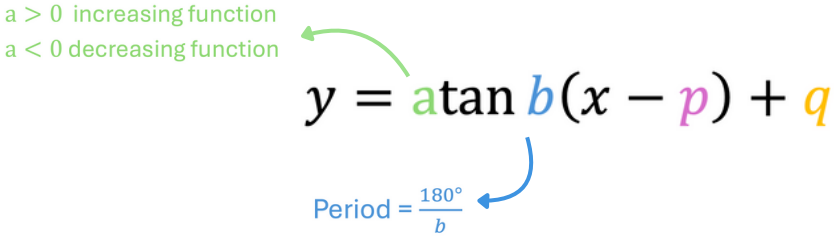
In order to achieve full marks for this section, you need to be able to:

- **draw** a sine, cosine or tangent graph with all parameter changes on any given domain (max 2 parameters at a time)
- **determine the equation** of any given graph
- **interpret** and make deductions from a given function (period, amplitude, domain, range, asymptotes, max/min values etc.)
- perform a **transformation** (translate/reflection) on any trigonometric function.

The parameters of the three trigonometric graphs:







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Terminology

Maximum value: the highest y-value on a sin or cos graph found at the maximum turning point. The tan graph does not have a maximum y-value.

Minimum value: the smallest y-value on a sin or cos graph found at the minimum turning point. The tan graph does not have a maximum y-value.

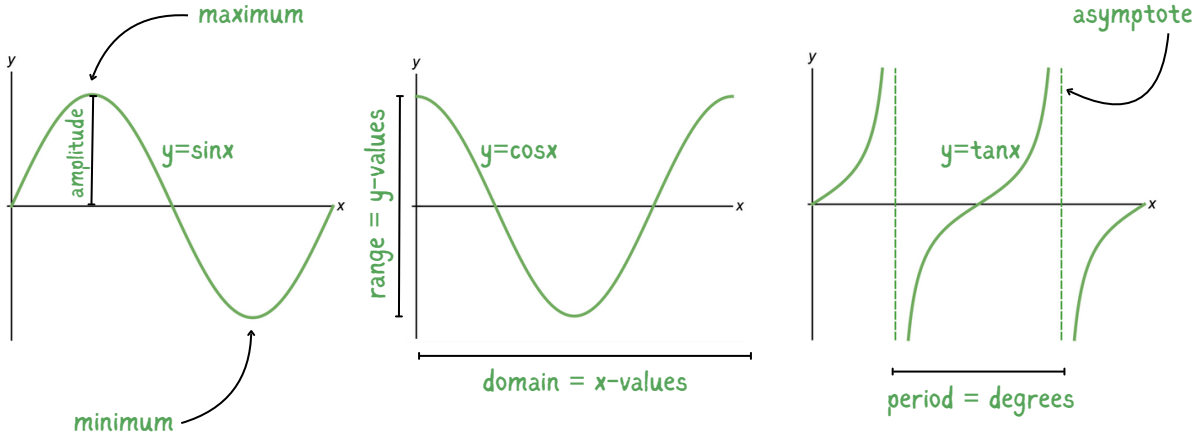
Domain: the domain is the x-values that exist on the graph. Since the graphs continue indefinitely, the given domain is a snapshot of a section of the graph.

Range: the domain is the y-values that exist on the graph, from the minimum y-value to the maximum y-value. The tan graph doesn't have a maximum or minimum so the range is always yER (y is an element of real values, therefore y can be any real number).

Amplitude: the amplitude is a measure of the graph's height. To calculate, find the total height (height = Max - Min) and divide by 2. Since the tan graph doesn't have a maximum or minimum, it also doesn't have an amplitude.

Asymptotes: the tan graph has asymptotes between each repetition of its curve. An asymptote is a line that the graph tends towards but will never touch or cross.

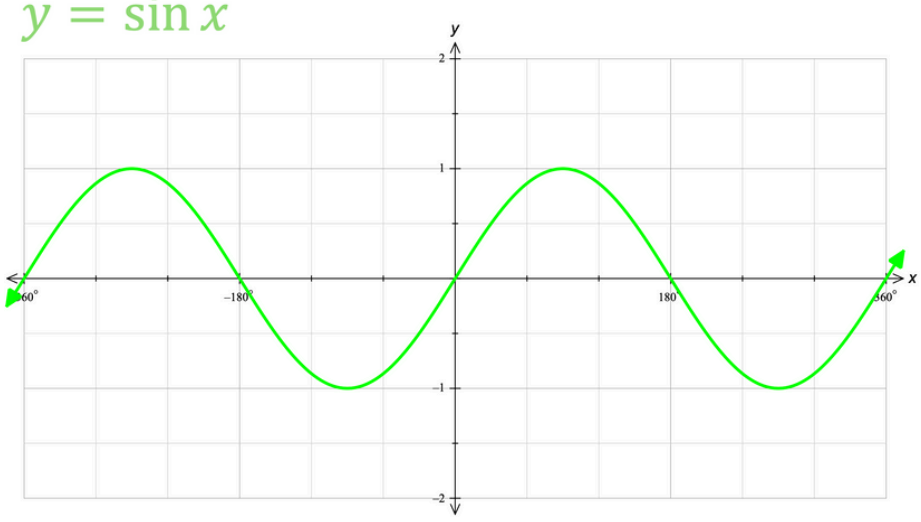
Period: the degrees it takes for a graph to repeat itself. All three of the trigonometry graphs repeat indefinitely and one cycle of the wave is called the period. The basic sin and cos graphs have a period of 360 and the basic tan graph has a period of 180.



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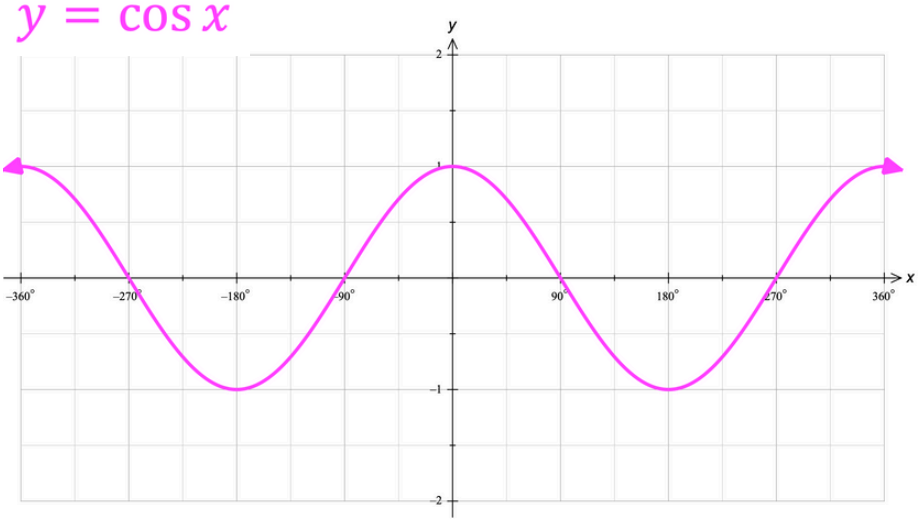
The basic trigonometric functions

$y = \sin x$



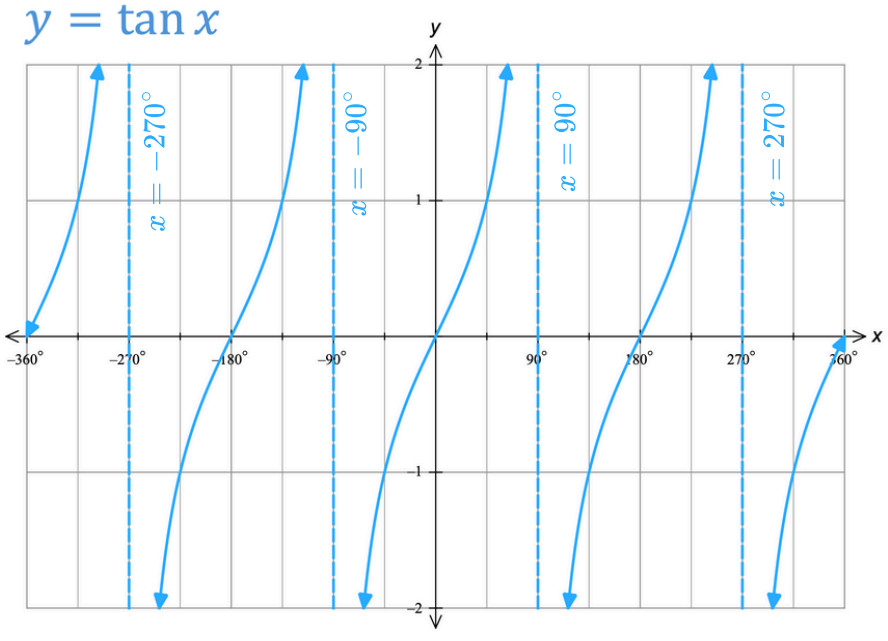
Period: 360°
Amplitude: 1
Range: $-1 \leq y \leq 1$
Asymptotes: none

$y = \cos x$



Period: 360°
Amplitude: 1
Range: $-1 \leq y \leq 1$
Asymptotes: none

$y = \tan x$



Period: 180°
Amplitude: none
Range: $y \in \mathbb{R}$
Asymptotes:
$x = -270°; -90°;$
$90°;\ 270°$

As you can see, the sinx and cosx graphs are very closely related, whereas the tanx graph has different features.

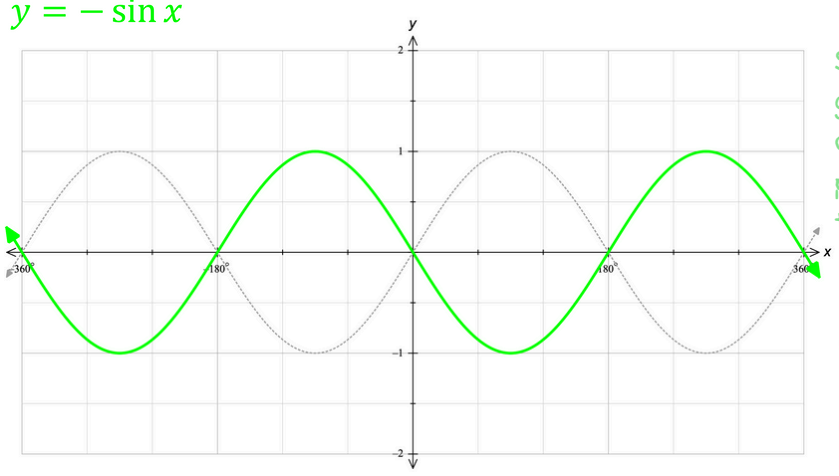
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How the parameter "a" affects the graphs

Since "a" is a vertical stretch, it will affect the amplitude and range of the sin and cos functions. It also, importantly, dictates whether the graphs are positive or negative. It does also stretch the tan graph, but since tan doesn't have a maximum or minimum value, it doesn't have an amplitude and its range will remain the same. Instead, for tan, we look at what is called the anchor point. The anchor point is used in place of turning points and is found at (45;1) in the basic tan graph.

The basic graph has been drawn as a dotted grey line for you to more easily compare and visualise the different affects of "a".

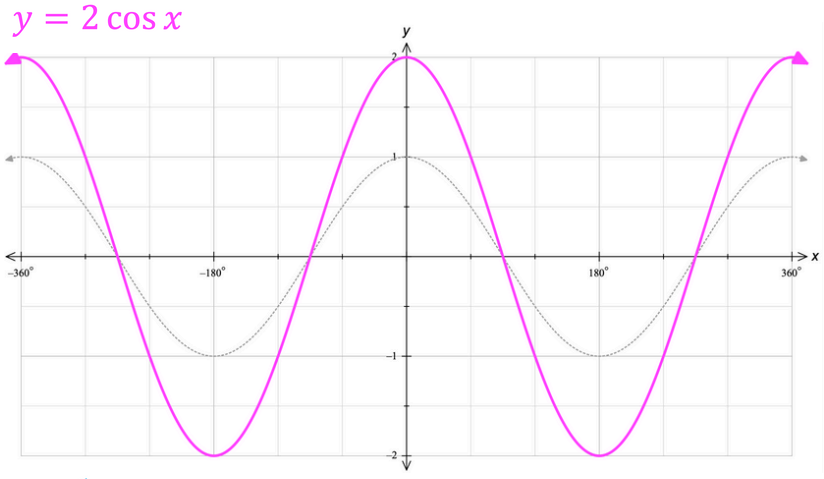
$y = -\sin x$



Since the "a" is negative, the sin graph has been reflected (flipped) over the x-axis. It's first turning point is now a minimum, rather than a maximum.

Period: 360°
Amplitude: 1
Range: $-1 \leq y \leq 1$
Asymptotes: none

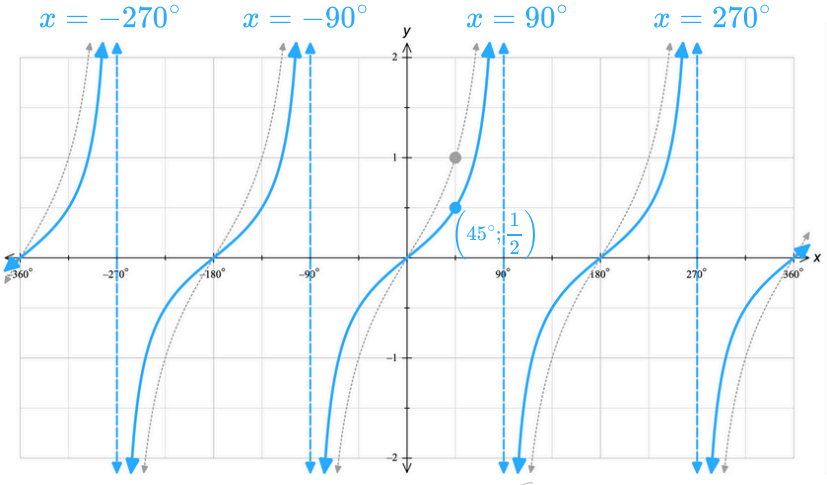
$y = 2\cos x$



The "a" value here is 2, therefore the graph is positive (first TP is a maximum) but the graph is double the height of the basic graph.

Period: 360°
Amplitude: 2
Range: $-2 \leq y \leq 2$
Asymptotes: none

$y = \frac{1}{2}\tan x$



The "a" value is now 1/2, therefore the graph is still positive (tan is increasing) but its anchor point is now at (45; 1/2). The asymptotes remain the same.

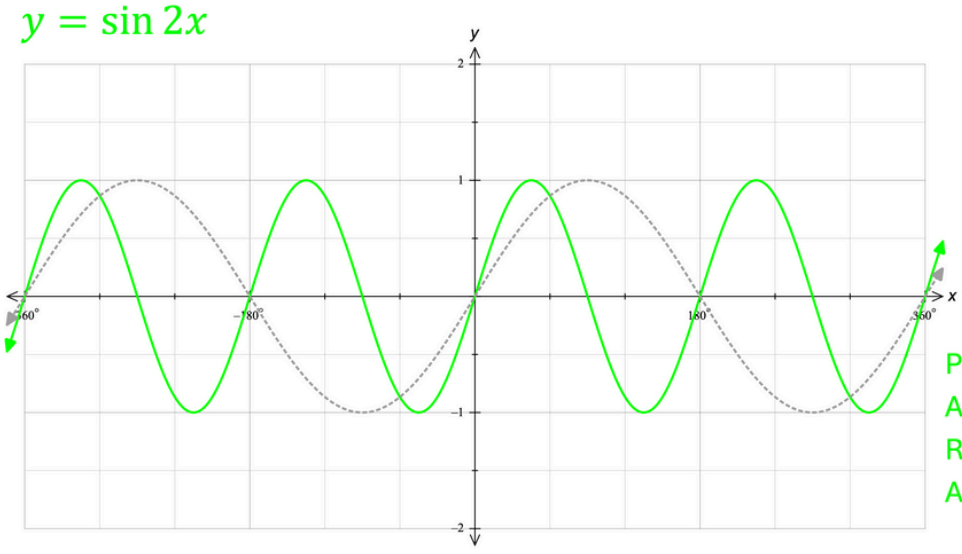
Period: 180°
Amplitude: none
Range: $y \in \mathbb{R}$
Asymptotes:
$x = -270°; -90°; 90°; 270°$

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How the parameter "b" affects the graphs

The "b" is a horizontal stretch, it will affect the period of all the graphs. The value of "b" is how many graphs fit in the usual period (360 for sin and cos and 180 for tan).

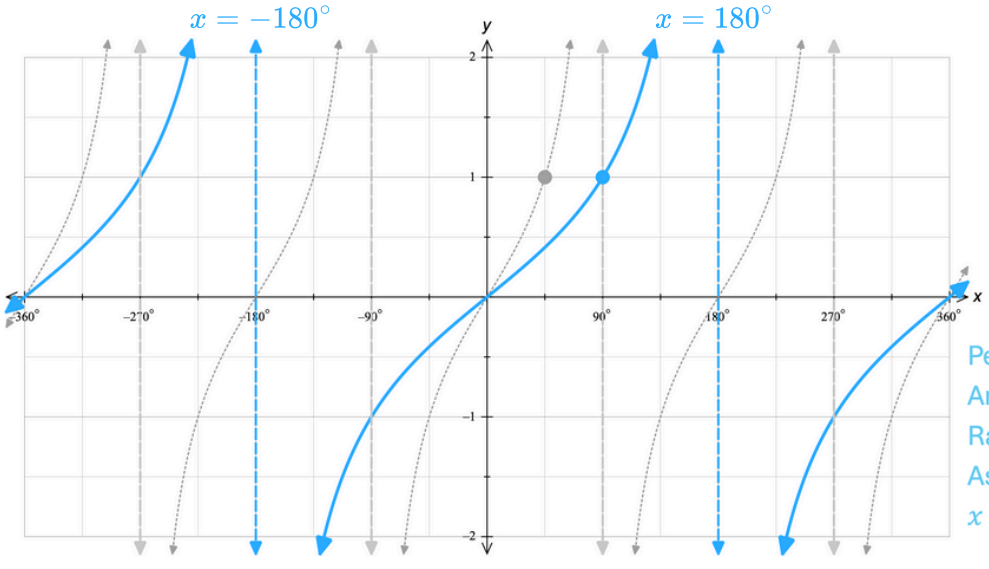
$y = \sin 2x$



Period: 180°
Amplitude: 1
Range: $-1 \leq y \leq 1$
Asymptotes: none

Since the "b" is 2, we can fit 2 graphs in the same space we would usually fit 1 for the basic sinx graph. This means the period is now $\frac{360^\circ}{2} = 180^\circ$

$y = \tan \frac{1}{2}x$



Period: 360°
Amplitude: none
Range: $y \in \mathbb{R}$
Asymptotes:
$x = -180°; x = 180°$

Since the "b" is $\frac{1}{2}$, we can fit half one full graph in the same space we would usually fit 1 for the basic tanx graph. This means the period is now $\frac{180^\circ}{\frac{1}{2}} = 360^\circ$. The easiest way to read off the period is to look at the asymptotes. Since the asymptotes are now $x = -180^\circ$ and $x = 180^\circ$ there is 360° between them, which is one period.

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How the parameter "p" affects the graphs

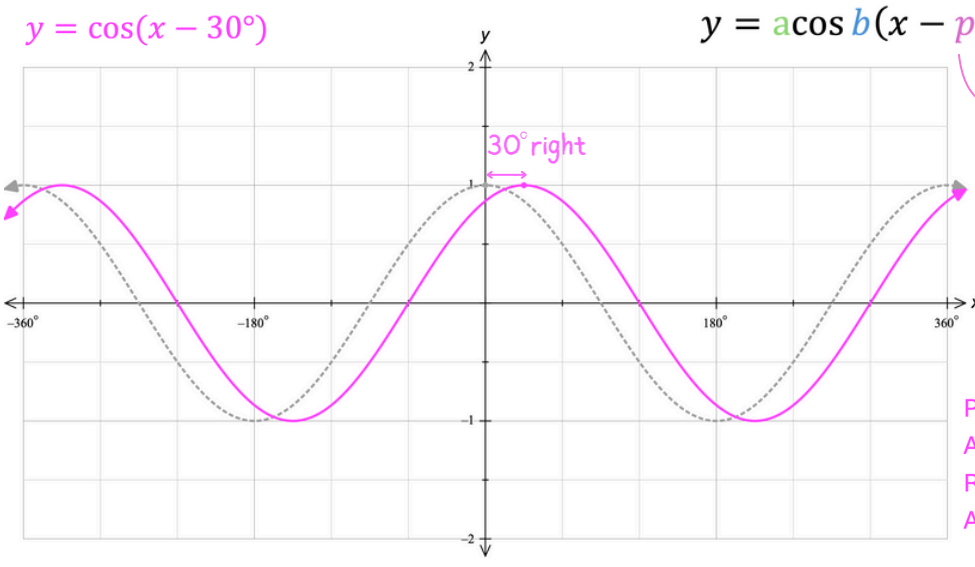
The value of "p" will shift the graph horizontally. If p > 0, the graph will shift right and if p < 0 the graph will shift left. The shape stays EXACTLY the same.

$y = \cos(x - 30°)$

$y = a\cos b(x - p) + q$

$p > 0$ shift right

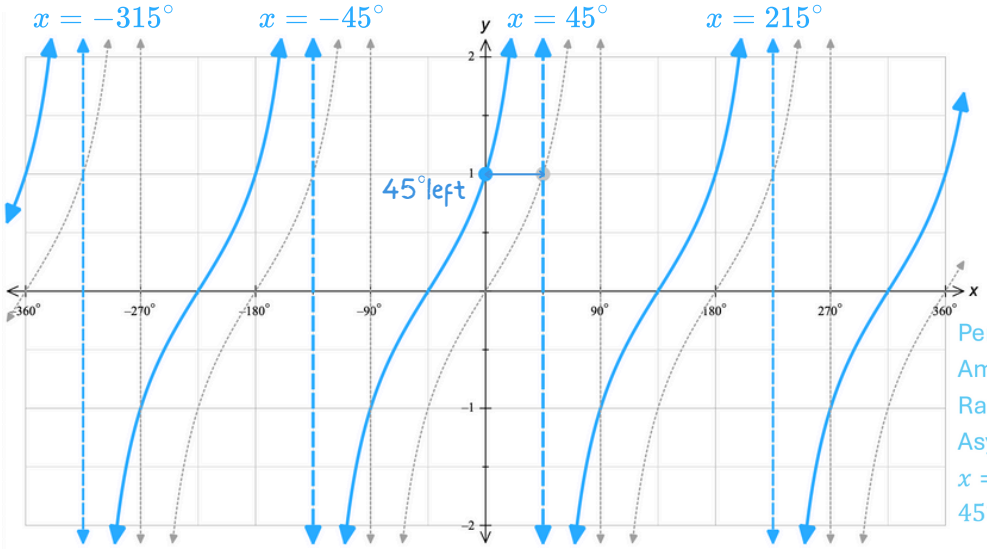
$p < 0$ shift left



Period: 360°
Amplitude: 1
Range: $-1 \leq y \leq 1$
Asymptotes: none

Here, the "p" value is +30 (remember the formula has a minus), therefore the graph has translated 30° to the right. Every co-ordinate has shifted, which keeps the shape exactly the same.

$y = \tan(x + 45°)$



Period: 180°
Amplitude: none
Range: $y \in \mathbb{R}$
Asymptotes:
$x = -315°; -135°; 45°; 225°$

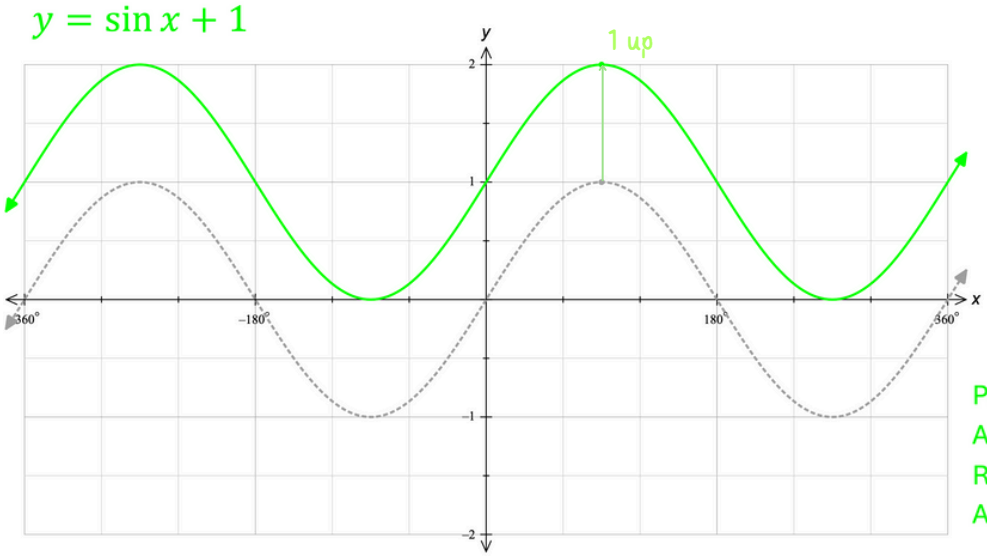
Here, the "p" value is -45 [$y = \tan(x - (-45°))$], therefore the graph has translated 45° to the left. Every co-ordinate has shifted, including the asymptotes, which keeps the shape exactly the same.

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How the parameter "q" affects the graphs

The value of "q" will shift the graph vertically. If q > 0, the graph will shift up and if q < 0 the graph will shift down. The shape stays **EXACTLY** the same.

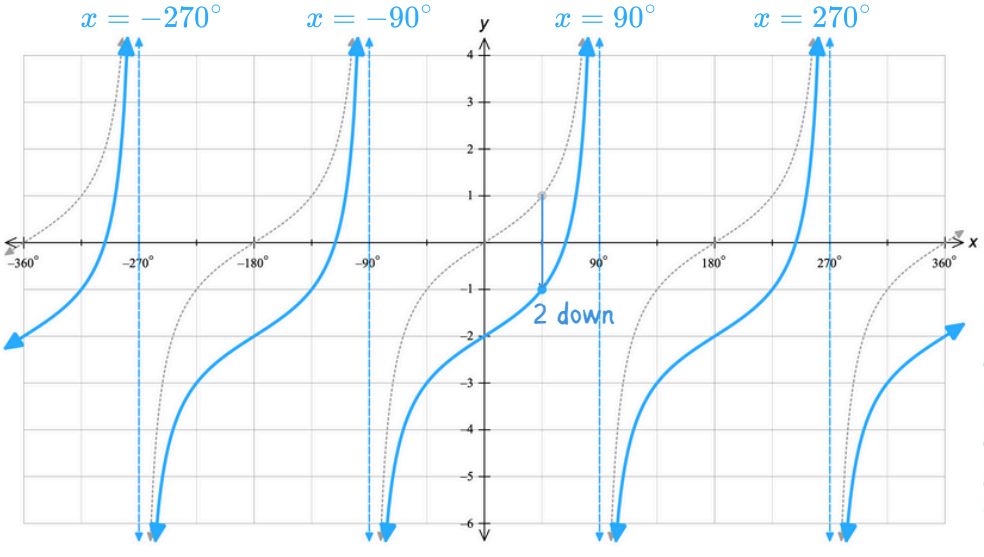
$y = \sin x + 1$



Period: 360°
Amplitude: 1
Range: $0 \leq y \leq 2$
Asymptotes: none

Since "q" value is +1, the graph has translated 1 unit up. Every co-ordinate has shifted, which keeps the shape exactly the same.

$y = \tan x - 2$



Period: 180°
Amplitude: none
Range: $y \in \mathbb{R}$
Asymptotes:
$x = -270°; -90°; 90°; 270°$

Since "q" value is -2, the graph has translated 2 units down. Every co-ordinate has shifted downwards, which keeps the shape exactly the same.

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Sketching trigonometric graphs

To draw a trigonometric graph, it's best to visualise the basic graph and the affect each given parameter will have on the graph. If in doubt, you can use the **TABLE** function on your calculator to give you a list of co-ordinates to help you sketch the graph.



How to use the TABLE function on your CASIO scientific calculator

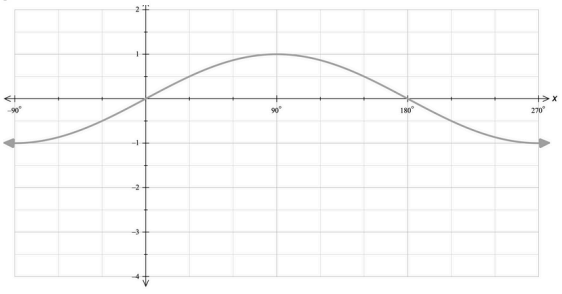
- Press [MODE] and then select the [TABLE] option.
- The screen should now say "f(X)=". Type in the equation of your graph using [ALPHA] then [ )] to type X. Press [=].
- The screen should now say "g(X)=". If you are drawing two graphs, you can type the other equation in now. If not, press [=] to skip.
- The screen should now say "Start?". This is the x-value where the domain begins (e.g. -360 or 0). Press [=].
- The screen should now say "Ends?". This is the x-value where the domain stop (e.g. 180 or 360). Press [=].
- The screen should now say "Step?". This is the the gap you want between the co-ordinates. It's best to use the width of the values on your x-axis (e.g. 30 or 45 for a co-ordinate every 30 or 45 degrees). Press [=].
- The screen should now show you a table of values. The first column is the numbering the co-ordinate (1, 2, 3, etc.), the second column is the x-coordinate, the third is the y-coordinate (and if you entered two graphs the fourth column is the y-coordinate of the second graph). Press the down arrow to see all the values in the table.
- If your calculator says "Insufficient memory", either you need to shorten the domain (start later or end sooner) or make the step wider to get fewer co-ordinates.

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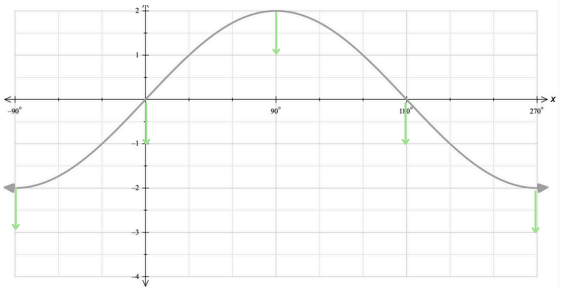
Worked example

R Draw the graph of $y = 2\sin x - 1$ on the set of axes below for the interval $x \in [-90°; 270°]$
Label all intercepts, start and endpoints.

$y = \sin x$



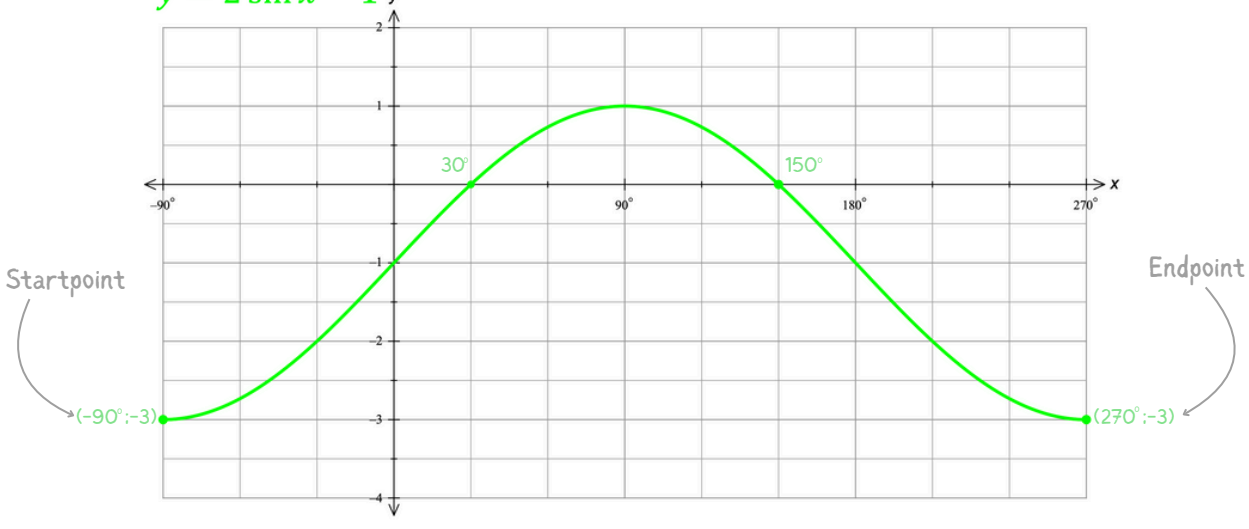
$y = 2\sin x$



First, picture the basic y=sinx graph on the given domain. → Now, apply the effect of "a" to the basic graph. Since a=2, the graph is stretched vertically to 2x the height.

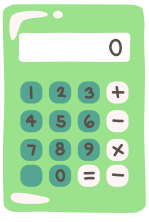
The green arrows show the effect of the "q" value. Since q=-1, the graph will shift down 1 unit.

$y = 2\sin x - 1$



Calculator steps:

f(x)=2sin(X)-1
Start? -90
End? 270
Step? 30



	X	f(X)
1	-90	-3
2	-60	-2.73...
3	-30	-2
4	0	-1
5	30	0
6	60	0.73...
7	90	1
....		

PRO-TIP

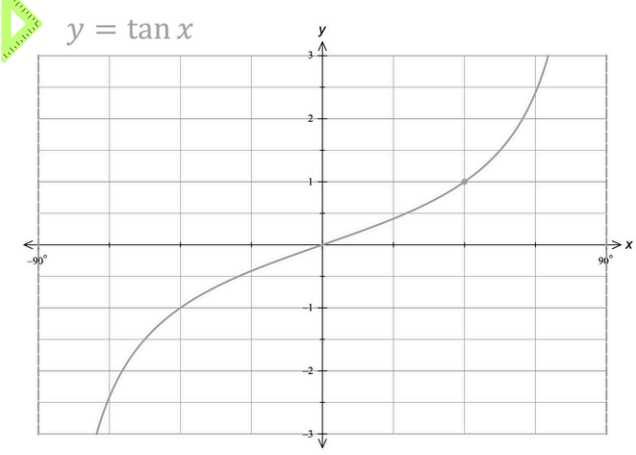
Most often, you will be given a grid to use for drawing your trig graph(s), in which case, use the width of the blocks along the x-axis as your step. If not, use the period of the graph and divide it by 4.

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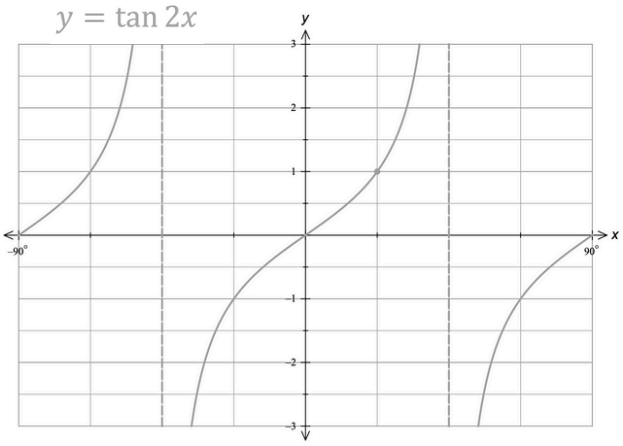
Worked example

R Draw the graph of $y = -\tan 2x$ on the set of axes below for the interval $x \in [-180°; 180°]$. Indicate and label all asymptotes and intercepts with axes.

$y = \tan x$

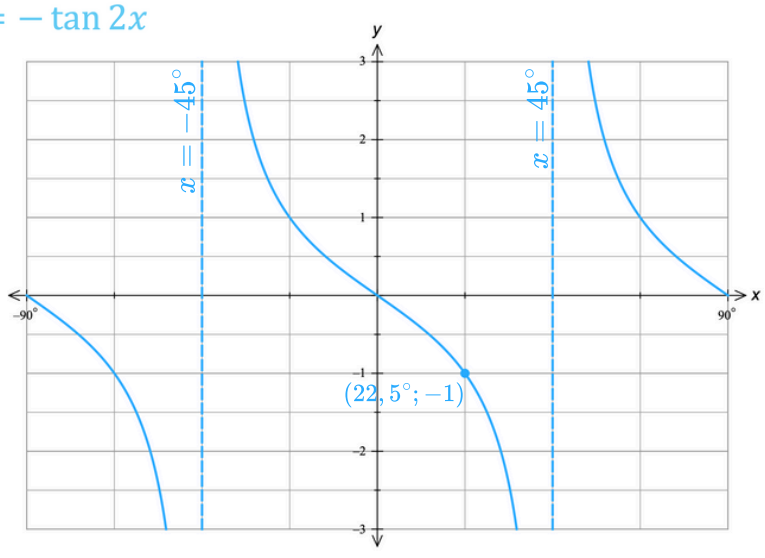


$y = \tan 2x$



First, picture the basic y=tanx graph on the given domain. → Now, apply the effect of "b" to the basic graph. Since b=2, the graph is compressed horizontally to fit 2 repetitions of the graph in the usual period.

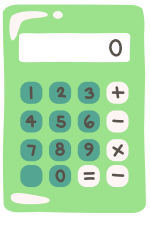
$y = -\tan 2x$



Lastly, we reflect the tan graph over the x-axis since the "a" value was negative. You can also look at the affect of "a" first and then "b" as all the parameters act independently.

Calculator steps:

f(x)=-tan(2X)
Start? -180
End? 180
Step? 45



	X	f(X)
1	-90	0
2	-67.5	-1
3	-45	ERROR
4	-22.5	1
5	0	0
6	22.5	-1
7	45	ERROR
....		

When you see "ERROR" in the table, it means your graph has an asymptote at that value, therefore x=-45 and x=45 are the asymptotes for y=-tan2x.

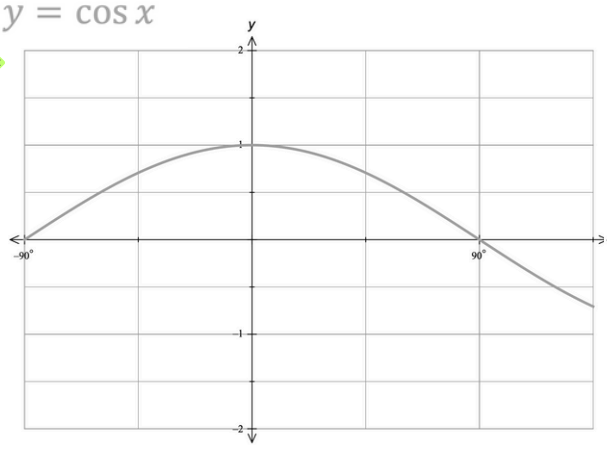
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Worked example

R Draw the graph of $y = \cos(2x + 90°)$ on the set of axes below for the interval $0° \leq x \leq 360°$. Indicate and label all asymptotes and intercepts with axes.

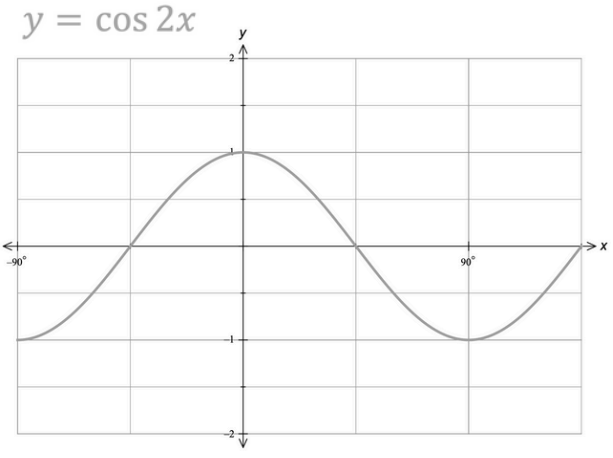
Before we can visualise the parameters, we need to rewrite the equation in standard form. To do this, we take out 2 as a common factor from the bracket to have the coefficient of x=1, to match standard form. Now we can also see the true "p" value is 45, not 90.

$y = \cos x$



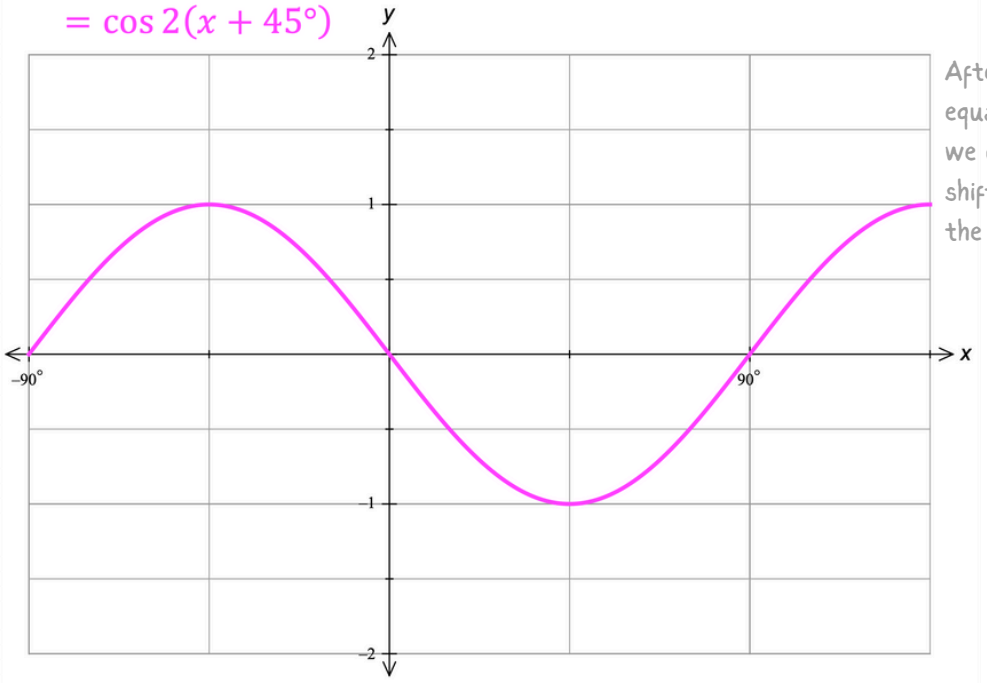
First, picture the basic y=cosx graph on the given domain.

$y = \cos 2x$



Now, apply the effect of "b" to the basic graph. Since b=2, the graph is compressed horizontally to fit 2 repetitions of the graph in the usual period.

$y = \cos(2x + 90°)$
$= \cos 2(x + 45°)$



After rewriting the equation in standard form, we can see the graph has shifted horizontally 45° to the left.

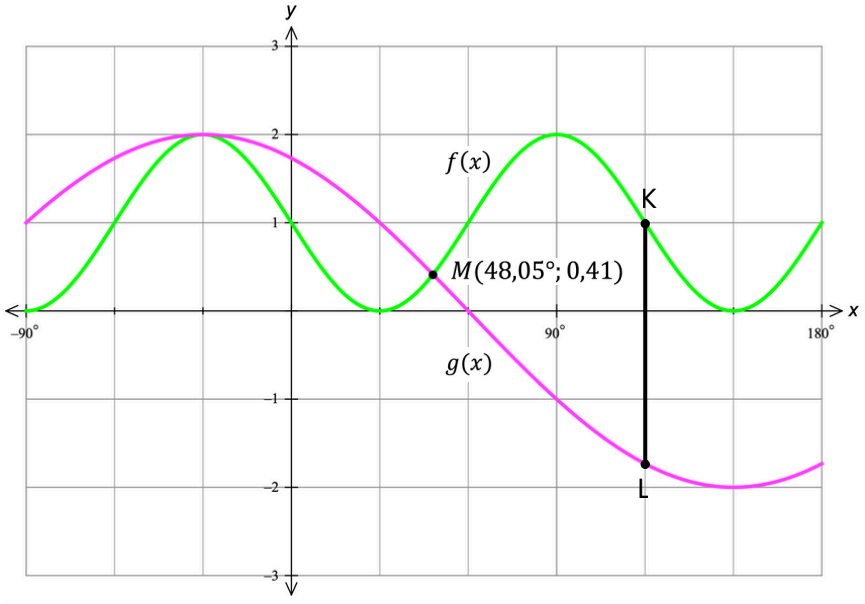
Looking at this graph, you could easily think it was a negative sinx graph rather than a shifted cosx graph. This is how closely related the sinx and cosx graphs are and why they are co-ratios of each other. Here, since the "b" value is 2, a shift of 45 degrees horizontally has the same effect as shifting the basic sinx or cosx graphs by 90 degrees.

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Determining the equation

Worked example

In the diagram below, the graphs of $f(x) = a\sin bx + c$ and $g(x) = d\cos(x + e)$ are shown for the interval $-90° \leq x \leq 180°$. $M(48{,}05°; 0{,}41)$ is the point of intersection between $f(x)$ and $g(x)$. KL is a vertical line with K on $f(x)$ and L on $g(x)$.



K Determine the values of a, b, c, d and e.

$a = -1$

$b = 3$

$c = 1$

$d = 2$

$e = 30$

To calculate "a" you need to take two things into account: the amplitude and whether the graph is positive or negative. To calculate the amplitude take the (MAX-MIN)/2, therefore amplitude = (2-0)/2 = 1. To determine whether the graph is positive or negative, look at the closest turning point to the y-axis. If the turning points are an equal distance from the y-axis, look after the y-axis. Since it is a minimum TP, the graph is negative, therefore a = -1.

To calculate "b", take the normal period of a sinx graph (360) and divide by the period of the given graph. The graph of f takes 120 degrees for a full wave, therefore b = 360/120=3.

To determine the "c" value, find the middle of the wave, (MAX+MIN)/2, therefore x = (2+0)/2 = 1.

To calculate "d" follow the same steps as "a" but now looking at the graph of g (the cos graph). Here, the closest turning point is a maximum TP before the y-axis, therefore the graph is positive. Amplitude = (MAX-MIN)/2=(2--2)/2=2, therefore d = +2.

"e" is the horizontal shift of g. Usually, a basic cosx graph would start at a maximum TP. Here, the maximum TP has been shifted 30 degrees to the left, therefore e = +30.

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R Write down the coordinates of the minimum turning point of $g(x)$ in the given interval.

$(150°; -2)$

- The graph of g only has one maximum and one minimum TP in the given interval.
- The x-axis is going up in 30s, therefore you can read off the minimum TP is at x=150 and y=-2. You must write your answer as a coordinate.

R Determine the length of KL, correct to one decimal place.

$K(120°; 1)$

$g(120°) = 2\cos(120° + 30°)$
$g(120°) = -\sqrt{3}$
$\therefore L(120°; -\sqrt{3})$
$\therefore KL = 1 - (-\sqrt{3}) = 2{,}7$

- The coordinate of K can be read off from the grid at x=120 and y=1. Since KL is a vertical line, L will have the same x-value but we cannot read off the y-value.
- By substituting x=120 into the graph of g, we can calculate the y-coordinate of L.
- The length is the difference between the y-values. Always top (K) minus bottom (L) as length must be a positive value.

R Write down the range of $3g(x) - 1$.

$y \in [-7; 5]$ ⟶ Interval notation

$-7 \leq y \leq 5$ ⟶ Set-builder notation

To find the range, first multiply the current range yE[-2;2] by 3, so now the range is yE[-6;6]. Then, subtract 1 from both values, therefore yE[-7;5].

C For which values of x is $f(x) \geq g(x)$?

$x = -30°$ or $48{,}05° \leq x \leq 270°$



The question is asking for the values of x where the graph of f has higher y-values (or equal) compared to the graph of g. You need to look for where the sin graph is on top of the cos graph. This is from the point of intersection (x=48,05) to the end of the given domain (x=270), including both since the question has equal to, and also where they are strictly equal, at x=-30.

C Determine the equation of $j(x)$ if $j(x)$ is a reflection of $f(x)$ in the y-axis. Simplify your answer.

$f(x) = -\sin 3x + 1$
$y = -\sin(-3x) + 1$
$h(x) = \sin 3x + 1$

To reflect any graph over the y-axis, we make x-negative. If the question hadn't specified that that the answer needed to be simplified, you could have left it as y=-sin(-3x)+1. Since we need to simplify, we need to reduce the negative angle. An angle of -3x is in quadrant 4, where sin is negative, therefore the graph becomes y=-(-sin3x)+1, which simplifies to y=sin3x+1 and we are told to call the new graph h(x).

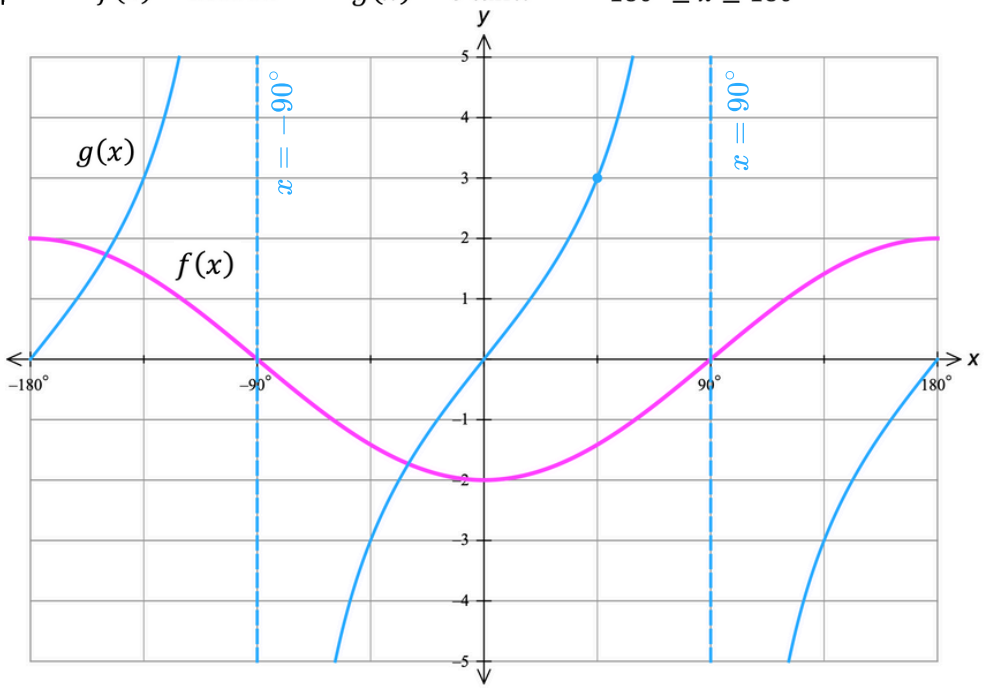
PRO-TIPS

You can give your answers in either interval notation, or set-builder notation, but you must be able to understand both since the question can also use either. You are also likely to get only one option in a memo to check your work.

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Worked example

The graphs of $f(x) = a\cos bx$ and $g(x) = c\tan x$ for $-180° \leq x \leq 180°$ are drawn below.



K Determine the values of a, b, and c.

$a = -2$
$b = 1$
$c = 3$

- Firstly, the amplitude of the cos graph is (2--2)/2 = 2, then because it starts on a minimum TP, the "a" value is negative, therefore a = -2
- From x = -180 to x = 180 is one full wave so the period is 360, therefore the b=1 (the same as the basic graph).
- For tan, we can't look at the maximum and minimum values to find the amplitude so we look at the anchor point. Here, when x = 45, y = 3, and the tan graph is increasing (positive), therefore c = +3.

C For which values of x is $f(x) = g(x)$ in the given interval?

$$-2\cos x = 3\tan x$$
$$-2\cos x = \frac{3\sin x}{\cos x}$$
$$-2\cos^2 x = 3\sin x$$
$$-2(1-\sin^2 x) = 3\sin x$$
$$-2 + 2\sin^2 x = 3\sin x$$
$$2\sin^2 x - 3\sin x - 2 = 0$$
$$(2\sin x + 1)(\sin x - 2) = 0$$
$$\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x \neq 2$$
$$ref \angle = 30°$$

Q3:

$$x = 180° + 30° + n.360°$$
$$x = 210° + n.360°; n \in \mathbb{Z}$$

Q4:

$$x = 360° - 30° + n.360°$$
$$x = 330° + n.360°; n \in \mathbb{Z}$$
$$\therefore x = -150°; -30°$$

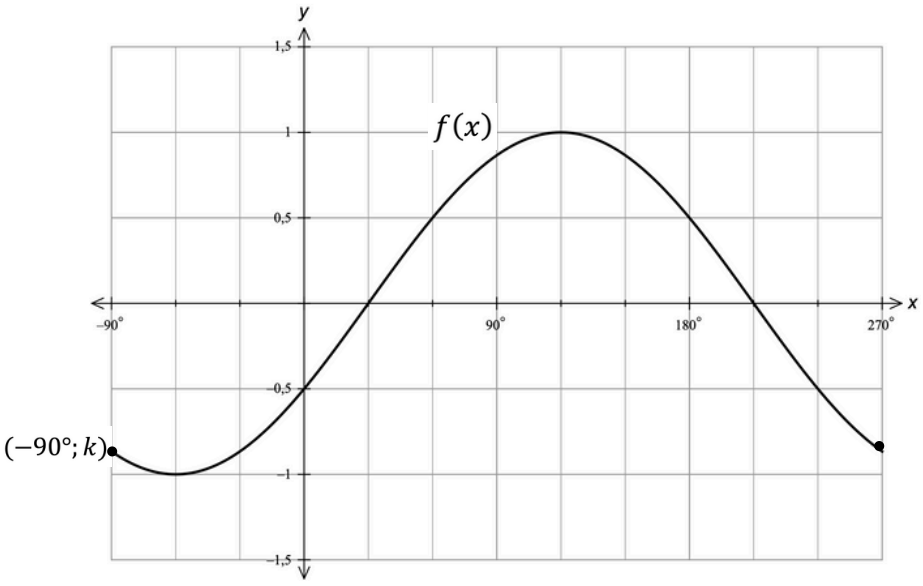
- Here, we cannot read off the points of intersection so we need to solve algebraically.
- First, use the tan identity to convert tan to sinx/cosx. Now, multiply both sides by cosx. We can now use the square identity to convert $\cos^2 x$ so we only have terms with sinx.
- Since we have a square, move everything to the same side of the equation so we can factorise the trinomial.
- We now have two factors, solve for sinx from each bracket.
- The first bracket has a negative value, therefore we find the reference angle (using a positive ratio) and find the value of the angles in the negative quadrants (Q3 and Q4).
- The second bracket does not give a solution since sinx has a maximum value of 1 (therefore it cannot = 2).
- Using the period, we can subtract 360 from each quadrant's general solution to get the points of intersection we can see in the sketch (210-360 = -150 and 330-360 = -30)

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Interpreting the graphs

Worked example

The graph of $f(x) = \sin(x - 30°)$ for $x \in [-90°; 270°]$ is drawn below.



K Write down the value of k, in surd form.

$k = \sin(-90° - 30°)$

$k = -\frac{\sqrt{3}}{2}$

- k is the y-value of the start point and has x=-90
- Therefore sub in x=-90 into the equation for f(x) to find the corresponding y-coordinate

R For what values of x is $f(x)$ strictly decreasing in the interval $x \in [-90°; 270°]$?

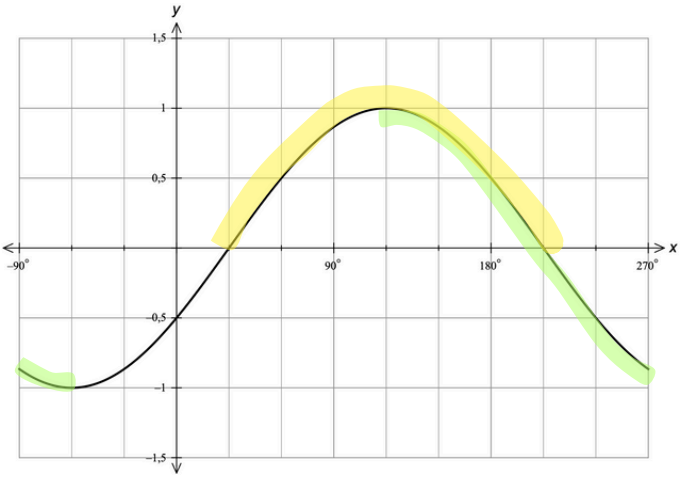
$-90° \leq x < -60°$ or $120° < x \leq 270°$

- A graph is decreasing when the y-values are getting smaller as the x-values are getting larger (the graph is going down from left to right - highlighted on the graph below).
- Since the graph is decreasing at the start and endpoints, we include those values but the graph ism't decreasing at the TPs so those values are excluded.

R For what values of x is $f(x) > 0$ in the interval $x \in [-90°; 270°]$.

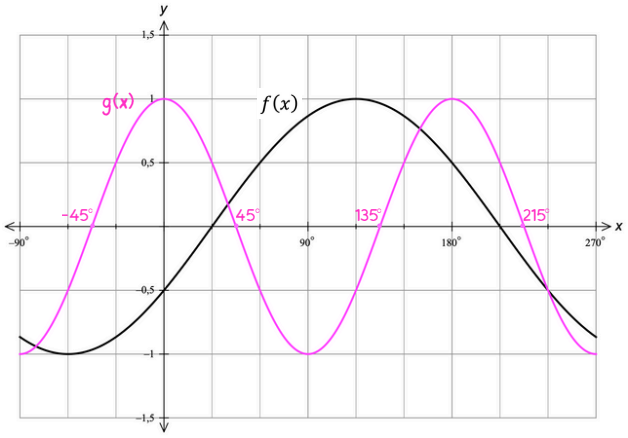
$30° < x < 210°$

- The graph is positive above the x-axis.
- Here, the x-intercepts can be read off, but if they couldn't you would need to make y = 0 and solve for x first.
- Since the x-axis goes up in 30 increments, the first x-intercept is 30 and the second is 210 and the graph is positive between these values.



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R Draw the graph of $g(x) = \cos 2x$ for $x \in [-90°; 270°]$ on the same set of axes, labelling all intercepts.



K Write down the period of $g(x)$

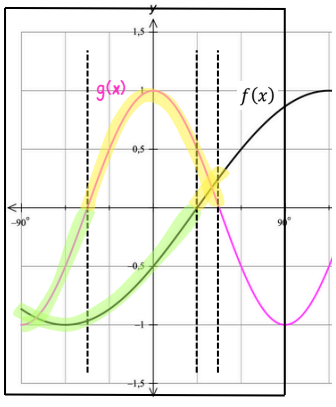
$180°$

- The new period = 360/b, therefore 360/2=180

C For what values of x is $\frac{f(x)}{g(x)} \geq 0$ in the interval $x \in [-90°; 90°]$.

$-90° \leq x < -45°$
or $30° \leq x < 45°$

- For the quotient (fraction) to have a positive value, either both graphs need to be positive, or they need to both be negative.
- Graphs change from positive to negative (or vice versa) when they cross the x-axis. Dotted lines have been drawn through the x-intercepts in the given interval.
- Since g(x) is in the denominator, it cannot be equal to zero, therefore exclude it's x-intercepts (x=-45 and x=45) from your answer.



C Calculate the values of x for which $f(x) = g(x)$ if $x \in [-90°; 270°]$.

$\sin(x - 30°) = \cos 2x$
$\sin(x - 30°) = \sin(90° - 2x)$
Q1:
$x - 30° = 90° - 2x$
$3x = 120° + n.360°$
$x = 40° + n.120°; n \in \mathbb{Z}$
Q2:
$x - 30° = 180° - (90° - 2x)$
$x - 30° = 180° - 90° + 2x$
$-x = 120° + n.360°$
$x = -120° + n.360°; n \in \mathbb{Z}$
$\therefore x = -80°; 40°; 160°; 240°$

- To solve for x, we use the co-ratio trick since we have sin(angle)=cos(different angle).
- Once you have sin on both sides, the angles must be equal but DO NOT SOLVE until the angles are in their correct quadrants.
- Since sin is positive in Q1, and Q2 we solve for x in each of these quadrants, using 180-(RHS angle) for Q2.
- When you divide by the coefficient of x, don't forget to divide everything including the period.
- Use the period to find the values of x for the given domain.

C For what values of x is $f(x + 30°) = g(x + 30°)$ in the interval $x \in [0°; 180°]$?

$x = 10°; 130°$

- (x+30) will shift the graphs (including their points of intersection) 30 to the left, therefore subtract 30 from all the x values in the previous question and only write the values between 0 and 180, as asked in the question.

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EUCLIDEAN GEOMETRY

The following strategy can be used when you are solving geometry questions:

1. **Read through the given information very carefully.**

Make sure that everything that is mentioned in writing is indicated on the diagram as well. If it isn't, add it into the diagram in **BLACK** or in pencil, so that you know that it was given information.

2. **Now analyse the diagram very carefully.**

Use different pencils, pens or highlighters to mark sides and/or angles that are equal, identify various parts of the diagrams, such as parallel lines, cyclic quads, tangents etc. and indicate all values that you can determine right at the start on the diagram as well.
Make sure to use a different colour for any values that were **NOT GIVEN**. You will have to state that value as well as a reason for it if you use it in a calculation.

3. **Analyse what you are asked to calculate/prove in the question.**

Ask yourself how you could do that, and which values or what other information you might need to be able to answer the question.

4. When writing the solution, make sure that you **write down all angles or sides that you are using** in order to get to the answer.
You also have to **write a reason for every statement.**

PRO-TIP

Marks are awarded for values shown on the diagram, even if you didn't write the values down, so make sure to indicate **EVERY** value that you possibly can!

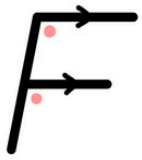
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Summary of ALL Euclidean Geometry knowledge from previous years

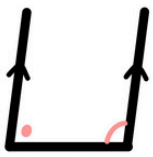
Note that all reasons are highlighted.
You may only use the reasons that are specified in the Grade 12 NSC Exam Guidelines. All reasons in this chapter are allowed to be used and some have acceptable alternatives listed in the guidelines.

AS SEEN IN THE GRADE 12 CHEAT SHEET

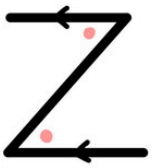
1. FUZYX:



corresponding angles are equal
(corresp. Ls; ll lines)



co-interior angles add up to 180°
(co-int. Ls; ll lines)



alternate angles are equal
(alt. Ls; ll lines)



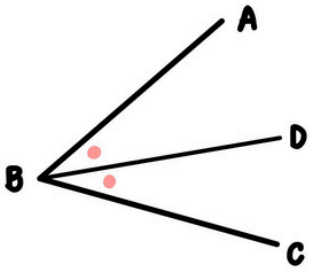
angles on a straight line add up to 180°
(Ls on a str line)



vertically opposite angles are equal
(vert opp Ls =)

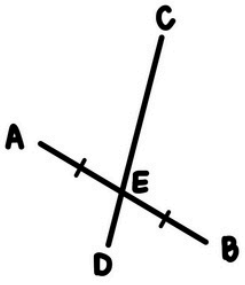
2. When angles or lines are bisected:

If BD bisects $A\hat{B}C$:



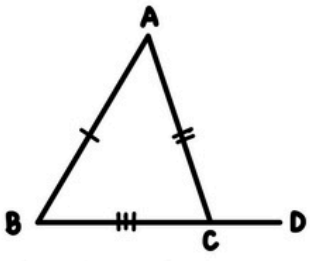
$A\hat{B}D = C\hat{B}D$

If CD bisects AB:



AE = BE and E is the midpoint of AB.

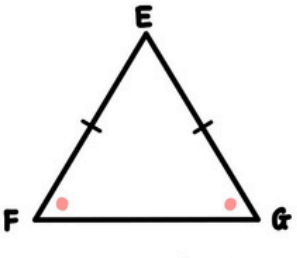
3. Triangle information:



$\hat{A} + \hat{B} + A\hat{C}B = 180°$
(L sum in Δ) or
(sum of Ls in Δ)

$A\hat{C}D = \hat{A} + \hat{B}$ (ext. L of Δ)

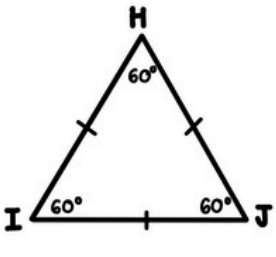
ΔABC is a scalene Δ.



If EF = EG : $\hat{F} = \hat{G}$
(Ls opp equal sides)

If $\hat{F} = \hat{G}$: EF = EG
(sides opp equal Ls)

ΔEFG is an isosceles Δ.

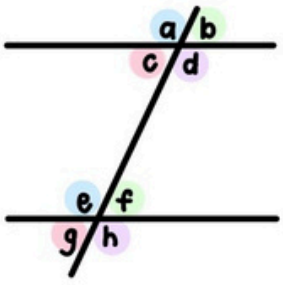


If HI = HJ = IJ :
$\hat{H} = \hat{I} = \hat{J} = 60°$

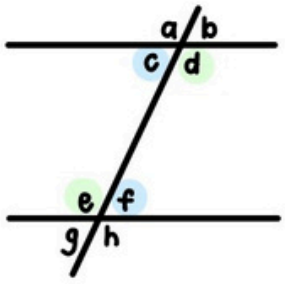
ΔHIJ is an equilateral Δ.

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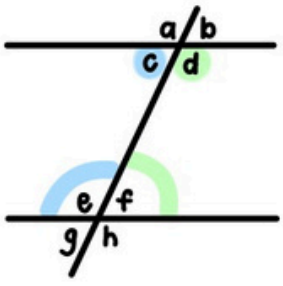
4. To prove that lines are parallel:



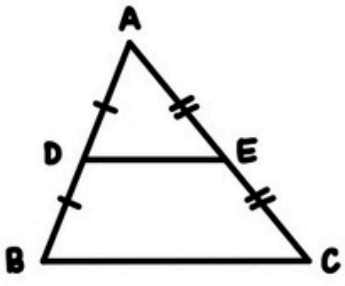
Prove that:
$a = e$ or $b = f$ or $c = g$ or $d = h$
(corresp. Ls equal)
because corresponding angles are only equal if they are formed by parallel lines.



Prove that:
$c = f$ or $d = e$
(alt. Ls equal)
because alternate angles are only equal if they are formed by parallel lines.



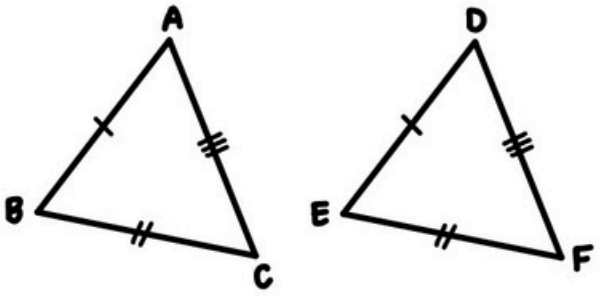
Prove that:
$c + e = 180°$ or $d + f = 180°$
(co-int. Ls supp)
because co-interior angles only add up to 180° if they are formed by parallel lines.



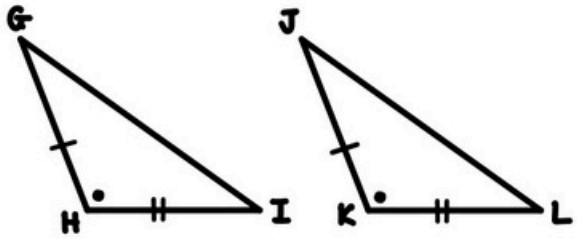
Prove that AD = BD and AE = CE
(Midpt theorem)
because a line from the midpoint of one side of a triangle to the midpoint of the second side, is parallel to the third side.

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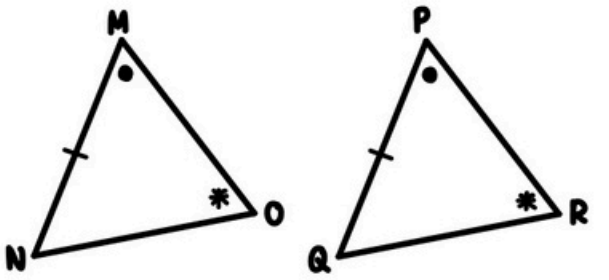
5. Ways of proving that two triangles are congruent:



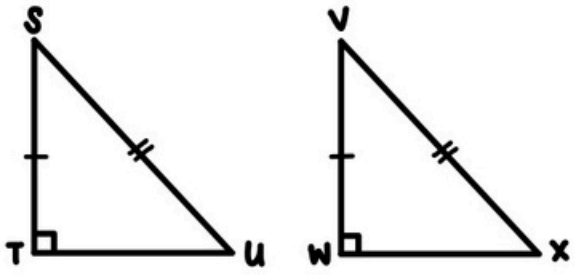
In ΔABC and ΔDEF:
1. $AB = DE$ (given)
2. $BC = EF$ (given)
3. $AC = DF$ (given)
$\therefore \Delta ABC \equiv \Delta DEF$ (SSS)



In ΔGHI and ΔJKL:
1. $GH = JK$ (given)
2. $\hat{H} = \hat{K}$ (given)
3. $HI = KL$ (given)
$\therefore \Delta GHI \equiv \Delta JKL$ (SAS)



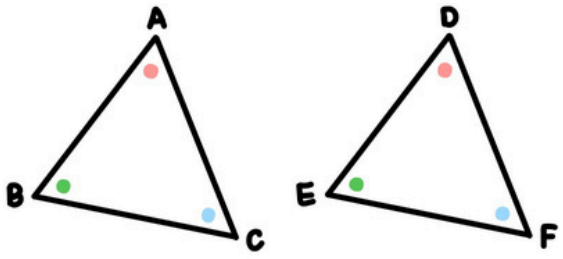
In ΔMNO and ΔPQR:
1. $MN = PQ$ (given)
2. $\hat{M} = \hat{P}$ (given)
3. $\hat{O} = \hat{R}$ (given)
$\therefore \Delta MNO \equiv \Delta PQR$ (AAS)



In ΔSTU and ΔVWX:
1. $ST = VW$ (given)
2. $\hat{T} = \hat{W} = 90°$ (given)
3. $SU = VX$ (given)
$\therefore \Delta STU \equiv \Delta VWX$ (RHS)

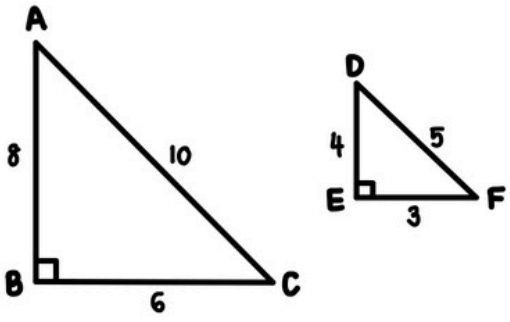
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6. Ways of proving that two triangles are similar:



In ΔABC and ΔDEF:
1. $\hat{A} = \hat{D}$ (given)
2. $\hat{B} = \hat{E}$ (given)
3. $\hat{C} = \hat{F}$ (given)

$\therefore \Delta ABC \,|||\, \Delta DEF$ (equiangular)

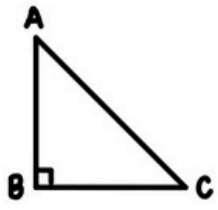


In ΔABC and ΔDEF

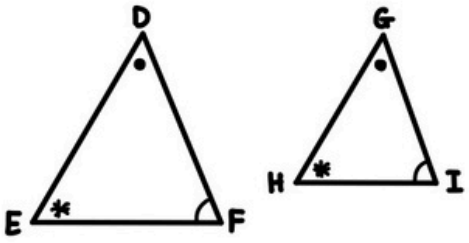
1. $\frac{AB}{DE} = \frac{8}{4} = 2$
2. $\frac{AC}{DF} = \frac{10}{5} = 2$
3. $\frac{BC}{EF} = \frac{6}{3} = 2$

$\therefore \Delta ABC \,|||\, \Delta DEF$ (sides in prop)

7. Ways of calculating the lengths of the sides of a triangle:

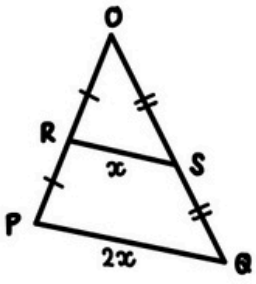


$AC^2 = AB^2 + BC^2$ (Pythagoras)

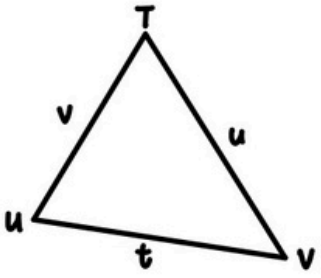


If $\Delta DEF \,|||\, \Delta GHI$:

$\frac{DE}{GH} = \frac{EF}{HI} = \frac{DF}{GI}$ (||| Δs)



If R is the midpoint of OP (OR = RP) and S is the midpoint of OQ (OS = SQ), then PQ = 2 × RS (Midpt theorem)



$$\frac{t}{\sin \hat{T}} = \frac{u}{\sin \hat{U}} = \frac{v}{\sin \hat{V}}$$

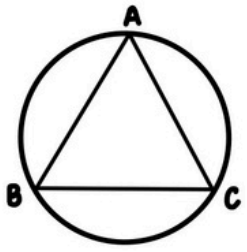
or $t^2 = u^2 + v^2 - 2uv.\cos \hat{T}$

$u^2 = t^2 + v^2 - 2tv.\cos \hat{U}$

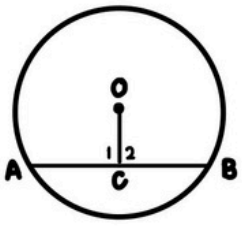
$v^2 = t^2 + u^2 - 2tu.\cos \hat{V}$

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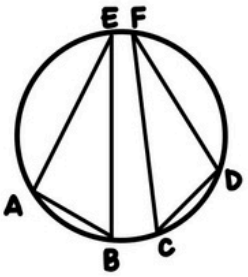
8. Ways of proving that two sides are equal:



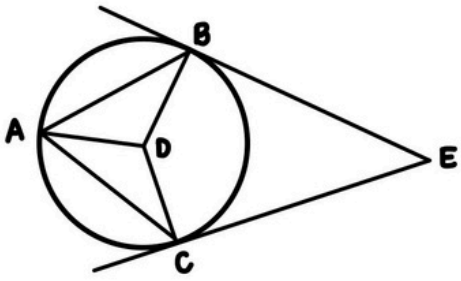
To prove AB = AC:
Prove that $A\hat{B}C = A\hat{C}B$
then AB = AC
(sides opp equal Ls)



To prove that AC = BC:
Prove that $\hat{C}_1 = 90°$
or $\hat{C}_2 = 90°$
then AC = BC
(line from centre ⊥ to chord)



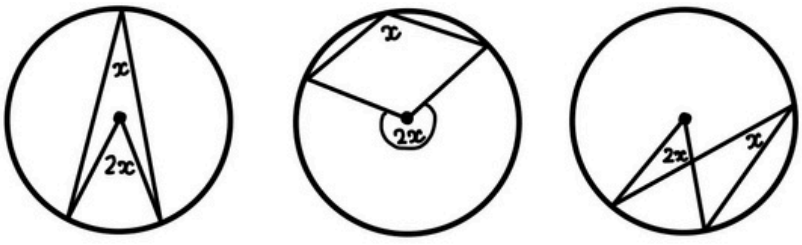
To prove that AB = CD:
Prove that $\hat{E} = \hat{F}$
then AB = CD
(equal Ls; equal chords)



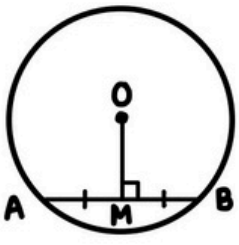
To prove that AD = CD:
Prove that ΔABD ≡ ΔACD
then AD = CD (≡ Δs)

If BE and CE are tangents,
then BE = CE (tans from common pt)

9. When the centre of a circle is given:

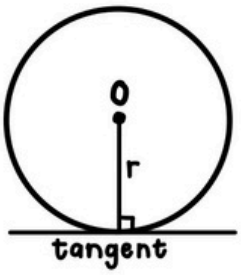


(∠ at centre = 2 × ∠ at circumference)

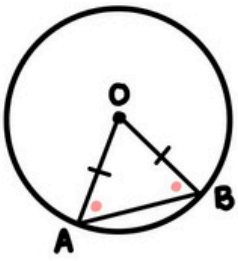


If OM ⊥ AB: AM = BM
(line from centre ⊥ to chord)

If AM = BM: OM ⊥ AB
(line from centre to midpt of chord)



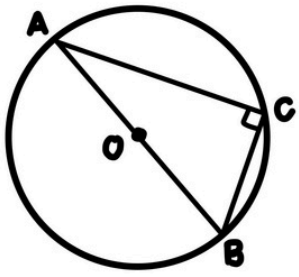
(radius ⊥ tangent)



OA = OB (radii)
$O\hat{A}B = O\hat{B}A$ (radii; Ls opp equal sides)

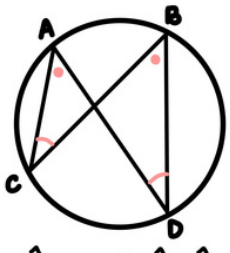
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10. When the diameter of a circle is given:

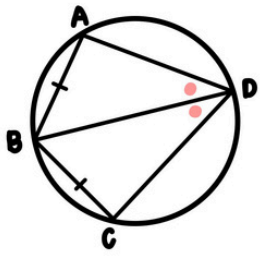


$A\hat{C}B = 90°$ (∠ in semicircle)
or (diameter subtends right ∠)

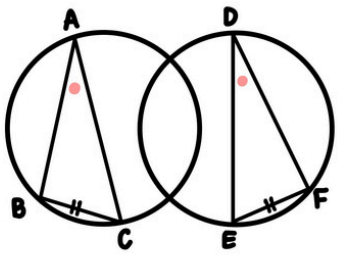
11. When angles formed at the circumference are given:



$\hat{A} = \hat{B}$ and $\hat{C} = \hat{D}$
(∠s in same segment)

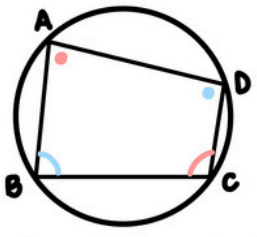


If AB = BC : $A\hat{D}B = C\hat{D}B$
(equal chords; equal ∠s)

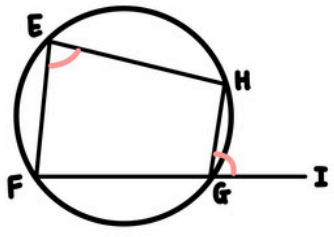


If ABC and DEF are equal circles
and BC = EF :
$\hat{A} = \hat{D}$ (equal chords; equal ∠s)

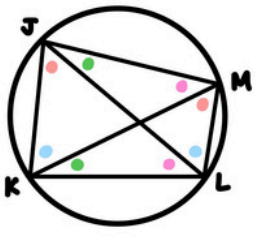
12. When a cyclic quadrilateral is given:



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$
(opp ∠s of cyclic quad)



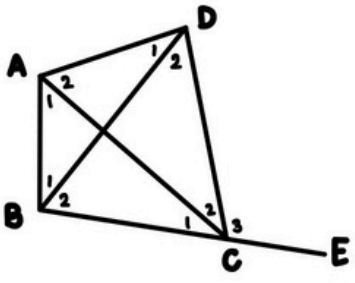
$H\hat{G}I = \hat{E}$
(ext. ∠s of cyclic quad)



$K\hat{J}L = K\hat{M}L$ and $L\hat{J}M = L\hat{K}M$
and $J\hat{K}M = J\hat{L}M$ and $J\hat{M}K = J\hat{L}K$
(∠s in same segment)

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13. Ways of proving that a quadrilateral is cyclic:

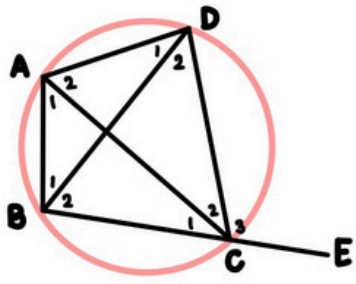


ABCD is a cyclic quadrilateral if any of the following conditions can be proved:

1. $\hat{A} + B\hat{C}D = 180°$ or $\hat{B} + \hat{D} = 180°$
   (converse opp $\angle$s of cyclic quad)

2. $D\hat{C}E = \hat{A}$ (converse ext. $\angle$ of cyclic quad)

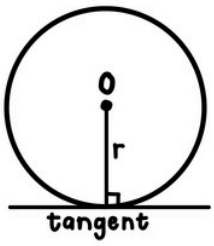
3. $\hat{A}_1 = \hat{D}_2$ or $\hat{A}_2 = \hat{B}_2$ or $\hat{C}_1 = \hat{D}_1$ or $\hat{C}_2 = \hat{B}_1$
   (converse $\angle$s in same segment)



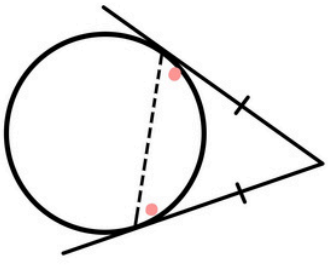
If it has been proven that ABCD is a cyclic quadrilateral, you can draw a circle passing through the vertices.

All properties of a cyclic quadrilateral can then be used.

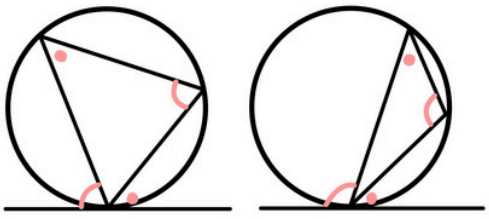
14. When tangents to a circle are given:



(tangent $\perp$ radius)

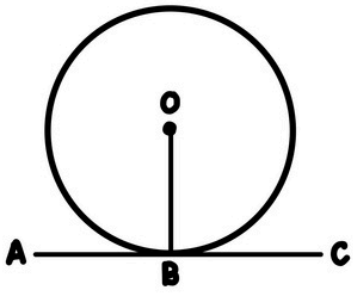


(tangents from common pt)

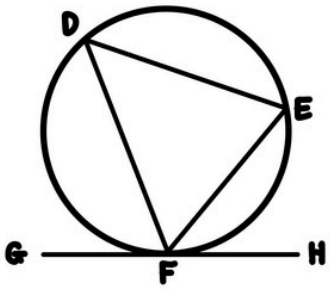


(tan chord theorem)

15. Ways of proving that a line is a tangent:



Prove that $O\hat{B}A = 90°$ or $O\hat{B}C = 90°$
(converse tangent $\perp$ radius)



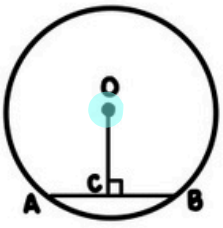
Prove $D\hat{F}G = \hat{E}$ or $E\hat{F}H = \hat{D}$
(converse tan chord theorem)

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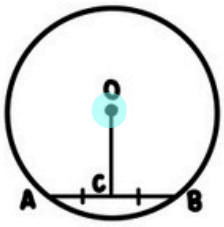
Summary of all Euclidean Geometry CIRCLE theorems:

It may seem overwhelming to look for all these shapes in a question but all the circle geometry reasons can be broken into three categories: Centre, Tangent and 4+ points on the circumference. If you do not have that feature, you can eliminate the entire category, which helps to reduce your options.

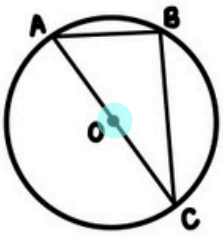
Centre



AC = BC
(line from centre ⊥ to chord)



OC ⊥ AB
(line from centre to midpt of chord)



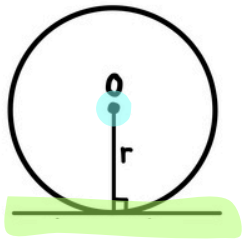
$A\hat{B}C = 90°$
(∠ in semicircle)



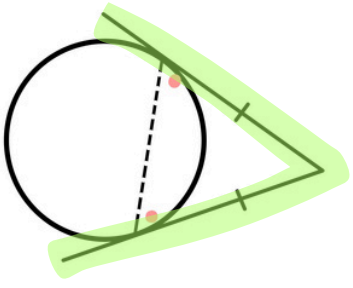
$A\hat{O}C = 2 \times A\hat{B}C$
(∠ at centre = 2 × ∠ at circumference)

Note: tan⊥rad is in both the centre and the tangent categories

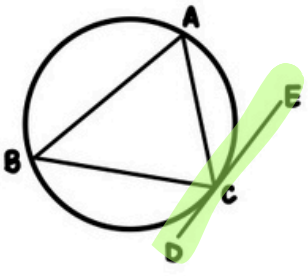
Tangent



(tangent ⊥ radius)

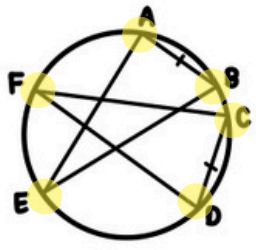


(tangents from common pt)

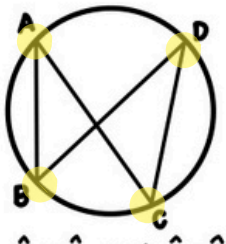


$A\hat{C}E = \hat{B}$ and $B\hat{C}D = \hat{A}$
(tan chord theorem)

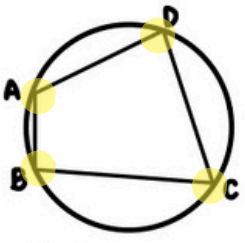
4+ points



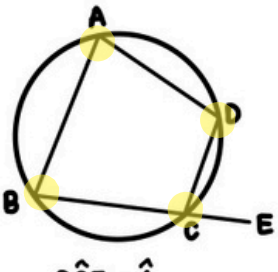
$\hat{E} = \hat{F}$
(equal chords; equal ∠s)



$\hat{A} = \hat{D}$ and $\hat{B} = \hat{C}$
(∠s in same segment)



$\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$
(opp ∠s of cyclic quad)



$D\hat{C}E = \hat{A}$
(ext. ∠ of cyclic quad)

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Euclidean Geometry CIRCLE theorem proofs

You can expect at least one proof in your final exam.

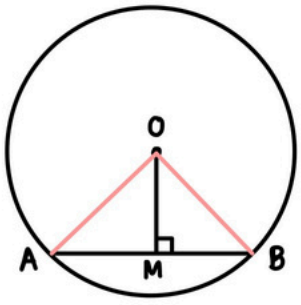
Make sure that you are able to write these proofs out in full, without leaving out any steps or reasons.

You also have to describe the construction, and draw it on the diagram.

AS SEEN IN THE GRADE 12 CHEAT SHEET

Theorem 1: Line from centre perpendicular to chord

A line from the centre of a circle, perpendicular to a chord, bisects the chord.



Given: A circle with centre O, line OM which is perpendicular to chord AB.

Required to prove: AM = MB

Construction: radii OA and OB

Proof:

In ΔAOM and ΔBOM

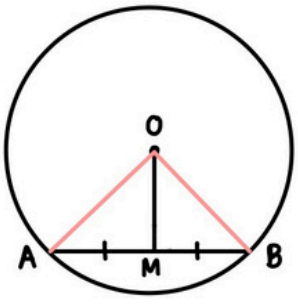
1. AO = BO (radii)
2. OM is common.
3. $\hat{OMA} = \hat{OMB}$ ($OM \perp AB$)

$\therefore \Delta AOM \equiv \Delta BOM$ (RHS)

$\therefore$ AM = BM (congruent Δs)

Converse of Theorem 1: Line from centre to midpoint of chord

A line from the centre of a circle to the midpoint of a chord, is perpendicular to that chord.



Given: A circle with centre O and line OM which bisects chord AB.

Required to prove: $OM \perp AB$

Construction: radii OA and OB

Proof:

In ΔAOM and ΔBOM:

1. AO = BO (radii)
2. AM = BM (given)
3. OM is common.

$\therefore \Delta AOM \equiv \Delta BOM$ (SSS)

$\therefore \hat{AMO} = \hat{BMO}$ (congruent Δs)

and $\hat{AMO} + \hat{BMO} = 180°$ (Ls on a straight line)

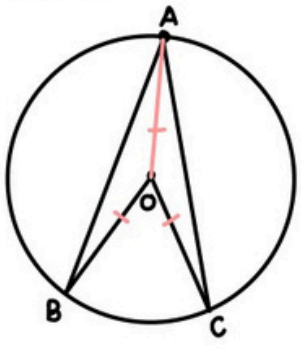
$\therefore \hat{AMO} = \hat{BMO} = 90°$

$\therefore OM \perp AB$

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Theorem 2: Angle at centre = 2 x angle at circumference

The angle subtended by an arc or chord at the centre of a circle is double the size of an angle subtended by the same arc or chord at the circumference of the circle.



Given: A circle with centre O and points A, B and C on the circumference.

Required to prove: $B\hat{O}C = 2B\hat{A}C$

Construction: radius OA

Proof:

Let $O\hat{A}B = x$ and $O\hat{A}C = y$

so $B\hat{A}C = x + y$

$O\hat{B}A = x$ and $O\hat{C}A = y$ (Ls opp equal sides; radii)

$B\hat{O}A = 180° - 2x$ and $C\hat{O}A = 180° - 2y$ (∠ sum in Δ)

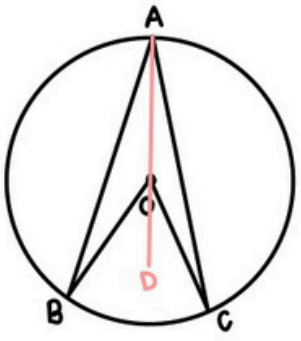
$\therefore B\hat{O}C = 360° - (180° - 2x) - (180° - 2y)$ (Ls around a point)

$= 360° - 180° + 2x - 180° + 2y$

$= 2x + 2y$

$= 2(x + y)$

$= 2B\hat{A}C$



Alternate proof:

Construction: Join AO and produce to D.

Proof:

Let $O\hat{A}B = x$ and $O\hat{A}C = y$

so $B\hat{A}C = x + y$

$O\hat{B}A = x$ and $O\hat{C}A = y$ (Ls opp equal sides; radii)

$B\hat{O}D = 2x$ and $C\hat{O}D = 2y$ (ext. ∠ of a Δ)

$\therefore B\hat{O}C = 2x + 2y$

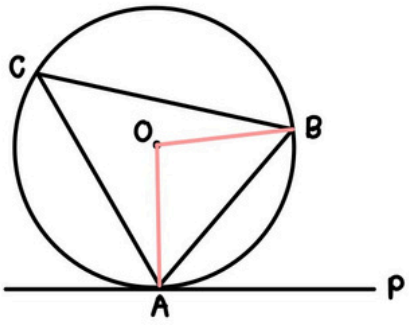
$= 2(x + y)$

$= 2B\hat{A}C$

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Theorem 3: Tan chord theorem

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle subtended by the same chord in the alternate segment.



Given: Circle with centre O. AP is a tangent to the circle at A and chord AB subtends $\hat{C}$ at the circumference.

Required to prove: $B\hat{A}P = A\hat{C}B$

Construction: radii OA and OB

Proof:

Let $A\hat{C}B = x$

$A\hat{O}B = 2x$ ($\angle$ at centre = $2 \times \angle$ at circumference)

$O\hat{A}B = 90° - x$ ($\angle$ sum in Δ; $\angle$s opp equal sides; radii)

and $O\hat{A}P = 90°$ (tangent $\perp$ radius)

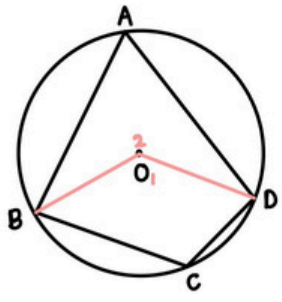
$\therefore B\hat{A}P = 90° - (90° - x)$

$= x$

$\therefore B\hat{A}P = A\hat{C}B$

Theorem 4: Opp angles of cyclic quad

The opposite angles of a cyclic quadrilateral are supplementary (add up to 180°).



Given: A circle with centre O and points A, B, C and D on the circumference of the circle.

Required to prove: $\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$

Construction: Draw BO and DO. Label the angles around point O as $\hat{O}_1$ and $\hat{O}_2$.

Proof:

Let $\hat{A} = x$

$\hat{O}_1 = 2x$ ($\angle$ at centre = $2 \times \angle$ at circumference)

$\hat{O}_2 = 360° - 2x$ ($\angle$s around a point)

$\therefore \hat{C} = 180° - x$ ($\angle$ at centre = $2 \times \angle$ at circumference)

$\therefore \hat{A} + \hat{C} = x + 180° - x$

$= 180°$

and $\hat{A} + \hat{C} + \hat{B} + \hat{D} = 360°$ ($\angle$ sum in quad)

$180° + \hat{B} + \hat{D} = 360°$

$\hat{B} + \hat{D} = 180°$

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HOW IS THIS SECTION TESTED?

Now that we've summarised everything you need to know for this section, we practice applying all of this knowledge.

There are 4 question types that you can expect in your final exams:

1. A basic question to find the sizes of angles or lengths of sides using Euclidean geometry theorems.
2. A more complex question requiring the use of variables.
3. A question that requires proving a property.
4. A question with multiple circles.

When approaching a circle geometry question we want to do the following steps:

- Read the information in the question and make sure everything has been **transferred to the diagram** (lines that are equal, lines that are parallel, equal angles etc.)
- If you don't know how to start answering the question, **see what you CAN find**. Can you find equal angles? Or the values of any angles or sides?
- Hopefully by filling in what you can, you have now found a way to answer the question but you probably don't need all the values you've found. Only **write down what you need** and give the **relevant geometry reason** for every statement, unless explicitly told not to provide reasons.
- You have to **answer** the questions **in the order** they were **asked**, you cannot go back and use an answer from a later question to answer an earlier question.

Remember to look for the 3 circle categories and other helpful information:

- Do you have a **centre**?
- Do you have a **tangent**?
- Do you have **4 or more points** on the circumference?
- Are there any **parallel lines**?
- Are there any **equal lines**?

If you were asked to prove something, but were unable to complete the question, you may use that it is now a given fact in any questions that follow. For example, if the question asks you to prove that a a line is a tangent, but you are unable to do so, you are allowed to use the fact the line is a tangent in any questions that come after, but not any that were asked before.

PRO-TIP

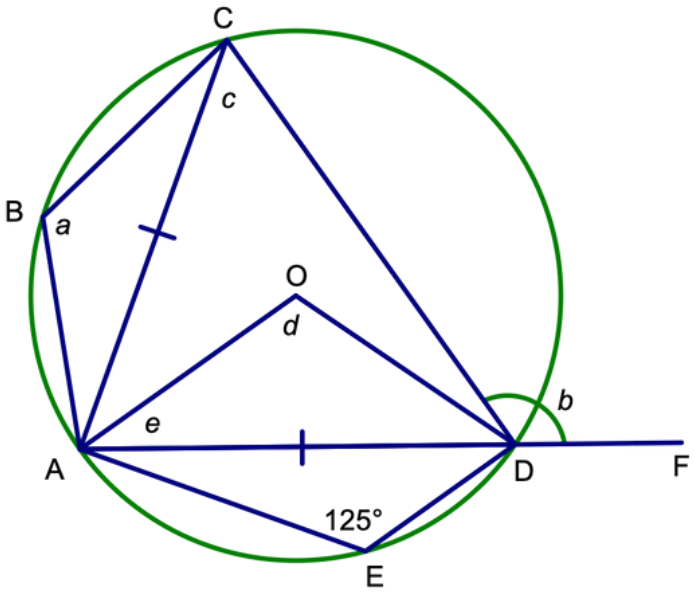
When **marking** your assessments, your teacher should look at what you've filled in on your diagrams. They may **award marks** for values that you've shown there, even if you haven't managed to write those values down as part of your solution.

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Question Type 1a:
Calculating angles using Euclidean Geometry theorems

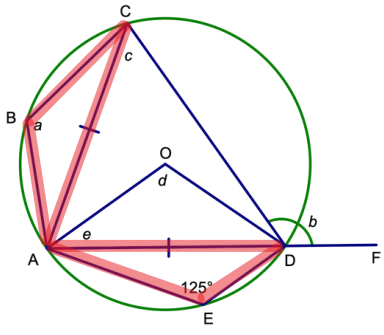
R Worked example

In the diagram, ABCDE are points around the circumference of a circle. AD is produced to F and AD = AC. Given $\angle AED = 125^{\circ}$, find the values of a, b, c, d, and e, with reasons.



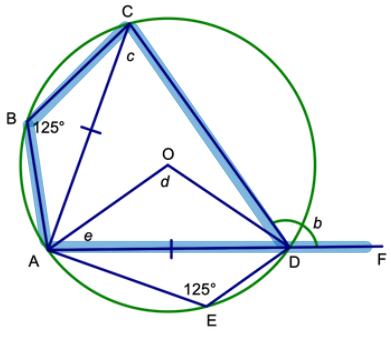
$a = 125°$ (*equal chords; equal angles*)

- We are told chords AC and AD are equal in length, this makes the angles opposite those chords equal in size as well.
- Since the angle opposite AD is given as 125° this will make the angle opposite AC, angle "a", equal to 125°



$b = 125°$ (*ext ∠ cyclic quad*)

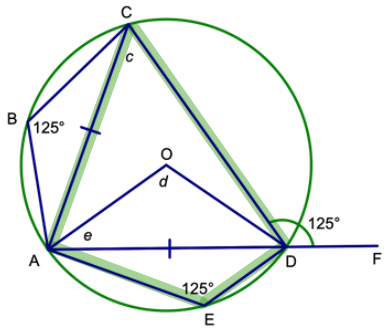
- Angle "b" is an exterior angle since it is between chord CD and extended chord AD.
- The exterior angle of a cyclic quadrilateral is equal in size to the interior angle opposite that vertex. In this case, the angle opposite vertex D is at $\hat{B}$.
- Since we found $\hat{B}$ is 125° in the previous question, this will mean b is also 125°



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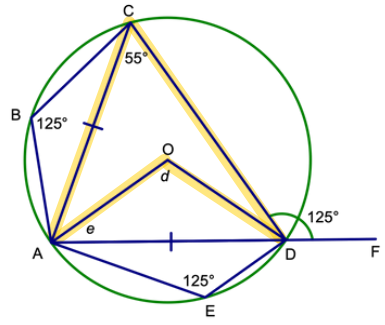
$c = 55°$ $(opp\ \angle s\ cyclic\ quad)$

- Opposite angles of a cyclic quadrilateral are supplementary, therefore c=180°-125°=55°



$d = 110°$ $(\angle\ at\ centre = 2 \times \angle\ at\ circumference)$

- The angle at the centre of the circle, AÔD, is subtended by chord AD.
- Chord AD also subtends an angle at the circumference at ACD.
- This means AÔD = 2 x AĈD = 2 x 55° = 110°
- Look for the "arrowhead" shape to find this relationship.

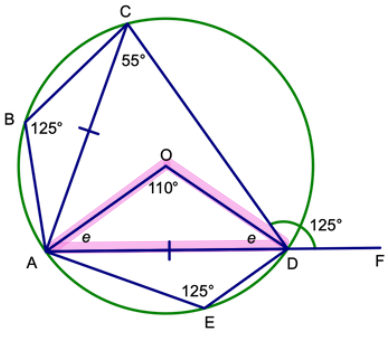


$OA = OD$ $(radii)$

$\therefore O\hat{D}A = e$ $(\angle s\ opp = sides)$

$\therefore e = 35°$ $(sum\ \angle s\ \Delta)$

- ΔOAD is an isosceles triangle since sides AO and OD are both radii.
- This means the base angles are also equal.
- Since all the angles in a triangle must add up to 180°, we have 110°+e+e=180°, therefore 2e=70° and finally divide the equation by 2, e=35°.



IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

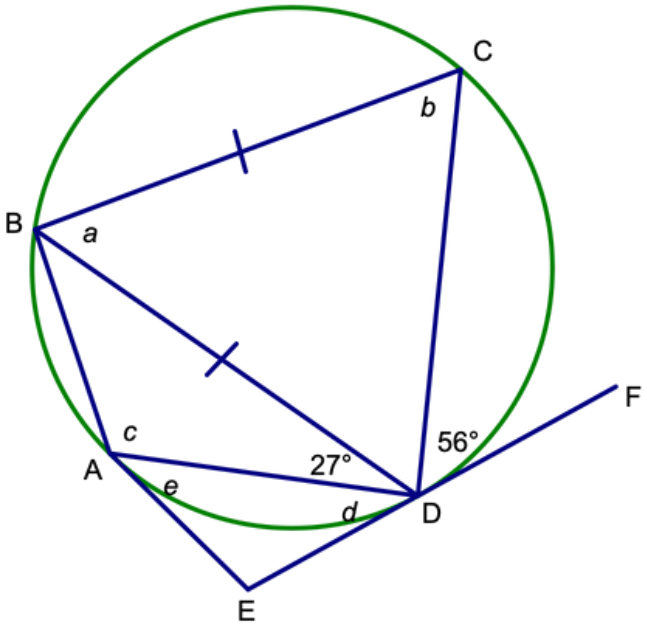
P **Problem solving:** Problems that are unfamiliar (unseen).

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Worked example

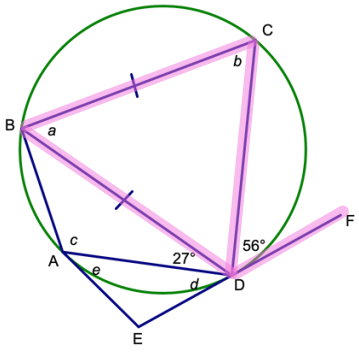
In the figure A, B, C and D lie on the circle. EF is a tangent at D and EA is a tangent at A. Chords BD and BC are equal in length. $C\hat{D}F = 56°$ and $B\hat{D}A = 27°$.

Determine the values of a, b, c, d, and e, showing all steps and reasons.



$a = 56°$ $(tan\ chord\ theorem)$

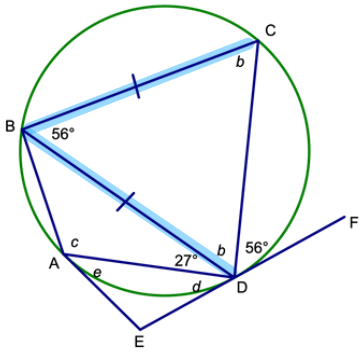
- The angle between a tangent and a chord, is equal to the angle opposite the chord.
- Using the chord, find the triangle that subtends at the circumference. Here, it is ΔDBC.



$B\hat{D}C = B\hat{C}D$ $(\angle s\ opp = sides)$

$b = 62°$ $(sum\ \angle s\ \Delta)$

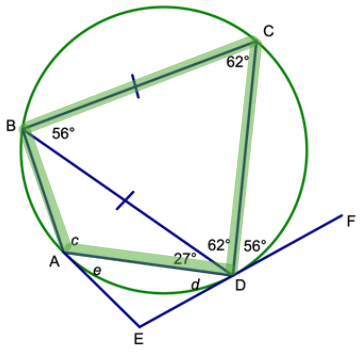
- ΔDBC is an isosceles triangle since BC=BD.
- This means the base angles are equal, therefore $B\hat{D}C$=b.
- Since the angles in a triangle add up to 180°, we can calculate b by using the equation a+b+b=180°, therefore 56°+2b=180°, so 2b=124°, and finally divide the equation by 2 to get b=62°.



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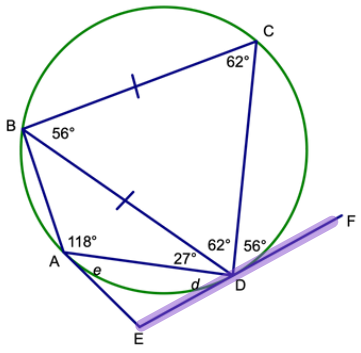
$c = 118°$ *(opp ∠s cyclic quad)*

- The opposite angles in a cyclic quad add up to 180°.
- Since we now know the angle opposite c = 62°, we can use the equation c = 180° - 62° = 118°.



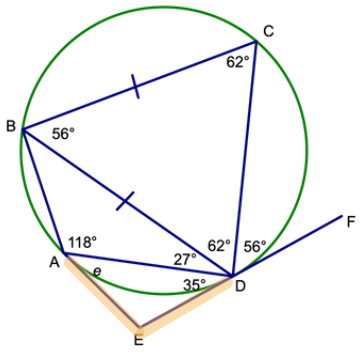
$d = 35°$ *(∠s on a straight line)*

- Since we know the values for all the other angles on the straight line EDF, we can subtract these values from 180° to calculate d.
- d = 180° - 56° - 62° - 27° = 35°



$AE = ED$ *(tans from common point)*
$\therefore e = 35°$ *(∠s opp = sides)*

- Tangents that meet at a common vertex are equal in length, therefore ΔAED is isosceles.
- This means the base angles, e and d, are also equal, therefore e = d = 35°



IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

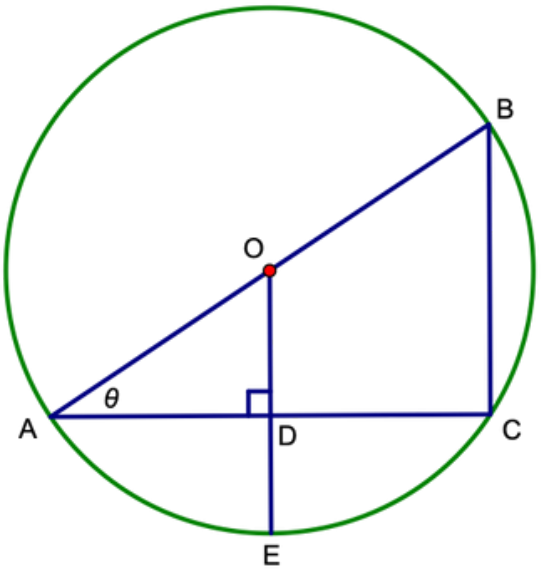
P **Problem solving:** Problems that are unfamiliar (unseen).

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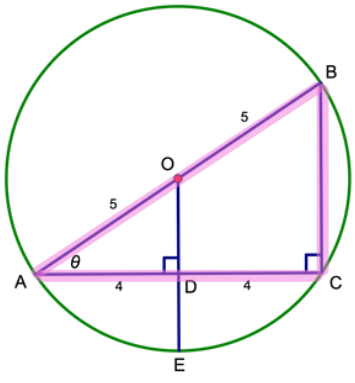
Question Type 1b:
Calculating sides using Euclidean Geometry theorems

Worked example

In the figure, O is the centre of the circle. A, B, and D lie on the circle with OE⊥AC at D. AB = 10, AD = 4, and $B\hat{A}C = \theta$.



R Find the length of BC.



$DC = 4\ cm$ $(line\ from\ centre\ \perp\ to\ chord)$

$A\hat{C}B = 90°$ $(\angle s\ in\ semi\ circle)$

$BC = \sqrt{10^2 - 8^2}$ $(pythag)$

$BC = 6\ cm$

- We are told OE⊥AC, and since OE is a line from the centre that is perpendicular to the chord AC, this will bisect the chord.
- Since AD = 4cm and this is half the length, DC will also be 4 cm.
- AB is a diameter, which makes the angle opposite the diameter, ACB, a right angle.
- ΔABC is right-angled, so we can use Pythagoras to find the length of the unknown side, BC.

R Calculate the size of θ.

$$\cos\theta = \frac{8}{10}$$

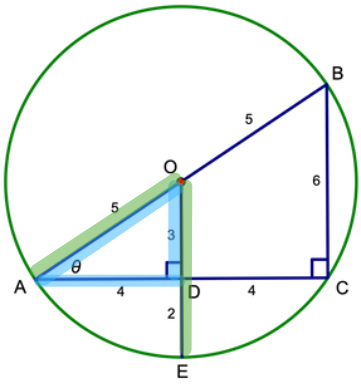
$$\theta = \cos^{-1}\left(\frac{8}{10}\right)$$

$$\theta = 36{,}87°$$

- Since ΔABC is right-angled, we can use trigonometric ratios to find the value of θ.
- We know the lengths of all the sides of the triangle, so we could use sinθ, cosθ or tanθ but the safest option is to use the given sides, AD (adjacent to θ) and AC (the hypotenuse) so here we have use cosθ.
- To find the value of θ use the [SHIFT] and [cos] buttons on the calculator and type in the ratio.

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Find the length of DE.



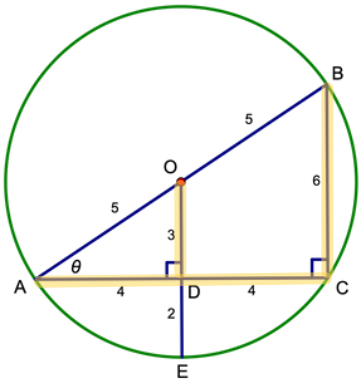
$OD = \sqrt{5^2 - 4^2} \quad (pythag)$
$OD = 3\ cm$
$\therefore DE = 5 - 3 = 2\ cm$

OR

$AO = OB \quad (radii) \quad \text{and} \quad AD = DC \quad (proven)$
$\therefore OD = 3\ cm \ (midpoint\ theorem)$

- ΔAOD is also right-angled, therefore we use Pythagoras again to calculate the value of OD.
- OA and OE are both radii and we know the length of a radius is 5 cm.
- DE = OE - OD = 5 - 3 = 2cm
- Alternatively, we can use the midpoint theorem since D is the midpoint of AC and O is the midpoint of AB.
- The midpoint theorem states that the side connecting the midpoints (OD) is half the length of the third side of the triangle, BC.
- Therefore OD = ½ BC = ½ (6) = 3 cm.

Prove OE||BC.



$A\hat{D}O = 90° \quad (given)$
$A\hat{C}B = 90° \quad (proven)$
$\therefore A\hat{D}O = A\hat{C}B$
$\therefore OE||BC \quad (corresp\ \angle s =)$

OR

$AO = OB \quad (radii) \quad \text{and} \quad AD = DC \quad (proven)$
$\therefore OE||BC \quad (midpoint\ theorem)$

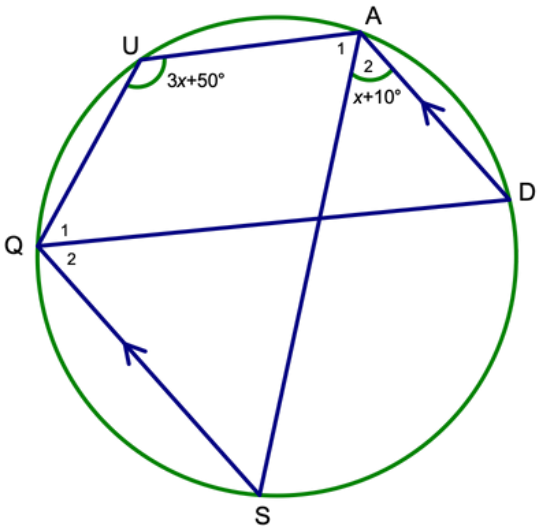
- We already know both AD̂O and AĈB are right angles.
- These two angles are corresponding angles, and since they are equal, the lines OD and BC must be parallel.
- OD and OE are the same straight line, therefore OE is parallel to BC.
- Alternatively, we can use midpoint theorem again.
- Midpoint theorem also says that the side connecting the midpoints (OD) is parallel to the third side of the triangle, BC.

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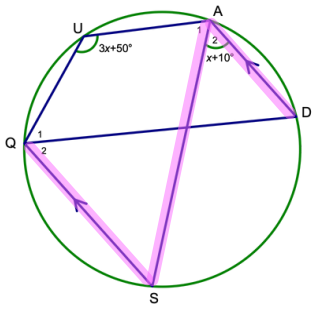
Question Type 2:
Complex questions that require working with variables

Worked example

Q, U, A, D and S are points on the circumference of the circle.
$\hat{A}_2 = x + 10°$ and $\hat{U} = 3x + 50°$. QS || AD.



C Determine the size of $Q\hat{S}A$ in terms of x.



$Q\hat{S}A = x + 10°$ $(alt\ \angle s; QS||AD)$

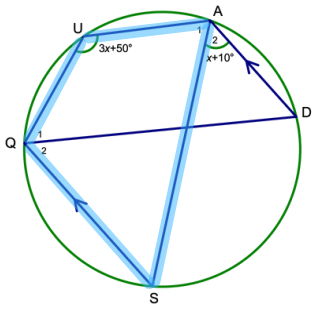
OR

$Q\hat{S}A = 180° - (3x + 50°)$ $(opp\ \angle s\ cyclic\ quad)$

$= 180° - 3x - 50°$

$= 130° - 2x$

C Calculate the value of x.



$Q\hat{S}A = 180° - (3x + 50°)$ $(opp\ \angle s\ cyclic\ quad)$

$= 180° - 3x - 50°$

$= 130° - 2x$

OR

$Q\hat{S}A = x + 10°$ $(alt\ \angle s; QS||AD)$

$x + 10° = 130° - 2x$

$3x = 120°$

$x = 40°$

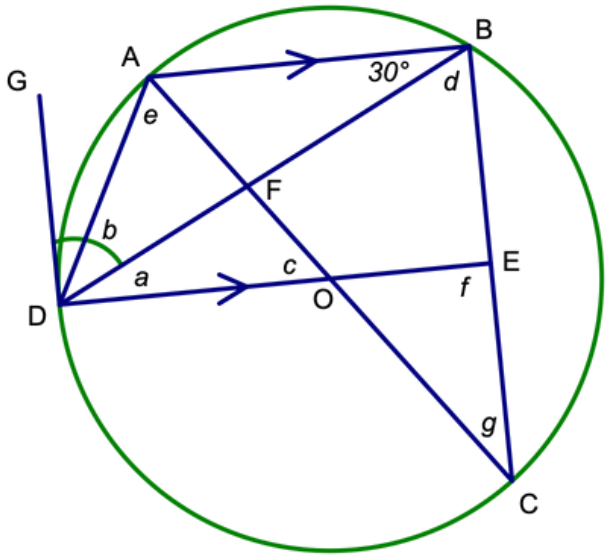
- There are two ways to find $Q\hat{S}A$ in terms of x.
- Either you can use the parallel lines to find $Q\hat{S}A$ using the alternate angles, which are equal.
- Or, you can use the cyclic quadrilateral QUAS and $Q\hat{S}A$ is opposite $Q\hat{U}A$, therefore the angles are supplementary.
- Whichever option you used to answer (a), you can use the other option to answer (b).
- Since there are two different ways to find $Q\hat{S}A$, we can use these two expressions, which will be equal in value, to create an equation and solve for x.

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Question Type 3a:
Proving a property using values

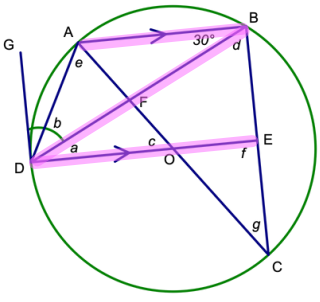
Worked example

In the figure, O is the centre of the circle. A, B, C and D lie on the circle with E on chord BC such that AB||DE. AC and BD intersect at F. DG is a tangent at D and $A\hat{B}D = 30°$.



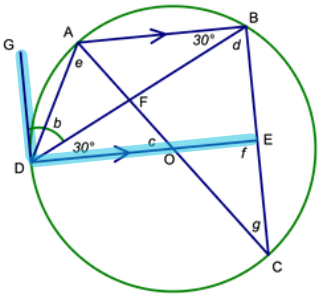
R Determine the values of $a, b, c, d, e, f,$ and g, showing all steps and reasons.

$a = 30°$ $(alt\ \angle s\ ; AB||DE)$



- Look for the Z shape using the parallel lines to see the alternate angles.
- Alternate angles on parallel lines are equal. therefore $A\hat{B}D = B\hat{D}E$.
- Don't forget to name the parallel lines used.

$b = 60°$ $(tan \perp rad)$

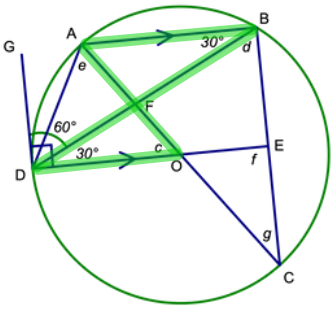


- The line OD is a radius that meets the tangent GD at D.
- This means $E\hat{D}G = 90°$ since a tangent is perpendicular to the radius of a circle.
- $b = 90° - 30° = 60°$

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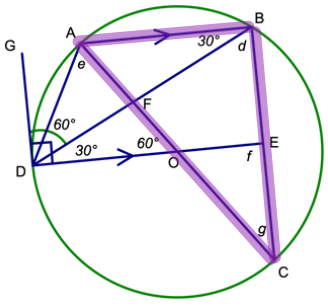
$c = 60°$ $(\angle \, at \, centre = 2 \times \angle \, at \, circumference)$

- The chord AD subtends an angle at the centre, $A\hat{O}D$, and the same chord subtends an angle at the circumference, $A\hat{B}D$.
- This means $A\hat{O}D = 2 \times A\hat{B}D = 2 \times 30^{\circ} = 60^{\circ}$



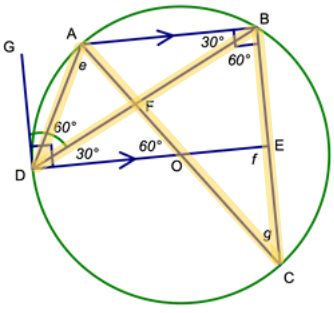
$d = 60°$ $(\angle s \; in \; semi \; circle)$

- Since AC passes through the centre, it is a diameter.
- The angle opposite a diameter is a right-angle, therefore $d = 90^{\circ} - 30^{\circ} = 60^{\circ}$



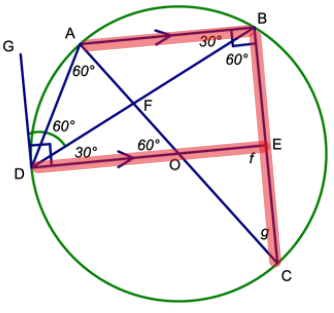
$e = 60°$ $(\angle s \; in \; same \; seg)$

- Look for the "bowtie" shape to find angles in the same segment.
- If you were to draw a chord from D to C, this chord would subtend an angle at $D\hat{B}C$ and also at $D\hat{A}C$. Since the chord must be equal in length, since it is the same chord, the angles must also be equal in size.
- $e = D\hat{B}C = 60^{\circ}$



$f = 90°$ $(corresp \; \angle s \; ; AB||DE)$

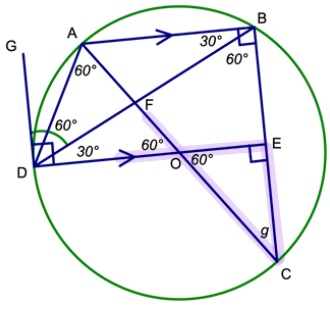
- Using the parallel lines again, look for the F shape to find the corresponding angles.
- $A\hat{B}C = D\hat{E}C$, therefore $f = 90^{\circ}$.



$E\hat{O}C = 60°$ $(vert \; opp \; \angle s)$

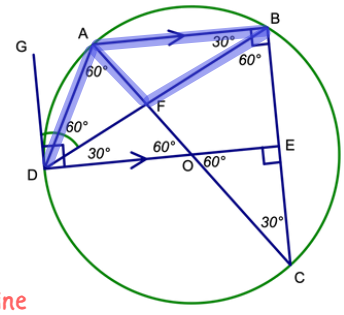
$g = 30°$ $(sum \; \angle s \; \Delta)$

- g is the last angle that needs to be found, so we already have a lot of information available to use. This means it is likely there are multiple alternatives to determine its value.
- All these options include finding another angle first, before the value of g can be determined.
- Here, we have found a second angle in ΔOEC using vertically opposite angles ($F\hat{O}D = E\hat{O}C = 60^{\circ}$), so we can now find g as the last unknown angle in this triangle.
- Alternatively, we could have found $B\hat{A}C = 60^{\circ}$ using alternate angles, to then find g using ΔABC.



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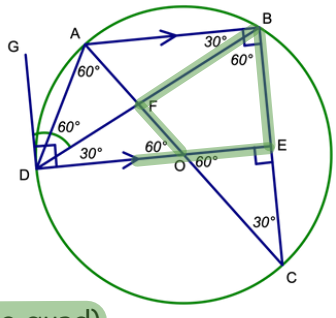
Determine whether AD would be a tangent to the circle through ABF.



$D\hat{A}F = 60°$ $(proven)$
$A\hat{B}F = 30°$ $(given)$
$\therefore D\hat{A}F \neq A\hat{B}F$
∴ AD is not a tangent

- There are two ways to prove whether a line that meets a circle is a tangent, or not (refer to page 94).
- Either, you need to prove the line that meets the potential tangent line is a radius (or a diameter) and the angle between these lines is 90°.
- Or, you need to prove the angle between the tangent and a chord, is equal to the angle opposite the chord.
- In this case, the angle between the potential tangent and the chord (D$\hat{A}$F) is not the same size as the angle opposite the chord, A$\hat{B}$F, therefore the line is not a tangent since tan chord theorem is not happening.

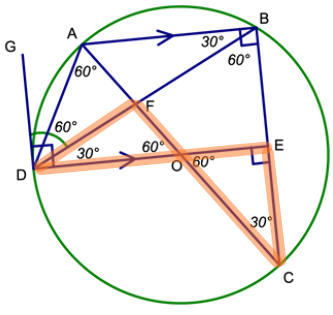
Prove BEOF is a cyclic quadrilateral.



$D\hat{O}F = 60°$ $(proven)$
$F\hat{B}E = 60°$ $(proven)$
$\therefore D\hat{O}F = F\hat{B}E$
∴ BEOF is a cyclic quadrilateral (converse exterior angle cyclic quad)

- There are three ways to prove whether or a quadrilateral is cyclic or not (refer to page 93).
- Here, there isn't a bowtie shape so using "converse angles in same segment" is very unlikely.
- Since we know the size of quite a few angles, we can try to prove the opposite angles of the quadrilateral are supplementary (add up to 180°) but without finding any more values, we can use angle F$\hat{O}$D, which would be the exterior angle of the quadrilateral since it is between an extended side and another side. Since F$\hat{O}$D is the same size as the opposite interior angle, F$\hat{B}$E, then BEOF must be a cyclic quadrilateral.

Prove DFEC is a cyclic quadrilateral.



$F\hat{D}O = 30°$ $(proven)$
$O\hat{C}E = 30°$ $(proven)$
$\therefore F\hat{D}O = O\hat{C}E$
∴ DFEC is a cyclic quadrilateral (converse ∠s in same seg)

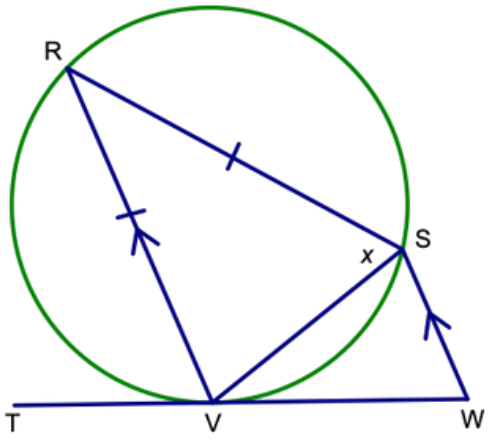
- As discussed in the previous question, there are three ways to prove whether or not a quadrilateral is cyclic (refer to page 94).
- Here, we do have a bowtie shape so we will try to use "converse angles in same segment".
- Either we can try to prove D$\hat{F}$O=C$\hat{E}$C (but we don't know D$\hat{F}$O yet) or we can use F$\hat{D}$O=O$\hat{C}$E since have already calculated both these angles are 30°.
- Since the angles are equal, angles in same seg is happening, therefore the quadrilateral must be cyclic.

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Question Type 3b:
Proving a property with variables

Worked example

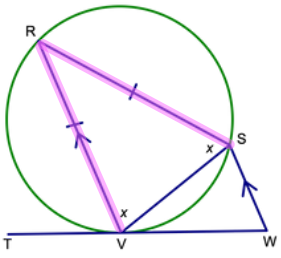
In the diagram below, TVW is a tangent to the circle at V with points R, S, and V on the circumference of the circle. RV||SW and RV = RS.



R If $R\hat{S}V = x$, name with reasons, FOUR other angles equal to x.

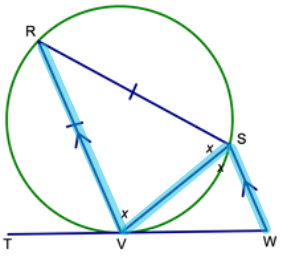
$R\hat{V}S = x$ $(\angle s\ opp = sides)$

- ΔVRS is isosceles, therefore the base angles of the triangle are equal.



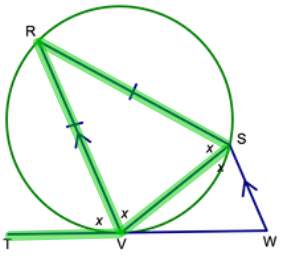
$V\hat{S}W = x$ $(alt\ \angle s; RV||SW)$

- Using the parallel lines, we look for the Z shape to find alternate angles.
- Since alternate angles on parallel lines are equal, we now have a second angle equal to x.



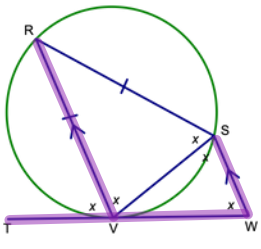
$R\hat{V}T = x$ $(tan\ chord\ theorem)$

- The angle between the tangent and the chord, $R\hat{V}T$, is equal to the angle opposite the chord, $R\hat{S}V$.
- This relationship uses the original given x, therefore it could have been found first.



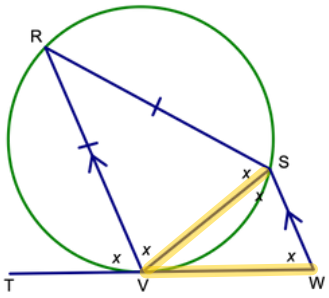
$S\hat{W}V = x$ $(corresp\ \angle s; SW||RV)$

- Using the parallel lines again, we can also find an F shape now that we have $R\hat{V}T = x$.
- The F shape gives us corresponding angles, which are equal on parallel lines ∴ $R\hat{V}T = S\hat{W}V = x$



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C Prove VS = VW.

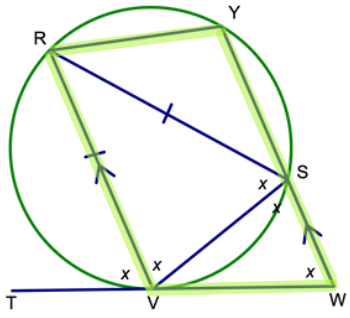


$R\hat{V}T = S\hat{W}V = x \quad (proven)$

$\therefore VS = VW$ $(sides\ opp = \angle s)$

- To prove the sides are equal in length, we need the angles opposite those sides to be equal in size.
- We have already proven R$\hat{V}$T and S$\hat{W}$V are both equal to x, so they must be equal to each other.
- This means the sides opposite those angles, are equal in length, and the triangle is isosceles.

P If WS is extended to meet the circle at Y and Y is joined to R, prove RYWV is a parallelogram.



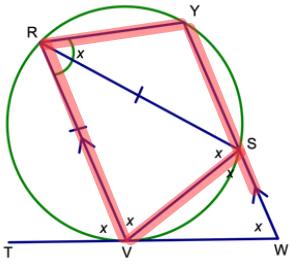
- First we need to understand what the question means. If WS is extended to meet the circle, it means the line stretches out until it reaches the circumference of the circle after S. Where is meets the circle, we label this as Y.
- Then Y and R are joined so we draw a line connecting them.
- Now, we have a cyclic quadrilateral named RYSV.

$V\hat{R}Y = x$ $(ext\ \angle\ cyclic\ quad)$

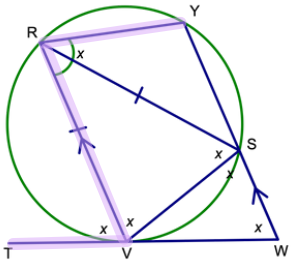
$\therefore T\hat{V}R = V\hat{R}Y = x$

$\therefore RY||TW$ $(alt\ \angle s =)$

$\therefore RYWV$ is a parallelogram $(2\ pairs\ opp\ sides\ ||)$



- To prove RYWV is a parallelogram first we look at what we already have: RV is parallel to WS (now WY).
- Either we want to prove RY is parallel to VW (then we would have two pairs of opposite sides parallel) or we want to prove RV is equal in length to YW (then we would have one pair of opposite sides are equal and parallel).
- Since SY is an extension of WS, the angle V$\hat{S}$W is an exterior angle of the cyclic quadrilateral RYSV.
- The exterior angle is equal to the opposite interior angle, therefore V$\hat{S}$W = V$\hat{R}$Y = x.

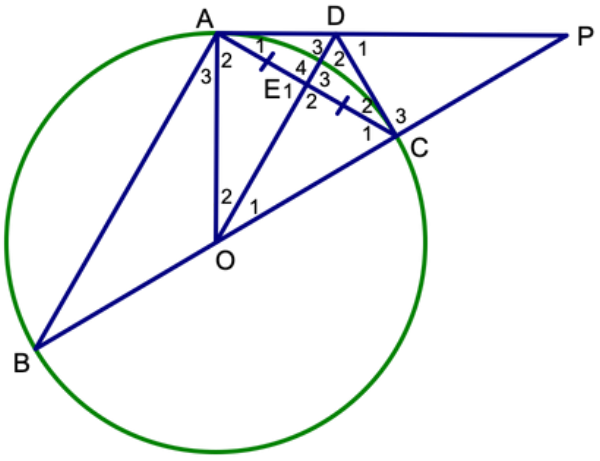


- Now, we can use that R$\hat{V}$T and V$\hat{R}$Y both equal x, and since they make a Z shape, they are alternate angles.
- If alternate angles are equal, they must be on parallel lines ∴ RY is parallel to TW.
- This proves RYWV is a parallelogram since it has two pairs of opposite sides parallel.

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Worked example

In the circle below, BOC is a diameter of the circle. AP is a tangent to the circle at A and AE = EC.



Prove that:

C BA||OD

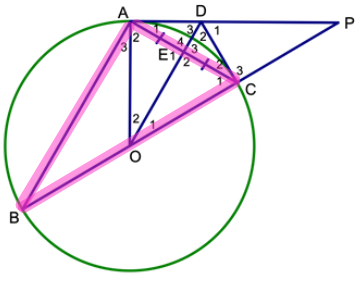
$B\hat{A}C = 90°$ $(\angle s \text{ in semicircle})$

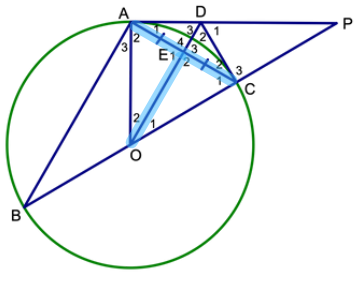
$\hat{E}_2 = 90°$ $(\text{line from centre to midpoint of chord})$

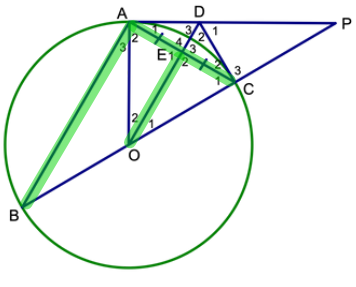
$\therefore B\hat{A}C = \hat{E}_2$

$\therefore BA||OD$ $(\text{corresp } \angle s =)$

- Since BC is a diameter, the angle opposite the diameter, $B\hat{A}C$, is a right-angle.
- EO is a line from the centre that meets chord AC at its midpoint, E. We know E is the midpoint since we are told AE = EC.
- That makes all the angles at E, right-angles.
- You can choose $\hat{E}1 = 90°$ and use co-interior angles are supplementary to prove BA is parallel to OD ($B\hat{A}E + A\hat{E}O = 180°$).
- You could choose $\hat{E}4 = 90°$ and use alternate angles to prove BA is parallel to OD ($B\hat{A}C = A\hat{E}D$).
- We chose $\hat{E}2 = 90°$ and since $B\hat{A}C$ and $\hat{E}2$ are corresponding angles that are equal, we can prove BA is parallel to OD.



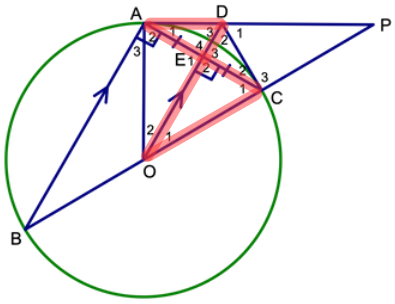




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C

AOCD is a cyclic quadrilateral



$\hat{A}_1 = \hat{B}$ $(tan\ chord\ theorem)$

$\hat{B} = \hat{O}_1$ $(corresp\ \angle s;\ BA||OD)$

$\therefore \hat{A}_1 = \hat{O}_1$

$\therefore AOCD$ is a cyclic quad $(converse\ \angle s\ in\ same\ seg)$

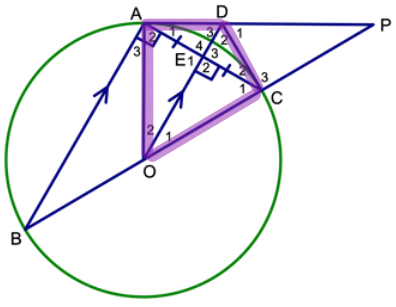
There are three ways to prove a quadrilateral is cyclic (refer to page 93).

- Prove prove any two pairs of angles in a "bowtie" shape are equal ($\hat{A}1 = \hat{C}1$. $\hat{D}3 = \hat{C}1$. $\hat{A}2 = \hat{D}2$ or $\hat{C}2 = \hat{C}2$.
- Prove the opposite angles in the quadrilateral add up to 180° (either $C\hat{A}D + D\hat{C}O = 180°$ or $A\hat{O}C + A\hat{D}C = 180°$)
- Prove the exterior angle of the quadrilateral is equal in value to the opposite, interior angle (either $P\hat{D}C = A\hat{O}C$ or $D\hat{C}P = C\hat{A}D$).

- Here we use the tangent AP to show $\hat{A}1 = \hat{B}$ since the angle between the tangent and the chord ($\hat{A}1$) is equal to the angle opposite the chord ($A\hat{B}C$).
- Using the parallel lines we prove in the previous question, $A\hat{B}C$ is corresponding to $\hat{O}1$, so these angles are also equal.
- Since $\hat{A}1 = \hat{B}$ and $\hat{B} = \hat{O}1$, then $\hat{A}1$ must be equal to $\hat{O}1$.
- $\hat{A}1$ and $\hat{O}1$ are a pair we can use to prove AOCD is cyclic as they are "tips" in the "bowtie" shape.

C

DC is a tangent to the circle at C.



$A\hat{O}D = 90°$ $(tan \perp rad)$

$D\hat{C}O = 90°$ $(opp\ \angle s\ cyclic\ quad)$

$\therefore DC$ is a tangent to the circle at C $(converse\ tan \perp rad)$

- There are two ways to prove DC is a tangent, either we need to prove OC⊥DC, then we can say DC is a tangent since it is perpendicular to the radius OC.
- Or, we need to prove the angle between the line DC and the chord AC ($D\hat{C}A$) is equal to the angle opposite the chord, $A\hat{B}C$, then tan chord theorem would be happening, therefore DC must be a tangent.

- We are told AP is a tangent, which means we can determine $C\hat{A}D$ is a right angle since OA is a radius that meets a tangent.
- We just proved AOCD is a cyclic quadrilateral, and since the opposite angles of a cyclic quadrilateral are supplementary, therefore $D\hat{C}O = 180° - 90° = 90°$
- Now we know the angle between the radius OC and the line DC is 90°, therefore DC must be a tangent since tan⊥rad is happening.

OR

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$\hat{C}_2 = \hat{O}_2$ $(\angle s\ in\ same\ seg)$

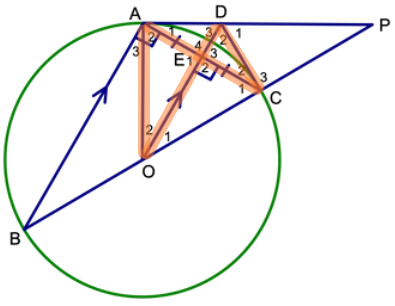
$\hat{O}_2 = \hat{A}_3$ $(alt\ \angle s;\ BA||OD)$

$OA = OB$ $(radii)$

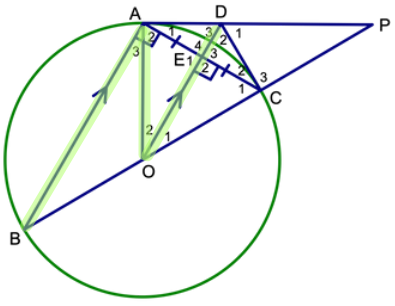
$\hat{A}_3 = \hat{B}$ $(\angle s\ opp = sides)$

$\therefore \hat{C}_2 = \hat{B}$

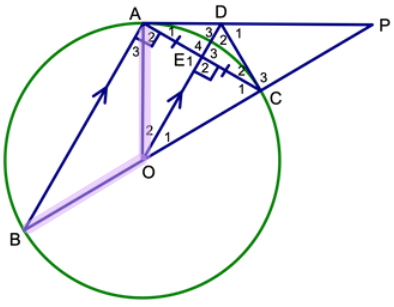
$\therefore DC$ is a tangent to the circle at C $(converse\ tan\ chord\ theorem)$



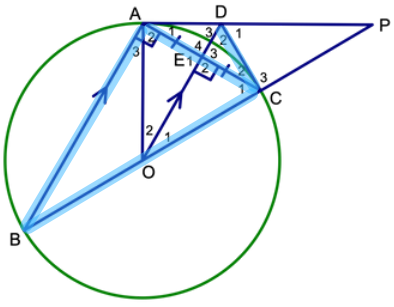
- In the previous question, we proved ADCO was a cyclic quad. We can now use the cyclic quad to find an angle equal to $\hat{C}2$ using angles in same seg.



- Using the parallel lines we proved previously, we can say $\hat{O}2=\hat{A}3$ since they are alternate angles.



- Since OA and OB are radii, we have an isosceles triangle making angle $\hat{B}$ and $\hat{A}3$ equal as they are the base angles opposite the equal sides. Some schools require you to specifically say OA=OB (radii) and some don't. It's always safest to write it, just in case.



- Now we have proven that angle $\hat{C}2$ and $\hat{B}$ are equal, which means the angle between the line DC and the chord AC is equal to the angle opposite the chord (angle B) so DC must be a tangent since tan chord theorem works.

Question source: DBE Grade 11 November 2016 Paper 2

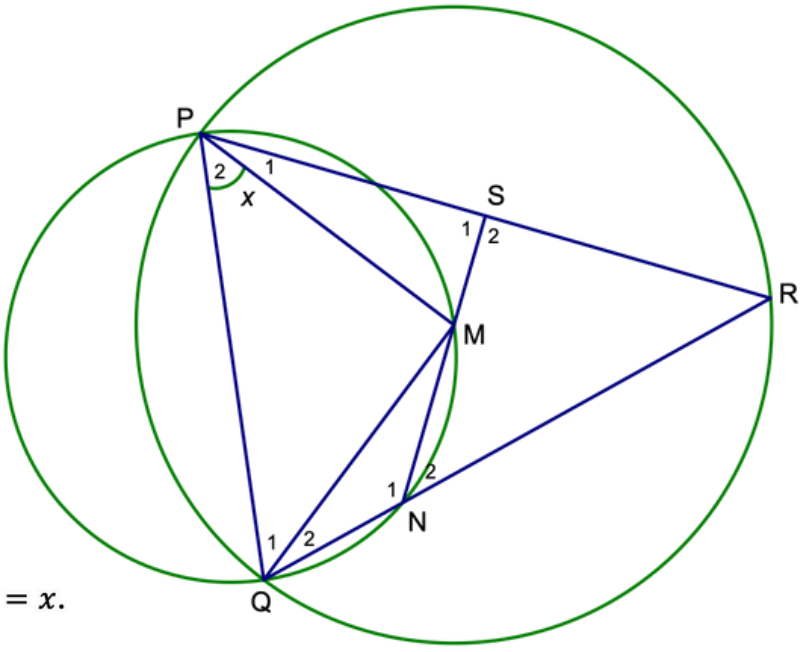
If we are proving a property, we say "converse" with the geometry reason.

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Question Type 4: Multiple circles

Worked example

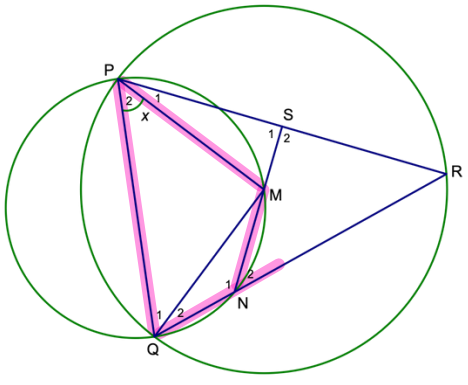
In the diagram, PQ is a common chord of the two circles. The centre, M, of the larger circle lies on the circumference of the smaller circle. PMNQ is a cyclic quadrilateral in the smaller circle. QN is produced to R, a point on the larger circle. NM produced meets the chord PR at S. $\hat{P}_2 = x$.



C Give a reason why $\hat{N}_2 = x$.

Ext ∠ of cyclic quad

- The smaller circle has cyclic quad PMNQ with an exterior angle at N. The exterior angle is equal to the opposite interior angle.

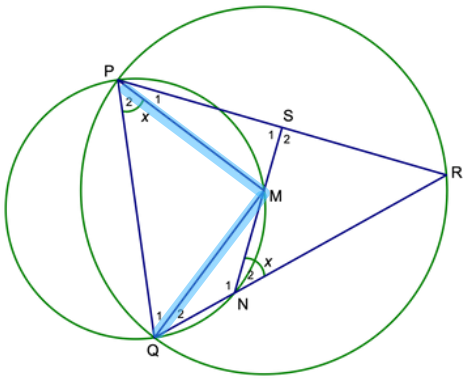


C Write down another angle equal in size to x. Give a reason.

$PM = MQ$ $(radii)$

$\hat{Q}_1 = x$ $(\angle s\ opp = sides)$

- M is the centre of the larger circle. This means PM and MQ are radii, creating an isosceles triangle. The base angles of the isosceles triangle are equal.

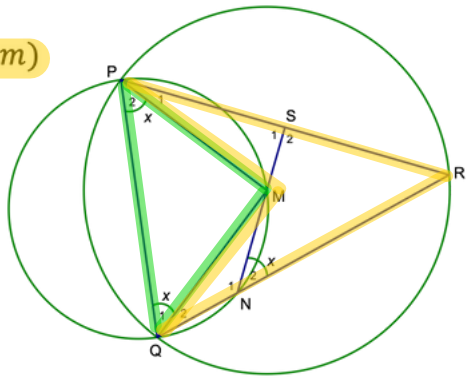


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C Determine the size of $\hat{R}$ in terms of x.

$P\hat{M}Q = 180° - 2x$ $(sum \angle \Delta)$

$\therefore \hat{R} = 90° - x$ $(\angle at\ centre = 2 \times \angle at\ circum)$

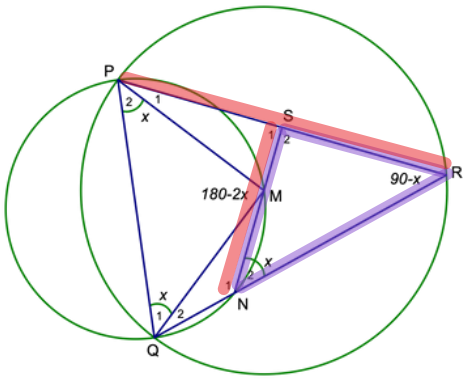


- Since you already have two angles in triangle PMQ, we can find $\hat{M}$ using angles in a triangle add up to 180°.
- Since M is the centre of the larger circle, the angle at M is twice the size of the angle at the circumference (R). Look for the "arrowhead" shape.
- $\therefore \hat{R} = \frac{1}{2}(180° - 2x) = 90° - x$.

C Prove that PS = SR.

$\hat{S}_2 = 90°$ $(sum \angle \Delta)$

$\therefore PS = SR$ $(line\ from\ centre\ \perp\ to\ chord)$



- We can find the value of the last angle in triangle SRN since we already know two of the angles.
- $\therefore \hat{S} = 180° - x - (90° - x) = 90°$
- Since $\hat{S}$=90°, this makes the line MS perpendicular to PR. When a line from the centre is perpendicular to a chord, it bisects the chord making PS=SR.

Question source: DBE Grade 11 November 2015 Paper 2

IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

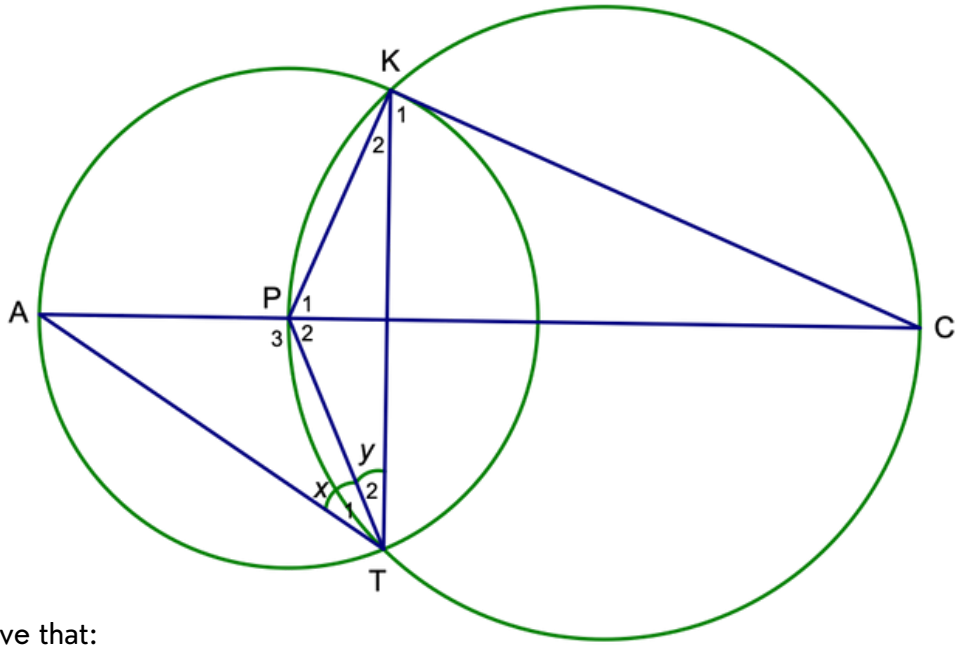
C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

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Worked example

A circle with centre P is drawn to intersect a larger circle at K and T with P on the circumference of the larger circle. CP bisects $\hat{KPT}$ and is produced to A.

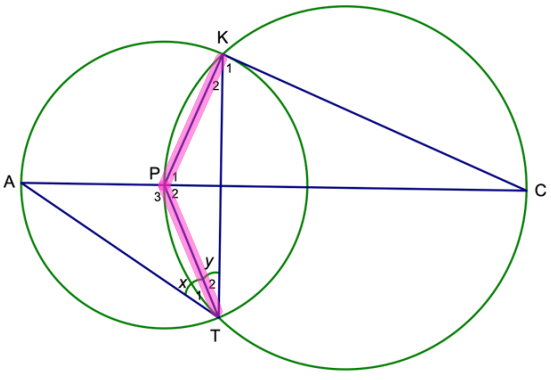


Prove that:

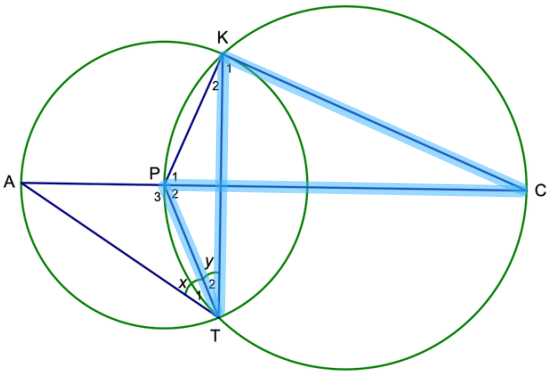
C

a) $\hat{K}_2 = \hat{C}$

$PT = PK$ *(radii)*
$\hat{K}_2 = y$ *(∠s opp = sides)*
$\hat{C} = y$ *(∠s in same seg)*
$\therefore \hat{K}_2 = \hat{C}$



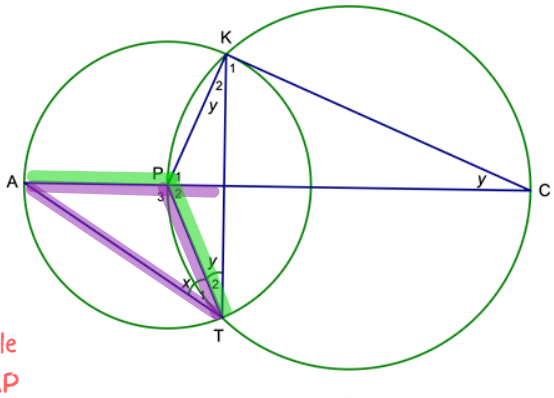
- We are given P is the centre of the smaller circle, which makes PT and PK radii.
- Since ΔKPT is isosceles, the base angles are equal, therefore $\hat{T}2 = \hat{K}2$.
- Now looking at the larger circle, we have the "bowtie" shape and the tips of the bowties are the same size, therefore $\hat{C} = \hat{T}2$.



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C

$\hat{K}_1 = 2\hat{T}_1$



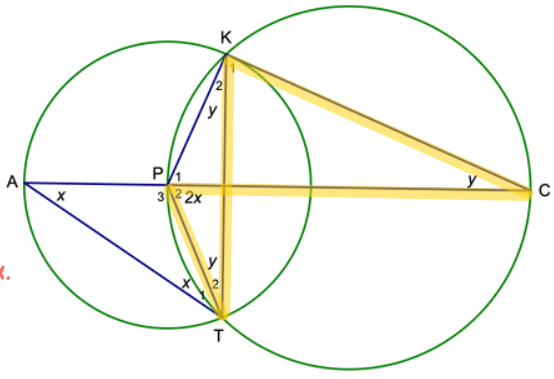
$AP = TP$ $(radii)$

$\hat{A} = x$ $(\angle s\ opp = sides)$

$\hat{K}_2 = 2x$ $(ext\ \angle\ of\ \Delta)$

$\hat{K}_2 = 2x$ $(\angle s\ in\ same\ seg)$

- Using P is the centre of the smaller circle again, but this time we are using radii AP and PT.
- Since ΔAPT is isosceles, the base angles are equal, therefore $\hat{A} = \hat{T}1$.
- $\hat{P}2$ is the exterior angle of ΔKPT, therefore $\hat{P}2 = \hat{A} + \hat{T}1 = x + x = 2x$.
- Now moving to the larger circle, we use the same "bowtie" to find $\hat{K}1 = \hat{P}2 = 2x$.



P

PC is a diameter of the larger circle.

$\hat{P}_2 = 2x$ $(CP\ bisects\ K\hat{P}T)$

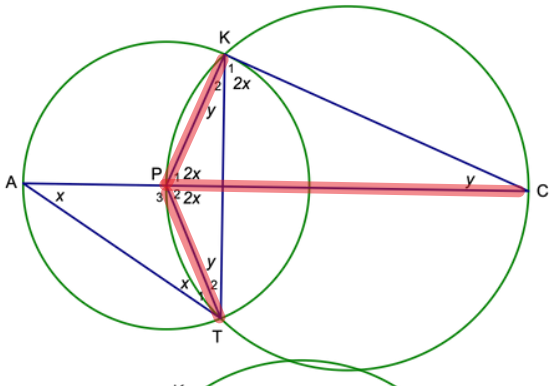
$\therefore 2x + y + 2x + y = 180°$ $(sum\ \angle s\ \Delta)$

$\therefore 2x + y = 90°$

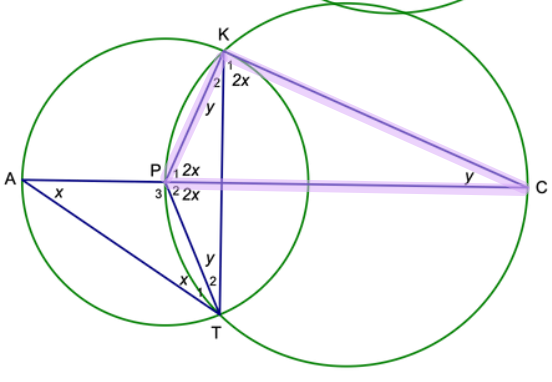
$\therefore P\hat{K}C = 2x + y = 90°$

$\therefore PC\ is\ a\ diameter$ $(converse\ \angle s\ in\ semi - circle)$

- We were told in the initial description that CP bisects KPT, this means $\hat{P}1 = \hat{P}2$.



- Working in ΔKCP, we have all the angles in the triangle, so we can make an equation since all the angles must add to 180°.
- We can then simplify this equation by dividing everything by 2, resulting in the equation $2x + y = 90°$.
- This simplified equation has the same values as $P\hat{K}C$ $(2x + y)$, therefore PKC also adds to 90°.
- Since the angle opposite the line PC is a right-angle, the line PC must be a diameter.



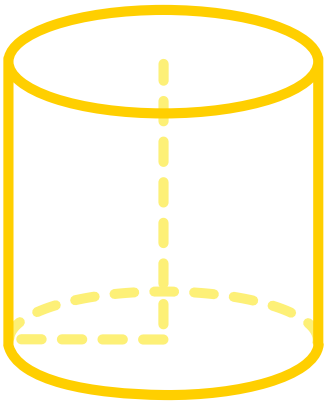
Question source: GDE Grade 11 June 2017 Paper 2

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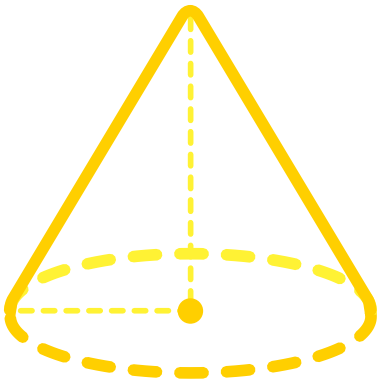
MEASUREMENT

<u>In order to achieve full marks for this section, you need to be able to:</u>

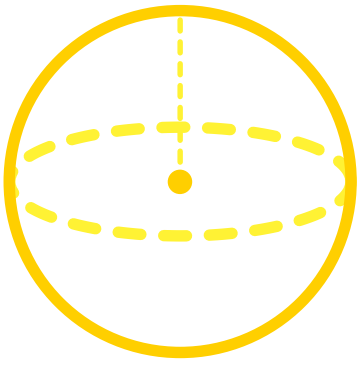
- **Identify** the shapes in order to **choose the correct formula.**
- **Calculate missing values** needed to substitute into the formulae.
- Calculate the **volume** and **total surface area** of **single shapes.**
- Calculate the **volume** and **total surface area** of **compound. shapes**.
- **Increase/decrease** the volume and total surface area **by a scale factor**.



Prism



Pyramid

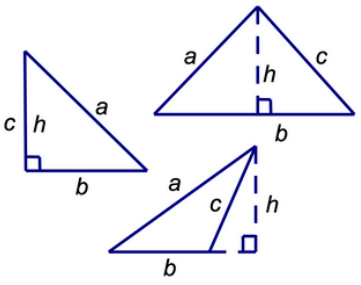
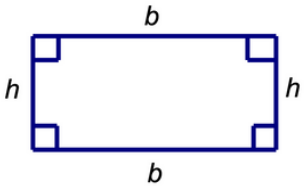
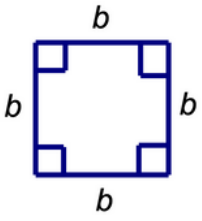
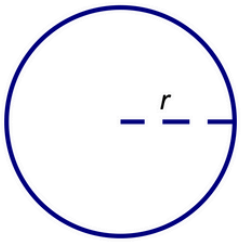
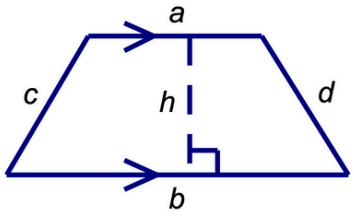


Sphere

This section is **revision** from **Grade 10**. There is no new content taught in Grade 11 and your school might not cover this section in class but may still expect you to know these formulae and how to perform the calculations for your examinations. You will also need to be familiar with this section to help with Calculus (optimisation) next year.

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2D Formulae

	SHAPE	AREA	PERIMETER
Triangle		$\frac{1}{2}b \times h$	$a + b + c$
Rectangle		$b \times h$	$2b + 2h$
Square		b^2	$4b$
Circle		πr^2	$2\pi r$ or πd
Trapezium		$\frac{1}{2}(a + b) \times h$	$a + b + c + d$

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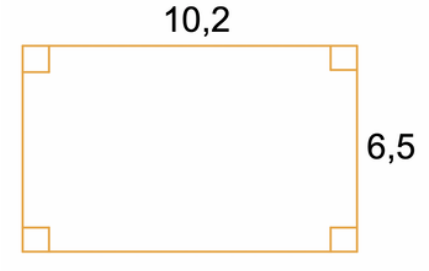
Area & Perimeter

In order to calculate the volume of a three-dimensional shape, you will need to know the area of its base and in order to calculate the total surface area of a three-dimensional shape, you will need to know how to calculate the perimeter of its base.

Worked example

Determine the area and perimeter of the following shapes:

K



- First identify the shape is a rectangle in order to choose the correct formula for Area (A) and Perimeter (P).
- Show substitution of the values into the formula. Do not go straight to the answer, otherwise you risk losing all your marks.
- If the units weren't given in the question you can say "units" in your answer (square units for area).

$A = b \times h$

$= 10{,}2 \times 6{,}5$

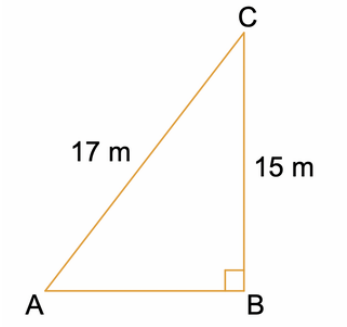
$= 66{,}3\ units^2$

$P = 2b + 2h$

$= 2 \times 10{,}2 + 2 \times 6{,}5$

$= 33{,}4\ units$

K



- In order to calculate the area of a right-angled triangle, we need the two sides that are perpendicular to each other, therefore we need to calculate AB.
- Since this is a right-angled triangle, we can use Pythagoras to calculate the missing side of the triangle. Don't forget to say (pythag) as your geometry reason.
- Once we have AB = 8, we have the perpendicular sides for the Area but also all the sides to calculate the Perimeter.

$AB^2 = 17^2 - 15^2 \quad (pythag)$

$\sqrt{AB^2} = \sqrt{64}$

$AB = 8\ m$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 8 \times 15$

$= 60\ m^2$

$P = a + b + h$

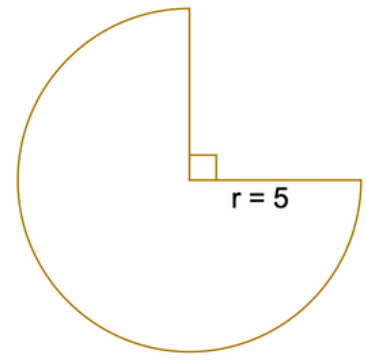
$= 8 + 15 + 17$

$= 40\ m$

PRO-TIP

When calculating the **area of any shape**, you can **ONLY** multiply measurements that are **perpendicular** to each other. If you are missing a perpendicular measure, you need to calculate it first.

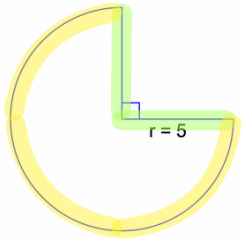
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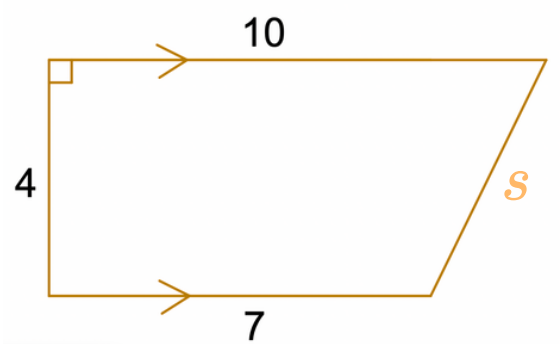


- The right-angle indicates that we have 270 degrees left of the circle
- Therefore, $\frac{270^\circ}{360^\circ} = \frac{3}{4}$ of a circle
- Both the Area (A) and the Perimeter (P) need to be multiplied by $\frac{3}{4}$ but in order to enclose the shape for the Perimeter, you also need to add the radius twice.
- In order to calculate the Perimeter, you can start wherever you want on the outline of the shape but you have to come back to where you started.

$$A = \frac{3}{4}\pi r^2$$
$$= \frac{3}{4}\pi(5)^2$$
$$= 58{,}90\ units^2$$

$$P = \frac{3}{4}(2\pi r) + r + r$$
$$= \frac{3}{4}\big(2\pi(5)\big) + 5 + 5$$
$$= 33{,}56\ units$$



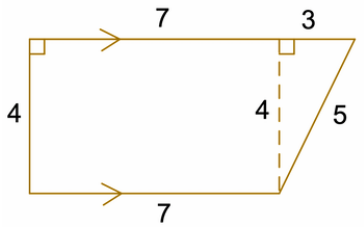


- This shape is a trapezium. Either we can use the formula for a trapezium, or we can break the shape up into a rectangle and a right-angled triangle, which are more familiar shapes.
- To calculate the Perimeter, we are missing a side of the trapezium (labelled s).
- Once the trapezium is broken into a rectangle and right-angled triangle we can see that the shorter sides of the triangle are 3 and 4 so we can use Pythagoras to calculate the hypotenuse.

$$A = \frac{1}{2}(a+b) \times h$$
$$= \frac{1}{2}(7+10) \times 4$$
$$= 34\ units^2$$

OR

$$A = 7 \times 4 + \frac{1}{2}(3 \times 4)$$
$$= 28 + 6$$
$$= 34\ units^2$$

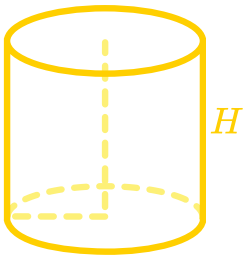
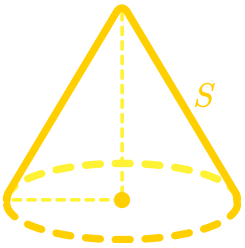
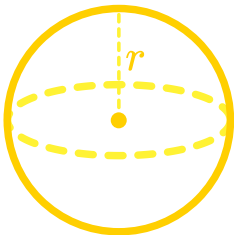


$$s^2 = 3^2 + 4^2 \quad (pythag)$$
$$\sqrt{s^2} = \sqrt{25}$$
$$s = 5\ units$$

$$P = a + b + h + s$$
$$= 7 + 10 + 4 + 5$$
$$= 26\ units$$

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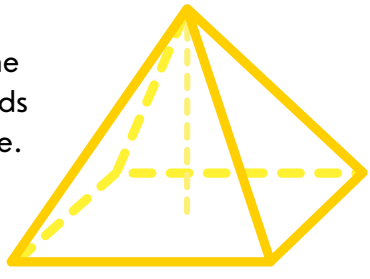
3D Formulae

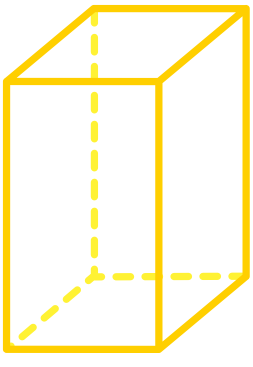
	SHAPE	VOLUME	TOTAL SURFACE AREA
Prism		$(area\ of\ base) \times H$	$2(area\ of\ base) + (perimeter\ of\ base) \times H$
Pyramid		$\frac{1}{3}(area\ of\ base) \times S$	$(area\ of\ base) + \frac{1}{2}(perimeter\ of\ base) \times S$
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$

$H = perpendicular\ height$ $S = slant\ height$ $r = radius$



A **pyramid** is a three-dimensional shape with sloping sides that meet in a point at the top. Pyramids are named after their bases e.g. rectangular pyramid and cone (a pyramid with a circular base). Pyramids can be identified by having a "hat" shape.





A **prism** is a three-dimensional shape which has identical faces that are parallel to each other. Prisms are also named by these identical faces (bases) e.g. rectangular prism, triangular and cylinder (a prism with a circular base). Prisms can be identified by having a "box" shape.

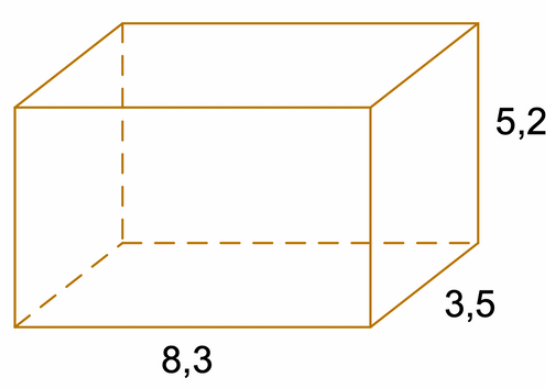
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Volume

Worked example

Determine the volume of the following shapes:

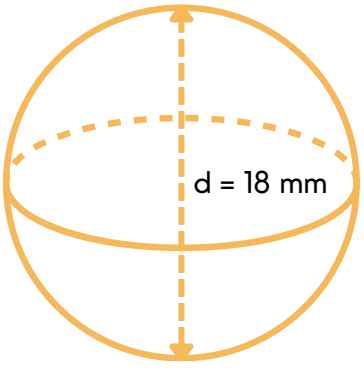
K



- Substitute the given values into the formula for a prism
- Round off your decimals to 2 places, unless told otherwise.
- The type of unit wasn't given so we say units cubed.

$$V = (area\ of\ base) \times H$$
$$= (8{,}3 \times 3{,}5) \times 5{,}2$$
$$= 151{,}06\ units^3$$

K



- To calculate the volume of a sphere we need the radius.
- In the diagram we were given the diameter, therefore we halve the diameter to get the length of the radius.
- Show substitution into the formula before writing down the answer.

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(9)^3$$
$$= 3053{,}63\ mm^3$$

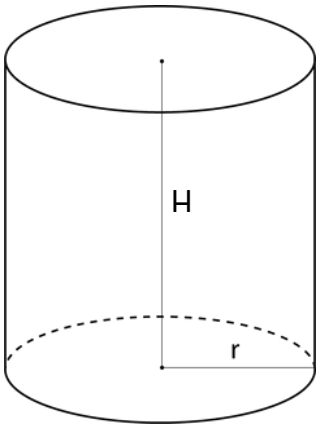
NOTE

- When rounding to 2 decimal places, take a look at the 3rd decimal.
- If the 3rd decimal is less than 5, round down.
- If the 3rd decimal is 5 or more, round up.
- Do not domino your rounding (unless you are rounding up a 9).

e.g. 3053.628059 (8 is 5 or more so round up)

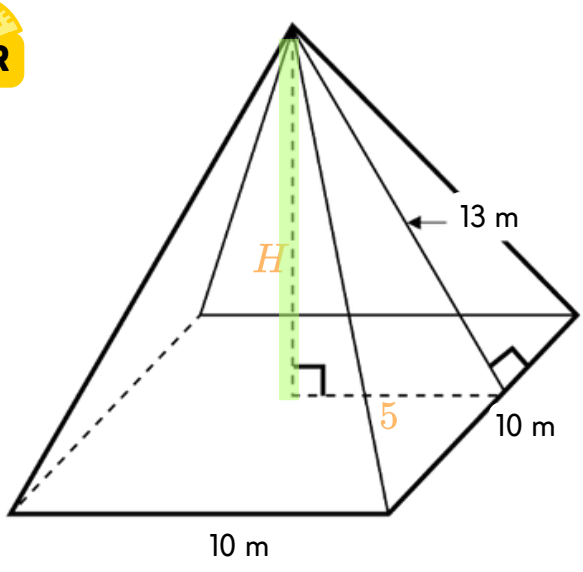
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Given the height is double the diameter and the radius is 3 cm.



- Remember, we were asked to calculate the volume at the beginning of this worked example.
- This shape is a prism ("box" not "hat")
- Since it has a circular base, we call this shape a cylinder.
- To calculate the volume (V) of a cylinder, we need the height (H) and the radius (r).
- We are given the radius, and instead of being given the height we are told the relationship between the radius and height is H = 2 x diameter (and diameter = 2 x radius), therefore H = 4 x radius = 4 x 3 = 12 cm
- Now we can substitute these values into the prism formula to calculate the volume (using the formula for the area of a circle as the base).
- The units given was cm, therefore we write our rounded answer with cm cubed.

$$V = (area\ of\ base) \times H$$
$$= (\pi r^2) \times H$$
$$= \pi(3)^2 \times 12$$
$$= 339{,}29\ cm^3$$



- First, we need to identify that this is a pyramid shape since all the sides meet at a singular point (it has a "hat" shape, not a "box" shape).
- To calculate the volume of a pyramid, we need the perpendicular height, which is found from the middle of the base to the tippy top of the highest point (highlighted on the diagram).
- To calculate the perpendicular height, we need two sides of the right-angled triangle.
- Since the base is a square (both sides are equal in length), we know the middle of the base is 5m from each side.
- Now we have two sides of the triangle to calculate the missing side using Pythagoras.

$$H^2 = 13^2 - 5^2 \quad (pythag)$$
$$\sqrt{H^2} = \sqrt{144}$$
$$H = 12\ m$$

A = area of base

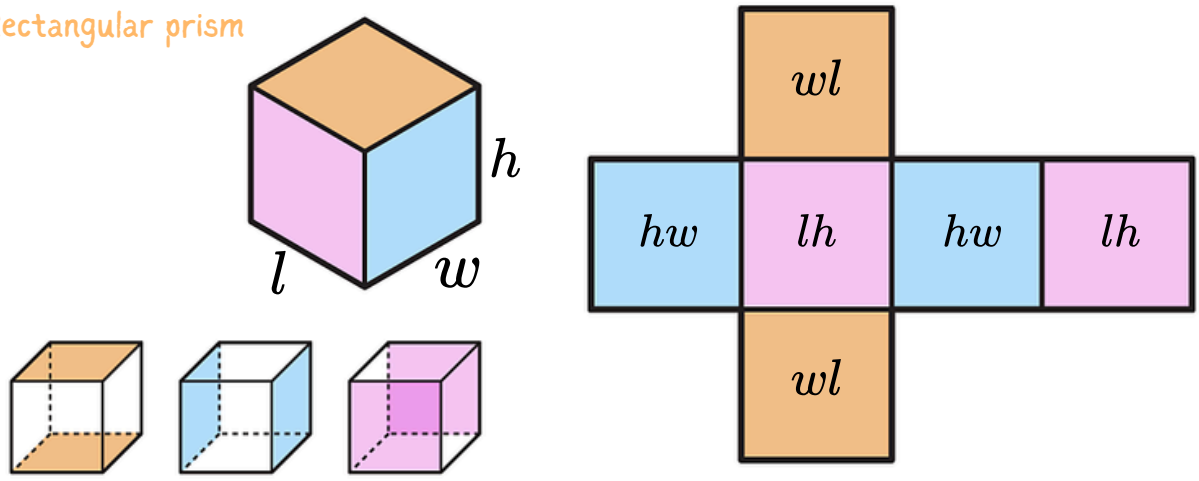
$$V = \frac{1}{3}(A) \times H$$
$$= \frac{1}{3}(10 \times 10) \times 12$$
$$= 400\ m^3$$

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Total Surface Area

The **Total Surface Area** is the sum of all the faces of the outside surfaces. If you needed to wrap a present perfectly, without any wrapping overlapping, it's the amount of wrapping paper you would need. Below are the nets of some prisms to help you visualise all the faces of each shape.

Rectangular prism

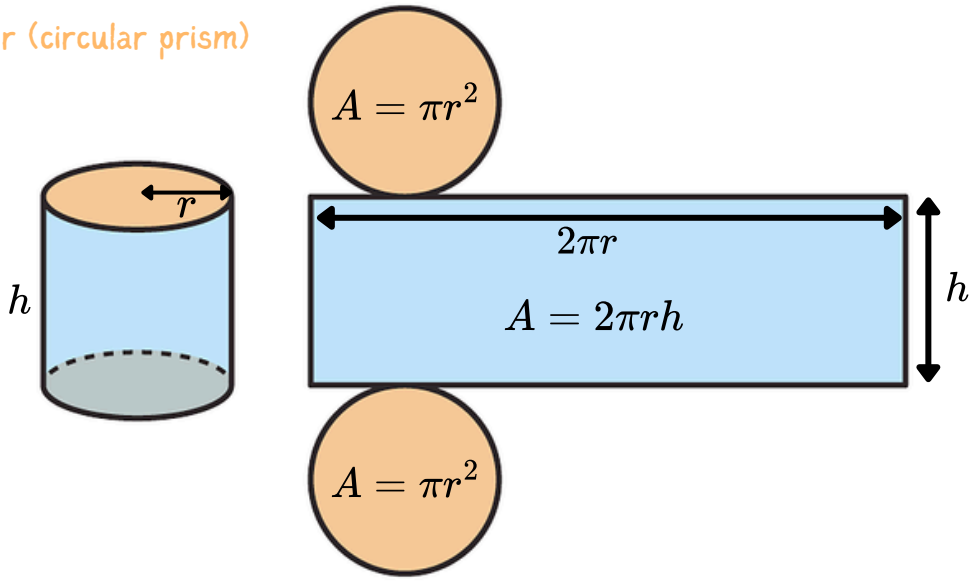


$$A = 2(area\ of\ base) + (perimeter\ of\ base) \times h$$

$$= 2(wl) + (2w + 2l) \times h$$

$$= 2wl + 2wh + 2lh$$

Cylinder (circular prism)



$$A = 2(area\ of\ base) + (perimeter\ of\ base) \times h$$

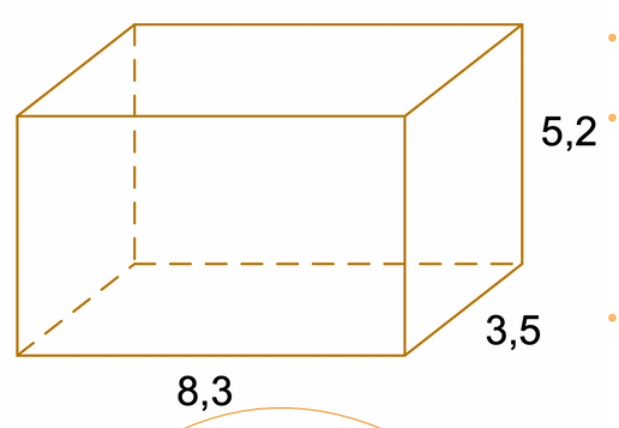
$$= 2\pi r^2 + 2\pi rh$$

Image source: Perimeter, area, volume - KS3 Maths - BBC Bitesize BBC News. Available at: https://www.bbc.co.uk/bitesize/topics/zrf3cdm (Accessed: 5 September 2025)

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Worked example

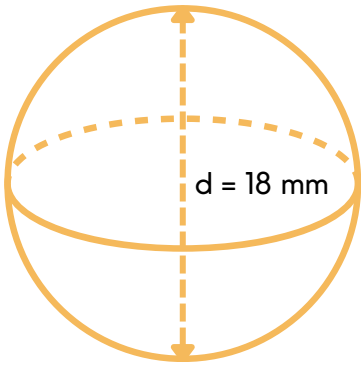
Determine the total surface area of the following shapes:



- First we identify that this shape is a rectangular prism.
- To calculate the total surface area (TSA) of a prism you can either use the formula, or you can calculate the area (A) of each face and add these areas together.
- Since the units weren't given, we use units squared for our answer.

P = perimeter of base

$$\begin{aligned} TSA &= 2(A) + (P) \times H \\ &= 2(b \times h) + (2b + 2h) \times H \\ &= 2(8{,}3 \times 3{,}5) + (2 \times 8{,}3 + 2 \times 3{,}5) \times 5{,}2 \\ &= 180{,}82\ units^2 \end{aligned}$$

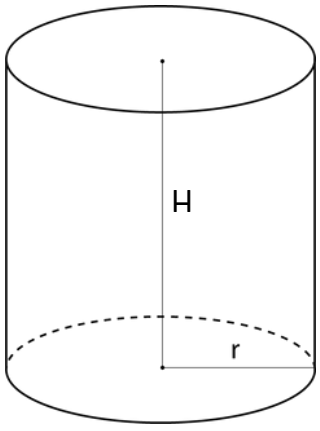


- To calculate the TSA of a sphere, we need the radius.
- In the diagram we were given the diameter, therefore we can halve the diameter to get the length of the radius (r).
- Show substitution into the formula before writing down the answer.

$$\begin{aligned} TSA &= 4\pi r^2 \\ &= 4\pi(9)^2 \\ &= 1017{,}88mm^2 \end{aligned}$$

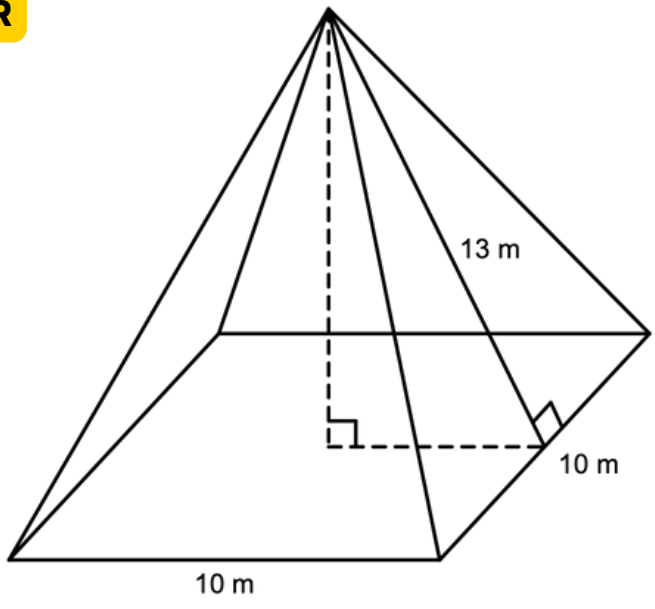
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Given the height is double the diameter and the radius is 3 cm.



- This is the same cylinder as on page 99 (refer to that page to see how to calculate the height).
- Now we can substitute H = 12 and r = 3 into the prism formula for TSA (using the formula for the area of a circle as the base).
- A = area of base and P = perimeter of base
- The units given was cm, therefore we write our rounded answer with cm squared.

$$\begin{aligned} TSA &= 2(A) + (P) \times H \\ &= 2\pi r^2 + 2\pi r \times H \\ &= 2\pi(3)^2 + 2\pi(3) \times 12 \\ &= 282{,}74\ cm^2 \end{aligned}$$



- In order to calculate the TSA of a pyramid, we need the slant height (S).
- In this diagram, the slant height is given as 13 m.
- Since the shape of the base is a square, we use the area (A) and perimeter (P) formulae for a square.
- Show substitution of the values from the diagram into the formula.
- All the values were given, so we do not need to calculate any missing values separately.
- The units given was metres, so our answer is in metres squared.

$$\begin{aligned} TSA &= (A) + \tfrac{1}{2}(P) \times S \\ &= b^2 + \tfrac{1}{2}(4b) \times S \\ &= 10^2 + \tfrac{1}{2}(4 \times 10) \times 13 \\ &= 360\ m^2 \end{aligned}$$

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Compound Shapes



A compound (or composite) shape combines two or more three-dimensional shapes together, for example: a house (rectangular prism with a triangular prism) or ice-cream (cone with a hemisphere).

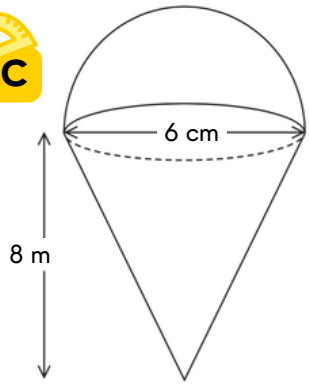
To calculate the volume of a compound shape, you can simply calculate each shape separately, and then add (or sometimes subtract) the volumes together.

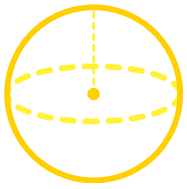
For total surface area, you need to pay close attention to how the shapes are joined as the faces where the shapes are joined together are not included in the final total surface area. If you consider the shapes are "glued" together, you need to subtract the "glue" areas from your final total. If you were to dip the entire composite shape in a bucket of paint, the "glued" area wouldn't get painted.

Worked example

Determine the volume and total surface area of the following shapes:

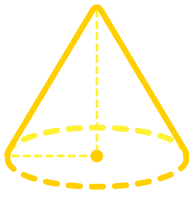
C





Sphere

$$V = \frac{4}{3}\pi r^3$$



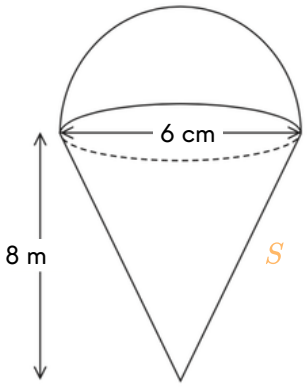
Cone

$$V = \frac{1}{3}(A) \times H$$

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \frac{1}{3}(\pi r^2) \times H$$
$$= \frac{1}{2}\left(\frac{4}{3}\pi(3)^3\right) + \frac{1}{3}(\pi(3)^2) \times 8$$
$$= 131{,}95\ cm^3$$

- To calculate the volume of the entire shape we break it down into a hemisphere (half a sphere) and a cone.
- The diameter of the cone and hemisphere is shared and given as 6cm, therefore the radius (r) is 3cm.
- The perpendicular height (H) of the cone is given as 8m.
- To calculate the volume of a hemisphere, we multiply the sphere formula by $\frac{1}{2}$.
- Substitute the values r = 3 and H = 8 into the formulae.
- Type the entire calculation into your calculator in one step to avoid any rounding errors.

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$$S^2 = 3^2 + 8^2 \quad (pythag)$$

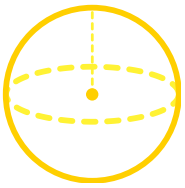
$$\sqrt{S^2} = \sqrt{73}$$

$$S = \sqrt{73}$$

$$TSA = \frac{1}{2}(4\pi r^2) + \frac{1}{2}(2\pi r) \times S$$

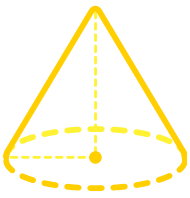
$$= \frac{1}{2}(4\pi(3)^2) + \frac{1}{2}(2\pi(3)) \times \sqrt{73}$$

$$= 137{,}07\ cm^2$$



Sphere

$$TSA = 4\pi r^2$$



Cone

$$TSA = (A) + \frac{1}{2}(P) \times H$$

- To calculate the TSA of the cone, we need to calculate the slant height (S).
- A horizontal (the radius) and a vertical line (the perpendicular height) will always meet at a right-angle, therefore we can use Pythagoras to find the slant height (S).
- Keep the value of S as a surd instead of rounding off to get the correct answer.
- Substitute the same value for r as discussed in the previous question (r=3 cm).
- A = area of base and P = perimeter of base
- Here we did not add the area of the base for the cone, since that is where the hemi-sphere is joined to the cone. If we were to use the formula as is, which includes adding the area of the base, we would need to subtract it again as this is the "glue" between the two shapes.

PRO-TIP

If you need to calculate a missing side before you can determine the V or TSA, **do NOT round off** the decimal places. Rather **store** answers on your calculator to get an accurate answer.

How to Store Answers: specifically on a CASIO fx-991ZA or fx-82ZA Plus II



1. The value you want to store, must be the current value on your screen.
2. Press **SHIFT** then **STO** (shown above the RCL key)
3. Now, press one of the keys with the letter you want (shown in red above some of the keys, e.g., D is above sin).

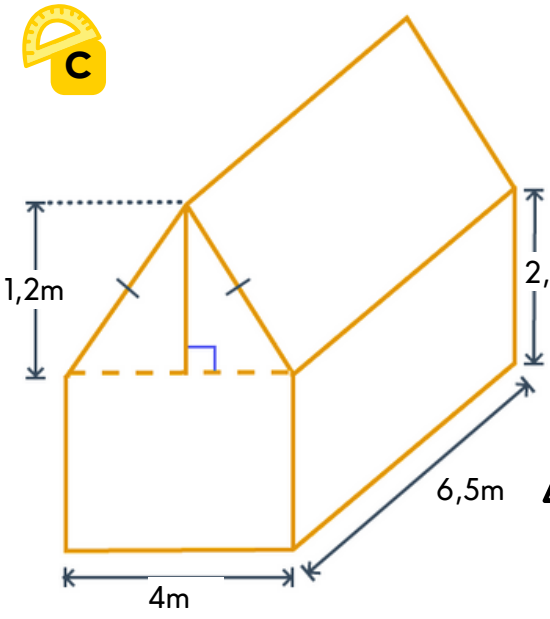
- The screen then briefly shows ANS->A, confirming the value is now in memory A (or B, C, D, etc.).
- Repeat steps 1–2 for any other answers, choosing a new letter each time. If you select a letter used previously, it will override the memory and save the new value over the old value.
- To use the stored value, click **ALPHA** and the desired letter.

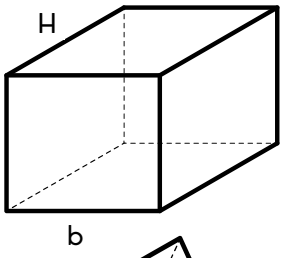


If you wish to CLEAR all memories **SHIFT CLR** → :Memory → =Wipes every lettered memory

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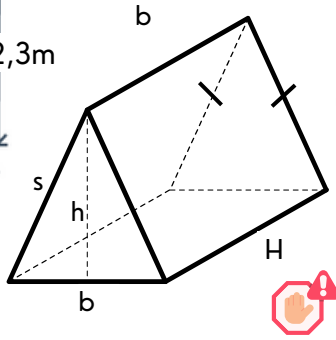
C





Rectangular Prism

$$TSA = 2(A) + (P) \times H$$
$$V = (A) \times H$$



Trigangular Prism

$$TSA = 2(A) + (P) \times H$$
$$V = (A) \times H$$

Be careful! Students frequently mistake the triangular prism for a pyramid.

$$V = (4 \times 2{,}3) \times 6{,}5 + \frac{1}{2}(4 \times 1{,}2) \times 6{,}5$$
$$= 75{,}4\ m^3$$

$$s^2 = 1{,}2^2 + 2^2 \quad (pythag)$$
$$\sqrt{s^2} = \sqrt{5{,}44}$$
$$s = \frac{2\sqrt{34}}{5}\ cm$$

$$TSA_{base} = 2(bh) + (2b + 2h) \times H + 2\left(\frac{1}{2}bh\right)$$
$$= 2(4 \times 2{,}3) + (2 \times 4 + 2 \times 2{,}3) \times 6{,}5$$
$$= 100{,}3\ m^2$$

$$TSA_{top} = (2s + b) \times H$$
$$= 2\left(\frac{1}{2} \times 4 \times 1{,}2\right) + 2\left(2 \times \frac{2\sqrt{34}}{5} + 4\right) \times 6{,}5$$
$$= 117{,}44\ m^2$$

$$TSA_{total} = TSA_{base} + TSA_{top} - 2(b \times H)$$
$$= 100{,}3\ldots + 117{,}44\ldots - 2(4 \times 6{,}5)$$
$$= 165{,}74\ m^2$$

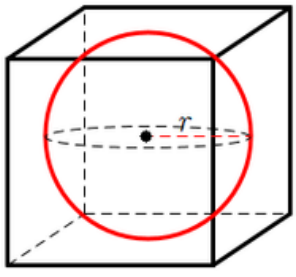
- This shape has a rectangular prism as the base and a triangular prism as the top to resemble a shape of a house.
- We have all the values we need to calculate the volume, which we can calculate separately and add together to get the total volume.
- Calculating TSA for this question is a bit more complicated so we can rather calculate the TSA of the base and the top separately and added together to get the total.
- We we add these areas together, we need to take away the "glue" twice – once for the "glue" on the base and once for the "glue" on the top.
- To calculate the TSA we need the perimeter of the triangular base of the top prism.
- Since the triangle is isosceles, we can split the triangle in half to create a right-angled triangle with a length of 2 m (half the total base), a height of 1.2 m (given) and the missing side (s).

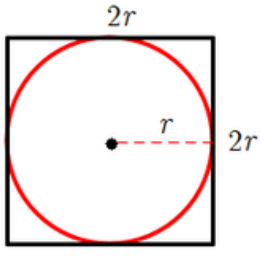
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P A high end cocktail bar is creating a spherical ice-cube mould from copper. If the mould uses $476{,}4\ cm^3$ of copper, what is the radius of largest ice cube that can be created?

$$
\begin{aligned}
V &= b^3 - \tfrac{4}{3}\pi r^3 \\
476{,}4 &= (2r)^3 - \tfrac{4}{3}\pi r^3 \\
476{,}4 &= 8r^3 - \tfrac{4}{3}\pi r^3 \\
476{,}4 &= r^3\left(8 - \tfrac{4}{3}\pi\right) \\
124{,}999\ldots &= r^3 \\
r &= 5\ cm
\end{aligned}
$$

- The largest sphere will be inscribed inside the cube. This means the sphere touches all the faces of the cubes (as shown in the diagram below).
- Since the sphere fills the cube, the diameter (2r) of the sphere is the same as the length of the cube.
- The volume of metal will be the volume of the cube, taking away (subtracting) the volume of the sphere.
- We set this volume equal to $476{,}4\ cm^3$ and solve for r.
- To isolate r, we factorise by taking out the highest common factor of r^3.
- To calculate r, we cube root both sides of the equation and round the answer to 2dp, which in this case rounded off to 5.00 so we can simply write 5.





IMPORTANT: WORKED EXAMPLE KEY

K **Knowledge**: These are meant to be your guaranteed marks.

R **Routine:** Familiar procedures and simple calculations.

C **Complex procedure:** Multiple step calculations that require higher-order thinking.

P **Problem solving:** Problems that are unfamiliar (unseen).

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INFORMATION SHEET

Familiarise yourself with the placement of the formulae that are given on the information/formula sheet by using this labelled formula sheet.

AS SEEN IN THE GRADE 12 CHEAT SHEET

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ quadratic formula

$A = P(1+ni)$ simple growth

$A = P(1-ni)$ simple decay/straight line depreciation

$A = P(1-i)^n$ compound decay/reducing balance depreciation

$A = P(1+i)^n$ compound growth

$T_n = a + (n-1)d$ arithmetic: term formula

$S_n = \frac{n}{2}[2a + (n-1)d]$ arithmetic: sum formula

$T_n = ar^{n-1}$ geometric: term formula

$S_n = \frac{a(r^n - 1)}{r-1}$; $r \neq 1$ geometric: sum formula

$S_\infty = \frac{a}{1-r}$; $-1 < r < 1$ sum to infinity

$F = \frac{x[(1+i)^n - 1]}{i}$ future value annuities

$P = \frac{x[1-(1+i)^{-n}]}{i}$ present value annuities

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ derivative by first principles

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ distance formula

$M\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$ midpoint formula

$y = mx + c$ straight line equations

$y - y_1 = m(x - x_1)$ straight line equations

$m = \frac{y_2 - y_1}{x_2 - x_1}$ gradient formula

$m = \tan\theta$ angle of inclination

$(x-a)^2 + (y-b)^2 = r^2$ equation of a circle

In ΔABC: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ sine rule

$a^2 = b^2 + c^2 - 2bc.\cos A$ cosine rule

area $\Delta ABC = \frac{1}{2}ab.\sin C$ area rule

$\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha.\cos\beta - \sin\alpha.\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta$

trig compound angle identities

$$\cos 2\alpha = \begin{cases} \cos^2\alpha - \sin^2\alpha \\ 1 - 2\sin^2\alpha \\ 2\cos^2\alpha - 1 \end{cases}$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

trig double angle identities

$\bar{x} = \frac{\Sigma x}{n}$ mean of a set of data values

$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$ variance of a set of data values

$P(A) = \frac{n(A)}{n(S)}$ probability of event A

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ probability identity

$\hat{y} = a + bx$ equation of least squares regression line

$b = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2}$ gradient of least squares regression line

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